Chapter 3

Cracks in electroelastic materials

3.1 Introduction

With increasingly wide application of piezoelectric materials in practical engineering, the study of crack problems in piezoelectric solids has received much interest. Early in 1975, Barnett and Lothe [1] generalized Stroh’s six-dimensional framework [2] to an eight-dimensional formalism to treat dislocations and line charges in anisotropic piezoelectric insulators. Deeg [3] developed a method of analysing piezoelectric cracks by generalizing a distributed dislocation method to include piezoelectric effects. Zhou et al. [4] proposed a multipole function representation and used an analogy theorem to obtain electroelasto-magnetic field equations for a finite piezoelectric body with defects. Sosa and Pak [5] studied a three-dimensional eigenfunction analysis of a semi-infinite crack in a piezoelectric material. Based on the complex potential approach, Sosa [6] obtained asymptotic expressions for coupled electroelastic fields at the tip of a crack in an infinite piezoelectric solid. The asymptotic expressions were obtained as the limiting case of an elliptical hole. Dunn [7] derived the electroelastic fields in and around an elliptical inclusion in a piezoelectric solid through use of the equivalent inclusion idea of Eshelby [8]. McMeeking [9] derived a path-independent integral for the problem of a crack filled with a conducting medium in an elastically deformable dielectric material that is subjected to both mechanical and electric loads. Suo et al. [10] studied cracks both in piezoelectric materials and on interfaces between piezoelectric and other materials such as metal electrodes or polymer matrices. Recently, Qin and Yu [11] studied the logarithmic singularities of crack-tip fields. Fracture and damage behaviours of a cracked piezoelectric solid under coupled thermal, mechanical and electrical loads have also been reported [12-16]. Here, we provide only a preliminary insight into the research in this field and do not provide a complete review. The readers may refer to the book [17] and articles [10,18,19] for more information in this field.

In this chapter, we consider some typical fracture problems in the mathematical theory of electroelasticity relating to cracks occurring commonly in practical engineering. The crack-tip singularities of a semi-infinite crack are investigated first. Then, solutions are provided for a semi-infinite crack, a Griffith crack opened under constant traction-charge, anti-plane crack, cracks in half-plane, cracks in bimaterials, interface cracks with a closed crack-tip model, macro- and micro-crack interactions. Examination of the problems of crack kink and crack deflection follows. Finally, the three-dimensional problem of elliptic cracks is discussed in detail. Some numerical results are presented to illustrate the application of the formulations discussed above.
3.2 Singularity of crack-tip fields for a semi-infinite crack

3.2.1 A general solution for crack-tip fields

The singularity of stress and electric displacement near the tip of a semi-infinite crack has been studied by Ting [20,21] for anisotropic elasticity, Qin and Yu [11] for electroelastic problems, and Yu and Qin [12] for thermoelectroelastic problems. In this section we follow the results given in [11].

Consider a semi-infinite crack along the negative x-axis. The SED singularities at tips of the crack can be determined by assuming the form of $f$ in eqns (1.52) and (1.57) in the following form [20]

$$ f(z_j) = \frac{z_j^{1-\eta}}{1-\eta} \quad (3.1) $$

where $\eta=a+ib$ is a complex constant with $a$ and $b$ being two real constants. Substituting eqn (3.1) into eqns (1.52) and (1.56) yields

$$ U = 2 \text{Re} \left\{ A \frac{z_\alpha^{1-\eta}}{1-\eta} q \right\} \quad (3.2) $$

$$ \Pi_2 = 2 \text{Re} \left\{ B \frac{z_\alpha^{-\eta}}{q} \right\} \quad (3.3) $$

If we use the polar coordinate system $(r, \theta)$ defined in Fig. 2.2, the complex variable $z_\alpha$ becomes

$$ z_\alpha = r(\cos \theta + p_\alpha \sin \theta) \quad (3.4) $$

We see that with the assumption of eqn (3.1) the SED given by eqn (3.3) is of the order $r^{-\eta}$. It is obvious that the SED is singular if the real part of $\eta$, i.e. $a$, is positive. For the potential energy to be bounded at the crack tip, we require that $a<1$. So we focus our attention on the internal $0<a<1$. Using the traction-charge free condition on the crack surfaces and noting that $z=r$ when $\theta=0$ and $z=re^{i\pi}$ when $\theta=\pm \pi$, we know that

$$ \Pi_2(\pi) = -r^{-a}(r^{-\theta} e^{-i\theta} Bq + r^{\theta} e^{i\theta} \overline{Bq}) = 0 \quad (3.5) $$

$$ \Pi_2(-\pi) = -r^{-a}(r^{-\theta} e^{i\theta} Bq + r^{\theta} e^{-i\theta} \overline{Bq}) = 0 \quad (3.6) $$

or in matrix form

$$ X(\eta)Q = 0 \quad (3.7) $$

where $Q=\begin{bmatrix} Bq & \overline{Bq} \end{bmatrix}^T$. To obtain a nontrivial solution for $Q$ we should let the determinant of $X$ vanish, i.e.
\[ |X| = 0 \] (3.8)

where the symbol \(|\cdot|\) denotes the determinant, which leads to

\[ b = 0, \quad (1 - e^{4 \pi a})^4 = 0. \] (3.9)

The solution of eqn (3.9) reads

\[ a = \frac{(1 - n)}{2}, \quad n = 0, 1, 2, \ldots. \] (3.10)

Hence, to satisfy \(0 < a < 1\), we should take \(n=0\), which is a fourfold root of eqn (3.9). The elastic displacement and electric potential, \(U\), and stress and electric displacement, \(\Pi\), may now be written in their asymptotic forms by combining eqns (3.2), (3.3) and (3.10) as

\[ U = 4r^{1/2} \text{Re}[A \langle \cos \theta + p_n \sin \theta \rangle^{1/2}]q], \] (3.11)

\[ \Pi_2 = 2r^{-1/2} \text{Re}[B \langle \cos \theta + p_n \sin \theta \rangle^{-1/2}]q]. \] (3.12)

### 3.2.2 Anisotropic degeneracy

The analyses presented so far tacitly assume that the eigenvalues \(p\)'s are distinct. When one of the \(p\)'s is a double root, one may or may not have four independent functions in eqns (3.11) and (3.12), and a set of additional solutions is required. It is not difficult to see that if eqns (3.11) and (3.12) are the solutions corresponding to the double root \(p_i\), so are

\[ U^{[21]} = 4r^{1/2} \text{Re} \left[ \frac{d}{dp_i} \langle A \langle \cos \theta + p_n \sin \theta \rangle^{1/2} \rangle q \right], \] (3.13)

\[ \Pi_2^{[21]} = 2r^{-1/2} \text{Re} \left[ \frac{d}{dp_i} \langle B \langle \cos \theta + p_n \sin \theta \rangle^{-1/2} \rangle q \right]. \] (3.14)

where \(dA/dp_i\) and \(dB/dp_i\) can be obtained by differentiating eqns (1.54) and (1.58) with respect to \(p_i\):

\[ \frac{d}{dp_i}(Da) = 0, \] (3.15)

\[ \frac{d}{dp_i}(Db) = \frac{d}{dp_i}[(R^T + pT)a] \] (3.16)

where \(D = Q + p(R + R^T) + p^2T\) is a 4\times4 matrix. The new solutions (3.13) and
(3.14) exist if the following equation holds true [22]:
\[ \frac{d^n}{d\eta^n} \left| \mathbf{D} \right|_{\eta=0} = 0, \quad n = N - M \]  

(3.17)

where \( N \) and \( M \) are the order and rank of \( \mathbf{D} \), respectively. However, it is found that the order of singularity is not changed in the presence of the new solution (3.14).

3.2.3 New solutions for \( \eta \) being a multiple root

If \( \eta \) is a multiple root of eqn (3.8), the components of \( \mathbf{q} \) may not be unique and one has to find other independent solutions. For a root of multiplicity \( m \), the new solutions are given by
\[ \Pi_2(\eta) = 2 \text{Re}[\mathbf{B}(z_a^{(0)} z_a^{(0)} z_a^{(0)} z_a^{(0)})^T], \quad i = 1, 2, \cdots, m-1 \]  

(3.18)

where \( z_a^{(0)} = z_a^- \left( - \ln z_a + \frac{\partial}{\partial \eta} \right) q_a \). Likewise, new solutions exist if
\[ \frac{d^n}{d\eta^n} \left| \mathbf{X} \right|_{\eta=1/2} = 0, \quad n = 8 - M \]  

(3.19)

holds true. Here \( M \) is the rank of matrix \( \mathbf{X} \). Since \( \eta=1/2 \) is a fourfold root (see eqn (3.9)), the SED singularities at the tips of a semi-infinite crack must occur in one of the following cases
\[ \Pi_2(r) = \begin{cases} 
O(r^{1/2}), & \text{only satisfying } \mathbf{X} \mathbf{Q} = 0, \\
O(r^{1/2} \ln r), & \text{satisfying } \frac{d}{d\eta} \mathbf{X} |_{\eta=1/2} = 0, \\
O(r^{1/2} \ln^2 r), & \text{satisfying } \frac{d^2}{d\eta^2} \mathbf{X} |_{\eta=1/2} = 0, \\
O(r^{1/2} \ln^3 r), & \text{satisfying } \frac{d^3}{d\eta^3} \mathbf{X} |_{\eta=1/2} = 0.
\end{cases} \]  

(3.20)

For a semi-infinite crack in an anisotropic piezoelectric medium, it is, therefore, shown that both stress and electric displacement at the crack tip may be in the order of \( r^{1/2} \), or \( r^{-1/2} \ln r, \ r^{-1/2} \ln^2 r, \ r^{-1/2} \ln^3 r \), as \( r \to 0 \), where \( r \) is the distance from crack tip to field point, depending on which boundary conditions are satisfied.

3.3 Reduction of a crack problem to a Hilbert problem

The crack problems discussed in the previous chapter can be reduced to the solution of a Hilbert problem. For a description of the Hilbert problem, see Muskhelishvili [23].

Suppose that we have a series of colinear cracks occupying the segments
of the $x$-axis $X$, and that the cracks are opened by application of traction on the crack faces. If we write

$$L = L_1 \cup L_2 \cup \cdots \cup L_n,$$

then the boundary conditions are the same as those of eqns (2.32)-(2.34) except that $|y| > c$ should be replaced by $x \in X - L$ and $|y| < c$ replaced by $L$. As was done in Section 2.4, using the subscripts “(1)” and “(2)” to identify variables associated with the domains $x > 0$ and $x < 0$, respectively, the continuity of traction, $t(x)$, across the whole $x$-axis requires that

$$B_{(1)}' f'_{(1)}(x) + B_{(2)}' f'_{(2)}(x) = B_{(2)}' f'_{(1)}(x) + B_{(1)}' f'_{(2)}(x).$$

(3.22)

Noting that one of the important properties of holomorphic functions used in the method of analytic continuation is that if the function $f(z)$ is holomorphic in the region $y > 0$ (or $y < 0$), then $\overline{f(z)}$ is holomorphic in the region $y < 0$ (or $y > 0$). In using this property, the above is rearranged in the form

$$B_{(1)} f_{(1)}(x) - \overline{B_{(2)} f_{(2)}}(x) = B_{(2)} f_{(1)}(x) - \overline{B_{(1)} f_{(2)}}(x).$$

(3.23)

Thus, the left-hand side of eqn (3.23) is the boundary value of a function analytic in the upper half-plane, and the right-hand side is the boundary value of another function analytic in the lower half-plane. Hence, both functions can be analytically continued into the entire plane. Since both vanish at infinity to conform to a zero far-field, they must vanish at any point $z$ from Liouville’s theorem. Thus,

$$f'_{(1)}(z) = \overline{B_{(2)} f_{(1)}}(z), \quad y > 0,$$

(3.24)

$$f'_{(2)}(z) = \overline{B_{(1)} f_{(2)}}(z), \quad y < 0$$

(3.25)

where the one-complex-variable approach presented by Suo [24] has been used. Suo introduced a vector function, $f(z) = [f_1(z) f_2(z) f_3(z) f_4(z)]^T$, $z = x + px_2$, $\text{Im}(p) > 0$, in his analysis of anisotropic elastic material, and then showed that the solution to any given boundary value problem can be expressed in terms of such a vector, regardless of the precise value of $p$. Once a solution of $f(z)$ is obtained, the variable $z$ in $f(z)$ should be replaced with the argument $z$, to compute the related fields. This approach allows standard matrix algebra to be used in conjunction with the techniques of analytic functions of one variable, and thus bypass some complexities arising from the use of four complex variables.

From eqns (1.64) and (3.24), the jump across the $x$-axis can be written as

$$i\Delta U'(x) = C[B_{(1)} f'_{(1)}(x) - B_{(2)} f'_{(2)}(x)]$$

(3.26)

where the matrix $C$ was defined in eqn (2.42). Continuity of the displacement and
electric potential across the x-axis, as inferred from eqn (3.26), implies that a function defined as

$$\Psi(z) = \begin{cases} B_1 f_1'(z), & y > 0, \\ B_2 f_2'(z), & y < 0 \end{cases}$$

(3.27)
is analytic in the entire plane except on the crack. The traction-charge boundary condition (2.33) leads to a non-homogeneous Hielbert problem

$$\Psi^+(x) + \Psi^-(x) = -\Pi^c, \quad x \in L.$$  (3.28)

This equation has many possible solutions. The auxiliary conditions that render a unique solution are: $\Psi(z) = O(z)$ as $|z| \to \infty$, $\Psi(x)$ is square root singular at the crack tips, and the net Burgers vector for every finite crack vanishes, i.e.,

$$\int_{L_x} [\Psi^+(x) - \Psi^-(x)] dx = 0, \quad i = 1, 2, \ldots, n.$$  (3.29)

As an illustration, we reconsider the Griffith crack problem described in Section 2.4. The solution is

$$\int \Psi(z) dz = \frac{1}{2} \Pi^c [I(z^2 - c^2)^{1/2} - 1].$$  (3.30)

Then, the functions $f_1(z)$ and $f_2(z)$ are obtained from definition (3.27). The field variables are obtained by replacing $z$ with $z_0$ in eqns (1.64) and (1.65). The results are the same as those given in eqns (2.49)-(2.52).

### 3.4 Solution of a semi-infinite crack

In Section 3.2, the singularity of stress and electric displacement at the tip of a semi-infinite crack was discussed. As a complement, we present a full-field solution of the same problem as in this section. For piezoelectric materials, Sosa and Pak [5] have considered the distribution of SED in the neighbourhood of a semi-infinite crack by way of a three-dimensional eigenfunction approach. Gao and Yu [25] studied the same problem based on the Stroh’s formalism and Green’s function approach. For the present, however, we restrict our attention to the problems of Gao and Yu [25].

The problem considered by Gao and Yu is an infinite piezoelectric solid with a semi-infinite crack $L$, which lies along the positive direction of the $x_1$-axis. The solid is subjected to a line force-charge $q_0$ at an arbitrary point $z_0(x_{0x}, x_{0y}, x_{0z})$, and the crack faces are assumed to be free of traction and external charge, but filled with air. In this case, the boundary conditions can be stated as

$$\Pi^c_{1j} = \Pi^c_{2j} = 0, \quad x_2 = 0, \quad x_1 \in L, \quad j = 1, 2, 3,$$  (3.31)

$$\Pi^c_{24} = \Pi^c_{24}, \quad E_i^+ = E_i^-, \quad x_2 = 0, \quad x_1 \in L$$  (3.32)
where the symbols “+” and “−” stand for the upper and lower crack faces.

We start by seeking a complex potential function \( f(z) \) with the form

\[
f(z) = \left( \ln(z_{\text{u}} - z_{\text{o}0}) \right) q + f_0(z)
\]

(3.33)

where \( z_{\text{o}0} = x_{\text{o}} + p_{\text{o}} x_{\text{m}} \), \( q \) is a constant vector (which is shown to be \( \mathbf{A}^T \mathbf{q}_0 \) in Section 3.8), and \( f_0(z) \) is a holomorphic function vector (with four components) up to infinity. Define the functions

\[
F(z) = \Gamma(z), \quad F_0(z) = f'_0(z), \quad G_0(z) = \mathbf{F}_0(z).
\]

(3.34)

Then, in a similar manner to that in Section 3.3, we have

\[
\mathbf{B} \mathbf{F}_0 - \overline{\mathbf{G}}_0 = 0.
\]

(3.35)

Similarly, from eqn (3.32)₂,

\[
\sum_{j=1}^{4} [A_{kj} F_{j0}(z_j) - \overline{A}_{kj} G_{j0}(z_j)] = 0.
\]

(3.36)

From eqns (3.35) and (3.36), we know that

\[
X_{j0}(z) = -\frac{1}{\sum_{j=4}^{4} \ln(A_{kj} A_{kj})} \sum_{j=4}^{4} \ln(A_{kj} A_{kj}) \left( \sum_{k=4}^{4} B_{jk} F_k(z_k) \right) X_{k0}(z)
\]

(3.37)

where \( \Lambda = \mathbf{B}^{-1} \), and \( X_{j0}(z) = \sum_{k=4}^{4} B_{jk} F_k(z_k) \) is a holomorphic function which will be used later.

To find the function \( X_{\text{r}0} \), rewrite the stress boundary condition (3.31) in the form

\[
2 \text{Re} \sum_{k=4}^{4} B_{jk} F_k(x_j) = 0, \quad x_j \in L, \quad (J = 1 - 3).
\]

(3.38)

Furthermore, introduce a function

\[
z_k = \zeta_k^2, \quad (\zeta_k = t + r_k \eta)
\]

(3.39)

where \( r_k \) are eigenvalues in the \( \zeta \)-plane, which maps the region \( S_k \) (\( S_k \) is obtained from the \( x_1-x_2 \) plane cut along \( L \), by the affine transformation. \( \zeta_k = x_1 + p_k x_2 \)) onto the upper half-plane in the \( \zeta \)-plane. Meanwhile, the crack faces are mapped on the real axis, \( t \), in the \( \zeta \)-plane. Thus, eqn (3.38) can be expressed in the form
\[ 2 \text{Re} \sum_{k=1}^{4} B_{jk} \tilde{F}_{k0}(t) = h_{j0}(t), \quad -\infty < t < \infty, \quad (J = 1 - 3) \quad (3.40) \]

where \( \tilde{F}_{k0}(\zeta_k) = F_{k0}[z_k(\zeta_k)] \), and

\[ h_{j0}(t) = -2 \text{Re} \sum_{k=1}^{4} B_{jk} q_k \left( \frac{1}{t + \zeta_{k0}} + \frac{1}{t - \zeta_{k0}} \right), \quad \zeta_{k0} = \sqrt{z_{k0}}. \quad (3.41) \]

Multiplying both sides of eqn (3.40) by \( dt/(t-\zeta) \), and integrating along the real axis \( t \), one obtains

\[ \sum_{j=1}^{4} B_{ij} \tilde{F}_{j0}(\zeta) = \tilde{X}_{j0}(\zeta), \quad J=1, 2, 3 \quad (3.42) \]

where

\[ \tilde{X}_{j0}(\zeta) = -\sum_{k=1}^{4} \frac{1}{2\zeta_j} \left[ B_{ij} q_j \right. + \left. \bar{B}_{ij} q_j \right]. \quad (3.43) \]

From eqns (3.37) and (3.39) and integrating eqn (3.42) with respect to \( \zeta_k \), we obtain

\[ Y_{j0}(z_k) = -\frac{1}{\sum_{j=1}^{4} \text{Im}(A_{ij}A_{jm})} \sum_{j=1}^{4} \text{Im} \left( \sum_{j=1}^{4} A_{ij} A_{jm} Y_{j0}(\zeta_k) \right) \quad (3.44) \]

where

\[ Y_{j0}(z) = \int X_{j0}(z_k) d\zeta_k = \sum_{k=1}^{4} B_{jk} f_{k0}(z_k), \quad J=1-4. \quad (3.45) \]

Substituting eqn (3.39) into eqn (3.43), and then into eqn (3.45) leads to

\[ Y_{j0}(z_k) = \sum_{k=1}^{4} \left[ B_{jk} q_k \ln(\sqrt{z_k} + \sqrt{z_{k0}}) + B_{jk} q_k \ln(\sqrt{z_k} - \sqrt{z_{k0}}) \right]. \quad J = 1 - 3. \quad (3.46) \]

Solving eqn (3.45) for \( f_{k0} \), one obtains

\[ f_{k0}(z_k) = \sum_{j=1}^{4} \Lambda_{kj} Y_{j0}(z_k). \quad (3.47) \]

Thus Green’s function for a semi-infinite crack in a piezoelectric material...
subjected to a line force-charge \( q_0 \), can be obtained by substituting eqn (3.47) into eqn (3.33), and then into eqns (1.64) and (1.65) as

\[
U = \frac{1}{\pi} \text{Im}[A \left( \ln(z_a - z_{a0}) \right) A^\top] q_0 + 2 \text{Re}[A f_0(z)],
\]

\[
\varphi = \frac{1}{\pi} \text{Im}[B \left( \ln(z_a - z_{a0}) \right) B^\top] q_0 + 2 \text{Re}[B f_0(z)].
\]

With this solution, the SED intensity factors \( K \) can be expressed as

\[
K = \{K_{II}, K_I, K_{III}, K_D\}^T = \lim_{x_1 \to 0} \sqrt{-2\pi x_1 \varphi}, \quad x_1 < 0
\]

where \( K=\{K_{II}, K_I, K_{III}, K_D\}^T \), in which \( K_I, K_{II}, K_{III} \) are the usual stress intensity factors, and \( K_D \) is the so-called "elastic displacement intensity factor".

### 3.5 Griffith crack opened under constant traction-charge

A Griffith crack under constant traction-charge has been discussed in Section 3.3. The derivation is based on the following boundary conditions on the crack surface:

\[
D_{2+} = D_{2-} = 0,
\]

which means that the impermeable assumption was used due to its much simpler mathematical treatment and the fact that the dielectric constants of a piezoelectric material are much larger than those of a vacuum (generally between 1000 and 3500 times larger). This is equivalent to assuming that the dielectric constant of the air vanishes. However, Hao and Shen [26] argued that even if the permittivity of a vacuum (or air) is quite small, the flux of electric fields can propagate through the crack gap, and the electric displacement should not be zero. They suggested therefore that the boundary condition (1.36) should be used. In this section we summarize the solution obtained by Hao and Shen [26] under such a crack face condition.

#### 3.5.1 The extended Lekhnitskii formalism

We begin by assuming that \( \sigma_i \) and \( D_i \) are independent of \( x_3 \). From Table 1.1, we know that

\[
u_{i3} = \varepsilon_3 = f_{3j} \sigma_j + g_{u3} D_u = D(x_1, x_2), \quad j=1-6, \; \alpha=1-3.
\]

Since the fields \( \sigma_i \) and \( D_i \) are independent of \( x_3 \), there exist the relations

\[
u_i = x_i D(x_1, x_2) + W_0(x_1, x_2), \quad u_{1,2} + u_{2,3} = \varepsilon_k = f_{4k} \sigma_j + g_{uk} D_u.
\]

From eqn (3.53), one obtains...
\[ u_2 = x_1 (f_{jj} \sigma_j + g_{a5} D_a) - \frac{x_1^2}{2} D_{12} - x_j W_{0i2} + V_\alpha(x_1, x_2). \] (3.54)

Similarly,
\[ u_1 = x_1 (f_{jj} \sigma_j + g_{a5} D_a) - \frac{x_1^2}{2} D_{11} - x_j W_{0i1} + U_\alpha(x_1, x_2). \] (3.55)

Because \( \varepsilon_i \) is independent of \( x_3 \) by assumption, we have
\[
\begin{align*}
  u_{41} &= \left\{ x_1 (f_{jj} \sigma_j + g_{a5} D_a)_1 - \frac{x_1^2}{2} D_{111} - x_j W_{0i1} + U_\alpha \right\} = 0, \\
  \text{which leads to} \\
  D_{11} &= 0 \quad \text{and} \quad (f_{jj} \sigma_j + g_{a5} D_a - W_{0i1})_1 = 0. 
\end{align*}
\] (3.57)

In a similar way, we know that
\[
\begin{align*}
  D_{22} &= 0, \quad (f_{jj} \sigma_j + g_{a5} D_a - W_{0i2})_2 = 0, \\
  D_{12} &= 0, \quad (f_{jj} \sigma_j + g_{a5} D_a - W_{0i1})_2 + (f_{jj} \sigma_j + g_{a5} D_a - W_{0i2})_1 = 0. 
\end{align*}
\] (3.58)

From eqns (3.55)-(3.58), we have
\[
\begin{align*}
  D &= A_i x_i + A_2 x_2 + A_3, \quad u_{11} = U_{01} = \varepsilon_1 = f_{jj} \sigma_j + g_{a5} D_a, \\
  u_{22} &= V_{02} = \varepsilon_2 = f_{jj} \sigma_j + g_{a2} D_a, \quad u_{12} + u_{21} = U_{01} + V_{01} = \varepsilon_3 = f_{jj} \sigma_j + g_{a5} D_a, \\
  u_{13} + u_{31} &= W_{01} + A_2 x_2 + \omega_1 = \varepsilon_4 = f_{jj} \sigma_j + g_{a5} D_a, \\
  u_{23} + u_{32} &= W_{02} - A_4 x_1 + \omega_2 = \varepsilon_5 = f_{jj} \sigma_j + g_{a5} D_a 
\end{align*}
\] (3.59)

where \( A_1, A_2, \omega_1 \text{ and } \omega_2 \) are integral constants to be determined. Following eqns (3.52), (3.59) and the constitutive relation in Table 1.1, we have
\[
\begin{align*}
  \sigma_j &= A_i x_i + A_2 x_2 + A_3 - (f_{jj} \sigma_j + g_{a5} D_a)/f_{jjj}, \\
  E_i &= -g_{j5} \sigma_j + \beta_{5j} D_a = -\phi_{3j}. 
\end{align*}
\] (3.60)

where the dumb index \( j=1, 2, 4, 5, 6 \), and \( i=1-3 \). Since \( E_3 \) is independent of \( x_3 \), we obtain
\[
\phi = -x_i E_i + \phi_0(x_1, x_2) \rightarrow E_\beta = \phi_\beta = -x_3 E_{3\beta} + \phi_{0\beta}, \quad \beta = 1, 2. 
\] (3.61)

Thus, we have
\[ E_{3j} = 0. \tag{3.62} \]

This results in \( E_3 = \text{constant}. \) For convenience we set \( E_3 = 0. \) Then, \( \sigma_3 \) and \( D_3 \) can be expressed as

\[
\sigma_3 = L(A_1 x_1 + A_2 x_2 + A_3) + F_j \sigma_j + G_u D_u,
\]

\[
D_3 = K(A_1 x_1 + A_2 x_2 + A_3) + H_j \sigma_j + J_u D_u
\tag{3.63}
\]

where the dumb index \( j = 1, 2, 4, 5, 6, \) and \( \alpha = 1, 2, \) with

\[
L = 1 - M_{33}, \quad M = \frac{g_{13}}{p_{33} f_{33}^2}, \quad F_j = -\frac{f_{j1}}{f_{33}} - M \left( g_{3j} - \frac{g_{33} g_{3j}}{f_{33}} \right), \quad K = M f_{33},
\]

\[
G_u = -\frac{g_{3u}}{f_{33}} - M \left( g_{33} g_{3u} - \beta_{3u} \right), \quad H_j = M \left( f_{33} g_{3j} - f_{3j} \right), \quad J_u = M \left( g_{3u} - \frac{f_{3j} \beta_{3u}}{g_{33}} \right).
\tag{3.64}
\]

Substituting eqn (3.63) into eqn (3.59), one obtains

\[
U_{0,1} = \rho_{j1} \sigma_j + \eta_{3u} D_u + \xi_j (A_1 x_1 + A_2 x_2 + A_3),
\]

\[
V_{0,2} = \rho_{j2} \sigma_j + \eta_{2u} D_u + \xi_j (A_1 x_1 + A_2 x_2 + A_3),
\]

\[
W_{0,1} = \rho_{j1} \sigma_j + \eta_{1u} D_u + \xi_j (A_1 x_1 + A_2 x_2 + A_3),
\]

\[
W_{0,2} = \rho_{j2} \sigma_j + \eta_{2u} D_u + \xi_j (A_1 x_1 + A_2 x_2 + A_3),
\]

\[
V_{0,1} + U_{0,2} = \rho_{j1} \sigma_j + \eta_{3u} D_u + \xi_j (A_1 x_1 + A_2 x_2 + A_3)
\tag{3.65}
\]

where the dumb index \( j = 1, 2, 4, 5, 6, \) and \( \alpha = 1, 2, \) with

\[
\rho_{j} = f_{ji} = \frac{f_{j1} f_{3j}}{f_{33}} + M \left[ \frac{g_{33} f_{3j} f_{3j}}{f_{33}^2} + \frac{f_{33} g_{3u} g_{3j}}{g_{33}} - (f_{3j} g_{3u} + f_{3j} g_{3j}) \right] = \rho_{ji},
\tag{3.66}
\]

\[
\eta_{ji} = g_{ji} + f_{ji} g_{3u} J_u, \quad \xi_j = f_{ji} L + g_{ji} K.
\tag{3.67}
\]

Substituting eqn (3.63) into the constitutive relation in Table 1.1, we have

\[
E_{\alpha} = -h_{ij} \sigma_j + \mu_{ij} D_i + \omega_{ij} (A_1 x_1 + A_2 x_2 + A_3), \quad \alpha, \beta = 1, 2; \quad j = 1, 2, 4, 5, 6
\tag{3.68}
\]

where

\[
h_{ij} = g_{ij} + g_{3i} F_j - \beta_{3j} H_j, \quad \mu_{ij} = \beta_{ij} - g_{ij} G_j + \beta_{ij} J, \quad \omega_{ij} = g_{ij} L + \beta_{ij} K.
\tag{3.69}
\]

Having obtained the above relations, we can now introduce a new shear function
\[ \psi, \text{ such that} \]
\[ \sigma_4 = -\psi_1, \quad \sigma_5 = \psi_2. \]  
(3.70)

and use the stress function \( F \) and induction \( V \) defined in eqn (1.94) for obtaining the final solution.

