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MESHLESS APPROACH AND ITS APPLICATION IN ENGINEERING PROBLEMS

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ABSTRACT

In this chapter, a meshless method is introduced for analyzing heat conduction, non-linear Poisson-type problems, thin plate bending on an elastic foundation, and functionally graded materials (FGM). In the meshless algorithm, the analog equation method (AEM) is used to obtain the equivalent homogeneous system to the original governing equation. After which MFS and RBF interpolation are used to construct the corresponding approximated particular part and complementary part, respectively. Finally, all unknowns are determined by satisfying boundary conditions and the governing equations in terms of potential, plate deflection, or displacement components at certain points, depending on the problem to be solved. Numerical examples are considered for square plate on a Winkler elastic foundation, different 2D structures made of FGM, and 2D standard Poisson equations. The numerical results are presented for each of the problems mentioned and comparison is made with those from other methods.

Keywords: Meshless method, radial basis function, virtual boundary collocation method, analog equation method

1. INTRODUCTION

The meshless numerical method has recently become an alternative to the finite element method (FEM) and the boundary element method (BEM), due to its advantages of avoiding meshing and remeshing, effective treatment of complicated load conditions, and avoidance of mesh distortion in large deformation problems. The meshless method is usually divided into two main categories: the boundary-type meshless method and the domain-type meshless
method. Since the method described in this chapter belongs to the boundary-type meshless method, developments corresponding to this direction only are briefly reviewed here.

Generally, BEM involves only discretization of the boundary of the structure, due to the governing differential equation being satisfied exactly inside the domain leading to a relatively smaller system size with adequate accuracy. This is an important advantage over 'domain' type solutions such as FEM or the finite difference method (FDM). This advantage exists only for problems without body forces and having explicit fundamental solutions. When a problem involves body force, a domain discretization is also required, which may cause some inconvenience in the implementation of BEM. In order to overcome this drawback, Nowak and Brebbia [1, 2] developed a multiple-reciprocity method (MRM) which can convert domain integrals into boundary integrals. Additionally, the dual reciprocity method (DRM) was introduced by Nardini and Brebbia [3] for transferring domain integrals to boundary integrals. Nowak and Partridge [4] compared these two methods and identified their advantages and drawbacks. The use of the analog equation method [5] makes it possible to treat such problems in which there are no available explicit fundamental solutions. From the discussion above it is evident that the methods mentioned still need to divide a boundary into elements. In order to overcome this disadvantage, many boundary collocation methods have been proposed in the past few decades. Sun et al [6,7] proposed a virtual boundary collocation method to avoid the singularity of fundamental solutions, applying this method successfully to solve many engineering problems. In fact, it can be viewed as a kind of fundamental method (MFS) [8]. However, they did not include a proper process for dealing with body sources. The appearance of radial basis functions (RBFs) provides the possibility of developing a real meshless method. Recently, Chen and Tanaka [9] presented a boundary knot method (BKM) based on the dual reciprocity principle and the nonsingular general solution. Later, Chen and Hon [10] presented numerical investigations into the convergence of BKM. In addition, Chen [11,12] developed the boundary particle method (BPM) based on the multiple reciprocity principle and the nonsingular general solution to analysis of potential problems. More recently, Wang and Qin [13] developed a new meshness method by combining the virtual boundary collocation method (VBCM) [6] with analog equation method (AEM) [5] and radial basis functions (RBF). RBFs here are used to approximate particular solutions related to a fictitious internal source which appears when the analog equation method is introduced, and the VBCM is used to compute corresponding homogeneous solutions (see section 3). The analog equation method makes it possible to use a simpler fundamental solution for the Laplacian operator to analyze complicated problems whose fundamental solutions are probably very complicated or difficult to obtain. Based on this method, they analyzed generalized Poisson-type pronlems [13], steady-state and transient heat conduction problems[14-16], thermo-mechanical problems of functionally graded materials[17], and thin plate bending[18]. In this chapter, however, we restrict our attention to the findings presented in [13-18].