Substituting eqns (1.94) and (3.70) into eqn (3.65) and eliminating \( U_0 \) and \( V_0 \), we obtain

\[ L_4 F + L_3 \psi + L_5 V = 0, \]
\[ L_4 F + L_3 \psi + L_6 V = -2A_4 - A_2 \xi_3 + A_4 \xi_4 \]  
(3.71)

where

\[ L_2 = \rho_{44} \frac{\partial^2}{\partial x_2^2} - 2\rho_{45} \frac{\partial^2}{\partial x_1 \partial x_2} + \rho_{35} \frac{\partial^2}{\partial x_2^2}, \]
\[ L_3 = -\rho_{24} \frac{\partial^3}{\partial x_1^3} + (\rho_{26} + \rho_{46}) \frac{\partial^3}{\partial x_1 \partial x_2^2} - (\rho_{14} + \rho_{46}) \frac{\partial^3}{\partial x_1 \partial x_2} + \rho_{15} \frac{\partial^3}{\partial x_2^2}, \]
\[ L_4 = \rho_{22} \frac{\partial^4}{\partial x_1^4} - 2\rho_{26} \frac{\partial^4}{\partial x_1 \partial x_2^3} + (2\rho_{12} + \rho_{46}) \frac{\partial^4}{\partial x_1 \partial x_2^2} - 2\rho_{16} \frac{\partial^4}{\partial x_1 \partial x_2} + \rho_{11} \frac{\partial^4}{\partial x_2^2}, \]
\[ L_5 = \eta_{11} \frac{\partial^3}{\partial x_2^3} - (\eta_{12} + \rho_{66}) \frac{\partial^3}{\partial x_1 \partial x_2^2} + (\eta_{21} + \eta_{62}) \frac{\partial^3}{\partial x_1 \partial x_2} - \eta_{22} \frac{\partial^3}{\partial x_1^3}, \]
\[ L_6 = \eta_{51} \frac{\partial^3}{\partial x_2^3} - (\eta_{52} + \rho_{44}) \frac{\partial^3}{\partial x_1 \partial x_2^2} + \rho_{42} \frac{\partial^3}{\partial x_1^3}. \]  
(3.72)

From eqns (1.94), (3.55) and (3.70), and considering \( E_{2,1} - E_{1,2} = 0 \), and then eliminating \( E_\nu \), we have

\[ L_7 F + L_8 \psi + L_9 V = A_2 \omega_1 - A_4 \omega_2 \]  
(3.73)

where

\[ L_7 = -h_{11} \frac{\partial^3}{\partial x_2^3} + (h_{16} + h_{21}) \frac{\partial^3}{\partial x_1 \partial x_2^2} - (h_{12} + h_{26}) \frac{\partial^3}{\partial x_1 \partial x_2} + h_{22} \frac{\partial^3}{\partial x_2^3}, \]
\[ L_8 = -h_{15} \frac{\partial^2}{\partial x_2^2} + (h_{25} + h_{14}) \frac{\partial^2}{\partial x_1 \partial x_2} - h_{24} \frac{\partial^2}{\partial x_1^2}, \]  
(3.74)
\[ L_9 = \mu_{11} \frac{\partial^2}{\partial x_2^2} - (\mu_{21} + \mu_{12}) \frac{\partial^2}{\partial x_1 \partial x_2} + \mu_{22} \frac{\partial^2}{\partial x_1^2}. \]
Equations (3.71) and (3.73) can be further written as

\[
(L_4L_6L_4 - L_4L_5L_2 + L_4L_3^2 - L_4L_3L_8 - L_2L_5L_7 + L_3L_8L_5)F = 0 .
\]  
(3.75)

As discussed in Section 1.8 for two-dimensional problems, eqn (3.75) can be solved by assuming a solution of \( f(z_j) \), where \( z_j = x_j + p_jy_j \). Thus the most general real solution is obtained from a linear combination of four arbitrary functions

\[
F = 2 \text{Re} \sum_{j=1}^{4} f_j(z_j) .
\]  
(3.76)

Substituting eqn (3.76) into eqn (3.71), we obtain

\[
\psi = 2 \text{Re} \sum_{j=1}^{4} a_jf_j(z_j) , \quad V = 2 \text{Re} \sum_{j=1}^{4} c_jf_j(z_j) .
\]  
(3.77)

Let \( L_4[f_j(z_j)] = l_4(p_j)d^5f_j(z_j)/dz^5_j \) with \( r_i \) being the order of \( L_4 \) such that

\[
L_4[f_j(z_j)] = \mu_{11} \frac{\partial^2 f_j}{\partial x_2^2} - (\mu_{21} + \mu_{12}) \frac{\partial^2 f_j}{\partial x_1\partial x_2} + \mu_{22} \frac{\partial^2 f_j}{\partial x_1^2}
\]  
(3.78)

Using this notation, \( a_j \) and \( c_j \) can be expressed as

\[
a_j = \frac{l_4(p_j)}{l_1(p_j)} \left[ \frac{l_4(p_j)l_2(p_j) - l_1^2(p_j)}{l_4(p_j)l_2(p_j) - l_1(p_j)l_4(p_j)} - \frac{l_1(p_j)}{l_5(p_j)} \right],
\]  
\[
c_j = \frac{l_4(p_j)l_2^2(p_j)}{l_5(p_j)[l_2(p_j) - l_1(p_j)l_4(p_j)]}.
\]  
(3.79)

Thus, one obtains

\[
\sigma_i = 2 \text{Re} \sum_{j=1}^{4} k_jf_j'(z_j) , \quad D_{ik} = 2 \text{Re} \sum_{j=1}^{4} d_{ik}f_j(z_j) , \quad k_{ij} = p_j^2 , \quad k_{2j} = 1 ,
\]  
(3.80)

\[
k_{ij} = -a_j , \quad k_{sj} = a_jp_j , \quad k_{e} = -p_j , \quad d_{ij} = c_jp_j , \quad d_{2j} = b_j .
\]

To obtain the explicit expression of \( (u_n - u_n^*) \) and \( (\phi^+ - \phi^-) \), we begin by considering the electric potential on the crack faces. From eqns (3.68) and (3.80), we have

\[
E_1 = -2h_{1r} \sum_{j=1}^{4} k_jf_j'(x_i) + 2\mu_{1a} \sum_{j=1}^{4} d_{1a}f_j^*(x_i) + \omega_1(Ax_i + C) , \quad x_2 = 0
\]  
(3.81)
where \( r=1,2,4,5,6, \alpha=1,2. \)

Noting that \( E_i = -\phi_{ji} \), we have

\[
\phi = 2\eta_{j1} \Re \left\{ \sum_{j=1}^{4} k_{ij} f'(x_j) - 2\mu_{ja} \sum_{j=1}^{4} d_{aj} f'(x_j) - \omega_1 x_i (Ax_i / 2 + C + C_j), \quad x_2 = 0 \right. \]

(3.82)

The displacement \( u_2 \) on the crack surface can be obtained by using eqns (3.59), (3.65) and (3.80) as

\[
u_2 = 2\Re \left\{ \sum_{j=1}^{4} (\rho_{j1} k_{ij} + \mu_{ja} d_{aj}) f'(x_j) / \rho_j + C' (x_j), \quad r = 1, 2, 4, 5, 6; \alpha = 1, 2; \quad x_2 = 0 \right. \]

(3.83)

where \( C'(x_j) \) is an arbitrary function of \( x_j \).

### 3.5.2 The solution of the Griffith crack

Consider a Griffith crack as shown in Fig. 2.1, subjected to uniform load:

\[
\sigma_2 = \sigma_4 = \sigma_6 = 0, \quad \text{i.e.} \quad F_{j1} = -\psi_{j1} = -F_{j2} = 0, \quad \text{on} \quad x_2 = 0, \quad |x_1| < c. \quad (3.84)
\]

We begin by seeking a complex potential function \( f_j(z_j) \) with the general form

\[
f_j'(z_j) = (X_{j1} + iX_{j2}) \ln z_j + (B_j + iC_j) z_j + f_{j0}'(z_j) \]

(3.85)

where \( X_{j1}, X_{j2}, B_j \) and \( C_j \) are real constants, and

\[
f_{j0}'(z_j) = a_{j0} + \sum_{n=1}^{\infty} a_{nj} / z_j^n \]

(3.86)

is a holomorphic function up to infinity. Single-valuedness of \( f_j \) on the crack surface under conditions (3.84) leads to \((X_{j1}+iX_{j2})=0\). Next, the constants \( B_j \) and \( C_j \) are determined by means of the far-field electro-mechanical loads:

\[
2\Re \sum_{j=1}^{4} k_{ij} (B_j + iC_j) = \sigma_i^\infty, \quad 2\Re \sum_{j=1}^{4} d_{aj} (B_j + iC_j) = D_a^\infty, \quad i = 1, 2, 4, 5, 6; \alpha = 1, 2.
\]

(3.87)

Because of the arbitrariness of the rotation at infinity [27], we can let \( C_i = 0 \). Thus the number of equations in eqn (3.87) are enough to determine the constants \( B_j \) and \( C_j \).

Integrating the boundary condition (3.84) and neglecting some constants which are of no use to the present problem, we obtain
\[ F_1 = \psi = F_2 = 0, \quad \text{on } x_2 = 0, \quad \left| x_1 \right| \leq c. \quad (3.88) \]

Noting that \( \kappa_u (\phi^- - \phi^+)/(\mu_2^i - \mu_2^{-}) \) is equal to a constant \( D_0 \) [26], considering that \( D_2 = -V_1 \), one has
\[ V = -D_0 x_1, \quad \text{on } x_2 = 0, \quad \left| x_1 \right| \leq c. \quad (3.89) \]

Substituting eqns (3.76) and (3.77) into eqns (3.88) and (3.89), yields
\[ 2 \Re \sum_{j=1}^4 f'_j(x_1) = 0, \quad 2 \Re \sum_{j=1}^4 p_j f'_j(x_1) = 0, \quad 2 \Re \sum_{j=1}^4 a_j f'_j(x_1) = 0, \quad (3.90) \]
\[ 2 \Re \sum_{j=1}^4 c_j f'_j(x_1) = -D_0 x_1, \quad x_2 = 0, \quad \left| x_1 \right| \leq c. \]

Now the four functions \( f_0 \) can be derived by means of eqn (3.90) together with the following conformal transformation which maps the exterior of the crack gap onto the exterior of a unit circle
\[ z_k = \frac{c}{2} \left( \zeta_k + \zeta_k^{-1} \right). \quad (3.91) \]

Through some algebraic manipulation we obtain the complex potentials as
\[ f'_j(z_k) = (B_j + iC_j)z_k + \frac{1}{c \Delta_0} p_{jk} e_k \left[ z_j - \sqrt{z_j^2 - c^2} \right], \quad k = 1 - 4 \quad (3.92) \]
where
\[ e_1 = \frac{1}{2} c \sigma_2^x, \quad e_2 = \frac{1}{2} c \sigma_6^x, \quad e_3 = \frac{1}{2} c \sigma_4^x, \quad e_4 = \frac{1}{2} c (D_x^c - D_0) \quad (3.93) \]
and \( \Delta_0 \) is the determinant of matrix \([p'_{ij}]\)
\[ [p'_{ij}] = \begin{bmatrix}
1 & 1 & 1 & 1 \\
p_1 & p_2 & p_3 & p_4 \\
a_1 & a_2 & a_3 & a_4 \\
c_1 & c_2 & c_3 & c_4
\end{bmatrix} \quad (3.94) \]
while \( p_{ij} \) is the algebraic complement of \([p'_{ij}]\) with the relations
\[ \sum_{j=1}^4 p_{ji} = \delta_{1i} \Delta_0, \quad \sum_{j=1}^4 p_j p_{ji} = \delta_{2i} \Delta_0, \quad \sum_{j=1}^4 a_j p_{ji} = \delta_{3i} \Delta_0, \quad \sum_{j=1}^4 c_j p_{ji} = \delta_{4i} \Delta_0. \quad (3.95) \]
From eqn (3.92), we have
\[ f'_j(z_j) - f'_j(z_j^-) = \frac{2i}{c \Delta_0} p_{j\alpha} e_i \sqrt{c^2 - x_i^2}, \quad k = 1-4, \quad x_2 = 0, \quad |x_1| \leq c. \quad (3.96) \]

Therefore, the quantities \((\phi^- - \phi^+)\) and \((u_x^- - u_x^+)\) can be expressed as
\[ \phi^+ - \phi^- = -2 \text{Re} \sum_{j=1}^{4} (a_{j1} k_{j\alpha} - \mu_{1\alpha} d_{\alpha j}) \frac{2i}{c \Delta_0} p_{j\alpha} e_i \sqrt{c^2 - x_i^2}, \quad (3.97) \]
\[ u_x^+ - u_x^- = -2 \text{Re} \sum_{j=1}^{4} (\beta_{21} k_{j\alpha} - \mu_{1\alpha} d_{\alpha j}) \frac{2i}{c \Delta_0} p_{j\alpha} e_i \sqrt{c^2 - x_i^2} \quad (3.98) \]

for \(k=1-4, \quad x_2=0\) and \(|x_1| \leq c\). With the usual definition, the stress and electric displacement intensity factors are given by
\[ K_I = \lim_{r \to c} \sqrt{2\pi(x_1 - c)} \sigma_2(x_i) = \frac{2\sqrt{\pi} e_i}{\sqrt{a}} \sum_{j=1}^{4} \frac{P_{j}}{\Delta_0} = \sigma_2^\infty \sqrt{\pi a}, \quad (3.99) \]
\[ K_{II} = \lim_{r \to c} \sqrt{2\pi(x_1 - c)} \sigma_4(x_i) = \frac{2\sqrt{\pi} e_i}{\sqrt{a}} \sum_{j=1}^{4} \alpha_j \frac{P_{j}}{\Delta_0} = \sigma_4^\infty \sqrt{\pi a}, \quad (3.100) \]
\[ K_{III} = \lim_{r \to c} \sqrt{2\pi(x_1 - c)} \sigma_6(x_i) = \frac{2\sqrt{\pi} e_i}{\sqrt{a}} \sum_{j=1}^{4} \beta_j \frac{P_{j}}{\Delta_0} = \sigma_6^\infty \sqrt{\pi a}, \quad (3.101) \]
\[ K_D = \lim_{r \to c} \sqrt{2\pi(x_1 - c)} D_2(x_i) = \frac{2\sqrt{\pi} e_i}{\sqrt{a}} \sum_{j=1}^{4} \gamma_j \frac{P_{j}}{\Delta_0} = (D_2^\infty - D_0) \sqrt{\pi a}. \quad (3.102) \]

Apparently, \(D_0\) has reduced the value of \(K_D\), but has no influence on stress intensity factors, except that it affects the value of \(\sigma_0\) near the crack tip.

### 3.6 Solution of elliptic hole problems

The fracture properties of a Griffith crack may be investigated based on an analysis of elliptic hole problems in piezoelectric media. In fact, a crack of length \(2c\) can be obtained by letting the minor axis, say \(b\) in this section, of the ellipse approach zero. The problem of determining the SED fields in the neighbourhood of an elliptic hole in a piezoelectric plate has been studied by Sosa and Khutoryansky [28]. Later, Lu et al. [29] and Qin [30] extended the analysis to thermopiezoelectric problems. Qin [30] used the exact boundary condition at the rim of an elliptic hole to avoid the common assumption of electric impermeability, and solved the problem by means of Lekhnitskii’s formalism.
The problem has also been considered by Gao and Fan [31] and others, but we shall follow the results given in [28,30] here.

3.6.1 Problem statement
Consider an infinite two-dimensional piezoelectric material containing an elliptic hole, where the material is transversely isotropic and coupling between in-plane stress and in-plane electric fields takes place. The contour \( \Gamma \) (see Fig. 3.1) of the hole is represented by

\[
x_1 = c \cos \theta, \quad x_2 = b \sin \theta
\]

(3.103)

where \( \theta \) is a real parameter and \( c, b \) are the half-lengths of the major and minor axes respectively of the ellipse (see Fig. 3.1). The hole is assumed to be filled with a homogeneous gas of dielectric constant \( \kappa_0 \), and is free of traction and surface-charge density. On the hole boundary the normal component of electric displacement and the electric potential are continuous. We, therefore, have boundary conditions on the contour \( \Gamma \):

\[
\sigma_{nn} = 0, \quad \sigma_{nm} = 0, \quad D_n = -\kappa_0 \frac{\partial \phi}{\partial n}, \quad \phi = \phi^c
\]

(3.104)

where \( n \) and \( m \), respectively, represent the normal and tangent to the hole boundary (Fig. 3.1). \( \sigma_{nn} \) and \( \sigma_{nm} \) are the normal and shear stresses along the boundary, \( D_n \) is the normal component of electric displacement vector, superscript "\( c \)" indicates the quantity associated with the hole medium.

When the piezoelectric material is subjected to a uniform remote SED \( \Pi^e (\Pi_1^e, \Pi_2^e) \), where \( \Pi_1^e = \{\sigma_{11}^e, \sigma_{12}^e, D_2^e\}^T, \Pi_2^e = \{\sigma_{21}^c, \sigma_{22}^c, D_3^c\}^T \) for the generalized plane strain problem under consideration, the SED condition at infinity is

\[
T \{\sigma, D\} = \{0,0\}
\]

for the generalized plane strain problem under consideration, the SED condition at infinity is
\[ \Pi \rightarrow \Pi^\circ, \quad \text{when } x_1^2 + x_2^2 \rightarrow \infty. \] (3.105)

From the definition in eqn (1.94), we know that
\[ F_1 = -\int_0^\tau t_1 \, ds, \quad F_2 = -\int_0^\tau t_2 \, ds, \quad V = -\int_0^\tau D_n \, ds \] (3.106)

where \( t_1 \) and \( t_2 \) are the rectangular Cartesian components of surface traction vector \( \mathbf{t} \), and \( ds \) is an element of arc length on \( \Gamma \). Substituting eqn (1.106) into eqn (3.104) and noting eqn (1.106), we have
\[ 2 \Re \sum_{k=1}^{3} \Phi_k = 0, \quad 2 \Re \sum_{k=1}^{3} \{ p_k \Phi_k \} = 0, \]  \[ 2 \Re \sum_{k=1}^{3} \{ \eta_k \Phi_k \} = -\kappa_0 \int_0^\tau \frac{\partial \Psi}{\partial n} \, ds, \quad 2 \Re \sum_{k=1}^{3} \{ t_k \Phi_k \} = \phi^c. \] (3.107)

Noting that
\[ \frac{\partial \Psi}{\partial n} (x_1 + i x_2) = -i \frac{\partial \Psi}{\partial m} (x_1 + i x_2), \] equation (3.107) can be further simplified as
\[ 2 \Re \sum_{k=1}^{3} \{ \phi_k \Phi_k \} = 2 \Re \{ i \kappa_0 F^c (z) \} \] (3.109)

where \( z = x_1 + i x_2 \) throughout this and the next sections, and
\[ \phi^c (z) = F^c (z) + \bar{F}^c (z). \] (3.110)

### 3.6.2 General solutions

Since conformal mapping is a fundamental tool used to find complex potentials, the transformation
\[ z_k = \frac{c - i p_k b}{2} \zeta_k + \frac{c + i p_k b}{2} \frac{1}{\zeta_k} \] (3.111)

will be used to map the region, \( \Omega \), occupied by piezoelectric material onto the outside of a unit circle in the \( \zeta \)-plane, since it has been shown that all the roots of equation \( dz_k / d\zeta_k = 0 \) for eqn (3.111) are located inside the unit circle \( |\zeta_k| = 1 \) [32]. To find the single-valued mapping of the region \( \Omega \), occupied by a vacuum
(or air), let

\[ z = \frac{c + b}{2} \zeta + \frac{c - b}{2} \frac{1}{\zeta}. \]  

(3.112)

The roots of \( dz/d\zeta = 0 \) are \( \zeta_{1,2} = \pm \sqrt{1 - e} \) \( \sqrt{1 + e} \), where \( e = b/c \). Thus, mapping of the region \( \Omega \) can be achieved by excluding a straight line \( \Gamma_0 \) along \( x_1 \) and of length \( 2a\sqrt{1 - e^2} \) from the ellipse (Fig. 3.1). In this case, the function (3.112) will transform \( \Gamma \) and \( \Gamma_0 \) into the ring of outer and inner circles with radii \( r_{out} = 1 \) and \( r_{in} = \sqrt{1 - e} \sqrt{1 + e} \), respectively.

The problem is now to find the complex potentials, \( \Phi_k (k=1-3) \) in the region \( \Omega \) and the function \( F' \) in the region \( \Omega_c \). To this end we assume a general solution of the form

\[ \Phi_k (z_k) = c_k z_k + c_{k0} \ln z_k + \Phi_{k0} (\zeta_k) \]  

(3.113)

where

\[ \Phi_{k0} (\zeta_k) = d_{k0} + \sum_{i=1}^{\infty} d_i z_k^{-i} \]  

(3.114)

is a holomorphic function up to infinity with constants \( d_{ki} \). The boundary conditions (3.107) require that \( c_{k0} = 0 \) for \( \Phi_k \) to be single valued. The constants \( c_k \) are determined from the far field loading conditions, as described at the end of this subsection.