2. POTENTIAL PROBLEMS

The generalized Poisson problem we discussed here can be written in the following form
\[ \nabla^2 u = f(x, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \text{ in } \Omega \] (1)

where \( u \) is the potential field variable, \( \Omega \) is the solution domain bounded by its boundary \( \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \) on which following boundary conditions may be prescribed:

\[ u = \bar{u} \text{ on } \Gamma_1 \] (2)

\[ q = \bar{q} \text{ on } \Gamma_2 \] (3)

\[ \beta_i u + \beta_3 q = \beta_3 \text{ on } \Gamma_3 \] (4)

where \( x \) is the position vector, \( \nabla^2 \) represents the Laplacian operator, and \( q \) stands for the boundary flux defined by \( q = -\frac{\partial u}{\partial n} \), \( n \) are components of the unit outward normal vector \( n \) to the boundary \( \Gamma \), \( \bar{u} \) and \( \bar{q} \) are specified values on the corresponding boundary, respectively, \( \beta_i \) (\( i = 1, 2, 3 \)) are known coefficients. Eq (1) may be linear or nonlinear depending on the right-hand term \( f \) function.

### 2.1. The Analog Equation Method

The boundary value problem defined by Eqs (1) –(4) can be converted into a standard Poisson type equation using the analog equation method [5]. For this purpose, suppose \( u(x) \) is the sought solution to the boundary value problem (BVP), which is a continuously differentiable function with up to two orders in \( \Omega \). If the Laplacian operator is applied to this function, namely,

\[ \nabla^2 u = b(x) \] (5)

then, Eq (5) indicates that the solution to Eq (1) can be obtained by solving the linear equation (5) subjected to the boundary conditions (2)-(4) too, if the fictitious source distribution \( b(x) \) is known. The solution procedure is detailed below.

Firstly, the solution of Eq (5) is written as a sum of the homogeneous solution \( u_h \) and the particular solution \( u_p \), that is

\[ u = u_h + u_p \] (6)

Accordingly, \( u_h \) and \( u_p \), respectively, satisfy
\[ \nabla^2 u_p(x) = b(x) \]  

and

\[
\begin{cases}
\nabla^2 u_h = 0 & \text{in } \Omega \\
u_h = \bar{u} - u_p & \text{on } \Gamma_1 \\
q_h = \bar{q} - q_p & \text{on } \Gamma_2 \\
\beta_1 u_h + \beta_2 q_h = \beta_3 - \beta_3 u_p - \beta_2 q_p & \text{on } \Gamma_3
\end{cases}
\]  

(8)

2.2. RBF Approximation for the Particular Solution \( u_p \)

The next step of the proposed approach is to evaluate the particular solution by RBF approximation. To this end, the right-hand term of Eq (5) is approximated by

\[ b(x) = \sum_{j=1}^{M} \alpha_j f_j(x) \]  

(9)

where \( M = I + B \), \( I \) and \( B \) are the number of interpolation points inside the domain and on its boundary, respectively, as shown in Figure 1. \( \alpha_j \) are coefficients to be determined and \( f_j \) are a set of approximating functions.

Similarly, the particular solution \( u_p \) is also written in the form

\[ u_p(x) = \sum_{j=1}^{M} \alpha_j \hat{u}_j(x) \]  

(10)

where \( \hat{u}_j \) are a corresponding set of particular solutions.

Correspondingly, boundary flux can be expressed as

\[ q_p(x) = -\sum_{j=1}^{M} \alpha_j \frac{\partial \hat{u}_j(x)}{\partial n} \]  

(11)

Figure 1. Interpolation points inside the domain and on its boundary.
Because the particular solution $u_p$ satisfy Eq (5), the key to this approximation is the assumption of a corresponding set of approximating particular solutions $f_j$, which, for the case of Laplacian operator, satisfy

$$\nabla^2 \hat{u}_j(x) = f_j(x)$$

(12)

The effectiveness and accuracy of the interpolation depends on the choice of the approximating functions $f_j$. In this section, the functions $f_j$ in Eq (9) are selected to be locally polynomial RBFs in terms of power series of a distance function $r_j$:

$$f_j(x) = 1 + r_j^3$$

(13)

where $r(x, x_j) = r_j(x) = |x - x_j|$ denotes the distance from the source point $x_j$ to the field point $x$. The functions in Eq. (13) have shown to be most convenient to implement into standard computer program.

### 2.3. VBCM for the Homogeneous Solution

To obtain a weak solution of Laplace problem (5), $N$ fictitious source points $y_i (i = 1, 2, \cdots N)$ on the virtual boundary and the same number collocation points on the physical one are selected, respectively (see Figure 2). The virtual boundary is outside the domain under consideration. Moreover, assume that there is a virtual source $\varphi_i$ ($1 \leq i \leq N$) at each fictitious source point.

![Figure 2. Illustration of a computational domain and point discretization on the real and virtual boundary.](image)
According to the superposition principle, the potential \( u_h \) and the boundary normal gradient at an arbitrary field point \( \mathbf{x} \) in the domain or on the boundary can be expressed by a linear combination of fundamental solutions in terms of fictitious source placed on the virtual boundary, respectively [7-11], that is

\[
    u_h(\mathbf{x}) = \sum_{i=1}^{N} u^*(\mathbf{x}, \mathbf{y}_i) \phi_i
\]  

(14)

\[
    q_h(\mathbf{x}) = -\frac{\partial u_h}{\partial n} = -\sum_{i=1}^{N} \phi_i \frac{\partial u^*(\mathbf{x}, \mathbf{y}_i)}{\partial n}
\]  

(15)

where \( u^* \) is the fundamental solution of the Laplacian operator,

\[
    u^*(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{r(\mathbf{x}, \mathbf{y})}
\]  

(16)

for a 2-D problem, and

\[
    u^*(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi r}
\]  

(17)

for a 3-D problem.

It should be mentioned that a boundary with a shape similar to that of the physical boundary, or simply a circular boundary for the 2D domain and a spherical boundary for 3D problems, are usually selected as the shape of the virtual boundary. A virtual boundary which mimicked the shape of the physical boundary was used by Sun et al [6] because of the advantage of maintaining the source at roughly the same magnitude from the physical boundary. However, the construction of a similar virtual boundary may be inconvenient, especially for complicated boundaries. A circular virtual boundary has good flexibility and can be applied easily to most problems. But in some cases, differences of large magnitude may be encountered which lead to ill-conditioned solutions.

The distance between the fictitious source point and the physical boundary is another interesting issue in the present meshless model. Theoretically, there are no rules for selecting the location of the virtual boundary. However, from the point of view of computation and considering the singularity of the fundamental solution, the accuracy of the result will degrade when the distance between the virtual and physical boundaries becomes very close [6]. Conversely, round-off error in C/Fortran floating point arithmetic may be a serious problem when the source points are far from the real boundary. In that case, the coefficient matrix of the system of equations is nearly zero [8]. For a virtual boundary that is similar in shape to the physical boundary, the location can be determined by defining the similarity ratio between the virtual and physical boundaries as
For a circular virtual boundary, the center of the circle may overlap with the center of the domain, and the radius is an important parameter to measure its location. In particular, for a rectangular domain, the real parameter $\beta$ (Figure 3), representing the similarity ratio of the similar virtual boundary, can measure the radius of the circular virtual boundary.

![Figure 3. Similar and circular virtual boundaries for a rectangular domain.](image)

2.4. The Construction of Solution System

Based on the procedure described above, the solution $u = u(x)$ we are seeking to Eq (5) can be obtained as

$$u(x) = \sum_{i=1}^{N} \varphi_{i} u^{*}(x, y_{i}) + \sum_{j=1}^{M} \alpha_{j} \hat{u}_{j}(x)$$  \hspace{1cm} (18)$$

$$q(x) = -\left[\sum_{i=1}^{N} \varphi_{i} \frac{\partial u^{*}}{\partial n}(x, y_{i}) + \sum_{j=1}^{M} \alpha_{j} \frac{\partial \hat{u}_{j}}{\partial n}(x)\right]$$  \hspace{1cm} (19)$$

which are also the solutions of Eq (1). Differentiating Eq (18) yields
\[
\frac{\partial \tilde{u}}{\partial x_m} = \sum_{i=1}^{N} \phi_i \frac{\partial \tilde{u}^* (x, y_i)}{\partial x_m} + \sum_{j=1}^{M} \alpha_j \frac{\partial \tilde{u}_j (x)}{\partial x_m} \tag{20}
\]
\[
\frac{\partial^2 u}{\partial x_m \partial x_n} = \sum_{i=1}^{N} \phi_i \frac{\partial \tilde{u}^* (x, y_i)}{\partial x_n \partial x_m} + \sum_{j=1}^{M} \alpha_j \frac{\partial \tilde{u}_j (x)}{\partial x_n \partial x_m} \tag{21}
\]

where \( m, n = 1, 2 \) for 2D plane problems and \( m, n = 1, 2, 3 \) for 3D cases.