Furthermore, the solution for \( F'(z) \), is assumed as

\[ F'(z) = s_0 + \sum_{k=1}^{\infty} s_k z^{-k} \]  

(3.115)

for a crack, and

\[ F'(z) = \sum_{k=0}^{\infty} s_k z^k \]  

(3.116)

for other situations, whose coefficients have the following relation [28]:

\[ s_{-k} = r_{-k}^2 s_k. \]  

(3.117)

Substituting eqns (3.111) and (3.113) into the left-hand side of eqn (3.107) yields the boundary conditions on the unit circle, namely
\[
\sum_{k=1}^{3} \Phi_{k,0}(\sigma) + \overline{\Phi_{k,0}(\sigma)} = \bar{l}_1 \sigma + l_1 \sigma^{-1}, \tag{3.118}
\]

\[
\sum_{k=1}^{3} p_k \Phi_{k,0}(\sigma) + p_k \overline{\Phi_{k,0}(\sigma)} = \bar{l}_2 \sigma + l_2 \sigma^{-1}, \tag{3.119}
\]

\[
\sum_{k=1}^{3} \alpha_k \Phi_{k,0}(\sigma) + \overline{\alpha_k \Phi_{k,0}(\sigma)} = \bar{l}_3 \sigma + l_3 \sigma^{-1} + i \kappa_0 [F^*(\sigma) - \bar{F}^*(\sigma)], \tag{3.120}
\]

\[
\sum_{k=1}^{3} t_k^* \Phi_{k,0}(\sigma) + t_k^* \overline{\Phi_{k,0}(\sigma)} = \bar{l}_4 \sigma + l_4 \sigma^{-1} + F^*(\sigma) + \bar{F}^*(\sigma) \tag{3.121}
\]

where \( \sigma = e^{i\theta} \) stands for the point located on the unit circle in the \( \zeta \)-plane, \( \theta \) is a polar angle, and

\[
l_1 = -a \sum_{k=1}^{3} [\text{Re}(c_k) + i e \text{Re}(c_k p_k)], \quad l_2 = -a \sum_{k=1}^{3} [\text{Re}(c_k p_k) + i e \text{Re}(c_k p_k^* \sigma_k)],
\]

\[
l_3 = -a \sum_{k=1}^{3} [\text{Re}(c_k \sigma_k) + i e \text{Re}(c_k p_k \sigma_k)], \quad l_4 = -a \sum_{k=1}^{3} [\text{Re}(c_k t_k^*) + i e \text{Re}(c_k p_k t_k^{*\prime})]. \tag{3.122}
\]

The substitution of eqns (3.113) and (3.115) [or (3.116)] into eqns (3.118)-(3.121) provides a new system of three equations in the unknowns \( d_k \):

\[
\sum_{k=1}^{3} d_k = \delta_{ij} l_i, \quad \sum_{k=1}^{3} p_k d_k = \delta_{ij} l_j, \quad \sum_{k=1}^{3} (\lambda_{ij} d_k + \bar{l}_k l_{ki}) = \delta_{ij} l_{ij}', \tag{3.123}
\]

where \( \delta_{ij} \) is the Kronecker delta, and

\[
\lambda_{ij} = \omega_{ij} - i \kappa_0 f_i^j, \quad l_i = l_i - i \kappa_0 d_{ki} \tag{3.124}
\]

\[
s_j = \sum_{k=1}^{3} t_k^* d_{ki} - \delta_{ij} l_{kj} \tag{3.125}
\]

for a crack, and

\[
\lambda_{ij} = \omega_{ij} - \frac{i \kappa_0}{1 - r_m^k} (1 + r_m^k) j_k, \quad \bar{l}_{kj} = \frac{2 i \kappa_0 r_m^{k'j'}}{1 - r_m^{k'}}, \tag{3.126}
\]

\[
l_{ij}' = l_i + \frac{i \kappa_0}{1 - r_m^k} [2 r_m^{k'j'} l_{kj} - (1 + r_m^k) l_{kj}].
\]
\[ s_i = \frac{1}{1 - \alpha_i} \left[ \sum_{k=1}^3 (\sigma_{i}^k \nu_{si} - \sigma_{is}^k \nu_{ki}) + \delta_{i} (I_i - r_{si}^2 l_i) \right] \] (3.127)

for other situations. Thus the unknowns \( d_{ij} \) can be solved from eqn (3.123). Once the constants \( d_{ij} \) are obtained, the complex potentials, \( \Phi_i(z_k) \), can be written in the form

\[ \Phi_i(z_k) = c_i z_k + \frac{d_{ki}}{c + ip_k b} \left[ z_k - \sqrt{z_k^2 - (c^2 + p_k^2 b^2)} \right]. \] (3.128)

through use of the relation (3.112) and the fact that all \( d_{ij} \) equal zero for \( j > 1 \).

To determine SED, the derivative of eqn (3.128) with respect to \( z_k \) is evaluated, which results in

\[ \Phi'_i(z_k) = c_i + \frac{d_{ki}}{c + ip_k b} \left[ 1 - \frac{z_k}{\sqrt{z_k^2 - (c^2 + p_k^2 b^2)}} \right]. \] (3.129)

To solve for \( c_k \) (\( k = 1 - 3 \)), one must invoke the remote electromechanical loading conditions. In the most general case, three mechanical and two electrical variables can be enforced. If these loads are the stress and electric displacement components, by making use of eqns (1.106), (3.105) and (3.129) we can generate a system of five equations for the six unknowns involved in the real and imaginary part of the constants \( c_k \). For the reason stated in Section 3.5, one can arbitrarily set one of these constants equal to zero. We will let Im(\( c_1 \)) = 0 in our analysis. Thus, using eqns (1.106), (3.105) and (3.129) and considering \( z_k \to \infty \), yields

\[ \begin{pmatrix} \sigma_{11}^e \\ \sigma_{22}^e \\ \sigma_{12}^e \end{pmatrix} = \Re \sum_{i=1}^3 \begin{pmatrix} \frac{p_i^2}{k} \\ 1 \\ -p_i \end{pmatrix} c_i \quad \text{and} \quad \begin{pmatrix} D_1^e \\ D_2^e \end{pmatrix} = \Re \sum_{i=1}^3 \begin{pmatrix} \varphi_i^0 p_i \\ -\varphi_i^0 \end{pmatrix} c_i \] (3.130)

which, compared with eqn (3.122), yields

\[ l_1 = -\frac{c}{2} (\sigma_{22}^e - i e \sigma_{12}^e), \quad l_2 = \frac{c}{2} (\sigma_{11}^e - i e \sigma_{12}^e), \quad l_3 = \frac{c}{2} (D_2^e - i e D_1^e). \] (3.131)

As was pointed out by Sosa and Khutoryansky [28], an electric field will be induced by the applied load at infinity in the piezoelectric material, whose value is obtained by inserting eqn (3.130) into eqn (1.91) giving

\[ \begin{pmatrix} E_1^0 \\ E_2^0 \end{pmatrix} = \Re \sum_{i=1}^3 \begin{pmatrix} 1 \\ p_i \end{pmatrix} t_i^i c_i \] (3.132)
which, compared with eqn (3.122), yields

\[ I_4 = -\frac{c}{2}(E_1^0 + iE_2^0). \quad (3.133) \]

Thus all unknown constants in eqn (3.129) have been completely determined.

3.6.3 Cracks

A crack of length \( 2c \) can be formed by letting the minor axis \( b \) of the ellipse approach zero. The solutions for a crack in an infinite piezoelectric plate can then be obtained from the formulation in subsection 3.6.2 by setting \( b = 0 \). In this case, eqn (3.128) can be simplified to

\[ \Phi_k(z_k) = c_k z_k + \frac{d_{k1}}{c}(z_k - \sqrt{z_k^2 - c^2}). \quad (3.134) \]

Thus, the asymptotic form of SED, \( \Pi_2 \), ahead of the crack tip along the \( x_1 \) axis can be given by

\[ \Pi_2 = \{\sigma_{12} \sigma_{22} D_2\}^T = \Pi_1 \frac{1}{\sqrt{x_1^2 - c^2}}. \quad (3.135) \]

where

\[ \Pi_1 = 2 \text{Re} \left[ \sum_{k=1}^{3} \begin{bmatrix} -p_k & 1 \\ 1 & -\sigma_{12} \end{bmatrix} d_{k1} \right]. \quad (3.136) \]

With the usual definition, the SED intensity factors are given by

\[ K = \lim_{x_1 \to \infty} \sqrt{2\pi(x_1 - a)}\Pi_2 = \Pi_1 \frac{\pi}{\sqrt{a}}. \quad (3.137) \]

3.7 Solutions of the anti-plane crack problem

Using linear piezoelectricity theory, the stress and electric fields around a crack under anti-plane loads are studied in this section.

Consider a mode III fracture problem for which a finite crack of length \( 2c \) is embedded in an infinite piezoelectric medium subjected to far-field mechanical and electric loads (see Fig. 3.2). This problem has been investigated by Pak [33], Zhang and Tong [34], and others. Pak [33] considered four possible cases of boundary conditions at infinity. The first case is a uniform shear traction, \( \sigma_{12} = \tau_a \), applied with a uniform electric displacement, \( D_1 = D_2 \); the second is a uniform shear strain, \( \gamma_{12} = \gamma_a \), with a uniform electric field, \( E_1 = E_2 \); the third is a uniform shear traction with a uniform electric field; and the fourth is a uniform
shear strain with a uniform electric displacement. The boundary conditions on the upper and lower surfaces of the crack are to be free of surface traction and surface charge for all cases of far-field loading conditions, i.e.,

\[
\sigma_{xy} = 0, \quad D_y = 0, \quad \text{on}\ |x| < c, \quad y = 0. \tag{3.138}
\]

Examination of eqn (1.46) reveals that the field equations are satisfied if \(u_x\) and \(\Phi\) are harmonic functions. This can be achieved by letting \(u_x\) and \(\Phi\) be the imaginary part of some analytic functions such that

\[
u_x = \text{Im}[X(z)], \quad \Phi = \text{Im}[\Phi(z)] \tag{3.139}
\]

where \(z = x + iy\). The SED can then be obtained from these functions

\[
\sigma_{xy} + i\sigma_{xz} = c_{44}X'(z) + e_{15}\Phi'(z), \quad D_y + iD_z = e_{15}X'(z) - \kappa_{15}\Phi'(z). \tag{3.140}
\]

Taking a semi-inverse approach by assuming \(X(z)\) and \(\Phi(z)\) to be

\[
X(z) = A(z^2 - c^2)^{1/2}, \quad \Phi(z) = -B(z^2 - c^2)^{1/2},
\]

we see that the governing equations (1.46) and the boundary conditions (3.138) are satisfied. In these equations, \(A\) and \(B\) are real constants. Substituting eqn (3.141) into eqn (3.140), we arrive at

\[
\sigma_{xy} + i\sigma_{xz} = c_{44} \frac{Az}{\sqrt{z^2 - c^2}} - e_{15} \frac{Bz}{\sqrt{z^2 - c^2}}, \tag{3.142}
\]

\[
D_y + iD_z = e_{15} \frac{Az}{\sqrt{z^2 - c^2}} + \kappa_{15} \frac{Bz}{\sqrt{z^2 - c^2}}.
\]

By applying the far-field loading conditions, the constants \(A\) and \(B\) are obtained as follows:

Case 1: \(\sigma_{xy} = \tau_{xy}\), and \(D_y = D_z\) as \(x^2 + y^2 \to \infty\)

\[
A = \frac{\kappa_{15}\tau_{xy} + e_{15}D_z}{c_{44}\kappa_{11} + e_{15}}, \quad B = \frac{c_{44}D_z - e_{15}\tau_{xy}}{c_{44}\kappa_{11} + e_{15}}. \tag{3.143}
\]

Case 2: \(\gamma_{xy} = \gamma_{xy}\), and \(E_y = E_z\) as \(x^2 + y^2 \to \infty\)

\[
A = \gamma_{xy}, \quad B = E_{xy}. \tag{3.144}
\]

Case 3: \(\sigma_{xy} = \tau_{xy}\), and \(E_y = E_z\) as \(x^2 + y^2 \to \infty\)

\[
A = \frac{\tau_{xy} + e_{15}E_z}{c_{44}}, \quad B = E_{xy}. \tag{3.145}
\]
Fig. 3.2: Far-field loads.

Case 4: $\gamma_y = \gamma_x$, and $D_y = D_x$ as $x^2 + y^2 \to \infty$

$$A = \gamma_x, \quad B = \frac{D_{xx} - e_{15} \gamma_w}{\kappa_{11}}.$$  \hspace{1cm} (3.146)

Thus, the near-tip fields are given by

$$\gamma_{yz} + i\gamma_{xz} = \frac{K_s}{\sqrt{2\pi r}} e^{-\frac{i\theta}{2}}, \quad \sigma_{yz} + i\sigma_{xz} = \frac{K_a}{\sqrt{2\pi r}} e^{-\frac{i\theta}{2}},$$  

$$E_y + iE_z = \frac{K_F}{\sqrt{2\pi r}} e^{-\frac{i\theta}{2}}, \quad D_y + iD_z = \frac{K_D}{\sqrt{2\pi r}} e^{-\frac{i\theta}{2}}.$$  \hspace{1cm} (3.147)

where $K_s = A\sqrt{\pi c}$ is the “strain intensity factor”, $K_F = B\sqrt{\pi c}$ the “electric field factor”, $K_a = c_{44} K_y - e_{15} K_F$ the “stress intensity factor”, and $K_D = e_{15} K_y + \kappa_{11} K_F$ the “electric displacement intensity factor”. For this particular piezoelectric fracture problem, all the field variables have the same crack-tip behaviour as in the classical mode III fracture problem.

The path-independent integral given in eqn (2.14) can now be evaluated to obtain the energy release rate for the mode III piezoelectric fracture problem under consideration. Using the solution obtained above, evaluating the integral in eqn (2.14) on a vanishingly small contour at a crack tip results in

$$J = \frac{K_u K_y - K_p K_F}{2}.$$  \hspace{1cm} (3.148)
Examining eqn (3.148) one can see that \( J \) may have negative values depending on the direction, magnitude, and type of electric load. Solving the roots of the quadratic equations in (3.148), the crack extension force can be shown to be positive when

\[
\begin{align*}
\text{Case 1:} & \quad -\frac{\sqrt{\varepsilon_{15}^2 + \kappa_{11}\varepsilon_{44}^2} - \varepsilon_{15}}{c_{44}} < \frac{D_{\infty}}{\tau_{\infty}} < \frac{\sqrt{\varepsilon_{15}^2 + \kappa_{11}\varepsilon_{44}^2} + \varepsilon_{15}}{c_{44}}, \\
\text{Case 2:} & \quad -\frac{\sqrt{\varepsilon_{15}^2 + \kappa_{11}\varepsilon_{44}^2} + \varepsilon_{15}}{\kappa_{11}} < \frac{E_{\infty}}{\gamma_{\infty}} < \frac{\sqrt{\varepsilon_{15}^2 + \kappa_{11}\varepsilon_{44}^2} - \varepsilon_{15}}{\kappa_{11}}, \\
\text{Case 3:} & \quad -\frac{1}{\sqrt{\varepsilon_{15}^2 + \kappa_{11}\varepsilon_{44}^2}} < \frac{E_{\infty}}{\tau_{\infty}} < \frac{1}{\sqrt{\varepsilon_{15}^2 + \kappa_{11}\varepsilon_{44}^2}}, \\
\text{Case 4:} & \quad -\frac{\sqrt{\varepsilon_{15}^2 + \kappa_{11}\varepsilon_{44}^2}}{\kappa_{11}} < \frac{D_{\infty}}{\gamma_{\infty}} < \frac{\sqrt{\varepsilon_{15}^2 + \kappa_{11}\varepsilon_{44}^2}}{\kappa_{11}}.
\end{align*}
\]

(3.149) (3.150) (3.151) (3.152)

This shows that, at certain ratios of electric load to mechanical load, crack arrest can be observed. It is also interesting to note that in the absence of mechanical load the crack extension force is always negative, indicating that a crack would not propagate under these conditions.

As an example, consider an anti-plane crack problem in PZT-5H material, with material properties as given by [33]: \( c_{44} = 3.53 \times 10^{10} \text{N/m}^2 \), \( \varepsilon_{15} = 17.0 \text{C/m}^2 \), \( \kappa_{11} = 1.51 \times 10^{-8} \text{C/(Vm)} \), \( J_{cr} = 5.0 \text{N/m} \), where \( J_{cr} \) is the critical energy release rate. The energy release rate for PZT-5H material with a crack of length \( 2c = 0.02 \text{m} \) is plotted in Figs 3.3 and 3.4 as a function of electrical load with the mechanical load fixed such that \( J = J_{cr} \) at zero electric load. As can be seen from Figs 3.3 and 3.4, the energy release rate can be made negative by setting electric loads outside the ranges given by (3.149)-(3.152).

3.8 Solution of half-plane problems

In this section, the problem of a crack embedded arbitrarily in a half-plane piezoelectric solid with traction-charge loading on the crack faces is considered. The boundary condition on the infinite straight boundary of the half-plane is free of surface traction-charge. Using Stroh’s formalism, simple explicit expressions of Green’s functions for a piezoelectric half-plane subjected to line force-charge and line dislocations are derived at first. Then, a system of singular integral equations for the unknown dislocation densities defined on the crack faces is formulated based on the Green’s function. The stress and electric displacement intensity factors are evaluated by numerically solving these singular integral equations.
3.8.1 Green’s function for infinite spaces
For an infinite domain subjected to a line force-charge \( q_0 \) and a line dislocation \( b \) both located at \( z_0(x_10, x_20) \), the solution in the form of eqns (1.64) and (1.65) is [35]
\[ U = \frac{1}{\pi} \text{Im}(A \{ \ln(z_u - z_{\alpha 0}) \} q), \quad \varphi = \frac{1}{\pi} \text{Im}(B \{ \ln(z_u - z_{\alpha 0}) \} q) \]  

(3.153)

where \( q \) is a complex vector to be determined. Since \( \ln(z_u - z_{\alpha 0}) \) is a multi-valued function we introduce a cut along the line defined by \( x_2 = x_{20} \) and \( x_1 \leq x_{10} \).

Using the polar coordinate system \((r, \theta)\) with its origin at \( z_0(x_{10}, x_{20}) \) and \( \theta = 0 \) being parallel to the \( x_1\)-axis, the solution (3.153) applies to

\[-\pi < \theta < \pi, \quad r > 0. \]

(3.154)

Therefore

\[ \ln(z_u - z_{\alpha 0}) = \ln r \pm i\pi, \quad \text{at } \theta = \pm \pi \text{ for } \alpha = 1 - 4. \]

(3.155)

Due to this relation, eqn (3.153) must satisfy the conditions

\[ U(\pi) - U(-\pi) = b, \quad \varphi(\pi) - \varphi(-\pi) = q_0 \]

(3.156)

which lead to

\[ 2 \text{Re}(Aq) = b, \quad 2 \text{Re}(Bq) = q_0. \]

(3.157)

This can be written as

\[
\begin{bmatrix}
A & \bar{A} \\
B & \bar{B}
\end{bmatrix}
\begin{bmatrix}
q \\
\bar{q}
\end{bmatrix}
= 
\begin{bmatrix}
b \\
q_0
\end{bmatrix}.
\]

(3.158)

It follows from eqn (1.75) that

\[
\begin{bmatrix}
q \\
\bar{q}
\end{bmatrix}
= 
\begin{bmatrix}
B^T & A^T
\end{bmatrix}
\begin{bmatrix}
b \\
q_0
\end{bmatrix}.
\]

(3.159)

Hence

\[ q = A^T q_0 + B^T b. \]

(3.160)

This result has been used in Sections 2.3 and 3.4 for eqns (2.15), (2.16) and (3.48).

### 3.8.2 Green’s function for half-spaces

Let the material occupy the region \( x_z > 0 \), and a line force-charge \( q_0 \) and a line dislocation \( b \) apply at \( z_0(x_{10}, x_{20}) \). To satisfy the boundary conditions on the infinite straight boundary of the half-plane, the solution should be modified as

\[ U = \frac{1}{\pi} \text{Im}(A \{ \ln(z_u - z_{\alpha 0}) \} q) + \sum_{\beta = 1}^{4} \frac{1}{\pi} \text{Im}(A \{ \ln(z_u - z_{\beta 0}) \} q_0 \hat{b} \beta). \]

(3.161)
where \( q \) is given in eqn (3.160) and \( q_\beta \) are unknown constants to be determined.

Consider first the case in which the surface \( x_2 = 0 \) is traction-charge free so that

\[
\varphi = 0 \quad \text{at} \quad x_2 = 0. \tag{3.163}
\]

Substituting eqn (3.162) into eqn (3.163) one obtains

\[
\varphi = \frac{1}{\pi} \text{Im} \{ B [ \ln(z_u - z_{uo})] q \} + \sum_{\beta=1}^{4} \frac{1}{\pi} \text{Im} \{ B [ \ln(z_u - z_{\beta 0})] q_\beta \} = 0. \tag{3.164}
\]

Noting that \( \text{Im}(f) = -\text{Im}(\tilde{f}) \), we have

\[
\text{Im} \{ B [ \ln(x_1 - z_{uo})] q \} = -\text{Im} \{ B [ \ln(x_1 - z_{uo})] \tilde{q} \}, \tag{3.165}
\]

and

\[
\langle \ln(x_1 - z_{uo}) \rangle = \sum_{\beta=1}^{4} \ln(z_u - z_{\beta 0}) I_\beta , \tag{3.166}
\]

where

\[
I_\beta = \langle \delta_u \rangle = \text{diag} \{ \delta_{u1}, \delta_{u2}, \delta_{u3}, \delta_{u4} \}. \tag{3.167}
\]

Equation (3.164) now yields

\[
q_\beta = B^{-1} B_\beta q = B^{-1} \tilde{B}_\beta (\tilde{A}^T q_0 + \tilde{B}^T b). \tag{3.168}
\]

If the boundary \( x_2 = 0 \) is a rigid surface, then

\[
U = 0, \quad \text{at} \quad x_2 = 0. \tag{3.169}
\]

The same procedure shows that the solution is given by eqns (3.161) and (3.162) with

\[
q_\beta = A^{-1} A I_\beta (\tilde{A}^T q_0 + \tilde{B}^T b). \tag{3.170}
\]

### 3.8.3 Inclined crack in a half-plane plate

Consider the problem of a straight crack with length \( 2c \) embedded in a piezoelectric half-plane plate. The geometrical configuration of the problem to be
solved is depicted in Fig. 3.5. The orientation of the crack is denoted by $\alpha$ and the depth of the centre of the crack from the infinite straight boundary of the half-plane is denoted by $d$. Uniform traction-charge is applied on both faces of the crack while the straight boundary of the half-plane is assumed to be traction-charge free. The mathematical statement of the boundary conditions of this problem can be stated more precisely as follows:

$$t^\pm = \pm \Omega^T(\alpha) \Pi_0, \quad \|c\| < c, \eta = 0^+, \quad (3.171)$$

$$\Pi_2 = 0, \quad x_2 \leq 0, \quad (3.172)$$

$$\Pi \to 0, \quad x_1^2 + x_2^2 \to \infty, \quad (3.173)$$

where $t_\eta$ is a traction-charge vector, $(\xi, \eta, x_3)$ the local coordinate system with the origin centred at the middle point of the crack, and the direction of $\xi$ is parallel to the crack faces (Fig. 3.5). The traction applied on the lower crack face is denoted by $-\Pi_0(-\tau_\eta, -\sigma_\eta, 0, -D_\eta)^T$ where the components are expressed in terms of the local coordinates $(\xi, \eta, x_3)$. The 4x4 matrix, $\Omega(\alpha)$, whose components are the cosine of the angle between the local coordinates and the global coordinates, is in the form [14]

$$\Omega(\alpha) = \begin{bmatrix}
\cos \alpha & \sin \alpha & 0 & 0 \\
-sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad (3.174)$$

Note that $\alpha$ defined here is positive when measured from the positive direction of the $x_1$-axis to the positive direction of the $\xi$-axis, as shown in Fig. 3.5.
The boundary condition given by eqn (3.171) can be satisfied by redefining the discrete dislocation function \( b \) in eqns (3.160) and (3.168) in terms of function \( b(\xi) \) defined along the crack line, \( z_k = p_k d + t \zeta_k, \quad z_{k0} = p_k d + \zeta_{k0}^* \), where \( \zeta_k = \cos \alpha + p_k \sin \alpha \). In this case, the load parameter \( q_0 \) appearing in eqns (3.160) and (3.168) should be zero. Enforcing satisfaction of the applied traction-charge conditions on the crack faces, a singular integral equation for the dislocation function \( b \) is obtained as

\[
\frac{L}{2\pi} \int_{-c}^{c} b(\zeta) d\zeta + \frac{1}{\pi} \int_{-c}^{c} K_0(t, \xi)b(\xi) d\xi = t_0(t), \quad |t| < c \tag{3.175}
\]

where

\[
K_0(t, \xi) = \frac{1}{\pi} \sum_{\beta=1}^{N} \{B_1(z_\alpha - z_{\beta0})^{-1}B_1^{-1}\bar{B}_\beta \bar{B}_\beta^T\} \tag{3.176}
\]

is a kernel function of the singular integral equations and is Hölder-continuous along \(-c \leq \zeta \leq c\).

For single valued displacements and electric potential around a closed contour surrounding the whole crack, the following conditions must also be satisfied

\[
\int_{-c}^{c} b(\zeta) d\zeta = 0 . \tag{3.177}
\]

For convenience, normalize the interval \((-c, c)\) by the change of variables, i.e., \( t = cs, \quad \xi = cs \). If we retain the same symbols for the new functions caused by the change of variables, eqns (3.175) and (3.177) can be rewritten as

\[
\frac{L}{2\pi} \int_{-1}^{1} b(s) ds + \frac{1}{\pi} \int_{-1}^{1} K_0(s_0, s)b(s) ds = t_0(s_0), \quad |s_0| < c . \tag{3.178}
\]

\[
\int_{-1}^{1} b(s) ds = 0 . \tag{3.179}
\]

The singular integral equation (3.178) for the dislocation density \( b \) combined with eqn (3.179) can be solved numerically [36]. Since the solution for the functions, \( b(s) \), has a square root singular at both crack tips, it is more efficient for numerical calculation to let

\[
b(\zeta) \approx \frac{\Theta(s)}{\sqrt{1-s^2}} = \sum_{l=1}^{n} \frac{E_l T_l(s)}{\sqrt{1-s^2}} \tag{3.180}
\]
where $\Theta(s)$ is a regular function defined in a closed interval $[s_1, s_2]$, $c$ are the real unknown constants, and $T_k(s)$ the Chebyshev polynomials. Thus the discretized form of eqns (3.178) and (3.179) may be written as [36]

$$\sum_{k=1}^{m} \frac{1}{n} \left[ \frac{L}{2(s_{n} - s_{k})} + K_n(s_{n}, s_{k}) \right] \Theta(s_{k}) = t_{n}(s_{n}).$$

(3.181)

where

$$s_{k} = \cos \left( \frac{2k-1}{2m} \pi \right), \quad k = 1, 2, \ldots, m, \quad s_{n} = \cos \left( \frac{rn}{m} \right), \quad r = 1, 2, \ldots, m-1. \quad (3.182)$$

Equation (3.181) provides a system of $4m$ linear algebraic equations to determine $\Theta(s_k)$. Once the function $\Theta(s_k)$ has been found from eqn (3.181), the stresses and electric displacements, $\Pi_n(t)$, in a coordinate system local to the crack line can be expressed in the form

$$\Pi_{n}(t) = \Omega(\alpha) t_{n} = \Omega(\alpha) \left[ \frac{L}{2\pi} \int_{-\infty}^{\infty} \frac{b(\xi)d\xi}{t-\xi} + \frac{1}{\pi} \int_{-\infty}^{\infty} K_n(t, \xi)b(\xi)d\xi \right]. \quad (3.183)$$

Using eqn (3.181) we can evaluate the SED intensity factors $K = \{K_{II}, K_{I}, K_{III}, K_{D}\}$ at the tips, e.g., at the right tip ($\xi = c$) of the crack by following definition:

$$K = \lim_{\xi \to c} \sqrt{2\pi(\xi - c)} \Pi_{n}(\xi). \quad (3.184)$$

Combining this with the results of eqn (3.183), one then obtains

$$K^* \approx \sqrt{\frac{\pi}{4c}} \Omega(\alpha) L \Theta(1). \quad (3.185)$$

Thus the solution of the singular integral equation enables direct determination of the stress intensity factors.