Finally, in order to determine unknowns \( \alpha_j \) and \( \phi_i \), Eq (18) should satisfy the governing differential equation (1) at \( M \) interpolation points. Besides, Eqs (18) and (19) should satisfy corresponding boundary conditions (2)-(4) at \( N \) collocation points on the physical boundary. Thus, we have

\[
\begin{align*}
\sum_{i=1}^{N} \phi_i \nabla^2 \tilde{u}^* (x, y_i) + \sum_{j=1}^{M} \alpha_j \nabla^2 \tilde{u}_j (x) &= f (\varphi_1, \alpha_j) \\
\sum_{i=1}^{N} \phi_i \tilde{u}^* (x, y_i) + \sum_{j=1}^{M} \alpha_j \tilde{u}_j (x) &= \bar{u} \\
\sum_{i=1}^{N} \phi_i \tilde{q}^* (x, y_i) + \sum_{j=1}^{M} \alpha_j \tilde{q}_j (x) &= \bar{q} \\
\sum_{i=1}^{N} \phi_i \left[ \beta_1 \tilde{u}^* (x, y_i) + \beta_2 \tilde{q}^* (x, y_i) \right] + \sum_{j=1}^{M} \alpha_j \left[ \beta_1 \tilde{u}_j (x) + \beta_2 \tilde{q}_j (x) \right] &= \beta_j
\end{align*}
\] (22)

As a result, a system of equations in terms of the unknown coefficients \( \alpha_j \) and \( \phi_i \) can finally be written as

\[
F (x) = 0
\] (23)

where the sought vector \( x \) contains the unknown coefficients \( \alpha_j \) and \( \phi_i \).

Once these unknown coefficients are determined, the potential field \( u \) and its normal derivative \( q \) at any point \( x \) inside the domain or on its boundary can be determined by using Eqs (18) and (19).

**Example 1: Nonlinear Poisson Problems**

Consider a generalized nonlinear Poisson problem in the \( 1 \times 1 \) rectangular domain whose governing equation is defined by...
\[ \nabla^2 u(x, y) + \left( \frac{\partial u}{\partial y} \right)^2 = 2y + x^4 \]

with following boundary conditions

\[ u(0, y) = 0 \quad \text{and} \quad u(1, y) = y + a \]

\[ u(x, 0) = ax \quad \text{and} \quad u(x, 1) = x(x + a) \]

The analytical solution of this problem is \( u(x, y) = x(xy + a) \).

Figure 4. Demonstration of fictitious source points, collocations and interpolation points.

Figure 5. Configuration of exact solutions and numerical results when parameter \( a \) is zero and \( N = 12, M = 25 \).
In our computation, the constant $a$ is selected to be 0 for the sake of simplify. Total collocation points on the physical boundary are selected to be 12 and the same number of fictitious source points is uniformly distributed on the virtual boundary (Figure 4). The number of internal interpolation points in the domain is from 9 to 25 (Figure 4). The initial guess of iteration is zero. The convergent results are achieved with about 8 iterations. Numerical results of potential $u$ are shown in Table 1 and Figure 5.

As was expected, it can be seen from both Table 1 and Figure 5 that the results from the proposed meshless method gradually converge to the exact values along with the increase in the number of interpolation points.

Table 1. Comparison of numerical results and exact ones

<table>
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<tr>
<th>Location</th>
<th>Numerical results</th>
<th>Exact solutions</th>
</tr>
</thead>
<tbody>
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<td>(0.00, 0.00)</td>
<td>(N=12, M=9)</td>
<td>(N=12, M=25)</td>
</tr>
<tr>
<td></td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>(0.25, 0.25)</td>
<td>0.0164</td>
<td>0.0159</td>
</tr>
<tr>
<td>(0.50, 0.50)</td>
<td>0.1271</td>
<td>0.1251</td>
</tr>
<tr>
<td>(0.75, 0.75)</td>
<td>0.4249</td>
<td>0.4222</td>
</tr>
<tr>
<td>(1.00, 1.00)</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(1.00, 0.50)</td>
<td>0.5000</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

3 STEADY-STATE HEAT CONDUCTION IN INHOMOGENEOUS MATERIALS

In this section, the steady-state heat conduction in both isotropic heterogeneous and anisotropic materials are considered. As was treated in Section 2, let $\Omega$ be the solution domain bounded with its piecewise smooth boundary $\Gamma$. The corresponding meshless formulation is presented based on the development in [14].

3.1. Governing Equation for Steady-State Heat Conduction in Isotropic Heterogeneous Media

Let us consider the general heat conduction problem in isotropic media

$$\nabla \cdot [k(x)\nabla u(x)] + f(x) = k(x)\nabla^2 u + \nabla k(x) \cdot \nabla u(x) + f(x) = 0$$

(24)

satisfying the boundary conditions (2), (3), and following convection boundary condition

$$q_u(x) = h_c (u - u_\infty) \quad x \in \Gamma_3$$

(25)

where $k(x)$ is spatially varying thermal conductivity, $u(x)$ is the temperature field, and the boundary heat flux arising in the boundary conditions is defined as $q = -k \frac{\partial u}{\partial x_i} n_i$. Here and after, repeated indices imply the summation convention of Einstein unless otherwise indicated.
\( f(\mathbf{x}) \) denotes the internal heat. The constant \( h_\infty \) is the convection coefficient and \( u_\infty \) is the temperature of environment. It should be mentioned that the convection boundary condition is actually a kind of Robin mixed boundary condition.