### 3.8.4 Direction of crack initiation

The strain energy density criterion [37] as well as the formulation presented above can be used to predict the direction of crack initiation in half-plane piezoelectric materials. To make the derivation tractable, the crack-tip fields are first studied. In doing this, a polar coordinate system $(r, \omega)$ centred at a crack tip, say the right tip of the crack, $(x_r, x_\omega) = (c \cos \alpha, \quad d + c \sin \alpha)$ and $\omega = 0$ along the crack line is used. Then, the variable $z_\omega$ becomes

$$z_\omega = p_{\omega} d + c(\cos \alpha + p_{\omega} \sin \alpha) + r[\cos(\alpha + \omega) + p_{\omega} \cos(\alpha + \omega)]. \quad (3.186)$$
With this coordinate system, SED fields near the crack tip can be evaluated by taking the asymptotic limit of eqn (3.183), and using expressions (1.56) and (3.162) as \( r \to 0 \):

\[
\Pi_1(r, \omega) \approx -\frac{1}{2cr} \text{Im} \left[ B \left( \frac{\partial}{\partial z^*_n(\alpha) z^*_n(\alpha + \omega)} \right) B^T \right] \Theta(1) = -\frac{1}{2cr} X_1(\omega), 
\]

\[
\Pi_2(r, \omega) \approx -\frac{1}{2cr} \text{Im} \left[ B \left( \frac{1}{\sqrt{z^*_n(\alpha) z^*_n(\alpha + \omega)}} \right) B^T \right] \Theta(1) = -\frac{1}{2cr} X_2(\omega), 
\]

where \( z^*_n(x) = \cos x + p_n \sin x \).

For a piezoelectric material, the strain density factor \( S(\omega) \) can be calculated by considering the related electroelastic potential energy \( W \). The relationship between the two functions is as follows:

\[
S(\omega) = rW = \frac{r}{2} \left\{ \begin{array}{c} \Pi_1 \\ \Pi_2 \end{array} \right\} \left\{ \begin{array}{c} X_1 \\ X_2 \end{array} \right\} 
\]

(3.189)

where the matrix \( F \) is the inverse of stiffness matrix \( E \), and is given in eqn (1.40). The substitution of eqns (3.187) and (3.188) into eqn (3.189) leads to

\[
S(\omega) = \frac{1}{4c} \left\{ X_1^T X_2^T \right\} F \left[ \begin{array}{c} X_1 \\ X_2 \end{array} \right]. 
\]

(3.190)

As shown in [37], the strain energy density criterion states that the direction of crack initiation coincides with the direction of the strain energy density factor \( S_{\min} \), i.e., the necessary and sufficient condition of crack growth in the direction \( \omega_h \), is that

\[
\frac{\partial S}{\partial \omega} \bigg|_{\omega=\omega_h} = 0 \quad \text{and} \quad \frac{\partial^2 S}{\partial \omega^2} \bigg|_{\omega=\omega_h} > 0. 
\]

(3.191)

Substituting eqn (3.190) into eqn (3.191), yields

\[
\left\{ X_1^T X_2^T \right\} F \frac{\partial}{\partial \omega} \left[ \begin{array}{c} X_1 \\ X_2 \end{array} \right] = 0. 
\]

(3.192)

Solving eqn (3.192), several roots of \( \omega \) may be obtained. The fracture angle will be the one satisfying eqn (3.191).
3.9 Solution of bimaterial problems

We now consider a bimaterial piezoelectric plate containing a crack either lying along the interface of the bimaterial or embedded in any one material, for which the upper half-plane \((x_2 > 0)\) is occupied by material 1, and the lower half-plane \((x_2 < 0)\) is occupied by material 2. They are rigidly bonded together so that

\[
U^{(1)} = U^{(2)}, \quad \varphi^{(1)} = \varphi^{(2)}, \quad \text{at } x_2 = 0 \tag{3.193}
\]

where the superscripts (1) and (2) label the quantities relating to materials 1 and 2. The equality of traction-charge continuity comes from the relation \(\partial \varphi / \partial s = t\), where \(t\) is the surface traction-charge vector on a curve boundary and \(s\) is the arc length measured along the curve boundary. When points along the crack faces are considered, integration \(t^{(1)} = t^{(2)}\) provides \(\varphi^{(1)} = \varphi^{(2)}\), since the integration constants can be neglected, which correspond to rigid motion. Furthermore, assume that an arbitrary and self-equilibrated loading is specified on the crack faces. This problem has been considered by many researchers [10,14,16]. Kuo and Barnett [38] investigated the character of the singularity at interface crack tips of piezoelectric material. Suo et al. [10] showed the possibility of the existence of real as well as oscillating singularities at an interface crack tip. Qin and Yu [39] presented a solution for an arbitrarily oriented crack terminating at the interface of a piezoelectric bimaterial. Qin and Mai [14,16] generalized these results to the case of thermo-electroelastic problems. In this section, we follow the results presented in [14,16].

3.9.1 Green’s function for bimaterials

For a bimaterial plate subjected to a line force-charge \(q_0\) and a line dislocation \(b\) both located in the upper half-plane at \(z_0(x_1, x_2)\), the solution may be assumed, in a similar way as treated in the half-plane problem, in the form

\[
U^{(1)} = \frac{1}{\pi} \text{Im} \left\{ A^{(1)} (\ln(z_a^{(1)} - z_m^{(1)})) q_0 + \sum_{\beta=1}^{4} \frac{1}{\pi} \text{Im} \left\{ A^{(1)} (\ln(z_a^{(1)} - z_m^{(1)})) \right\} q_0^{(1)} \right\}, \tag{3.194}
\]

\[
\varphi^{(1)} = \frac{1}{\pi} \text{Im} \left\{ B^{(1)} (\ln(z_a^{(1)} - z_m^{(1)})) q_0 + \sum_{\beta=1}^{4} \frac{1}{\pi} \text{Im} \left\{ B^{(1)} (\ln(z_a^{(1)} - z_m^{(1)})) \right\} q_0^{(1)} \right\} \tag{3.195}
\]

for material 1 in \(x_2 > 0\) and

\[
U^{(2)} = \sum_{\beta=1}^{4} \frac{1}{\pi} \text{Im} \left\{ A^{(2)} (\ln(z_a^{(2)} - z_m^{(2)})) q_0^{(2)} \right\}, \tag{3.196}
\]

\[
\varphi^{(2)} = \sum_{\beta=1}^{4} \frac{1}{\pi} \text{Im} \left\{ B^{(2)} (\ln(z_a^{(2)} - z_m^{(2)})) q_0^{(2)} \right\} \tag{3.197}
\]
for material 2 in $x_2<0$. The value of $q$ is given in eqn (3.160) and $q^{(i)}_\beta$, $q^{(2)}_\beta$ are unknown constants which are determined by substituting (3.194)-(3.197) into (3.193). Following the derivation in subsection 3.8.2, we obtain

$$
A^{(i)}q^{(i)}_\beta + A^{(2)}q^{(2)}_\beta = A^{(i)}_p \bar{q}, \quad B^{(i)}q^{(i)}_\beta + B^{(2)}q^{(2)}_\beta = B^{(i)}_p \bar{q}.
$$

(3.198)

Solving eqn (3.198) yields

$$
q^{(i)}_\beta = B^{(i)-1}(I - 2(M^{(i)-1} + \overline{M}^{(2)-1})^{-1}L^{(i)-1})B^{(i)}_p \bar{q},
$$

(3.199)

$$
q^{(2)}_\beta = 2B^{(2)-1}(\overline{M}^{(1)-1} + M^{(2)-1})^{-1}L^{(1)-1}B^{(1)}_p \bar{q},
$$

(3.200)

where $M^{(i)} = -iB^{(i)}A^{(i)-1}$ is the surface impedance matrix.

### 3.9.2 Solution for an inclined crack in bimaterials

The geometrical configuration of the problem to be solved is depicted in Fig. 3.6, showing a crack with an orientation angle $\alpha$ and length $2c$ near an interface between material 1 and material 2. The boundary conditions of the problem are given by eqns (3.171), (3.173) and (3.193). The crack can be modelled by the Green’s functions of eqn (3.195). To this end, the discrete dislocation $b$ represents generalized distributions along a given crack line. Expressing the crack line as $\xi^{(i)} = \zeta^{(i)} + \xi^{(i)}$, where $\zeta^{(i)} = x_0 + \beta_0 x_0$ and $\zeta^{(i)} = \cos \alpha + \beta_0 \sin \alpha$, the dislocation function $b$ becomes $b(\zeta)$. At this point, $b(\zeta)$ is unknown, and will be determined such that the surface traction-charge is equal to the given value. Thus, the boundary condition (3.171) requires that

$$
\frac{L^{(i)}}{2\pi} \int_{t-\xi} b(\zeta) d\zeta + \frac{1}{\pi} \int_{-\infty}^{t} \mathbf{K}_b(t, \zeta)b(\zeta) d\zeta = t_n(t), \quad ||\xi|| < c
$$

(3.201)

where

$$
\mathbf{K}_b(t, \zeta) = \frac{1}{\pi} \sum_{i=1}^{4} \left\{ B^{(i)} \left( \zeta^{(i)} - \zeta^{(i)}_b \right)^{-1} B^{(i)^T} \right\},
$$

(3.202)

with

$$
\mathbf{B} = B^{(i)-1}(I - 2(M^{(i)-1} + \overline{M}^{(2)-1})^{-1}L^{(i)-1}B^{(i)}).$

(3.203)

The singular integral equation (3.201) together with eqn (3.177) can be used to determine the unknown $b$, and then to evaluate the SED intensity factors.

### 3.9.3 Multiple cracks in bimaterials

In what follows, formulations will be derived for an infinite piezoelectric plate of bimaterial with $N$ arbitrary located cracks of length $2c_i$ ($i = 1, 2, \cdots, N$) in the
Fig. 3.6: Geometry of a crack in a bimaterial solid.

Fig. 3.7: Geometry of the multiple crack system.

plane \((x_1, x_2)\) and subjected to remote traction-charge \(\mathbf{P}_{20}\). The configuration of the crack system is shown in Fig. 3.7.

Assume that all cracks are located in material 1 (upper half plane) and that material 2 has no cracks. These assumptions are made for simplifying the ensuing writing; extension to the case of cracks in the whole plane is straightforward. The central point of the \(i\)th crack is denoted as \((x_{i1}, x_{i2})\) and the orientation angle is denoted as \(\alpha_i\) (see Fig. 3.7). The cracks are initially assumed to remain open and hence be free of tractions and charges. The corresponding boundary conditions are, then, as follows:
On the faces of each crack i:
\[ t_{ni}^{(i)} = -\Pi_{1}^{(i)} \sin \alpha_i + \Pi_{2}^{(i)} \cos \alpha_i = 0, \quad (i = 1, 2, \cdots, N) . \quad (3.204) \]

At infinity:
\[ \Pi_i^+ = 0, \quad \Pi_i^- = \Pi_{20} , \quad (3.205) \]

where \( n \) stands for the normal direction to the lower face of a crack, and \( t_{ni}^{(i)} \) is the surface traction and charge vector acting on the \( i \)th crack.

It is convenient to represent the solution as the sum of a uniform remote SED loading in an unflawed solid and a corrective solution in which the boundary conditions are

On the faces of each crack i:
\[ t_{ni}^{(i)} = -\Pi_{1}^{(i)} \sin \alpha_i + \Pi_{2}^{(i)} \cos \alpha_i = -\Pi_{20} \cos \alpha_i , \quad (i = 1, 2, \cdots, N) . \quad (3.206) \]

At infinity:
\[ \Pi_i^+ = \Pi_i^- = 0 . \quad (3.207) \]

Using the principle of superposition [40], the boundary value problem defined above is decomposed into \( N \) sub-problems, each of which contains only one crack. The boundary conditions (3.205) can be satisfied by replacing the discrete dislocation vector \( b \) by the distributing dislocation density \( \mathbf{b}(\xi) \) defined along a given crack line such as crack \( i \), \( \xi_{i}^{(i)} = z_{i0}^{0} + t_{i} z_{i}^{*} \), \( \xi_{j}^{(i)} = z_{i0}^{0} + \hat{z}_{i} z_{j}^{*} \), where \( z_{i0}^{0} = x_{i} + p_{i0}^{0} x_{j} \), \( \hat{z}_{i} = \cos \alpha_{i} + p_{i}^{0} \sin \alpha_{i} \), and \( \alpha_{i} \) is shown in Fig. 3.7. Enforcing the satisfaction of the applied traction-charge conditions on each crack face, a system of singular integral equations for the dislocation density \( \mathbf{b}(\xi) \) is obtained as

\[
\frac{1}{2\pi} \int_{c_{i}}^{e_{i}} \frac{\mathbf{b}(\xi_{j})d\xi_{j}}{t_{j} - \xi_{i}} + \frac{1}{\pi} \int_{c_{i}}^{e_{i}} \left\{ \sum_{j,j'}^{N} \mathbf{K}_{\alpha j}(t_{i}, \xi_{j}) + \sum_{j,j'}^{N} \mathbf{G}_{\alpha j}(t_{i}, \xi_{j}) \right\} \mathbf{b}_{j}(\xi_{j})d\xi_{j} = -t_{ni}^{(i)},
\]

where \( |c_{i}| < c_{i} , \quad i = 1, 2, \cdots, N \) \hspace{1cm} (3.208)

\[
\mathbf{K}_{\alpha j}(t_{i}, \xi_{j}) = \mathbf{B}^{(i)}(\xi_{j}^{0} + t_{i} z_{i}^{*} - \hat{z}_{i} \xi_{j}^{*})^{-1}\mathbf{B}^{(i)T},
\]

\[
\mathbf{G}_{\alpha j}(t_{i}, \xi_{j}) = \sum_{\beta=1}^{4} \mathbf{B}^{(i)}(\xi_{j}^{0} + t_{i} z_{i}^{*} - \hat{z}_{i} \xi_{j}^{*})^{-1}\mathbf{B}^{(i)T} \mathbf{1}_{\beta} \mathbf{B}^{(i)T}.
\]

(3.209) \hspace{1cm} (3.210)
For single valued displacements and electric potential around a closed contour surrounding the whole $i$th crack, the following conditions must also be satisfied

$$\int_{-\epsilon_i}^{\epsilon_i} b_i(\xi) d\xi = 0, \quad (i = 1, 2, \cdots, N). \quad (3.211)$$

If we express the function $b_i(\xi_i)$ in the form

$$b_i(\xi) = \frac{\Theta(\xi)}{\sqrt{c_i^2 - \xi^2}}, \quad (3.212)$$

then, in a similar manner to that in Section 3.8, we have

$$K^*(c_i) \approx \frac{\pi}{4c_i} \Omega(\alpha_i) L^{(i)} \Theta_i(c_i), \quad (3.213)$$

and the near-tip fields as $r \to 0$:

$$\Pi_i(r, \omega) \approx -\frac{1}{2c_i r} \text{Im} \left[ \begin{bmatrix} B^{(i)} \left( \frac{p^{(i)}_u}{\sqrt{z_u^r(\alpha_i)z_u^r(\alpha_i + \omega)}} \right) B^{(i)T} \end{bmatrix} \Theta_i(c_i) \right], \quad (3.214)$$

$$\Pi_i(r, \omega) \approx \frac{1}{2c_i r} \text{Im} \left[ \begin{bmatrix} B^{(i)} \left( \frac{1}{\sqrt{z_u^r(\alpha_i)z_u^r(\alpha_i + \omega)}} \right) B^{(i)T} \end{bmatrix} \Theta_i(c_i) \right], \quad (3.215)$$

where $z_u^r(x)$ and the coordinate system $(r, \omega)$ are defined in subsection 3.8.4.

### 3.9.4 SED jump across interface

It should be mentioned that for interface problems the normal stress, $\sigma_{11}$, in the $x_1$-direction and the horizontal component of electric displacement are discontinuous across the interface. This implies that it is not possible to have the parallel stress and electric displacement vanish in both materials: perpendicular loading will give rise to parallel SED. The SED that should be used in an infinite region analysis in a finite body with uniform (typically zero) parallel loading is thus not as obvious as in the simple case of homogeneous materials. Thus, it is necessary to distinguish $(\sigma^1_{11})$ and $(D^2_1)$ in the region $x_2 > 0$, i.e., material 1, from $(\sigma^2_{11})$ and $(D^1_1)$ in the region $x_2 < 0$. Following the procedure of Rice and Sih [41], they may be derived by considering the equilibrium of a differential element occupying both regions $x_2 > 0$ and $x_2 < 0$, $x_2 = 0$ being the interface. The stress $\sigma_{11}$ and the electric displacement $D_1$ are taken to be discontinuous across the interface and the strain $\varepsilon_{11}$ and the electric field $E_1$ to be continuous along such an interface, i.e.,
(\varepsilon_{11})_1 = (\varepsilon_{11})_2, \quad (E_i)_1 = (E_i)_2. \quad (3.216)

It follows from the relation (1.91) that

\[
(\sigma_{11}^e)_i = \frac{1}{f_{i1}^{(2)}} \left[ f_{i1}^{(2)}(\sigma_{11}^e)_2 + (f_{i1}^{(2)} - f_{i2}^{(2)})\sigma_{22}^e + (p_{21}^{(2)} - p_{21}^{(1)})D_z^e \right].
\]

\[
(D_z^e)_i = \frac{1}{P_{i1}^{(2)}} \left[ P_{i1}^{(2)}(D_z^e)_2 - (P_{i1}^{(2)} - P_{i1}^{(1)})\sigma_{11}^e \right]. \quad (3.217)
\]

It is obvious from eqn (3.217) that there is a jump for \( \sigma_{11} \) and \( D_z \) across the interface.

### 3.10 A closed crack-tip model for an interface crack

In this section, a closed crack-tip model for analysing an interface crack is presented to treat the oscillatory singularity of SED crack-tip fields. Previous studies [42,43] have indicated that the full open crack-tip model will cause stress oscillatory singularity at crack tips. As was pointed out by England [43], a physically unreasonable aspect of the oscillatory singularities is that they lead to overlapping near the ends of the crack. To correct this unsatisfactory feature, Comninou [44] introduced a closed crack-tip model for analysing interface cracks in isotropic elastic materials. Qin and Mai [45] extended this model to the case of thermopiezoelectric materials. We follow the formulations presented in [45] in this section.

#### 3.10.1 Problem description

Consider a crack of length \( 2c \) lying in the interface of dissimilar anisotropic piezoelectric media. Let the crack be partially closed with frictionless contact in the intervals \((-c, -a)\) and \((b, c)\) and opened in the interval \((-a, b)\) (see Fig. 3.8). We assume that the media is subjected to far-field mechanical and electric loads, say \( \Pi^e_2 \). The surface of the crack is traction- and charge-free. For the present problem, it suffices to consider the associated problem in which the crack surface is subjected to the conditions:

\[
b_k(x_i) = U'_k(x_i, 0^+) - U'_k(x_i, 0^-), \quad |x_i| < c, \quad K = 1, 3, 4, \quad (3.218)
\]

\[
b_i(x_i) = U'_i(x_i, 0^+) - U'_i(x_i, 0^-), \quad -a < x_i < b, \quad (3.219)
\]

\[
\Pi_{2k}^{(1)} = \Pi_{2k}^{(2)} = -\Pi_{2k}^e, \quad |x_i| < c, \quad K = 1, 3, 4, \quad (3.220)
\]

\[
\Pi_{32}^{(1)} = \Pi_{32}^{(2)} = -\Pi_{32}^e, \quad -a < x_i < b \quad (3.221)
\]

and the continuous conditions on the interface excluding the interval \([-c, c]\) are given by eqn (3.193).
### 3.10.2 Full open crack-tip model

In a manner similar to that discussed in Section 3.3, define

\[
\Phi(z) = \begin{cases} 
    MB_f(z), & z \in \text{material 1}, \\
    MB_f'(z), & z \in \text{material 2},
\end{cases}
\]  

(3.222)

where

\[
M = M_1 + M_2.
\]  

(3.223)

![Geometry of a partially closed interface crack.](image)

Fig. 3.8: Geometry of a partially closed interface crack.

It should be pointed out that the physical meaning of \( \Phi \) is quite obvious and can be seen by noting the dislocation vector \( b \) defined by

\[
b = \frac{\partial}{\partial x_1} (U^+ - U^-),
\]  

(3.224)

and checking eqn (3.222), with which we have

\[
b = (\Phi^+ - \Phi^-).
\]  

(3.225)

The Plemelj-Sokhotskii formulae for the Cauchy integral yields

\[
\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{b}{x_1 - z} dx_1.
\]  

(3.226)

Using the results (3.24), (3.25) and (3.222), the conditions (3.220) and (3.221) can be expressed as

\[
N\Phi^+(x_i) + N\Phi^-(x_i) = -\Pi^c(x_i)
\]  

(3.227)
where \( \mathbf{N} = \mathbf{M}^{-1} \), for a full open interface crack model, whose solution is [45]

\[
\Phi(z) = -\frac{\mathbf{X}_0(z)}{2\pi i} \int_{\mathcal{C}} \mathbf{M}^{-1}[\mathbf{X}_0(s)]^{-1} ds + \mathbf{X}_0(z) \mathbf{p}_t(z)
\]  

(3.228)

where \( \mathbf{p}_t(z) \) is a vector of linear function of \( z \), and \( \mathbf{X}_0(z) \) a matrix of the basic Plemelj function defined as

\[
\mathbf{X}_0(z) = \mathbf{\Xi} \Gamma(z), \quad \mathbf{\Xi} = \text{diag}[\delta_1, \lambda_2, \lambda_3, \lambda_4], \quad \Gamma(z) = \left\{ (z + c)^{-\alpha} (z - c)^{\beta} \right\},
\]  

(3.229)

while \( \delta_u \) and \( \lambda_u \) are the eigenvalues and eigenvectors of

\[
(\mathbf{M} - e^{2\pi i \mathbf{D}}) \mathbf{\lambda} = 0.
\]  

(3.230)

The explicit solution of eqn (3.230) has been obtained by Suo et al. [10] as

\[
\delta_{1,2} = -\frac{1}{2} \pm \imath \varepsilon, \quad \delta_{3,4} = -\frac{1}{2} \pm \kappa,
\]  

(3.231)

where

\[
\varepsilon = \frac{1}{\pi} \tanh^{-1} [(r^2 - s)^{-1/2} - r]^{1/2}, \quad \kappa = \frac{1}{\pi} \tan^{-1} [(r^2 - s)^{-1/2} + r]^{1/2},
\]  

(3.232)

with

\[
b = \frac{1}{4} \text{tr}[\mathbf{D}^{-1} \mathbf{W}], \quad c = \left\| \mathbf{D}^{-1} \mathbf{W} \right\|,
\]  

(3.233)

where “tr” stands for the trace of the matrix, and

\[
\mathbf{D} = \text{Re}(\mathbf{M}) = \mathbf{L}_1 + \mathbf{L}_2, \quad \mathbf{W} = \text{Im}(\mathbf{M}) = \mathbf{S}_1 \mathbf{L}_1 - \mathbf{S}_2 \mathbf{L}_2
\]  

(3.234)

are two real matrices.

### 3.10.3 Closed crack-tip model

Introducing the following symbols:

\[
\mathbf{N}_0 = \begin{bmatrix}
N_{11} & N_{13} & N_{14} \\
N_{31} & N_{33} & N_{34} \\
N_{41} & N_{43} & N_{44}
\end{bmatrix}, \quad \mathbf{\Phi}_0 = \begin{bmatrix}
\Phi_1 \\
\Phi_3 \\
\Phi_4
\end{bmatrix}, \quad \mathbf{\Pi}_0 = \begin{bmatrix}
-\Pi_{21}^* \\
-\Pi_{23}^* \\
-\Pi_{24}^*
\end{bmatrix}
\]  

(3.235)

\[
\mathbf{W}_0 = \begin{bmatrix}
\tilde{W}_{21} & \tilde{W}_{23} & \tilde{W}_{24}
\end{bmatrix}, \quad \mathbf{D}_0 = \begin{bmatrix}
\tilde{D}_{21} & \tilde{D}_{23} & \tilde{D}_{24}
\end{bmatrix}.
\]  

(3.236)
\[ f_x = -W_0^\dagger (\Phi_1^* - \Phi_1) - iP_0^\dagger (\Phi_2^* + \Phi_2), \quad f_{x^2} = W_0 (\Phi_1^* - \Phi_1) - iP_0 (\Phi_2^* + \Phi_2) \]  
(3.237)

and using the boundary conditions (3.220) and (3.221), eqn (3.227) can be rewritten as

\[
N_1 \Phi_1^* + \overline{N}_1 \Phi_1 = \Pi_1 + f_1, \quad |x_1| < c, \tag{3.238}
\]

\[
N_{x_{1}} \Phi_{x_1}^* + \overline{N}_{x_1} \Phi_{x_1} = \Pi_{x_1} + f_{x_1}, \quad -a < x_1 < b \tag{3.239}
\]

where the real matrices \( \overline{W} \) and \( \overline{D} \) are defined by [35]

\[
(\overline{D} \pm i\overline{W})(\overline{D} \pm i\overline{W}) = I. \tag{3.240}
\]

Equation (3.238) is a generalized Hilbert problem if \( \Phi_1 \) is taken as a known function. The Hilbert problem can be solved by way of the technique of Clements [46]. In doing this, multiplying eqn (3.238) by \( Y_i \) and summing over \( i \) results in

\[
Y_i N_{x_{1}} \Phi_{x_1}^* + Y_i \overline{N}_{x_{1}} \Phi_{x_1} = Y_i (\Pi_{x_{1}} + f_{x_{1}}), \quad |x_1| < c. \tag{3.241}
\]

The \( Y_i \) are chosen such that

\[
Y_i N_{x_{1}} = V_j, \quad Y_i \overline{N}_{x_{1}} = \lambda V_j. \tag{3.242}
\]

Eliminating \( V_j \) from eqn (3.242), one has

\[
(\overline{N}_{x_{1}} - \lambda N_{x_{1}})Y = 0. \tag{3.243}
\]

For a non-trivial solution it is necessary that

\[
\| N_{x_{1}} - \lambda N_{x_{1}} \| = 0. \tag{3.244}
\]

Let \( \lambda, (\gamma = 1, 2, 3) \) denote the roots of eqn (3.244) and the corresponding values of \( Y_i \) and \( V_j \) be expressed by \( Y_\gamma \) and \( V_\gamma \). Equation (3.241) may then be recast as

\[
Y_\gamma \Phi_{x_1}(x_1) - \lambda_\gamma Y_\gamma \Phi_{x_1}(x_1) = Y_\gamma (\Pi_{x_{1}}(x_1) + f_{x_{1}}(x_1)), \quad |x_1| < c. \tag{3.245}
\]

Equation (3.245) is a standard Hilbert problem and its solution may be given by

\[
\Phi_{\gamma}(z) = \sum_{k=1}^{3} (V^{-1})_\alpha \frac{X_\gamma(z)}{2\pi i} \left[ \int_{\tilde{x}}^{x} \frac{Y_\gamma \Pi_{x_1}(x)}{X^*_\gamma(x) - x - z} dx + \int_{-\infty}^{b} \frac{Y_\gamma f_{x_1}(x)}{X^*_\gamma(x) - x - z} dx \right] \tag{3.246}
\]

where
\[ X_k(z) = (z - c)^{n_k}(z + c)^{-\nu_k}, \quad \delta_k = \frac{1}{2\pi i} \ln \lambda_k. \]  \tag{3.247}