### 3.2. Governing Equation for Steady-State Heat Conduction in Anisotropic Media

Unlike in isotropic materials whose thermal conductivity is a scalar function, thermal conductivity in general anisotropic materials is represented by a second-order tensor which contains nine components in the 3D situation. Thus the governing equation in the Cartesian coordinate system is written as, in this case,

\[
\nabla \cdot [K(\mathbf{x})\nabla u(\mathbf{x})] + f(\mathbf{x}) = \frac{\partial u}{\partial x_j} \frac{\partial k_{ij}}{\partial x_j} + k_{ij} \frac{\partial^2 u}{\partial x_j \partial x_i} + f(\mathbf{x}) = 0 \quad (26)
\]

with the same boundary conditions as those of Eqs (2), (3), and (25), where \( K \) is the thermal conductivity tensor, whose components should obey Onsager’s reciprocity relation \( k_{ij} = k_{ji} \).

The boundary heat flux \( q(\mathbf{x}) \) here is defined as \( q = -k_{ij} \frac{\partial u}{\partial x_j} n_i \).

In particular we have

\[
K = \begin{bmatrix}
k_{11} & k_{12} \\
k_{12} & k_{22}
\end{bmatrix}
\quad (27)
\]

for the case of a 2D domain.

Eq (26) can be further written as

\[
\nabla \cdot [K(\mathbf{x})\nabla u(\mathbf{x})] + f(\mathbf{x}) = \left( \frac{\partial k_{11}}{\partial x_1} + \frac{\partial k_{12}}{\partial x_2} \right) \frac{\partial u}{\partial x_1} + \left( \frac{\partial k_{12}}{\partial x_1} + \frac{\partial k_{22}}{\partial x_2} \right) \frac{\partial u}{\partial x_2} + \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2 \partial x_1} + \frac{\partial^2 u}{\partial x_2^2} + f(\mathbf{x})
\]

\[
= \frac{\partial^2 u}{\partial x_1 \partial x_2} + k_{11} \frac{\partial^2 u}{\partial x_1^2} + k_{22} \frac{\partial^2 u}{\partial x_2^2} + f(\mathbf{x}) = 0 \quad (28)
\]

If the coefficients \( k_{ij} \) are assumed to be independent of the space variables, Eq (28) can be simplified as

\[
\nabla \cdot [K(\mathbf{x})\nabla u(\mathbf{x})] + f(\mathbf{x}) = k_{11} \frac{\partial^2 u}{\partial x_1^2} + k_{22} \frac{\partial^2 u}{\partial x_2^2} + 2k_{12} \frac{\partial^2 u}{\partial x_1 \partial x_2} + f(\mathbf{x}) = 0 \quad (29)
\]
3.3. IMPLEMENTATION OF THE MESHLESS METHOD

Since the problem to be studied is a linear problem, a general linear differential operator $L$ is introduced for the sake of writing convenience. Thus, Eqs (24) and (26) can be replaced by

$$L[u(x)] + f(x) = 0$$

where

$$L = k \nabla^2 + \nabla k \cdot \nabla$$

for isotropic materials and

$$L = \frac{\partial}{\partial x_j} \left( k_j \frac{\partial}{\partial x_j} \right) + k_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

for anisotropic materials.

The application of the analog equation method to Eq (24) or Eq (26) leads to the same form as that of Eq (5), except for the term $b$ being different.

As was done in Section 2, $u$ can be written in a sum of the homogeneous solution $u_h$ and the inhomogeneous solution or particular solution $u_p$ (see Eq(6)). It leads to same equation as those of Eqs (7) and (8) except that the fourth equation of Eq (8) is replaced by

$$h_x u_h(x) - q_h(x) = h_x u_o - h_x u_p + q_p \quad x \in \Gamma_3$$

3.4. The Virtual Boundary Collocation Method for the Homogeneous Solution

To obtain a weak solution of Laplace problem (8) and (33), $N$ nodal points $x_j \ (j = 1, 2, \ldots, N)$ on the real boundary and $N$ fictitious source points $y_i \ (i = 1, 2, \ldots, N)$ on the virtual boundary are again selected, respectively (see Figure 2).