In eqn (3.246) the branches of \( X_k(z) \) are chosen such that \( zX_k(z) \to 1 \), as \( |z| \to \infty \), and the argument of \( \lambda_k \) is selected to lie between 0 and \( 2\pi \). Substitution of eqn (3.246) into eqn (3.239) leads to

\[ A(x) b_2(x) + \frac{1}{\pi} \int_a^b \frac{B(x,t)}{t-x} b_2(t) \, dt + \frac{1}{\pi} \int_a^b \frac{C(x,t)}{t-x} \left[ \frac{1}{\pi} \int_0^\infty \frac{b_1(t)}{t-\tau} \, d\tau \right] \, dt = D(x) \]  \tag{3.248}

where

\begin{align*}
A(x) &= \sum_{i=1,3,4} \sum_{k=1}^2 (V^{-1})_{ik} N^*_i \tilde{W}_i, \\
B(x,t) &= \tilde{D}_{22} + \sum_{i=1,3,4} \sum_{k=1}^2 (V^{-1})_{ik} N^*_i \tilde{W}_i - X_i(x) \tilde{N}^*_i \tilde{D}_i, \\
C(x,\tau) &= -\sum_{i=1,3,4} \sum_{k=1}^2 (V^{-1})_{ik} X_i(x) \tilde{N}^*_i \tilde{D}_i, \\
D(x) &= \Pi_{22} + \sum_{i=1,3,4} \sum_{k=1}^2 (V^{-1})_{ik} Y_{ik} \Pi_{ik} \left\{ \tilde{W}_i - \frac{1}{\pi} X_i(x) \tilde{D}_i \int_x^e \frac{dt}{X_i^*+(t)(t-x)} \right\} \\
\end{align*}

with

\begin{align*}
N^*_i &= Y_{i1} \tilde{W}_{1i} + Y_{i2} \tilde{W}_{2i} + Y_{i3} \tilde{W}_{3i}, \quad N^*_{ik} = Y_{i1} \tilde{D}_{1i} + Y_{i2} \tilde{D}_{2i} + Y_{i3} \tilde{D}_{3i}. \tag{3.253}
\end{align*}

For convenience, normalizing the interval \((-a, b)\) by the change of variables;

\begin{align*}
t &= \frac{b-a}{2} + \frac{b+a}{2} \xi, \quad x &= \frac{b-a}{2} + \frac{b+a}{2} s, \quad \tau &= \frac{b-a}{2} + \frac{b+a}{2} \xi, \tag{3.254}
\end{align*}

If we retain the same symbols for the new functions, eqn (3.248) can be rewritten as

\[ A(s_0) b_2(s_0) + \frac{1}{\pi} \int_{s-s_0}^{s_0} B(s_0,\xi) b_2(s) \, ds + \frac{1}{\pi} \int_{s-s_0}^{s_0} C(s_0,\xi) \left[ \frac{1}{\pi} \int_{s-s_0}^{s_0} b_1(s) \, ds \right] ds_i = D(s_0). \]  \tag{3.255}

In addition to eqn (3.255), the single-valuedness of elastic displacements and electric potential around a closed contour surrounding the whole crack requires

\[ \int_{s}^{s_0} b_1(s) \, ds = 0. \]  \tag{3.256}
Finally, the separation condition and the condition of unilateral constraint require that

\[ u^{(1)}_s(x_1, 0) - u^{(2)}_s(x_1, 0) \geq 0, \quad \text{for} \ -a \leq x_1 \leq b, \]

\[ \sigma_{22}(x_1, 0) \leq 0, \quad \text{for} \ -c < x_1 \leq -a, \ b \leq x_1 < c. \]  \hspace{1cm} (3.257)

From eqn (3.255) we know that

\[ \sigma_{22} = A(s_0) b_2(s_0) + \frac{1}{\pi} \int_{-1}^{1} \frac{B(s_0, s)}{s - s_0} b_2(s) \, ds \]

\[ + \frac{1}{\pi} \int_{-1}^{1} C(s_0, s_\tau) \left[ \frac{1}{\pi} \int_{-1}^{1} \frac{b_2(s) \, ds}{s - s_\tau} \right] \, ds_\tau - D(s_0) \]  \hspace{1cm} (3.258)

for \ -c < x_1 < a \ and \ b < x_1 < c. \ Since \ the \ contact \ is \ smooth, \ \sigma_{22} \ should \ be \ equal \ to \ zero \ at \ s_0 = \pm 1, \ which \ provides \ two \ conditions \ for \ determining \ the \ unknowns, \ a \ and \ b. \ Using \ eqn \ (3.180), \ the \ discretized \ form \ of \ eqns \ (3.255) \ and \ (3.256) \ may \ be \ written \ as \ [36]

\[ A(s_{0r}) \Theta_2(s_{0r}) + \sum_{k=1}^{n} \left[ \frac{B(s_{0r}, s_j)}{s_j - s_{0r}} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{C(s_{0r}, s_j)(1 - s_j^2)}{(s_j - s_{0r})(s_j - s_q)} \right] \Theta_2(s_j) = D(s_{0r}), \]

\[ \sum_{k=1}^{n} \Theta_2(s_k) = 0, \]  \hspace{1cm} (3.259)

where

\[ s_k = \cos \left( \frac{(k - 1)\pi}{2n} \right), \quad k = 1, 2, \ldots, n, \quad s_{0r} = \cos \left( \frac{r\pi}{n} \right), \quad r = 1, 2, \ldots, n - 1, \]

\[ s_q = \cos \left( \frac{(j - 1)\pi}{2m} \right), \quad k = 1, 2, \ldots, m, \quad m \neq n. \]  \hspace{1cm} (3.260)

Equation (3.259) provides a system of \( n \) linear algebraic equations to determine \( \Theta_2(s_k) \). Once the function \( \Theta_2(s_k) \) has been found from eqn (3.259), the stresses and electric displacements, \( \Pi_2(x_1) \), can be given from eqn (1.56) in the form

\[ \Pi_2(x_1) = \tilde{W}b + \frac{\tilde{D}}{\pi} \int_{-1}^{1} \frac{b(s) \, ds}{x_1 - s}, \]  \hspace{1cm} (3.261)

Thus, we can evaluate the SED intensity factors by the following equation:

\[ K^* \approx \lim_{x_1 \to \infty} \sqrt{2\pi(x_1 + c)} \left[ \tilde{W}b + \frac{\tilde{D}}{\pi} \int_{-1}^{1} \frac{b(s) \, ds}{x_1 - s} \right]. \]  \hspace{1cm} (3.262)
The energy release rate can be evaluated by considering the relative displacements $\Delta U$, which are obtained from the definition $\Phi$ in eqn (3.226) as

$$\Delta U_j(x) = \sum_{k=1}^{3} (W^{-1})_k \left[ 2Y_j \Pi_j + N_k^c \sqrt{x^2 - x^2} \sum_{l=1}^{m} \frac{E_{k,l}^x}{k} T_{k,l}(x_1/c) + \left( \sum_{l=1}^{m} \frac{E_{k,l}^x}{k+1} T_{k,l}(x_1/c) \right) \right]$$

where $T_j(x)$ are Chebyshev polynomials of the second kind. The energy release rate $G$ can then be evaluated by substituting eqns (3.261) and (3.263) into eqn (2.17).

### 3.11 Interaction between macro- and micro-cracks

The effect of microcracks on the stress and electric displacement near the tips of a main crack has been considered by Qin [47]. In that study, the Green’s function solution and principle of superposition are used to formulate a system of singular integral equations for solving the unknown dislocation density of elastic displacement-electric potential. The residual stress and electric displacement on the microcrack are evaluated directly from the near-tip field of the main crack.

Fig. 3.9: Geometry of the crack-microcrack problem.

#### 3.11.1 Statement of the problem

The geometric configuration of the problem is depicted in Fig. 3.9, showing a main crack of length $2c_0$ and the $N$ microcracks of length $2c_i$ ($i=1, 2, \cdots, N$) near the main crack tip. The central point of the $i$th microcrack is denoted by $(x_i, y_i)$. The orientation angle is $\alpha_i$. The cracks are initially assumed to remain open with free traction-charge. The corresponding boundary conditions are
For the $i$th crack:

$$t_{ni} = -\Pi_1 \sin \alpha_i + \Pi_2 \cos \alpha_i = 0, \quad (i=0, 1, 2, \cdots, N).$$  

(3.264)

At infinity:

$$\Pi_i^\alpha = \Pi_{30}, \quad \Pi_i^\beta = 0.$$  

(3.265)

The main crack is assumed to be much larger than the $i$th microcrack. Moreover, the distance between the right tip of the main crack and the centre of the $i$th microcrack is such that $e_0 >> c_i, e_0 >> d_i$ ($e_0, c_i$ and $d_i$ are shown in Fig. 3.9). Under this assumption, the residual stress and electric displacement are calculated approximately by the near-tip solutions of the main crack.

3.11.2 Singular integral equations

The problem shown in Fig. 3.9 can be considered as a superposition of $N+1$ single crack problems with unknown dislocation densities on each crack face. The boundary conditions for each single problem are still defined by eqns (3.264) and (3.265), except that

$$t_{ni} = -\Pi_1 \sin \alpha_i + \Pi_2 \cos \alpha_i = \Pi_{ni}^\alpha, \quad (i=1, 2, \cdots, N)$$  

(3.266)

where $\Pi_{ni}^\alpha$ is the residual stress and electric displacement, which is evaluated by the near-tip fields of the main crack. For a crack of length $2c_0$ subjected to the far-field $\Pi_{30}$, the near-tip fields are given in eqns (2.12) and (2.13). With these two equations, the residual SED $\Pi_{ni}^\alpha$ can be expressed as

$$\langle \Pi_{ni}^\alpha \rangle_K = (\Pi_{30})_1 \sqrt{\frac{e_0}{2r}} \text{Re} \sum_{p=0}^4 \{B_{kp} B_{pi}^{-1} \sqrt{\cos \alpha_i + p_i \sin \alpha_i} \}$$  

(3.267)

where $r_i^2 = (x_i - c_0)^2 + x_{2i}^2$.

Using the solution (3.153) and the principle of superposition [40], the electroelastic solution to the problem shown in Fig. 3.9 can be expressed as follows:

$$\Pi_1 = -\frac{1}{\pi} \sum_{i=0}^N \text{Im}[B P (z_i - z_{00})^{-1}] B^T \mathbf{b}_i,$$  

(3.268)

$$\Pi_2 = \frac{1}{\pi} \sum_{i=0}^N \text{Im}[B (z_i - z_{00})^{-1}] B^T \mathbf{b}_i$$  

(3.269)

where
\[ \hat{\xi}_{aw} = z_{aw}^0 + \hat{\xi} \xi_{aw} = x_{aw} + p_a x_{aw} + \xi (\cos \alpha_i + p_a \sin \alpha_i). \]  

(3.270)

Using eqns (3.268) and (3.269), the boundary condition (3.266) can be formulated as

\[ \frac{L}{2\pi} \int_{-\xi^0}^{\eta^0} \frac{\mathbf{b}_0(\xi)d\xi}{\eta - \xi} + \frac{1}{\pi} \sum_{\eta^j}^{\eta^j} \text{Im} \int_{-\xi}^{\xi} K_{ij}^{(0)}(\eta, \xi) b_j(\xi)d\xi = 0, \]

(3.271)

\[ \frac{L}{2\pi} \int_{-\xi^0}^{\eta^0} \frac{\mathbf{b}_0(\xi)d\xi}{\eta - \xi} + \frac{1}{\pi} \sum_{\eta^j}^{\eta^j} \text{Im} \int_{-\xi}^{\xi} K_{ij}^{(1)}(\eta, \xi) b_j(\xi)d\xi = -\mathbf{T}_i^{(0)}(\eta), \]

(3.272)

where

\[ K_{ij}^{(0)}(\eta, \xi) = \mathbf{B}^T(z_{aw}^*(\eta_{aw}^0 + \xi_{aw}^0)) \mathbf{B} \]

(3.273)

with

\[ z_{aw} = \eta_{aw} + z_{aw}^0. \]

(3.274)

Moreover, the single-valued condition requires that

\[ \int_{-\xi}^{\xi} b_i(\xi)d\xi = 0, \quad (i = 0, 1, 2, \ldots, N). \]

(3.275)

The singular integral equations (3.271) and (3.272) together with eqn (3.275) can be used to determine the unknown \( \mathbf{b} \), and then to evaluate the SED intensity factors.

### 3.12 Anti-plane crack in piezoelectric ceramic strip [48,49]

This section deals with the determination of stress and electric displacement fields of a piezoelectric ceramic strip containing an anti-plane crack. This problem has been considered by Shindo et al. [48, 49] and Narita and Shindo [50] by way of the Fourier transform method. These researchers considered a piezoelectric body in the form of an infinitely long strip containing a finite crack subjected to external loads as shown in Fig. 3.10. A set of Cartesian coordinates \((x, y, z)\) is attached to the centre of the crack for reference purposes. The poled piezoelectric ceramic strip with \( z \) as the poling axis occupies the region \((-\infty < x < \infty, -h \leq y \leq h)\), and is thick enough in the \( z \)-direction to allow a state of anti-plane shear. The crack is situated along the plane \((-c < x < c, y=0)\). Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for \( 0 \leq x \leq \infty, \ 0 \leq y \leq h \) only. Four possible cases of loading conditions at the edges of the strip are considered. They are
Case 1: \[ \sigma_{yx}(x, h) = \tau_0, \quad D_y(x, h) = D_0, \] (3.276)

Case 2: \[ \gamma_{yx}(x, h) = \gamma_0, \quad E_y(x, h) = E_0, \] (3.277)

Case 3: \[ \sigma_{yx}(x, h) = \tau_0, \quad E_y(x, h) = E_0, \] (3.278)

Case 4: \[ \gamma_{yx}(x, h) = \gamma_0, \quad D_y(x, h) = D_0, \] (3.279)

where \( \tau_0 \) is a uniform shear traction, \( \gamma_0 \) a uniform shear strain, \( D_0 \) a uniform electric displacement and \( E_0 \) a uniform electric field. The boundary conditions are given by

\[ \sigma_{yx}(x, 0) = 0, \quad 0 \leq x < c, \] (3.280)

\[ u_x(x, 0) = 0, \quad c \leq x < \infty, \] (3.281)

\[ \phi(x, 0) = 0, \quad c \leq x < \infty, \] (3.282)

\[ E_y(x, 0) = E'_y(x, 0), \quad 0 \leq x < c, \] (3.283)

\[ D_y(x, 0) = D'_y(x, 0), \quad 0 \leq x < c \] (3.284)

where the superscript “c” stands for the electric quantities in the crack gap.

In a vacuum, the constitutive equation (1.45) and the governing equation (1.46) become

\[ D_y = \kappa_0 E_y, \quad D'_y = \kappa_0 E'_y, \] (3.285)

Fig. 3.10: A piezoceramic strip with a crack. Definition of geometry and loading.
\[ \phi_{xx} + \phi_{yy} = 0 \]  

(3.286)

where \( \kappa_0 \) is the dielectric constant of gas.

The solution for this problem can be obtained by way of the Fourier transform technique. To this end, a Fourier transform is applied to eqn (1.46) with \( f_{15} = q_{10} = 0 \), and the results are

\[ u_{i}(x, y) = \frac{2}{\pi} \int_{0}^{\infty} \left[ A_{i}(\alpha) \cosh(\alpha y) + B_{i}(\alpha) \sinh(\alpha y) \right] \cos(\alpha x) d\alpha + a_{i0} y, \quad (3.287) \]

\[ \phi(x, y) = \frac{2}{\pi} \int_{0}^{\infty} \left[ B_{1}(\alpha) \cosh(\alpha y) + B_{2}(\alpha) \sinh(\alpha y) \right] \cos(\alpha x) d\alpha - b_{0} y \quad (3.288) \]

where \( A_{i}(\alpha), B_{i}(\alpha) (i = 1, 2) \) are the unknowns to be determined and \( a_{i0}, b_{0} \) are real constants, which will be determined from the edge loading conditions. A simple calculation leads to

\[ \sigma_{xy}(x, y) = \frac{2}{\pi} \int_{0}^{\infty} \alpha \left[ c_{44}A_{i}(\alpha) + e_{15}B_{i}(\alpha) \right] \sinh(\alpha y) \]

\[ \times \left[ c_{44}A_{2}(\alpha) + e_{15}B_{2}(\alpha) \right] \cos(\alpha x) d\alpha + c_{0}, \quad (3.289) \]

\[ D_{xy}(x, y) = \frac{2}{\pi} \int_{0}^{\infty} \alpha \left[ e_{15}A_{i}(\alpha) - \kappa_{11}B_{i}(\alpha) \right] \sinh(\alpha y) \]

\[ \times \left[ e_{15}A_{2}(\alpha) - \kappa_{11}B_{2}(\alpha) \right] \cos(\alpha x) d\alpha + d_{0} \quad (3.290) \]

where

\[ c_{0} = c_{44}a_{0} - e_{15}b_{0}, \quad d_{0} = e_{15}a_{0} + \kappa_{11}b_{0}. \quad (3.291) \]

Application of a Fourier transform to eqn (3.286) yields

\[ \phi^{*}(x, y) = \frac{2}{\pi} \int_{0}^{\infty} C(\alpha) \sinh(\alpha y) \cos(\alpha x) d\alpha, \quad 0 \leq x < c \quad (3.292) \]

where \( C(\alpha) \) is also an unknown function.

By applying the edge loading conditions, the constants \( a_{0} \) and \( b_{0} \) are obtained as

\[ a_{0} = \frac{\kappa_{11}^{*} + e_{15}D_{0}}{c_{44} \kappa_{11} + e_{15}^{*}}, \quad b_{0} = \frac{c_{44}D_{0} - e_{15}^{*} \kappa_{11}}{c_{44} \kappa_{11} + e_{15}^{*}}, \quad \text{for case 1,} \quad (3.293) \]

\[ a_{0} = \gamma_{0}, \quad b_{0} = E_{0}, \quad \text{for case 2,} \quad (3.294) \]
\[ a_0 = \frac{\tau_0 + e_{13} E_2}{c_{44}}, \quad b_0 = E_0, \quad \text{for case 3,} \quad \text{(3.295)} \]
\[ a_0 = \gamma_0, \quad b_0 = \frac{D_0 - e_{15} \gamma_0}{\kappa_{11}}, \quad \text{for case 4.} \quad \text{(3.296)} \]

The boundary conditions of eqns (3.276)-(3.279) lead to the following relations between unknown functions

\[ A_1(\alpha) = -\tanh(\alpha h) A_4(\alpha), \quad B_2(\alpha) = -\tanh(\alpha h) B_4(\alpha). \quad \text{(3.297)} \]

Satisfaction of the four mixed boundary conditions (3.280)-(3.283) leads to two simultaneous dual integral equations of the following form

\[ \int_0^\infty \alpha \tanh(\alpha h) A_1(\alpha) \cos(\alpha x) d\alpha = \frac{\pi c_0}{2c_{44}}, \quad 0 \leq x < c, \quad \text{(3.298)} \]
\[ \int_0^\infty A_1(\alpha) \cos(\alpha x) d\alpha = 0, \quad c \leq x < \infty, \]
\[ \int_0^\infty \alpha B_1(\alpha) \sin(\alpha x) d\alpha = 0, \quad 0 \leq x < c, \quad \text{(3.299)} \]
\[ \int_0^\infty B_1(\alpha) \cos(\alpha x) d\alpha = 0, \quad c \leq x < \infty. \]

The set of two simultaneous dual integral eqns (3.298) and (3.299) may be solved by using new functions \( \Phi_1(\xi) \) and \( \Phi_2(\xi) \) defined by

\[ A_1(\alpha) = \frac{\pi c_0^2}{2} \int_0^1 \sqrt{\xi} \Phi_1(\xi J_0(\alpha \xi) d\xi, \quad B_1(\alpha) = \frac{\pi c_0^2}{2} \int_0^1 \sqrt{\xi} \Phi_2(\xi J_0(\alpha \xi) d\xi \quad \text{(3.300)} \]

where \( J_0() \) is the zero order Bessel function of the first kind. The function \( \Phi_1(\xi) \) is the solution to the following Fredholm integral equation of the second kind:

\[ \frac{c_0}{c_0} \Phi_1(\xi) + \int_0^1 K(\xi, \eta) \left[ \frac{c_0}{c_0} \Phi_1(\eta) \right] d\eta = \sqrt{\xi} \quad \text{(3.301)} \]

where the kernel function \( K(\xi, \eta) \) is expressed as

\[ K(\xi, \eta) = \sqrt{\xi} \int_0^1 \alpha [\tanh(\alpha h) - 1] J_0(c \alpha \xi) J_0(c \alpha \eta) d\alpha. \quad \text{(3.302)} \]

The function \( \Phi_2(\xi) = 0. \)

Thus, the singular parts of the SED fields in the neighbourhood of the crack
tip are

\[ \gamma_{sc} = -\frac{K_{III}}{c_{44}} \sin\left(\frac{0}{2}\right), \quad \gamma_{sc} = \frac{K_{III}}{c_{44}} \cos\left(\frac{0}{2}\right), \quad E_x = E_y = 0, \quad \sigma_{sc} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin\left(\frac{0}{2}\right), \quad \sigma_{sc} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\left(\frac{0}{2}\right), \quad D_x = -\frac{e_{15}K_{III}}{c_{44}} \sin\left(\frac{0}{2}\right), \quad D_x = \frac{e_{15}K_{III}}{c_{44}} \cos\left(\frac{0}{2}\right) \]

(3.303) (3.304) (3.305) (3.306)

where \( r \) and \( \theta \) are the polar coordinates defined in Fig. 2.2, and \( K_{III} \) is the so-called stress intensity factor defined as

\[ K_{III} = \lim_{x \to c} \sqrt{2\pi(x-c)} \sigma_{sc}(x,0) = c_{44} \sqrt{\pi c} \Phi_1(1). \]

(3.307)

With the expressions (3.303)-(3.306), the generalized path-independent \( J \)-integral (2.14) can be evaluated as

\[ J = \frac{1}{2c_{44}} K_{III}^2. \]

(3.308)

Writing the energy release rate expression \( J \) in terms of the mechanical and electric loads, we obtain

\[ J = \frac{\pi c}{2} c_{44} \left( \frac{\epsilon_0}{\epsilon_0} \right)^2 \Phi_1^2(1), \quad \text{for cases 1 and 3}, \]

(3.309)

\[ J = \frac{\pi c}{2} c_{44} \left( \frac{\epsilon_0 \gamma_0 - e_{15} E_0}{\epsilon_0} \right)^2 \Phi_1^2(1), \quad \text{for case 2}, \]

(3.310)

\[ J = \frac{\pi c}{2} c_{44} \left( \frac{(e_{44} \kappa_{11} + e_{15}^2 \gamma_0 - e_{15} D_0}{\kappa_{11} \epsilon_0} \right)^2 \Phi_1^2(1), \quad \text{for case 4}. \]

(3.311)

It can be seen from eqn (3.309) that the electric fields have no effect on the value of \( J \) in cases 1 and 3, and thus they do not affect crack propagation.

To illustrate the effect of electroelastic interactions on the stress intensity factor and the energy release rate, the numerical results obtained by Shindo et al. [49] are presented here. In the analysis, they take PZT-5H, PZT-6B and BaTiO\(_3\) ceramics as an example, for which the engineering material constants are listed in Table 3.1.
Table 3.1: Material properties of piezoelectric ceramics

<table>
<thead>
<tr>
<th>Ceramics</th>
<th>PZT-5H</th>
<th>PZT-6B</th>
<th>BaTiO$_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{44}(\times10^{10}\text{N/m}^2)$</td>
<td>3.53</td>
<td>2.71</td>
<td>4.3</td>
</tr>
<tr>
<td>$e_{15}(\text{C/m}^2)$</td>
<td>17.0</td>
<td>4.6</td>
<td>11.6</td>
</tr>
<tr>
<td>$\kappa_{11}(\times10^{-10}\text{C/Vm})$</td>
<td>151.0</td>
<td>36.0</td>
<td>112.0</td>
</tr>
<tr>
<td>Critical energy release rate $J_{cr}(\text{N/m})$</td>
<td>5.0</td>
<td>5.0</td>
<td>4.0</td>
</tr>
</tbody>
</table>

![Fig. 3.11: Energy release rate of PZT-5H (Case 2).](image1)

![Fig. 3.12: Energy release rate of piezoelectric ceramics (Case 2).](image2)
The normalized energy release rate $J/J_{cr}$ of a PZT-5H ceramic is shown in Fig. 3.11 as a function of the applied electric field $E_0$ and the $c/h$ ratio for a crack length of $2c=0.02$ m and $\gamma_0 = 9.5 \times 10^{-5}$ (case 2). In case 2, as the magnitude of the electric load is increased from zero, $J$ can be made to increase or to decrease depending on the directions of the load. However, once the minimum value of $J$ is reached, further increase in the electric load will monotonically increase $J$. Fig. 3.12 displays the normalized energy release rate $J/J_{cr}$ of the above-mentioned three types of piezoelectric ceramics against the applied electric field $E_0$ for case 2 and $c/h=1.5$. The mechanical loads are fixed such that $J=J_{cr}$ at zero electric loads for $c/h\rightarrow 0$. $J$ is found to depend on the piezoelectric material constants.

### 3.13 Crack kinking problem

The problems of crack kinking in a bimaterial system with various material combinations is treated in this section. The combinations may be piezoelectric-piezoelectric (PP), piezoelectric-anisotropic conductor (PAC), piezoelectric-isotropic dielectric (PID), or piezoelectric-isotropic conductor (PIC). The focus is on the effect of electric field on the path selection of crack extension.

There are many analytical studies of crack kinking in solid materials, for example, the work of Atkinson [51], Lo [52], Hayashi and Nemat-Nasser [53], He and Hutchinson [54] for isotropic materials; Miller and Stock [55], Atkinson et al. [56] for anisotropic solids, and Qin and Mai [57] for piezoelectric materials. However, we restrict our attention here to the formulations and numerical results presented in [57].

![Fig. 3.13: Geometry of the kinked crack system.](image)

The geometrical configuration analysed is depicted in Fig. 3.13. A main crack of length $2c$ is shown lying along the interface between two dissimilar piezoelectric materials, with a branch of length $a$ and angle $\theta$ kinking upward into
material 1 or downward into material 2. The length $a$ is assumed to be small compared to the length $2c$ of the main crack. To solve the branch problem, we first derive the Green’s function satisfying the traction-charge free conditions on the faces of the main crack by way of the Stroh formalism and the solution for bimaterials due to an edge dislocation. Competition between crack extension along the interface and kinking into the substrate is investigated with the aid of the integral equations developed and the maximum energy release rate criterion. The formulations for the energy release rate and the stress and electric displacement (SED) intensity factors to the onset of kinking are expressed in terms of dislocation density function and branch angle. The numerical results for the energy release rate versus loading condition and branch angle are presented to study the relationships among the kinking angle, loading condition and energy release rate.

### 3.13.1 Crack-dislocation interaction in various bimaterial systems

#### (a) Crack-dislocation interaction in a homogeneous material (HM)

Consider the problem of a piezoelectric branched crack in a two-dimensional infinite medium (see Fig. 3.13). The interaction between crack and dislocation can be analysed by first studying the Green function due to a dislocation line, since a crack may be viewed as a continuous distribution of dislocation singularities. For an infinite plane subject to a line dislocation, say $\mathbf{b}$, located at $z_0(x_1, x_2)$, the electroelastic solution is

$$
\mathbf{f}_0(z) = \frac{\ln(z - z_0)}{2\pi i} \mathbf{B}^\dagger \mathbf{b}.
$$

(3.312)

The expression for $\mathbf{f}_0(z)$ is obtained by imposing zero net traction-charge in a circuit around the dislocation, and by demanding that the jumps in EDEP be given by $\Delta \mathbf{u} = [\Delta u_1, \Delta u_2, \Delta u_3, \Delta \varphi]^T$, where $\Delta u$ is the jump for the quantity $u$ across the dislocation line. In the presence of crack, however, the solution for an edge dislocation is no longer given by $\mathbf{f}_0(z)$ alone. In addition to the conditions imposed at $z_0$, the boundary conditions for the crack must be satisfied. This can be accomplished by evaluating the traction-charge on the crack face due to the dislocation singularity and introducing an appropriate function $\mathbf{f}_1(z)$ to cancel the traction-charge. Substituting for $\mathbf{f}_0(z)$ from eqn (3.312) into eqn (1.65), and then into eqn (1.56), the SED on the crack surface induced by the edge dislocation alone at $z_0$ is of the form

$$
\mathbf{t}(x) = \mathbf{Bf}'_0(x) + \overline{\mathbf{Bf}'}_0(x) = \frac{1}{2\pi i} \mathbf{B} \left[ \frac{1}{x - z_{0u}} \right] \mathbf{B}^\dagger \mathbf{b} - \frac{1}{2\pi i} \mathbf{B} \left[ \frac{1}{x - z_{0u}} \right] \overline{\mathbf{B}}^\dagger \mathbf{b}.
$$

(3.313)

The prescribed traction-charge, $-\mathbf{t}(x_1)$, leads to the following Hilbert problem [23]:

$$
\mathbf{Bf}'_1(z) = -\frac{\chi'(z)}{2\pi i} \int^c -\mathbf{t}(x)dx + \mathbf{e}\chi(z)
$$

(3.314)
where \( \chi(z) \) is the basic Plemelj function defined as
\[
\chi(z) = (z^2 - c^2)^{-1/2}.
\]  
(3.315)

Using contour integration one is finally led to
\[
f'_0(z) = -\frac{1}{2\pi i}\mathbf{F}_c(z,z_0)\mathbf{B}'\mathbf{b} + \frac{1}{2\pi i}\mathbf{B}^{-1}\mathbf{F}_c(z,z_0)\mathbf{B}'\mathbf{b}, \quad c = -\frac{\mathbf{Lb}}{4\pi}
\]  
(3.316)

where
\[
\mathbf{F}_c(z,z_0) = \frac{1}{2} \left[ \frac{1}{z_u - z_{0u}} \right. \left[ 1 - \frac{\chi(z_u)}{\chi(z_{0u})} \right] \right].
\]  
(3.317)

The functions defined in eqns (3.312) and (3.316) together provide a Green’s function for the crack problem:
\[
f'(z) = f'_0(z) + f'_b(z) = \frac{1}{2\pi i} \left[ \left( \frac{1}{z_u - z_{0u}} \right) - \mathbf{F}_c(z,z_0) \right] \mathbf{B}'\mathbf{b} + \mathbf{B}^{-1}\mathbf{F}_c(z,z_0)\mathbf{B}'\mathbf{b} \right].
\]  
(3.318)

It is obvious that the singularity is contained in \( f_b(z) \) and the crack interactions are accounted for by \( f_b(z) \).

(b) Crack-dislocation interaction in a PP bimaterial
The solution for this bimaterial group has been provided in Section 3.9. Using those solutions, the functions \( f_0(z^{(i)}) \) are expressed as
\[
f_0(z^{(i)}) = \frac{1}{2\pi i} \left[ \ln(z_u^{(i)} - z_{0u}^{(i)}) \mathbf{B}^{0T} + \sum_{k=1}^{4} \ln(z_u^{(i)} - z_{0k}^{(i)}) \mathbf{B}_k \right] \mathbf{b},
\]  
(3.319)

for material 1 \((x_2 = y > 0)\), and
\[
f_0(z^{(2)}) = \frac{1}{2\pi i} \sum_{k=1}^{4} \ln(z_u^{(2)} - z_{0k}^{(1)}) \mathbf{B}_k \mathbf{b},
\]  
(3.320)

for material 2 \((y < 0)\), where
\[
\mathbf{B}_k = \mathbf{B}^{(i-1)}(\mathbf{I} - 2(\mathbf{M}^{(i-1)} + \mathbf{M}^{(2i-1)} - \mathbf{L}^{(i-1)}))\mathbf{B}^{0T}\mathbf{I}\mathbf{B}^{0T},
\]  
(3.321)
\[
\mathbf{B}_k^{**} = \mathbf{B}^{(2i-1)}(\mathbf{M}^{(i-1)} + \mathbf{M}^{(2i-1)})^{-1}\mathbf{L}^{(i-1)}\mathbf{B}^{0T}\mathbf{I}\mathbf{B}^{0T}.
\]  
(3.322)

The interface SED induced by the dislocation \( \mathbf{b} \), then, is of the form
\[ t(x) = B^{(2)} f^{(2)}_0(x) + \mathcal{B}^{(2)} F^{(2)}_0(x) = [F_{m}(x) + \mathcal{F}_{m}(x)] b \]  

(3.323)

where

\[ F_{m}(x) = \frac{1}{2\pi i} \sum_{k=0}^{4} \frac{B^{(2)} B^m_{0k}}{x - z^{(i)}_{0k}}. \]  

(3.324)

This traction-charge vector, \( t(x) \), can be removed by superposing a solution with the traction-charge, -\( t(x) \), induced by the potential \( f_{i}(z) \). The solution for \( f_{i}(z) \) can be obtained by setting

\[ h(z) = \begin{cases} B^{(i)} f^{(i)}_0(z), & \text{in material 1,} \\ \mathcal{H}^{(i)} B^{(2)} f^{(2)}_0(z), & \text{in material 2,} \end{cases} \]  

(3.325)

where \( \mathcal{H} = -\mathcal{M}^{(i)} - \overline{\mathcal{M}}^{(i)} \).

The function \( h(z) \) defined above together with prescribed traction-charge, -\( t(x) \), on the crack faces, yields the following non-homogeneous Hilbert problem:

\[ h^*(x) + \mathcal{H}^{-1} \mathcal{H}^* h^*(x) = -t(x). \]  

(3.326)

Writing the quantities \( h(x) \) and \( t(x) \) in their components in the sense of the eigenvectors \( w, w_3 \) and \( w_4 \) as [10]:

\[ h(x) = h_1(x)w + h_2(x)\overline{w} + h_3(x)w_3 + h_4(x)w_4, \]

\[ t(x) = t_1(x)w + t_2(x)\overline{w} + t_3(x)w_3 + t_4(x)w_4, \]  

(3.327)

one obtains

\[ h_1(z) = \frac{\chi(z)}{2\pi i} \int_{-\epsilon}^{\epsilon} \frac{t(x)dx}{x - \chi'(x)(x - z)} + c_{i} \chi(z), \]

\[ h_2(z) = -\frac{\overline{\chi}(z)}{2\pi i} \int_{-\epsilon}^{\epsilon} \frac{\overline{t}(x)dx}{x - \overline{\chi}'(x)(x - z)} + c_{i} \overline{\chi}(z), \]

\[ h_3(z) = \frac{\chi_3(z)}{2\pi i} \int_{-\epsilon}^{\epsilon} \frac{t_3(x)dx}{x - \chi_3(x)(x - z)} + c_{i} \chi_3(z), \]

\[ h_4(z) = -\frac{\chi_4(z)}{2\pi i} \int_{-\epsilon}^{\epsilon} \frac{t_4(x)dx}{x - \chi_4(x)(x - z)} + c_{i} \chi_4(z) \]  

(3.328)

where \( w, w_3 \) and \( w_4 \) are eigenvectors and have been defined in [10], and

\[ \chi(z) = (z + c)^{-3/2 - \kappa} (z - c)^{-1/2 + \kappa}, \]

\[ \chi_3(z) = (z + c)^{-3/2 - \kappa} (z - c)^{-1/2 + \kappa}, \]  

\[ \chi_4(z) = (z + c)^{-3/2 + \kappa} (z - c)^{-1/2 - \kappa}. \]  

(3.329)
The functions $h_i$ and $t_i$ are evaluated by taking inner products with $w, w_3$ and $w_4$, respectively, i.e.

$$h_i(x) = \frac{w^T H h_i(x)}{w^T H w}, \quad t_i(x) = \frac{w_i^T H t_i(x)}{w_i^T H w_i}$$

$$t = \frac{w^T H t(x)}{w^T H w}, \quad t_3 = \frac{w_i^T H t(x)}{w_i^T H w_i}, \quad t_4 = \frac{w_i^T H t(x)}{w_i^T H w_i} \quad (3.330)$$

The contour integration of eqn (3.328) provides

$$h_i(z) = h_i^*(z) \cdot b, \quad (i=1, 2, 3, 4), \quad (3.331)$$

$$h_i^*(z) = \frac{w^T H}{w^T H w} \sum_{k=1}^{4} [F(\epsilon, \chi, z, \epsilon_{ik}^{(1)}, B^{(2)}_k, B^{(2)''}_k) + F(\epsilon, \chi, z, \epsilon_{ik}^{(1)}, \bar{B}^{(2)}, \bar{B}^{(2)''})]. \quad (3.332)$$

$$h_i^*(z) = \frac{w^T H}{w^T H w} \sum_{k=1}^{4} [F(-\epsilon, \chi, z, \epsilon_{ik}^{(1)}, B^{(2)}_k, B^{(2)''}_k) + F(-\epsilon, \chi, z, \epsilon_{ik}^{(1)}, \bar{B}^{(2)}, \bar{B}^{(2)''})], \quad (3.333)$$

$$h_i^*(z) = \frac{w^T H}{w^T H w_3} \sum_{k=1}^{4} [F(-i\epsilon, \chi, z, \epsilon_{ik}^{(1)}, B^{(2)}_k, B^{(2)''}_k) + F(-i\epsilon, \chi, z, \epsilon_{ik}^{(1)}, \bar{B}^{(2)}, \bar{B}^{(2)''})]. \quad (3.334)$$

$$h_i^*(z) = \frac{w^T H}{w^T H w_4} \sum_{k=1}^{4} [F(i\epsilon, \chi, z, \epsilon_{ik}^{(1)}, B^{(2)}_k, B^{(2)''}_k) + F(i\epsilon, \chi, z, \epsilon_{ik}^{(1)}, \bar{B}^{(2)}, \bar{B}^{(2)''})]. \quad (3.335)$$

$$c_i = \frac{-w^T H \begin{bmatrix} X_1 \\ 1 + e^{-i\omega_k} \end{bmatrix}}{w^T H w} \quad c_2 = \frac{w^T H \begin{bmatrix} X_1 \\ 1 + e^{i\omega_k} \end{bmatrix}}{w^T H w} \quad c_3 = \frac{w_i^T H \begin{bmatrix} X_1 \\ 1 + e^{-i\omega_k} \end{bmatrix}}{w_i^T H w_i} \quad c_4 = \frac{w_i^T H \begin{bmatrix} X_1 \\ 1 + e^{i\omega_k} \end{bmatrix}}{w_i^T H w_i} \quad (3.336)$$

where

$$F(r, \chi, z, \epsilon_{ik}, X, Y) = \frac{-XY}{(1 + e^{-2i\omega_k})} \left[ 1 - \frac{\chi(z)}{\chi(\epsilon_{ik})} \right] \frac{1}{z - \epsilon_{ik}}. \quad (3.337)$$

$$X_i = \sum_{k=1}^{4} (B^{(2)}_k B^{(2)''}_k + \bar{B}^{(2)} \bar{B}^{(2)''}) b.$$
use of eqns (3.325) and (3.327).

It should be pointed out that the above solutions do not apply when the dislocation is located on the interface. For an edge dislocation \( \mathbf{b} \) located on the interface, say \( x_0 \), the solution can be obtained by setting
\[
\begin{align*}
\mathbf{f}_0^{(1)}(z) &= \langle \ln(x_u^{(1)} - x_0) \rangle \mathbf{q}_1, \quad \text{for } y > 0, \\
\mathbf{f}_0^{(2)}(z) &= \langle \ln(x_u^{(2)} - x_0) \rangle \mathbf{q}_2, \quad \text{for } y < 0.
\end{align*}
\]
(3.338)

The continuity condition on the interface and the condition related to the dislocation give
\[
\begin{align*}
\mathbf{q}_1 &= \frac{1}{4\pi \mu} (\mathbf{A}^{(1)} - \overline{\mathbf{A}}^{(2)} \overline{\mathbf{B}}^{(2)-1} \mathbf{B}^{(1)}) \mathbf{b}, \quad \mathbf{q}_2 = -\mathbf{B}^{(2)-1} B^{(1)} \overline{\mathbf{q}}_1.
\end{align*}
\]
(3.339)

The corresponding interface traction-charge vector is
\[
\mathbf{t}(x) = \frac{1}{x - x_0} (\mathbf{B}^{(1)} \mathbf{q}_1 + \overline{\mathbf{B}}^{(1)} \overline{\mathbf{q}}_1) = \frac{1}{x - x_0} (\mathbf{Y}_s + \overline{\mathbf{Y}}_s) \mathbf{b}
\]
(3.340)

where \( \mathbf{Y}_s = \frac{\mathbf{B}^{(1)}}{4\pi \mu} (\mathbf{A}^{(1)} - \overline{\mathbf{A}}^{(2)} \overline{\mathbf{B}}^{(2)-1} \mathbf{B}^{(1)}) \).

As was done previously, let
\[
\mathbf{t}(x) = \frac{1}{x - x_0} (\mathbf{w} + i \mathbf{w} + t_3 \mathbf{w}_3 + t_4 \mathbf{w}_4).
\]
(3.341)

Similarly, the related functions \( h_i(x) \) are as follows:
\[
\begin{align*}
h_i^1(z) &= \frac{\mathbf{w}^T \mathbf{H}}{w^T \mathbf{H} w} \mathbf{F}(-i\kappa, z, x_0, \mathbf{I}, \mathbf{Y}_s + \overline{\mathbf{Y}}_s), \\
h_i^2(z) &= \frac{\mathbf{w}^T \mathbf{H}}{w^T \mathbf{H} w} \mathbf{F}(i\kappa, z, x_0, \mathbf{I}, \mathbf{Y}_s + \overline{\mathbf{Y}}_s), \\
h_i^3(z) &= \frac{\mathbf{w}^T \mathbf{H}}{w^T \mathbf{H} w} \mathbf{F}(\kappa, z, x_0, \mathbf{I}, \mathbf{Y}_s + \overline{\mathbf{Y}}_s), \\
h_i^4(z) &= \frac{\mathbf{w}^T \mathbf{H}}{w^T \mathbf{H} w} \mathbf{F}(-\kappa, z, x_0, \mathbf{I}, \mathbf{Y}_s + \overline{\mathbf{Y}}_s).
\end{align*}
\]
(3.342)

\[
\begin{align*}
c_1 &= -\frac{\mathbf{w}^T \mathbf{H} \mathbf{(Y}_s + \overline{\mathbf{Y}}_s) \mathbf{b}}{w^T \mathbf{H} w + e^{-2\mu}}, \quad c_2 = -\frac{\mathbf{w}^T \mathbf{H} \mathbf{(Y}_s + \overline{\mathbf{Y}}_s) \mathbf{b}}{w^T \mathbf{H} w + e^{2\mu}}.
\end{align*}
\]
(3.346a)
\[ c_1 = \frac{w_i^2 Y (Y + \overline{Y}) b}{w_i^2 H w_3 \left( 1 + e^{-2i\omega t} \right)}, \quad c_4 = \frac{w_i^2 Y (Y + \overline{Y}) b}{w_i^2 H w_4 \left( 1 + e^{-2i\omega t} \right)} \quad (3.346b) \]

(c) Crack-dislocation interaction in a PAC bimaterial

In (b) the solutions for a PP bimaterial due to a single dislocation were presented. Here we extend them to the case of a PAC bimaterial. When one of the two materials, say material 2, is an anisotropic conductor, the interface condition (3.193) should be changed to

\[
\begin{bmatrix}
\mu_1^{(1)} \\
\mu_2^{(1)} \\
\mu_1^{(2)} \\
0
\end{bmatrix}
= 
\begin{bmatrix}
\phi_1^{(1)} \\
\phi_2^{(1)} \\
\phi_1^{(2)} \\
0
\end{bmatrix}
, 
\quad
\begin{bmatrix}
\Phi_1^{(1)} \\
\Phi_2^{(1)} \\
\Phi_1^{(2)} \\
0
\end{bmatrix}
= 
\begin{bmatrix}
\phi_1^{(1)} \\
\phi_2^{(1)} \\
\phi_1^{(2)} \\
0
\end{bmatrix}.
\quad (3.347)
\]

Further inside the conductor, we have

\[ D_i^{(2)} = E_i^{(2)} = 0. \quad (3.348) \]

The formulations presented in (b) are still available if the following modifications are made:

\[ A^{(2)} = \begin{bmatrix} A_{(3 \times 3)} & 0 \\ 0 & 1 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} B_{(3 \times 3)} & 0 \\ 0 & 1 \end{bmatrix}, \quad (3.349) \]

\[ p_i^{(2)} = i, \quad q_{\beta}^{(2)} = \begin{bmatrix} q_{\beta 1}^{(2)} \\ q_{\beta 2}^{(2)} \\ q_{\beta 3}^{(2)} \end{bmatrix}, \quad (\beta = 1, 2, 3, 4) \quad (3.350) \]

where \( A_{(3 \times 3)} \) and \( B_{(3 \times 3)} \) are well-defined material matrices for anisotropic elastic materials (see [35], for example).

(d) Crack-dislocation interaction in a PID bimaterial

In this subsection we consider a PID bimaterial for which the upper half-plane \( (x_2 > 0) \) is occupied by piezoelectric material (material 1), and the lower half-plane \( (x_2 < 0) \) is occupied by isotropic dielectric material (material 2). For simplicity a plane strain deformation only is considered in this subsection. Thus the related interface condition is reduced to

\[
\begin{bmatrix}
\mu_1^{(1)} \\
\mu_2^{(1)} \\
\phi
\end{bmatrix}
= 
\begin{bmatrix}
\mu_1^{(2)} \\
\mu_2^{(2)} \\
\phi
\end{bmatrix}, 
\quad
\begin{bmatrix}
\Phi_1^{(1)} \\
\Phi_2^{(1)} \\
\Phi_3^{(1)} \\
\Phi_3^{(2)}
\end{bmatrix}
= 
\begin{bmatrix}
\Phi_1^{(2)} \\
\Phi_2^{(2)} \\
\Phi_3^{(2)}
\end{bmatrix}, \quad \text{at } x_2 = 0. \quad (3.351) \]
General solutions for isotropic dielectric material

The general solutions for isotropic dielectric material can be assumed in the form

\[ u_i = 2 \text{Re}[v_i f(z)], \quad \varphi_i = 2 \text{Re}[h_i f(z)], \quad z = x_1 + px_2 \]  

(3.352)

where \( v(=\{v_i\}) \) and \( h(=\{h_i\}) \) are determined by

\[ [Q + p(R + R^T) + p^2T]v = 0, \quad h = (R^T + pT)v. \]  

(3.353)

For isotropic dielectric material, the \( 3 \times 3 \) matrices \( Q, R \) and \( T \) are in the form

\[
Q = \begin{bmatrix}
\lambda + 2G & 0 & 0 \\
0 & G & 0 \\
0 & 0 & -\varepsilon_0
\end{bmatrix}, \quad R = \begin{bmatrix}
0 & \lambda & 0 \\
G & 0 & 0 \\
0 & 0 & G
\end{bmatrix}, \quad T = \begin{bmatrix}
G & 0 & 0 \\
0 & \lambda + 2G & 0 \\
0 & 0 & -\varepsilon_0
\end{bmatrix}
\]  

(3.354)

where \( \lambda \) and \( G \) are the Lamé constants [59] defined by

\[
\lambda = \frac{E\mu}{(1+\mu)(1-2\mu)}, \quad G = \frac{E}{2(1+\mu)}.
\]

Since \( p=i \) is a double root, a second independent solution can be written as

\[ u_i = 2 \text{Re}\left\{ \frac{\partial}{\partial p} [v_i f(z)] \right\}, \quad \varphi_i = 2 \text{Re}\left\{ \frac{\partial}{\partial p} [h_i f(z)] \right\}, \]  

(3.355)

where

\[ v = v_p = \frac{1}{2G} [i \ 0], \quad h = h_p = \{i \ 0\} \]  

(3.356)

for plane elastic deformation, and

\[ v = v_e = -\frac{1}{\varepsilon_0} [0 \ 0 \ 1], \quad h = h_e = \{0 \ 0 \ i\} \]  

(3.357)

for electric field. Following the method of Ting and Chou [59], the complete general solutions can be expressed as

\[ U = Aqf(z) + \frac{z - z_0}{2i} v q_f f'(z), \quad \varphi = Bqf(z) + \frac{z - z_0}{2i} h q_f f'(z). \]  

(3.358)
\[
A = \frac{1}{2G} \begin{bmatrix}
1 & (1-2k)i & 0 \\
i & -2k & 0 \\
0 & 0 & -2G\varepsilon_0^{-1}
\end{bmatrix}, \quad B = \begin{bmatrix}
i & 0 & 0 \\
-1 & -i & 0 \\
0 & 0 & i
\end{bmatrix}
\]

where \( \varepsilon_0 \) is the dielectric constant, \( \mathbf{q} = [q_1, q_2, q_3]^T \) are constants to be determined, and

\[
k = \frac{\lambda + 2G}{\lambda + G}.
\]

(d2) Green’s function for homogeneous isotropic materials

For an infinite domain subject to a line dislocation \( \mathbf{b} \) located at \( z_0 = x_{10} + ix_{20} \), the function \( f(z) \) in eqn (3.358) can be assumed as

\[
f(z) = \ln(z - z_0).
\]

The condition

\[
\int_C d\mathbf{U} = \mathbf{b}, \quad \text{for any closed curve} \ C \ \text{enclosing the point} \ z_0
\]

yields

\[
\mathbf{q} = \mathbf{q}_0 = [q_{10}, q_{20}, q_{30}]^T = \frac{1}{2\pi i} A^\star \mathbf{b},
\]

where

\[
A^\star = \frac{1}{1 - 4k} \begin{bmatrix}
-4kG & -(2-4k)iG & 0 \\
-2iG & 2G & 0 \\
0 & 0 & -\varepsilon_0(1-4k)
\end{bmatrix}.
\]

(d3) Green’s function for PID bimaterials

(d3.1) \( z_0 \) located in isotropic dielectric material (material 2). Consider a PID bimaterial for which a line dislocation \( \mathbf{b} \) is applied at \( z_0 = x_{10} + ix_{20} \). The interface condition (3.193) suggests that the solution can be assumed in the form:

\[
\mathbf{U}^{(1)} = \frac{1}{\pi} \ln[\mathbf{A}^{(1)}(\ln(z_0 - z_0)]\mathbf{q}_1], \quad \varphi^{(1)} = \frac{1}{\pi} \ln[\mathbf{B}^{(1)}(\ln(z_0 - z_0)]\mathbf{q}_1]
\]

for material 1 in \( x_2 > 0 \) and
U^{(2)} = \frac{1}{\pi} \text{Im} \left[ A^{(2)} \mathbf{b} \ln(z - z_0) + \frac{\pi(z - \overline{z}) v \nu_{20}}{\overline{(z - z_0)}} + A^{(2)} \mathbf{q}_2 \ln(z - \overline{z}_0) \right],
\(3.366\)

\[ \phi^{(2)} = \frac{1}{\pi} \text{Im} \left[ B_2 \mathbf{A} \mathbf{b} \ln(z - z_0) + \frac{\pi(z - \overline{z}) h \nu_{20}}{\overline{(z - z_0)}} + B^{(2)} \mathbf{q}_2 \ln(z - \overline{z}_0) \right] \]

for material 2 in \(x_2 < 0\), where \(\mathbf{q}_1 \) and \(\mathbf{q}_2 \) are unknowns to be determined. The substitution of eqns (3.365) and (3.366) into eqn (3.193), yields

\[ \mathbf{A}^{(1)} \mathbf{q}_1 + A^{(2)} \mathbf{q}_2 = \mathbf{A}^{(2)} \mathbf{A}^* \mathbf{b}, \quad \mathbf{B}^{(1)} \mathbf{q}_1 + B^{(2)} \mathbf{q}_2 = \mathbf{B}^{(2)} \mathbf{A}^* \mathbf{b}. \quad (3.367) \]

Solving eqn (3.367), we have

\[ \mathbf{q}_1 = \mathbf{B}^* \mathbf{b}, \quad \mathbf{q}_2 = \mathbf{B}^* \mathbf{b}, \quad (3.368) \]

where

\[ \mathbf{B}^a = (\overline{\mathbf{B}}^{(2)+1} \mathbf{B}^{(1)} - \overline{\mathbf{A}}^{(2)+1} \mathbf{A}^{(1)} - \overline{\mathbf{A}}^{(2)+1} \mathbf{A}^{(2)} \mathbf{A}^*), \]

\[ \mathbf{B}^b = (\overline{\mathbf{B}}^{(1)+1} \mathbf{B}^{(2)} - \overline{\mathbf{A}}^{(1)+1} \mathbf{A}^{(2)} - \overline{\mathbf{A}}^{(1)+1} \mathbf{A}^{(1)} \mathbf{A}^*). \quad (3.369) \]

(d3.2) \(z_0 \) located in piezoelectric material (material 1). When the single dislocation \(\mathbf{b} \) is applied at \(z_0\) with \(x_2 > 0\), the electroelastic solution can be assumed in the form

\[ U^{(1)} = \frac{1}{\pi} \text{Im} [\mathbf{A}^{(1)} \mathbf{b} \ln(z_a - z_{0b})] + \frac{1}{\pi} \text{Im} \sum_{\mathbf{b} = 1}^{3} [\mathbf{A}^{(1)} \mathbf{b} \ln(z_a - z_{0b}) \mathbf{q}_b^{(1)}], \quad (3.370) \]

\[ \phi^{(1)} = \frac{1}{\pi} \text{Im} [\mathbf{B}^{(1)} \mathbf{b} \ln(z_a - z_{0b})] + \frac{1}{\pi} \text{Im} \sum_{\mathbf{b} = 1}^{3} [\mathbf{B}^{(1)} \mathbf{b} \ln(z_a - z_{0b}) \mathbf{q}_b^{(1)}] \quad (3.371) \]

for material 1 in \(x_2 > 0\) and

\[ U^{(2)} = \frac{1}{\pi} \text{Im} \sum_{\mathbf{b} = 1}^{3} [\mathbf{A}^{(2)} \mathbf{b} \ln(z - z_{0b}) \mathbf{q}_b^{(2)}] \quad (3.372) \]

\[ \phi^{(2)} = \frac{1}{\pi} \text{Im} \sum_{\mathbf{b} = 1}^{3} [\mathbf{B}^{(2)} \mathbf{b} \ln(z - z_{0b}) \mathbf{q}_b^{(2)}] \quad (3.373) \]

for material 2 in \(x_2 < 0\). Substituting eqns (3.370)–(3.373) into eqn (3.193), we find the solution has the same form as that given in (b) except that all matrices here are \(3 \times 3\) matrices.
The interface SED induced by the dislocation \( \mathbf{b} \), then, is of the form

\[
\mathbf{t}(x) = \mathbf{\varphi}^{(1)} = 2 \text{Re}[\mathbf{B}^{(1)} \mathbf{f}_0^{(1)}] = \frac{1}{\pi} \text{Im} \left[ \frac{1}{x - z_0} \mathbf{B}^{(1)} \mathbf{B}^* \mathbf{b} \right]
\]

(3.374) for \( z_0 \) in material 2 and

\[
\mathbf{t}(x) = \mathbf{\varphi}^{(2)} = 2 \text{Re}[\mathbf{B}^{(2)} \mathbf{f}_0^{(2)}] = \frac{1}{\pi} \text{Im} \left[ \sum_{p=1}^{3} \mathbf{B}^{(2)} \left( \frac{1}{x - z_{0p}} \right) \mathbf{q}_b^{(2)} \right] = [\mathbf{F}_v(x) + \mathbf{F}_w(x)] \mathbf{b}.
\]

(3.375)

for \( z_0 \) in material 1.