Apply the superposition principle, the potential $u_h$ is given by Eq (14) and the boundary heat flux at field points $x$ in the domain or on the boundary can be expressed by a linear combination of fundamental solutions in terms of fictitious sources located on the virtual boundary:

$$q_h(x) = -k \left[ \sum_{i=1}^{N} \varphi_i \frac{\partial u^*(x,y_i)}{\partial x_m} \right] n_m$$

for isotropic materials and
Meshless Approach and its Application in Engineering Problems

\[ q_n(x) = -k_{mn} \frac{\partial u_n}{\partial x_m} n_m = -k_{mn} \left[ \sum_{i=1}^{N} \varphi_i \frac{\partial u^*(x, y_i)}{\partial x_m} \right] n_m \]  

(35)

for anisotropic materials.

### 3.5. The Construction of Solving Equations

According to the above analysis, the solution \( u = u(x) \) we are seeking to Eqs (30) and (2), (3), (25) can be written as Eqs (18) and (19). For anisotropic materials, Eq (19) is replaced by

\[ q = -k_{mn} \left[ \sum_{i=1}^{N} \varphi_i \frac{\partial u^*(x, y_i)}{\partial x_m} + \sum_{j=1}^{M} \alpha_j \frac{\partial \hat{u}_j}{\partial x_m} \right] n_m \]  

(36)

Similar to that in Section 1, a system of \( N + I + B \) linear equations can be constructed as

\[
\begin{align*}
\sum_{i=1}^{N} \varphi_i \left[ k \nabla^2 u^* + \nabla k \cdot \nabla u^* \right] + \sum_{j=1}^{I+B} \alpha_j \left[ k \nabla^2 \hat{u}_j + k \nabla \hat{u}_j \right] &= -f(x) \\
\sum_{i=1}^{N} \varphi_i u^* + \sum_{j=1}^{I+B} \alpha_j \hat{u}_j &= \bar{u} \\
\sum_{i=1}^{N} \varphi_i \left( -k \frac{\partial u^*}{\partial x_m} n_m \right) + \sum_{j=1}^{I+B} \alpha_j \left( -k \frac{\partial \hat{u}_j}{\partial x_m} n_m \right) &= \bar{q}_n \\
\sum_{i=1}^{N} \varphi_i \left( h_i u^* + k \frac{\partial u^*}{\partial x_m} n_m \right) + \sum_{j=1}^{I+B} \alpha_j \left( h_i \hat{u}_j + k \frac{\partial \hat{u}_j}{\partial x_m} n_m \right) &= h_i \bar{u}_n
\end{align*}
\]

(37)

for isotropic materials and

\[
\begin{align*}
\sum_{i=1}^{N} \varphi_i \left( \frac{\partial \hat{u}^*}{\partial x_m} + k_{mn} \frac{\partial u^*}{\partial x_m} \right) + \sum_{j=1}^{I+B} \alpha_j \left( \frac{\partial \hat{u}_j}{\partial x_m} + k_{mn} \frac{\partial \hat{u}_j}{\partial x_m} \right) &= -f(x) \\
\sum_{i=1}^{N} \varphi_i u^* + \sum_{j=1}^{I+B} \alpha_j \hat{u}_j &= \bar{u} \\
\sum_{i=1}^{N} \varphi_i \left( -k_{mn} \frac{\partial u^*}{\partial x_m} n_m \right) + \sum_{j=1}^{I+B} \alpha_j \left( -k_{mn} \frac{\partial \hat{u}_j}{\partial x_m} n_m \right) &= \bar{q}_n \\
\sum_{i=1}^{N} \varphi_i \left( h_i u^* + k_{mn} \frac{\partial u^*}{\partial x_m} n_m \right) + \sum_{j=1}^{I+B} \alpha_j \left( h_i \hat{u}_j + k_{mn} \frac{\partial \hat{u}_j}{\partial x_m} n_m \right) &= h_i \bar{u}_n
\end{align*}
\]

(38)
for anisotropic materials, from which the unknown coefficients $\alpha_j$ and $\varphi_i$ can be determined.

### 3.6. Numerical Assessment

In order to demonstrate the efficiency and accuracy of the formulation presented in this section, two benchmark numerical examples are considered and their results are compared with the analytical results.