Similarly, \( \mathbf{t}(x) \) can be removed by superposing a solution with the traction-charge, \( -\mathbf{t}(x) \), induced by the potential \( \mathbf{f}_0(z) \). The solution for \( \mathbf{f}_0(z) \) can also be obtained by setting

\[
\mathbf{h}(z) = \begin{cases} \mathbf{B}^{(1)} \mathbf{f}_0^{(1)}(z), & \text{in material 1}, \\ \mathbf{H}^{-1} \mathbf{B}^{(2)} \mathbf{f}_0^{(2)}(z), & \text{in material 2}. \end{cases}
\]

(3.376)

The subsequent derivation is the same as that in (b) except that

\[
\chi(z) = (z + c)^{-1/2 - i\epsilon} (z - c)^{-1/2 + i\epsilon},
\]

\[
\chi_3(z) = (z + c)^{-1/2} (z - c)^{-1/2}
\]

where \( \epsilon \) is determined in a similar manner to that of Suo et al. [10].

(e) Crack-dislocation interaction in a PIC bimaterial

As treated in (c), the solutions for a PIC bimaterial due to a single dislocation can be obtained based on the results given in (d). When one of the two materials, say material 2, is an isotropic conductor, the interface condition (3.193) should be changed to

\[
\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \phi_1^{(1)} \end{bmatrix} = \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ \phi_2^{(2)} \end{bmatrix}, \quad \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ \phi_2^{(2)} \end{bmatrix} = \begin{bmatrix} \phi_1^{(2)} \\ \phi_2^{(2)} \\ 0 \end{bmatrix}, \quad \text{at } x_2 = 0.
\]

(3.378)

The formulations presented in subsection (d) are still available if the following modification is made:

\[
\mathbf{q}_2 = \{q_{21}, q_{22}, 0\}, \quad (\beta = 1, 2, 3).
\]

(3.379)
3.13.2 Singular integral equations
For the sake of conciseness, all formulations developed in this section are only for the material combinations HM, PP and PAC. The related singular integral equations for PID and PIC bimaterials can be obtained straightforwardly, as it is necessary only to choose the appropriate dislocation density function.

(a) Electroelastic fields induced by remote uniform load
To satisfy the boundary conditions for the branch crack, it is necessary to know the electroelastic fields induced by the remote uniform load \( \Pi_2^\infty \), as the Green’s functions derived above do not satisfy the condition of uniform SED at infinity. Consequently, set

\[
f'(z) = f'_0(z) + f'_1(z) + f'_2(z)
\]

where \( f'_0(z) \) is the solution corresponding to the remote load. It is convenient to represent the solution due to \( \Pi_2^\infty \) as the sum of a uniform SED in an unflawed solid and a corrective solution in which the main crack is subject to \(-\Pi_2^\infty\). The solution to the latter is [10]:

\[
f'_0(z) = \frac{1}{2} \left[ (z^2 - c^2)^{3/2} - 1 \right] \mathbf{B}^{-1} \Pi_2^\infty
\]

(3.380)

for homogeneous piezoelectric material, and

\[
f'_{0\text{PP}}(z) = \mathbf{B}^{(1)} (h_{1w}(z) w + h_{2w}(z) \bar{w} + h_{3w}(z) w_3 + h_{4w}(z) w_4)
\]

\[
f'_{0\text{PAC}}(z) = \mathbf{B}^{(2)} \mathbf{H}^{-1} \mathbf{H} (h_{1w}(z) w + h_{2w}(z) \bar{w} + h_{3w}(z) w_3 + h_{4w}(z) w_4)
\]

(3.382)

for PP or PAC bimaterials, where

\[
h_{1w}(z) = \frac{1}{1 + e^{2\pi k}} \left[ \left( \frac{z - c}{z + c} \right)^{i k} (z^2 - c^2)^{3/2} (z + 2ic\varphi) - 1 \right] \frac{w^* \Pi_2^\infty}{w \bar{w}},
\]

\[
h_{2w}(z) = \frac{1}{1 + e^{2\pi k}} \left[ \left( \frac{z - c}{z + c} \right)^{i k} (z^2 - c^2)^{3/2} (z - 2ic\varphi) - 1 \right] \frac{w^* \Pi_2^\infty}{w \bar{w}},
\]

\[
h_{3w}(z) = \frac{e^{i \pi k}}{1 + e^{2\pi k}} \left[ \left( \frac{z - c}{z + c} \right)^{i k} (z^2 - c^2)^{3/2} (z + 2i\kappa) - 1 \right] \frac{w^* \Pi_2^\infty}{w \bar{w}_3},
\]

\[
h_{4w}(z) = \frac{e^{i \pi k}}{1 + e^{2\pi k}} \left[ \left( \frac{z - c}{z + c} \right)^{i k} (z^2 - c^2)^{3/2} (z - 2i\kappa) - 1 \right] \frac{w^* \Pi_2^\infty}{w \bar{w}_4}.
\]

(3.383)

So far the function \( f(z) \) satisfies the conditions at infinity and on the main crack face. It remains only to ensure that the traction-charge free conditions for the branch crack be met.
Singular integral equations and solution methods

The remaining boundary conditions for the branch can be satisfied by redefining the discrete dislocation, \( b \), in terms of distributed dislocation densities, \( b(\xi) \), defining along the line, \( z = c + \xi z^* \), \( z_0 = c + \xi z^* \), where \( z^* = \cos \theta + p \sin \theta \) emanating from the main crack tip. For simplicity we first consider the case of a branch in a homogeneous solid. Enforcing satisfaction of traction-charge free condition on the branch, a system of singular integral equations for the dislocation \( b(\xi) \) is obtained as

\[
\frac{L}{2\pi} \int_0^\infty \frac{b(\xi)d\xi}{\eta - \xi} + \int_0^\infty Y_\theta(\eta, \xi)b(\xi)d\xi + Q_\alpha(\eta) = 0 \tag{3.384}
\]

where \( Y_\theta(\eta, \xi) \) is a kernel matrix function of the singular integral equation and is \( \text{Holder-continuous} \) along \( 0 \leq \xi < a \), and \( Q_\alpha(\eta) \) is a known function vector corresponding to the SED field induced by the external load. \( Y_\theta(\eta, \xi) \) and \( Q_\alpha(\eta) \) are expressed as

\[
Y_\theta(\eta, \xi) = \text{Im}\{ \tilde{B}(z^*_a B(\xi, z_0) - B(z^*_a B(\xi, z_0) B^T ) \}, \tag{3.385}
\]

\[
Q_\alpha(\eta) = \text{Re}\left\{ B(z^*_a \left[ \frac{z_a}{(z_a^2 - c^2)^{1/2}} - 1 \right] B^{-1} \Pi^\circ_\alpha \right\}. \tag{3.386}
\]

in which \( z^*_a = \cos \theta + p_a \sin \theta \).

For the purpose of numerical solution, the following normalized quantities are introduced

\[
s_0 = \frac{2\eta - a}{a}, \quad s = \frac{2\xi - a}{a}. \tag{3.387}
\]

If we retain the same symbols for the new functions caused by the change of variables, the singular integral equation (3.384) can then be rewritten in the form

\[
\frac{L}{2\pi} \int_{s_0}^{s_1} \frac{b(s)ds}{s_s} + \int_{s_0}^{s_1} Y_\theta(s, s_0)b(s)ds + Q_\alpha(s_0) = 0. \tag{3.388}
\]

Similarly, for a branch in a PP or PAC bimaterial solid, the corresponding singular integral equation can be obtained as

\[
\frac{L^{(1)}}{2\pi} \int_{s_0}^{s_1} \frac{b(s)ds}{s_s} + \int_{s_0}^{s_1} \{ Y_\theta(s, s_0)b(s) + Y_\alpha(s, s_0) \}ds + Q_\alpha(s_0) = 0 \tag{3.389}
\]

where

\[
Y_\theta(s, s_0) = \text{Im}\sum_{i=1}^k \left[ B^{(1)} \left[ \frac{z^{(1)}_a}{(s_0 + 1)z^{(1)}_a - (s + 1)z^{(1)}_a} \right] B^T_i \right], \tag{3.390}
\]

\( z^{(1)}_a = \cos \theta + p_a \sin \theta \).
\[ Y_i(s, s_0) = \pi a \text{Re} \{ B^{(i)} \left( \frac{s}{s_0} \right) B_i^* h(s, s_0) \}, \]  
\[ Q_{se}(s_0) = 2 \text{Re} \{ B^{(i)} \left( \frac{s}{s_0} \right) \bar{f}^{(i)}(s_0) \}. \]  

(c) Numerical scheme
Both the integral equations (3.388) and (3.389) can be solved numerically using a method developed by Erdogan and Gupta [36]. First we write the unknown density function, \( b(s) \), in eqn (83) in the form [54]:

\[ b(s) = \hat{b}(s) \sqrt{1 - s^2} = \sum_{k=1}^{m} \hat{b}_k T_j(s) \]  

where \( \hat{b}(s) \) is a regular function vector defined in the interval \( |s| \leq 1 \), and \( \hat{b}(-1) = 0 \) [54], since the SED behaves like \( d^{-3/2} \), where \( d \) measures distance from the branched crack tip, and the SED at the vertex behaves as \( d^{-\alpha} \), \( \alpha < 1/2 \) for a SED-free wedge with a wedge angle less than \( \pi \). \( \hat{b}_j \) are the real unknown constant vectors, and \( T_j(s) \) the Chebyshev polynomials of first kind. Thus the discretized form of eqn (3.388) (or (3.389)) together with the condition, \( b(-1) = 0 \), may be written as [36]

\[ \sum_{k=1}^{m} \frac{1}{n} \left[ \frac{L}{2(s_{0k} - s_k)} + Y_6(s_0, s_k) \right] \hat{b}(s_k) + Q_{se}(s_0) = 0, \]  
\[ \sum_{k=1}^{m} \hat{b}_k T_j(-1) = 0, \]  

for a homogeneous piezoelectric solid, and

\[ \sum_{k=1}^{m} \frac{1}{n} \left[ \frac{L^{(i)}}{2(s_{0k} - s_k)} + Y_i(s_0, s_k) \right] \hat{b}(s_k) + \hat{Y}_i(s_0, s_k) = 0, \]  
\[ \sum_{k=1}^{m} \hat{b}_k T_j(-1) = 0, \]  

for a PP or PAC bimaterial solid, where

\[ Y_i(x, y) = \frac{\hat{Y}_i(x, y)}{\sqrt{1 - s^2}}, \]
\[ s_k = \cos \left( \frac{(2k-1)\pi}{2m} \right), \quad (k = 1, 2, \ldots, m), \quad (3.397) \]
\[ s_m = \cos \left( \frac{\pi}{m} \right), \quad (r = 1, 2, \ldots, m-1). \]

Equation (3.394) (or (3.395)) provides a system of 4\(m\) equations for the determination of the 4\(m\) values of \( k_{jb} \hat{b} \). Once the function \( \hat{b}(s) \) has been found from eqn (3.394) (or (3.395)), the SED, \( \Pi^{(1)}(\eta) \), in a coordinate local to the crack branch line can be expressed in the form

\[
\Pi^{(1)}(s_0) = \Omega(\theta) \left[ \frac{L}{2\pi} \int_{s_0 - s}^{1} \frac{b(s)}{s} ds + \int_{1}^{s_0} \left\{ y_0(s, s_0) b(s) + y_1(s, s_0) \right\} ds + \tilde{Q}_e(s_0) \right] \]

(3.398)

for a homogeneous solid, and

\[
\Pi^{(1)}(s_0) = \Omega(\theta) \left[ \frac{L^{(1)}}{2\pi} \int_{s_0 - s}^{1} \frac{b(s)}{s} ds + \frac{1}{\pi} \int_{1}^{s_0} \left\{ y_0(s, s_0) b(s) + y_1(s, s_0) \right\} ds + \tilde{Q}_e(s_0) \right] \]

(3.399)

for a PP or PAC bimaterial solid.

(d) SED intensity factors and energy release rate

The SED intensity factors at the right tip of the branch crack are of interest and can be derived by first considering the traction and surface charge (TSC) on the direction of the branch line and considering the TSC very near the tip \( (s_0 \to 1) \) which is given by eqn (3.398) (or (3.399)) as

\[
\Pi^{(1)}(r) \approx \Omega(\theta) \frac{L^{(1)} \hat{b}(1)}{\sqrt{8(s_0 - 1)}} = \frac{1}{4} \sqrt{\frac{2\pi r}{r}} \Omega(\theta) L^{(1)} \hat{b}(1) \]

(3.400)

where \( r \) is a distance ahead of the branch tip, \( L^{(1)} = L \) for a homogeneous material, and \( L^{(1)} = L \) for a bimaterial.

Using eqn (3.400) we can calculate the SED intensity factors at the right tip of the branch by the following usual definition:

\[
K = \{ K_{II}, K_{III}, K_{I}, K_{II} \}^T = \lim_{r \to 0} \sqrt{2\pi r} \Pi^{(1)}(r). \]

(3.401)

Combining with the results of eqn (3.400), one obtains

\[
K \approx \sqrt{\frac{a\pi}{8} \Omega(\theta) L^{(1)} \hat{b}(1)}. \]

(3.402)
The energy release rate $G^*$ can be computed by the closure integral [60]:

$$G^* = \lim_{x \to 0} \frac{1}{2x} \int_0^x \Pi_u^{(1)}(r) \Delta U_u(x-r) dr$$

(3.403)

where $x$ is the assumed crack extension, $\Delta U$ is a vector of elastic displacement and electric potential jump across the branch crack. Noting that

$$\Delta U(r) = \int_0^r b(\alpha - \xi) d\xi,$$

(3.404)

we have

$$G^* = \lim_{x \to 0} \frac{1}{2x} \int_0^x \Phi_u^{(1)}(r) \frac{\partial \Delta U_u(x-r)}{\partial x} dr$$

(3.405)

where

$$\Phi_u^{(1)}(r) = \int_0^r \Pi_u^{(1)}(x) dx.$$  

(3.406)

During the derivation of eqn (3.405), the following conditions are employed:

$$\Phi_u^{(1)}(0) = 0, \quad \Delta U(x-r)|_{x=0} = 0.$$  

(3.407)

Substituting eqn (3.393) into eqn (3.404), and eqn (3.400) into eqn (3.406), later into eqn (3.405), one obtains

$$G^* = \frac{a\pi}{16} \hat{b}^T (1) L^T [\Omega^T (0)]^2 \hat{b}(1).$$

(3.408)

### 3.13.3 Criterion for crack kinking

An interface crack can advance either by continued growth in the interface or by kinking out of the interface into one of the adjoining materials. This competition can be assessed by comparing the ratio of the energy release rate for kinking out of the interface and for interface cracking, $G_{\text{local}}/G_i$, to the ratio of substrate toughness to interface toughness, $\Gamma_s/\Gamma_i$. In essence, if

$$\frac{G_{\text{local}}}{G_i} > \frac{\Gamma_s}{\Gamma_i},$$

(3.409)

kinking is favoured. Conversely, if the inequality in eqn (3.409) is reversed, the flaw meets the condition for continuing advance in the interface at an applied
load lower than that necessary to advance the crack into the substrate, where 
\( \Gamma \) and \( \Gamma' \) are substrate and interface toughness, respectively. For a given load condition, \( \Pi \), the fracture toughness, \( \Gamma \), is defined as the energy release rate \( G \) at the onset of the crack growth and has been discussed in detail elsewhere [61]. \( G_{\text{kink}} \) and \( G'_i \) are the energy release rates for kinking out of the interface and for interface cracking, respectively, in which \( G'_i \) can be derived by setting \( \theta = 0 \) in the previous sections, and \( G_{\text{kink}} \) is related to SED intensity factors, \( K \), by

\[
G_{\text{kink}} = \frac{1}{2} DK \tag{3.410}
\]

where \( D = \Omega^2 \left( \frac{\sigma}{\sigma_2} \right) L \Omega^2 \left( \frac{\sigma}{\sigma_2} \right) \).

### 3.13.4 Numerical results

Since the main purpose of this section is to outline the basic principles of the proposed method, assessment is limited to a PP bimaterial plate with an interface crack of length \( 2c \), a branch crack of length \( a \) and its interface coinciding with the \( x_1 \)-axis as shown in Fig. 3.13. In all calculations, the ratio of \( a/c \) is taken to be 0.1. The upper and lower materials are assumed to be PZT-5H and PZT-5 [39], respectively. The material constants for the two materials are as follows:

1. **Material properties for PZT-5H:**
   - \( c_{11} = 117 \text{ GPa} \), \( c_{12} = c_{13} = 53 \text{ GPa} \), \( c_{22} = c_{33} = 126 \text{ GPa} \), \( c_{44} = 35.5 \text{ GPa} \),
   - \( c_{23} = 55 \text{ GPa} \), \( c_{55} = c_{66} = 35.3 \text{ GPa} \), \( e_{12} = e_{13} = -6.5 \text{ C/m}^2 \), \( e_{11} = 23.3 \text{ C/m}^2 \),
   - \( e_{33} = e_{26} = 17 \text{ C/m}^2 \), \( \kappa_{11} = 130 \times 10^{-10} \text{ C/Vm} \), \( \kappa_{22} = \kappa_{33} = 151 \times 10^{-10} \text{ C/Vm} \).

2. **Material properties for PZT-5:**
   - \( c_{11} = 111 \text{ GPa} \), \( e_{12} = c_{13} = 75.2 \text{ GPa} \), \( c_{22} = c_{33} = 121 \text{ GPa} \), \( c_{44} = 22.8 \text{ GPa} \),
   - \( c_{23} = 75.4 \text{ GPa} \), \( c_{55} = c_{66} = 21.1 \text{ GPa} \), \( e_{12} = e_{13} = -5.4 \text{ C/m}^2 \), \( e_{11} = 15.8 \text{ C/m}^2 \),
   - \( e_{33} = e_{26} = 12.3 \text{ C/m}^2 \), \( \kappa_{11} = 73.46 \times 10^{-10} \text{ C/Vm} \), \( \kappa_{22} = \kappa_{33} = 81.7 \times 10^{-10} \text{ C/Vm} \).

The ratio \( G_{\text{kink}}/G \) versus kink angle \( \theta \) with \( \sigma_{23} = D^2 \) shown is Fig. 3.14 for a number of loading combinations measured by \( \psi = \tan^{-1}(K'_1/K'_2) \), where \( K'_1 \) and \( K'_2 \) are the conventional mode I and mode II stress intensity factors. The curve clearly shows that the maximum energy release rate occurs at different kink angle \( \hat{\theta} \) for different loading phases \( \psi \), where \( \hat{\theta} \) is the critical kink angle for which the maximum energy release rate will occur under a given loading condition. For example, \( \hat{\theta} = 0^\circ \) for curves 4 and 5, \( \hat{\theta} = 11^\circ \) for curve 3, \( \hat{\theta} = 13^\circ \) for curve 2 and \( \hat{\theta} = 18^\circ \) for curve 1.
Figure 3.15 shows the ratio of $G_{\text{kink}} / G_i$ as a function of kink angle $\theta$ for several values of remote loading $D^\infty_2$ with $\sigma_{22}^\infty = 4 \times 10^6 \text{N/m}^2$ and $\sigma_{ii}^\infty = 0$. The calculation indicates that the energy release rate can have negative value depending on the direction and magnitude of the remote loading $D^\infty_2$. For this particular problem, the energy release rate $G_i$ will be positive when $-1.8 \times 10^3 \text{C/m}^2 < D^\infty_2 < 1.8 \times 10^3 \text{C/m}^2$. Figure 3.15 also shows that the critical kink angle is affected by the load $D^\infty_2$. Generally, the critical kink angle will increase along with the increase of absolute value of $D^\infty_2$.

Fig. 3.14: Variation of $G_{\text{kink}} / G_i$ with kink angle $\theta$ for several loading combinations measured by $\psi = \tan^{-1}(K_{ii}^\infty / K_i^\infty)$.

Fig. 3.15: Variation of $G_{\text{kink}} / G_i$ with kink angle $\theta$ for several loadings $D^\infty_2$ ($\sigma_{22}^\infty = 4 \times 10^6 \text{N/m}^2$).
Fig. 3.16: Variation of $G_{k\theta}/G_0$ with remote loading $D_2^\infty$ for several kink angles $\theta$ ($\sigma_{22}^\infty = 4 \times 10^6$ N/m$^2$).

Figure 3.16 shows the variation of $G_{k\theta}/G_0$ as a function of $D_2^\infty$ for several values of kink angle $\theta$ and again for $\sigma_{22}^\infty = 4 \times 10^6$ N/m$^2$ and $\sigma_{12}^\infty = \sigma_{21}^\infty = 0$. Here $G_0$ is the value of $G$ at zero electric loads. It can be seen from this figure that the ratio of $G_{k\theta}/G_0$ will mostly decrease along with an increase in the magnitude of electric load.

The applied electric displacement can either promote or retard crack extension. This conclusion may be useful in designing piezoelectric components with an interface.

### 3.14 Crack deflection in bimaterial systems

In the previous section, crack kinking problems were considered in piezoelectric materials in which the main crack is assumed to lie along the interface. When the main crack is arbitrarily oriented and terminated at the interface, the formulation developed above cannot be used to solve this kind of problem. In this section, we show how to solve this crack problem, namely, the problem of crack deflection. The geometrical configuration analysed is depicted in Fig. 3.17. An inclined finite crack of length $2c$ in material 2 is shown impinging on an interface between two dissimilar piezoelectric materials, with a branch of length $a$ and angle $\theta_1$ between the branch and the interface. The length $a$ is assumed to be small compared to the length $2c$ of the inclined finite crack. Based on the Green’s functions developed in the previous section, the branch crack problem is modelled by applying distributed dislocation along the cracks and placing a concentrated dislocation at the joint of the finite crack and the branch. Then, a set of singular integral equations is developed and solved numerically. Generally, a crack impinging on an interface joining two dissimilar materials may arrest or may advance by either penetrating the interface or deflecting into the interface. Competition between deflection and penetration is investigated using the integral equations developed.
and the maximum energy release rate criterion. Numerical results are presented to demonstrate the role of remote electroelastic loads on the path of crack extension.

3.14.1 Singular integral equations
For the sake of conciseness, all formulations developed in this section are for the material combinations PP or PAC. The related singular integral equations for PID and PIC bimaterials can be obtained straightforwardly, by choosing the appropriate dislocation density functions.

![Fig. 3.17: Geometry of the branch crack.](image)

(a) **Boundary conditions**
Consider a finite crack with length $2c$ and a branch with length $a$ ($a < c$) embedded in a bimaterial plane subjected to the remote load $\Pi_0$ (see Fig. 3.17). The cracks are initially assumed to remain open and hence to be free of tractions and charges. The corresponding boundaries are, then, as follows:

**On the faces of each crack $i$:**

$$t_n^{(i)} = -\Pi_1^{(i)} \sin \theta_i + \Pi_2^{(i)} \cos \theta_i = 0 \quad (i = 1, 2) . \quad (3.411)$$

**At infinity:**

$$\Pi_1^\infty = \Pi_0, \quad \Pi_2^\infty = 0, \quad (3.412)$$

where $n$ is the normal direction to the lower face of a crack, and $t_n^{(i)}$ is the surface traction and charge vector acting on the faces of the $i$th crack.

To study the effect of cracks on fracture behaviour, it is convenient to represent the solution as the sum of a uniform SED in an unflawed solid and a corrective solution in which the boundary conditions are
On the faces of each crack $i$:

$$t^{(i)} = -\Pi^{(i)}_1 \sin \theta + \Pi^{(i)}_2 \cos \theta = -\Pi^{(i)}_0 \cos \theta, \quad (i = 1, 2) \quad (3.413)$$

At infinity:

$$\Pi^{\infty}_x = \Pi^{\infty}_y = 0. \quad (3.414)$$

(b) Singular integral equations

The singular integral equations for the problem described above can be formulated by considering the stress and electric displacement fields on the prospective site of the branch crack configuration due to the distributed dislocation caused by all the cracks. The assumed distribution of dislocation consists of two parts: (i) a concentrated dislocation with intensity, $\tilde{B}$, applied at the joint of the finite crack and the branch (the origin is shown in Fig. 3.17); (ii) distributed dislocation densities $b_i (= b_{a_1}, b_{a_2}, b_{b_2}, b_{b_0}; \quad i=1,2)$ applied along the $i$th crack.