**Example 2:** Consider an isotropic disc whose radius is 0.1 m. Here, the thermal conductivity is assumed to vary bi-quadratically as

$$k(x, y) = (2x + y + 2)^2$$

An analytical expression for the temperature field satisfying the governing heat conduction equation can be given as

$$u(x, y) = \frac{6x^2 - 6y^2 + 20xy + 30}{2x + y + 2}$$

This temperature profile is also used to impose the boundary conditions shown in Figure 6.

![Figure 6. Geometry of the domain and boundary conditions.](image)

This non-homogeneous problem is solved using 36 fictitious source points outside of the domain (shown in Figure 7) and 181 interpolation points (shown in Figure 8). The similarity ratio is equal to 3.0. The distribution of temperature on the boundary of the circle inside the disc is plotted in Figure 9, while the boundary heat fluxes are plotted in Figure 10. It is clear from these figures that good agreement is achieved with relatively few collocation points.
Example 3: In order to illustrate typical numerical results of heat conduction in anisotropic materials, we consider a 2D anisotropic medium with the thermal conductivity tensor given by

\[ k_{11} = 5.0, \ k_{22} = 1.0, \ k_{12} = k_{21} = 2.0 \]

The Dirichlet problems are solved in the plane domain \( \Omega = \{(x, y) : x^2 + y^2 < 1\} \), i.e. the 2D disc of radius unity. The analytical temperature distribution to be retrieved is given by

\[ u(x, y) = \frac{x^3}{5} - x^2 y + xy^2 + \frac{y^3}{5} \]

which imposes the boundary conditions.
In computation, the similarity ratio is selected to be equal to 3.0. The number of fictitious source points on the virtual boundary is 36 and there are 33 interpolation points to be used during RBF approximation, including 25 internal points and 8 boundary points. Figure 11 shows the distribution of temperature at the points on the circles whose radii are equal to 0.5 and 1.0, respectively. We can see that good agreement is obtained between the numerical results and the exact solutions.
Figure 11. Distribution of temperature on boundary of circles (radius=0.5 and 1.0) within a disc.

Figure 12 depicts the distribution of heat flux on the boundary and we can see that the numerical results also agree well with the exact results.

Figure 12. Distribution of heat flux on the boundary of a disc.
4. TRANSIENT HEAT CONDUCTION IN FUNCTIONALLY GRADED MATERIALS

In Sections 2 and 3, application of meshless method to generalized potential problems and steady-state heat conduction is discussed. Extension to transient heat conduction in FGM is presented in this section.

4.1. Basic Formulas of Transient Heat Conduction

Consider a transient heat conduction problem, occupying an arbitrarily shaped region \( \Omega \) bounded by its boundary \( \Gamma \). In the Cartesian coordinates system, the transient temperature field in a FGM or a heterogeneous isotropic medium is governed by the diffusion equation

\[
\nabla \cdot (k \nabla u) + f(x, t) = \kappa \nabla^2 u + \nabla \cdot \nabla u + f(x, t) = \rho c \frac{\partial u(x, t)}{\partial t}
\]

(39)

with the boundary conditions (2), (3), and (25), where \( t \) denotes time \((t > 0)\), \( \rho \) is the mass density and \( c \) is the specific heat. \( f(x, t) \) stands for the internal heat source generated per unit volume.

For functionally graded materials (FGM), for instance, assume that the thermal conductivity varies exponentially in one Cartesian coordinate, i.e.

\[
k(x) = k_0 e^{\lambda x}
\]

(40)

where \( \lambda \) is the non-homogeneity parameter. Specially, for homogeneous and isotropic materials, the thermal conductivity \( k \) is a constant. In this case, Eq. (39) can be simplified as a popular hyperbolic Poisson equation,

\[
k \nabla^2 u + f(x, t) = \rho c \frac{\partial u(x, t)}{\partial t}
\]

(41)

which has been studied by many researchers.

Finally, the initial condition must be supplied

\[
u(x, t) \big|_{t=0} = g(x)
\]

(42)

4.2. Meshless Formulation

A meshless model for solving the hyperbolic BVP defined by Eqs (2), (3), (25), (39)-(42) can be obtained in a way similar to that described in Section 2. We start by converting Eq. (39) into a simple Poisson equation using the analog equation method and then consider the RBF
approximation of the fictitious loading term induced in the converting process. Finally a virtual boundary collocation formulation is given for two-dimensional problems.