Using the principle of superposition [40], the problem shown in Fig. 3.17 is decomposed into two subproblems, each of which contains only one single crack. The boundary conditions in eqn (3.413) can be satisfied by redefining the discrete Green’s function $b_i$ given in Section 3.13 in terms of distributing Green’s function $b_i (\xi)$ defined along each crack line, $\tilde{e}_1^{(i)} = \eta \tilde{e}_1^{(0)}$, $\tilde{e}_2^{(i)} = \xi \tilde{e}_2^{(0)}$, $\tilde{z}_k^{(i)} = \cos \theta_i + \tilde{p}_k^{(0)} \sin \theta_i, \quad (i=1, 2)$. Enforcing satisfaction of the applied stress conditions on each crack face, a system of singular integral equations for Green’s function is obtained as

$$\int_0^{2\pi} \left( \frac{L^{(1)}}{2(\eta - \xi)} + K_{11}(\eta, \xi) \right) b_1(\xi) d\xi + \int_0^{2\pi} K_{12}(\eta, \xi) b_2(\xi) d\xi + \frac{V}{\eta} \tilde{B} = -\pi \Pi^{(i)}_0 \cos \theta_i \quad (3.415)$$

$$\int_0^{2\pi} \left( \frac{L^{(2)}}{2(\eta - \xi)} + K_{22}(\eta, \xi) \right) b_2(\xi) d\xi + \int_0^{2\pi} K_{21}(\eta, \xi) b_1(\xi) d\xi - \frac{V}{\eta} \tilde{B} = -\pi \Pi^{(i)}_0 \cos \theta_2 \quad (3.416)$$

where $K_{ij}$ are $4 \times 4$ known kernel matrices and are regular within the related interval, which can be derived from eqns (1.56), (3.194)-(3.197). They are

$$K_{11} = \text{Im} \left[ \sum_{\beta=1}^{4} B^{(1)} \left( \frac{\tilde{z}_m^{(1)}}{\eta \tilde{e}_a^{(1)} - \tilde{z}_m^{(1)}} \right) \left( B^{*} \Pi_{\beta} B^{(1)} \right) \right], \quad (3.417)$$

$$K_{12} = \text{Im} \left[ \sum_{\beta=1}^{4} B^{(1)} \left( \frac{\tilde{z}_m^{(1)}}{\eta \tilde{e}_a^{(1)} - \tilde{z}_m^{(1)}} \right) \left( B^{*} \Pi_{\beta} B^{(1)} \right) \right], \quad (3.418)$$

$$K_{22} = \text{Im} \left[ \sum_{\beta=1}^{4} B^{(1)} \left( \frac{\tilde{z}_m^{(1)}}{\eta \tilde{e}_a^{(1)} - \tilde{z}_m^{(1)}} \right) \left( B^{*} \Pi_{\beta} B^{(1)} \right) \right], \quad (3.419)$$

$$K_{21} = \text{Im} \left[ \sum_{\beta=1}^{4} B^{(1)} \left( \frac{\tilde{z}_m^{(1)}}{\eta \tilde{e}_a^{(1)} - \tilde{z}_m^{(1)}} \right) \left( B^{*} \Pi_{\beta} B^{(1)} \right) \right], \quad (3.420)$$
The singular integral equations (3.415), (3.416) and (3.423) are too complicated to be evaluated exactly and consequently a numerical evaluation is necessary. To this end, let

\[
 b_i(\xi) = \left(\frac{\xi - \xi_i}{d_i - \xi_i}\right)^{1/2} B_i(\xi_i) \quad (3.424)
\]

where \( B_i(\xi_i) \) are regular functions defined on the related interval \( 0 \leq \xi_i \leq d_i \) and \( d_1 = a, \ d_2 = 2c \).

Employing the semi-open quadrature rule [62], eqns (3.415), (3.416) and (3.423) can be rewritten as

\[
 \sum_{m=1}^{M} \left[ \frac{L^{(1)}}{\eta_{1k} - \xi_{2m}} + K_{11}(\eta_{1k}, \xi_{2m}) \right] A_{1m} B_{1}(\xi_{2m}) + K_{12}(\eta_{1k}, \xi_{2m}) A_{2m} B_{2}(\xi_{2m})
\]

\[
 + \frac{V}{\eta_{1k}} \hat{B} = t_n(\eta_{1k}),
\]

\[
 \sum_{m=1}^{M} \left[ \frac{L^{(2)}}{\eta_{2k} - \xi_{2m}} + K_{21}(\eta_{2k}, \xi_{2m}) \right] A_{2m} B_{1}(\xi_{2m}) + K_{22}(\eta_{2k}, \xi_{2m}) A_{2m} B_{2}(\xi_{2m})
\]

\[
 - \frac{V}{\eta_{2k}} \hat{B} = t_n(\eta_{2k}).
\]

(3.425)
\[
\sum_{m=1}^{M} [A_{im} B_{ij} (\xi_{im}) + A_{im} B_{ik} (\xi_{km})] = \hat{B}
\] (3.427)

where

\[A_{im} = \frac{\pi d}{M} \sin^2 \frac{m\pi}{2M} \quad (i = 1, 2; m = 1, 2, \ldots, M - 1),\]

\[A_{im} = \frac{\pi d}{2M}, \quad A_{im} = \frac{\pi d}{2M}.
\]

\[\xi_{im} = d_i \sin^2 \frac{m\pi}{2M}, \quad \eta_{ik} = d_i \sin^2 \frac{(k - 0.5)\pi}{2M}, \quad (i = 1, 2; m, k = 1, 2, \ldots, M).
\]

Equations (3.425)-(3.427) provide a system of 4(2M+1) algebraic equations to determine \(\hat{B}, B_{ij}(\xi_{im})\) and \(B_{ik}(\xi_{km})\). Once these unknowns have been found from eqns (3.425)-(3.427), the SED, \(\Pi^{(i)}(\eta)\), in a coordinate local to the crack branch line can be expressed in the form

\[
\Pi^{(i)}(\eta) = \frac{1}{\pi} \mathbf{\Omega}(\theta) \left[ \int_0^1 \left( \frac{L^{(i)}(\xi)}{2(\eta - \xi)} + K_{ij}(\eta, \xi) \right) b_i(\xi) d\xi + \int_0^1 \frac{\gamma}{\eta} K_{ij}(\eta, \xi) b_j(\xi) d\xi + \frac{V}{\eta} \hat{B} + \pi \Pi_0 \cos \theta \right].
\] (3.428)

### 3.14.2 SED intensity factors and energy release rate

The SED intensity factors at the right tip of the branch crack are of interest and can be derived by first considering the traction and surface charge (TSC) on the direction of the branch line and considering the TSC very near the tip \((\eta \to a)\), which is given, from eqn (3.428), as

\[
\Pi^{(i)}(r) \approx \mathbf{\Omega}(\theta_i) \left[ \frac{a L^{(i)}(\xi)}{2(\eta - \xi)} + \frac{a}{2} \sqrt{\frac{a}{\pi \rho}} \mathbf{\Omega}(\theta_i) L^{(i)}(\xi) B_{ij}(a) \right] = \frac{1}{2} \sqrt{\frac{a}{\pi \rho}} \mathbf{\Omega}(\theta_i) L^{(i)}(\xi) B_{ij}(a)
\] (3.429)

where \(r\) is a distance ahead of the branch tip.

Using eqn (3.429) we can calculate the SED intensity factors at the right tip of the branch by the following usual definition

\[
\mathbf{K} = \{K_{ij}, K_{ij}, K_{ij}, K_{ij}\}^T = \lim_{r \to 0} \sqrt{2\pi r} \Pi^{(i)}(r).
\] (3.430)

Combining with the results of eqn (3.429), one obtains
\[ K \approx \sqrt{\frac{a}{2}} \Omega(0) L^B \mathbf{B}_a(a). \]  

(3.431)

The energy release rate \( G^* \) can be computed by the closure integral

\[ G^* = \lim_{s \to 0} \frac{1}{2x} \int_0^s \Phi_n^{(1)}(r) \Delta U_s(x - r) dr \]  

(3.432)

where \( x \) is the assumed crack extension, and \( \Delta U \) is a vector of elastic displacement and electric potential jump across the branch crack. Noting that

\[ \Delta U(r) = \int_0^r b_1(a - \xi) d\xi , \]  

(3.433)

we have

\[ G^* = \lim_{s \to 0} \frac{1}{2x} \int_0^s \Phi_n^{(1)}(r) \frac{\partial \Delta U_s(x - r)}{\partial x} dr , \]  

(3.434)

where \( \Phi_n^{(1)}(r) \) is defined in eqn (3.406).

Substituting eqn (3.424) into eqn (3.433), and eqn (3.428) into eqn (3.406), later into eqn (3.434), one obtains

\[ G^* = \frac{a \pi}{4} B_v^L(a) \mathbf{L}^L \bigl[ \Omega_i(0) \bigr] \mathbf{B}_a(a). \]  

(3.435)

3.14.3 Criterion for crack propagation

A crack impinging an interface joining two dissimilar materials may advance by either penetrating the interface or deflecting into the interface. This competition can be assessed by comparing the ratio of the energy release rate for penetrating the interface and for deflecting into the interface, \( G_p/G_d \), to the ratio of the mode I toughness of material 1 to the interface toughness, \( \Gamma'_I/\Gamma_v \). In essence, if

\[ \frac{G_p}{G_d} > \frac{\Gamma'_I}{\Gamma_v}, \]  

(3.436)

the impinging crack is likely to penetrate the interface, since the condition for penetrating across the interface will be met at a lower load than that for propagation in the interface. Conversely, if the inequality in eqn (3.436) is reversed, the crack will tend to be deflected into the interface, where \( \Gamma_v \) and \( \Gamma'_I \) are the interface toughness and mode I toughness of material 1, respectively. For a given load condition, say \( \Pi_{\Gamma}^v \), the fracture toughness, \( \Gamma \), is defined as the
energy release rate $G$ at the onset of crack growth and has been discussed in detail elsewhere [61]. $G_d$ and $G_p$ are the energy release rates for deflecting into the interface and for penetrating the interface, respectively.

3.14.4 Numerical results

Since the main purpose of this section is to outline the basic principles of the proposed method, the assessment has been limited to a PP bimaterial plate with an inclined crack of length $2c$, a branch crack of length $a$ and its interface coinciding with the $x_1$-axis as shown in Fig. 3.17. In all calculations, the ratio of $a/c$ is taken to be 0.1. The upper and lower materials are assumed to be PZT-5H and PZT-5 [39], respectively. The material constants for the two materials are given in Section 3.13.

The ratio $G_p/G_d$ versus kink angle $\theta_1$, with $\theta_2 = -90^\circ$ and $\sigma_{22}^o = D_2^o = 0$, is shown in Fig. 3.18 for a number of loading combinations measured by $\psi = \tan^{-1}(K_{II}^o / K_I^o)$, where $K_I^o$ and $K_{II}^o$ are the conventional mode I and mode II stress intensity factors. The curve clearly shows that the maximum energy release rate occurs at a different kink angle $\hat{\theta}$ for different loading phases $\psi$, where $\hat{\theta}$ is the critical kink angle for which the maximum energy release rate will occur under a given loading condition. For example, $\hat{\theta} = 43^\circ$ for curve 1; $\hat{\theta} = 34^\circ$ for curve 2; $\hat{\theta} = 28^\circ$ for curve 3; $\hat{\theta} = 25^\circ$ for curve 4 and $\hat{\theta} = 22^\circ$ for curve 5. It is also evident from Fig. 3.18 that the ratio $G_p/G_d$ will increase along with the increase of phase angle $\psi$, which means that the possibility of the crack penetrating the interface will increase.

To study the effect of remote loading $D_2^o$ on the path selection of crack propagation, the numerical results of the ratio $G_p/G_d$ have been obtained and are plotted in Fig. 3.19 as a function of kink angle $\theta_1$ for several values of remote loading $D_2^o$ with $\theta_2 = 180^\circ$, $\sigma_{22}^o = 4 \times 10^6$ N/m$^2$ and $\sigma_{11}^o = \sigma_{23}^o = 0$. From the

![Fig. 3.18: Variation of $G_p/G_d$ with kink angle $\theta_1$ for several loading combinations specified by $\psi = \tan^{-1}(K_{II}^o / K_I^o)$.

```
calculations we find that the energy release rate may have a negative value depending on the direction and magnitude of the remote loading $D_2^\infty$. For this particular problem, the energy release rate $G_i$ will be positive when $-1.8 \times 10^{-3} \text{ C/m}^2 < D_2^\infty < 1.8 \times 10^{-3} \text{ C/m}^2$ and negative for other values of $D_2^\infty$.

### 3.15 Problems of elliptic cracks

In this section, we consider three-dimensional problems in the mathematical theory of electroelasticity relating to elliptic cracks. In the analysis, we find that, unlike in the two-dimensional case, the method of curvilinear coordinates and the method of complex potential functions play central roles.

Based on the potential functions developed by Wang and Zheng [63], Zhao et al. [64,65] presented some Green’s function solutions for penny-shaped cracks. The problem of determining SED intensity factors for an elliptic crack has been considered by Wang and Huang [66,67]. They considered an infinite transversely isotropic piezoelectric medium containing an elliptic crack with semi-axes $a$ and $b$ ($a > b$), as shown in Fig. 3.20. The crack plane lies in the $xy$-plane, being parallel to the plane of isotropy, while the $z$-axis is directed normal to the crack. Furthermore, equal and opposite uniform pressure $p_0$ and electric displacement $q_0$ are applied to the upper and lower crack surfaces. The boundary conditions of the problem can be stated as follows:

\[
\sigma_z = -p_0, \quad D_z = q_0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \text{ and } z = 0, \tag{3.437}
\]

\[
w = \phi = 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1 \text{ and } z = 0. \tag{3.438}
\]
Fig. 3.20: Polar coordinates at crack tip.

\[ \tau_{r\theta} = \tau_{\theta r} = 0, \quad z = 0 \]  \hspace{1cm} (3.439)

where \( w \) is the elastic displacement in the \( z \)-direction, and all disturbances vanish as \( (x^2 + y^2 + z^2) \to \infty \). That is,

\[ \Pi \to 0. \]  \hspace{1cm} (3.440)

To find solutions for the problem defined above, the general solutions expressed by the potential functions are adopted (see Section 2.5). Considering the variety of eqn (2.62), namely

\[ \frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_j}{\partial y^2} = -\frac{\partial^2 \psi_j}{\partial z_j^2} \]  \hspace{1cm} (3.441)

where \( z_j = x_j \) throughout this section, and using the expressions (2.63), \( \sigma_z \) and \( D_z \) can be given as

\[ \sigma_z = \sum_{j=1}^{3} A_j \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad D_z = \sum_{j=1}^{3} C_j \frac{\partial^2 \psi_j}{\partial z_j^2} \]  \hspace{1cm} (3.442)

where

\[ A_j = -c_{ij} + (c_{ij} r_j + e_{ij} r_j) s_j, \quad C_j = -e_{ij} + (e_{ij} r_j - \kappa_{ij} r_j) s_j, \quad j = 1-3. \]  \hspace{1cm} (3.443)

Without loss of generality, \( \psi_4 (= \chi) \) here may be set to zero on account of symmetry. The boundary conditions (3.439) give

\[ \sum_{j=1}^{3} A_j \frac{\partial^2 \psi_j}{\partial z_j} = -p_0, \quad \sum_{j=1}^{3} C_j \frac{\partial^2 \psi_j}{\partial z_j} = -q_0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \text{ and } z = 0. \]  \hspace{1cm} (3.444)
while eqns (3.438) and (3.439) yield the respective equations

$$\sum_{j=1}^{3} \tau_{j} \frac{\partial^{2} \psi_{j}}{\partial z_{j}^{2}} = 0, \quad \sum_{j=1}^{3} \tau_{j} \frac{\partial^{2} \psi_{j}}{\partial z_{j}^{2}} = 0,$$

and

$$\sum_{j=1}^{3} B_{j} \frac{\partial^{2} \psi_{j}}{\partial z^{2}} = 0, \quad \sum_{j=1}^{3} B_{j} \frac{\partial^{2} \psi_{j}}{\partial z^{2}} = 0, \quad z=0$$

where $B_{j} = c_{44}(1 + \tau_{j}) + e_{ij} \tau_{j}$. It can be found from eqn (3.446) that

$$B_{1} \psi_{1} \big|_{z=0} = -(B_{2} \psi_{2} + B_{3} \psi_{3}) \big|_{z=0}.$$

This implies that $\psi_{1}$, $\psi_{2}$, and $\psi_{3}$ are not completely independent. Let

$$B_{1} \psi_{1} \big|_{z=0} = -(B_{2} \psi_{2} + B_{3} \psi_{3}) \big|_{z=0}.$$ (3.448)

Substituting eqn (3.448) into eqn (3.444) yields

$$\left. \frac{\partial^{2} \psi_{3}}{\partial z^{2}} \right|_{z=0} = \frac{N_{1} p_{0} - M_{1} q_{0}}{M_{1} N_{2} - M_{2} N_{1}}, \quad \left. \frac{\partial^{2} \psi_{1}}{\partial z^{2}} \right|_{z=0} = -\frac{N_{1} p_{0} - M_{1} q_{0}}{M_{1} N_{2} - M_{2} N_{1}},$$

where

$$M_{1} = A_{2} - \frac{A_{1} B_{2}}{B_{1}}, \quad M_{2} = A_{3} - \frac{A_{1} B_{3}}{B_{1}}, \quad N_{1} = C_{2} - \frac{C_{1} B_{2}}{B_{1}}, \quad N_{2} = C_{3} - \frac{C_{1} B_{3}}{B_{1}}.$$ (3.450)

It is expedient to introduce the ellipsoidal coordinates $(\xi, \eta, \zeta)$, which are related to the Cartesian coordinates $(x, y, z)$ as

$$a^{2}(a^{2} - b^{2}) x^{2} = (a^{2} + \xi)(a^{2} + \eta)(a^{2} + \zeta),$$

$$b^{2}(b^{2} - a^{2}) y^{2} = (b^{2} + \xi)(b^{2} + \eta)(b^{2} + \zeta), \quad a^{2} b^{2} z^{2} = \xi \eta \zeta$$

where $\xi, \eta, \zeta$ are some roots $v$ of the ellipsoidal equation

$$\frac{x^{2}}{a^{2} + v} + \frac{y^{2}}{b^{2} + v} + \frac{z^{2}}{v} - 1 = 0,$$ (3.452)

and

$$-a^{2} \leq \zeta \leq -b^{2}, \quad 0 \leq \xi < \infty.$$ (3.453)
while \( z=0, \xi=0 \) indicates points inside the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \) and \( z=0, \eta=0 \) represents points outside the ellipse. It can be proved that the function [67]

\[
\psi_k(x, y, z_j) = \frac{1}{2} H_k \int_{h_k}^\infty \left[ \frac{x^2}{a^2 + v_k} + \frac{y^2}{b^2 + v_k} + \frac{z_j^2}{v_k} - 1 \right] \frac{dv_k}{\sqrt{Q(v_k)}}, \quad k = 2, 3
\]  (3.454)

satisfies eqn (2.62), where

\[
Q(v_k) = v_k (a^2 + v_k)(b^2 + v_k)
\]  (3.455)

and \( H_k(k = 2, 3) \) stand for

\[
H_2 = \frac{ab^2(q_oM_z - p_oN_z)}{2E(k)(M_zN_z - M_zN_z)}, \quad H_3 = \frac{ab^2(q_oM_z - p_oN_z)}{2E(k)(M_zN_z - M_zN_z)}.
\]  (3.456)

The complete elliptic integral of the second kind is denoted by \( E(k) \) with the argument

\[
k = \sqrt{\frac{a^2 - b^2}{a^2}}.
\]  (3.457)

Differentiation of eqn (3.454) involves the quantities

\[
I_k = \frac{1}{2} \int_{h_k}^\infty \left[ \frac{x^2}{a^2 + v_k} + \frac{y^2}{b^2 + v_k} + \frac{z_j^2}{v_k} - 1 \right] \frac{dv_k}{\sqrt{Q(v_k)}},
\]  (3.458)

whose derivatives are

\[
\frac{\partial^2 I_j}{\partial z_j} = \frac{2z_j^{\mid j\mid} \left[ (a^2b^2 - \eta_j\xi_j) - a^2b^2(\eta_j + \xi_j) - (a^2 + b^2)\eta_j\xi_j \right]}{a^2b^2(\xi_j - \eta_j)(\xi_j - \xi_j)(a^2 + \xi_j)^{\frac{1}{2}}(b^2 + \xi_j)^{\frac{1}{2}}}
\]

\[
= \left[ E(u_j) - \frac{\text{snu}_j\text{cnu}_j}{d\text{n}_j} \right]
\]  (3.459)

where \( u, \text{snu}, \text{cnu} \) and \( d\text{n}_j \) are all elliptic functions. With \( j=2 \) and \( j=3 \) in eqn (3.459), and making use of eqn (2.62) with the recognition that if \( \xi=0 \), then \( u_2 = u_3 = \frac{\pi}{2} \) and hence \( E(u_j) = E(u_j) = E(k) \), which is the complete elliptic integral of the second kind with argument \( k = (1 - (b/a)^2)^{\frac{1}{2}} \), and \( \text{snu}_j \text{cnu}_j / d\text{n}_j = 0 \), \( H_2 \) and \( H_3 \) in eqn (3.456) are determined. Owing to the fact that
while \( z=0, \ z_j = 0 \), it is evident from eqn (3.454) that the boundary conditions (3.445) are satisfied.

To sum up, when eqn (3.456) is substituted into eqn (3.454), the solutions of the potential functions \( \psi_j \) are fully determined.

The stress and electric displacement intensity factors at the crack tip can be found by referring to a system of local coordinates \((r, \theta)\) as indicated in Fig. 3.20 [66]. The local coordinate system is related by

\[
x = a \cos \theta + r \cos \theta \cos \beta,
\]

\[
y = b \sin \theta + r \cos \theta \sin \beta,
\]

\[
z = r \sin \theta,
\]

or

\[
\zeta = -a^2 \sin^2 \theta + b^2 \cos^2 \theta
\]

\[
\zeta = \frac{2abr}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}} \left(\cos \frac{\theta}{2}\right)^2,
\]

\[
\eta = \frac{2abr}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}} \left(\sin \frac{\theta}{2}\right)^2,
\]

where \( \Theta \) is the angle in the parametric equations of the ellipse and \( \beta \) is defined as the angle between the outer normal of the elliptical crack border and \( x \)-axis.

Substituting the above-mentioned solutions of potential functions (3.454) and eqn (3.459) into eqn (3.442) with the help of eqns (3.461) and (3.462), yields

\[
\sigma_z = F(0, \Theta, r, A_1, A_2, A_3), \quad D_z = F(0, \Theta, r, C_1, C_2, C_3),
\]

\[
F(0, \Theta, r, y_1, y_2, y_3) = \sqrt{ba^{-1}} \left[ a^2 \sin^2 \Theta + b^2 \cos^2 \Theta \right]^{3/4} \left\{ \sqrt{2E(k)}[A_1(B_1C_3)
\]

\[
- B_2C_1 + B_i(C_iA_2 - C_2A_i) + C_i(A_iB_2 - A_2B_i)] \right\}^{-1}
\]

\[
\times \left( A_i[(B_iC_3 - B_3C_i)p_0 - (A_iB_2 - A_2B_i)q_0]s^r - 1/2 \right)
\]

\[
+ A_i[(B_iC_3 - B_3C_i)p_0 + (A_iB_2 - A_2B_i)q_0]s^r - 1/2
\]

\[
+ A_i[(B_iC_2 - B_2C_i)p_0 + (A_iB_2 - A_2B_i)q_0]s^r - 1/2 \right) \cos \frac{\theta}{2}.
\]

Thus, the SED intensity factors can be evaluated from the conventional
definitions (3.99) and (3.102) as

\[ K_I = \sqrt{2\pi F(0, 0, 1, A_1, A_2, A_3)} \]
\[ K_D = \sqrt{2\pi F(0, 0, 1, C_1, C_2, C_3)}. \]  (3.465)

The remaining stress and electric displacement components near the crack tip can be found similarly, which we do not intend to elaborate here. In order to indicate application of the solutions obtained, a numerical example for piezoelectric ceramic PZT-6B is considered, whose physical constants of material are [66]

\[ c_{11} = 16.8 \times 10^{10} \text{Nm}^{-2}, \quad c_{33} = 16.3 \times 10^{10} \text{Nm}^{-2}, \quad c_{44} = 2.71 \times 10^{10} \text{Nm}^{-2}, \]
\[ c_{12} = c_{13} = 6.0 \times 10^{10} \text{Nm}^{-2}, \quad e_{15} = 4.6 \text{Cm}^{-2}, \quad e_{31} = -0.9 \text{Cm}^{-2}. \]
\[ e_{31} = 7.1 \text{Cm}^{-2}, \quad \kappa_{11} = 36 \times 10^{-10} \text{C(Vm)}^{-1}, \quad \kappa_{33} = 34 \times 10^{-10} \text{C(Vm)}^{-1}. \]

Once the eigenvalues \( \sigma_i, (i=1-3) \) are found from eqn (2.59), \( s_i \) can be calculated by the relation \( s_i^2 = \sigma_i^{1/2} \). They are \( s_1 = 0.505, \quad s_2 = 1.02 - 0.475i, \quad s_3 = 1.02 + 0.475i \). With these results, the SED intensity factors \( K_I \) and \( K_D \) can be further expressed as

\[ K_I = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/4}}{E(k)} \sqrt{\frac{b}{a} [1.28 p_0 + (6.55 \times 10^4) q_0]}, \]  (3.466)
\[ K_D = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{1/4}}{E(k)} \sqrt{\frac{b}{a} [(1.64 \times 10^{-11}) p_0 - 0.7 q_0]}. \]  (3.467)

Fig. 3.21: Variation of normalized stress intensity factor with angle \( \theta \).
Displayed graphically in Fig. 3.21 is a plot of the normalized intensity factor \( \beta_1 \)

\[
\beta_1 = \frac{K_I}{\rho_0 \sqrt{\pi b}} = 1.28 \sqrt{\alpha} \frac{(\alpha^2 \sin^2 \theta + \cos^2 \theta)^{1/4}}{E(k)},
\]

with the parametric angle \( \theta \) for \( q_0=0 \) and several values of \( \alpha \), where \( \alpha=b/a \). It can be seen from Fig. 3.21 that as the angle \( \theta \) increases from 0° to 90°, \( \beta_1 \) tends to increase with the lowest value at the border intersecting the major axis to a maximum at the minor axis. The rate of increase becomes more pronounced for small ratios of \( \alpha \) corresponding to narrower elliptical cracks.

References