Making use of the analog equation method, Eq. (39) is converted into the form

\[ \nabla^2 u' = b' (x) \]  

As was done in Section 2, the fictitious source \( b' (x) \), here, is evaluated using global RBF approximation. To this end, decomposing \( u' \) at any particular time \( t \) into two major parts: the homogeneous solution \( u'_h \) and the inhomogeneous solution or particular solution \( u'_p \)

\[ u' = u'_h + u'_p \]  

where \( u'_h \) and \( u'_p \) satisfy respectively

\[ \nabla^2 u'_h (x) = b' (x) \]  

and

\[
\begin{align*}
\nabla^2 u'_p (x) &= 0 & \mathbf{x} \in \Omega \\
u'_h (x) &= \overline{u'} - u'_p (x) & \mathbf{x} \in \Gamma_1 \\
q'_h (x) &= \overline{q'} - q'_p (x) & \mathbf{x} \in \Gamma_2 \\
h_n u'_h (x) - q'_h (x) &= h_n u_n - h_n u'_p + q'_p & \mathbf{x} \in \Gamma_3
\end{align*}
\]

Applying RBF approximation to the particular solution \( u'_p \), we have

\[ b' (x) = \sum_{j=1}^{M} a'_j f_j (x) \]  

\[ u'_p (x) = \sum_{j=1}^{M} \alpha'_j \hat{u}_j (x) \]  

\[ q'_p (x) = -k \left[ \sum_{j=1}^{M} \alpha'_j \frac{\partial \hat{u}_j}{\partial n} \right] \]  

where \( \alpha'_j \) are coefficients to be determined.

Making use of VBCM, the potential \( u'_h \) and the boundary heat flux at field points \( x \) in the domain or on the boundary can be expressed by a linear combination of fundamental solutions.
in terms of fictitious sources located on the virtual boundary, as presented in the previous sections:

\[ u'_i(x) = \sum_{j=1}^{N} \phi'_{ij} u^*(x, x'_j) \] (50)

\[ q'_i(x) = -k \left[ \sum_{j=1}^{N} \phi'_{ij} \frac{\partial u^*}{\partial n} \right] \] (51)

in which \( u^* \) is defined in Eq (16) or (17).

4.3. The Backward Time Stepping Scheme

Based on the discussion above, the solution \( u'(x) \) to Eqs (2), (3), (25), and (39) at a particular time \( t \) can be written as

\[ u'(x) = \sum_{j=1}^{N} \phi'_{ij} u^*(x, x'_j) + \sum_{j=1}^{M} \alpha'_{ij} \hat{u}_j \] (52)

\[ q'(x) = -k \left( \sum_{j=1}^{N} \phi'_{ij} \frac{\partial u^*}{\partial n} + \sum_{j=1}^{M} \alpha'_{ij} \frac{\partial \hat{u}_j}{\partial n} \right) \] (53)

Differentiating Eq. (52) with respect to \( x \) or \( y \) yields

\[ u'_{,xx} = \sum_{j=1}^{N} \phi'_{ij} u^*_{,xx}(x, x'_j) + \sum_{j=1}^{M} \alpha'_{ij} \frac{\partial \hat{u}_j}{\partial x} \] (54)

\[ u'_{,yy} = \sum_{j=1}^{N} \phi'_{ij} u^*_{,yy}(x, x'_j) + \sum_{j=1}^{M} \alpha'_{ij} \frac{\partial \hat{u}_j}{\partial y} \] (55)

\[ u'_{,x} = \sum_{j=1}^{N} \phi'_{ij} u^*_{,x}(x, x'_j) + \sum_{j=1}^{M} \alpha'_{ij} \frac{\partial \hat{u}_j}{\partial x} \] (56)

\[ u'_{,y} = \sum_{j=1}^{N} \phi'_{ij} u^*_{,y}(x, x'_j) + \sum_{j=1}^{M} \alpha'_{ij} \frac{\partial \hat{u}_j}{\partial y} \] (57)
It is noted that the formulations given in Section 4.2 are in terms of a particular time \( t \). In order to obtain the temperature field and its flux at any time, the time domain is divided into elements and a simple backward time stepping scheme is used, i.e.,

\[
\frac{\partial u}{\partial t} \bigg|_{t+\Delta t} = \frac{u^{t+\Delta t} - u^{t}}{\Delta t}
\]

(58)

where \( \Delta t \) is the time step.

Substituting Eq. (58) into Eqs (2), (3), (25), and (39), we have

\[
k\nabla^2 u^{t+\Delta t} + \nabla k \cdot \nabla u^{t+\Delta t} - \frac{\partial c}{\partial t} u^{t+\Delta t} = -\frac{\partial c}{\partial t} u'(X) - f^{t+\Delta t} (X)
\]

(59)

with boundary conditions

\[
u^{t+\Delta t} (x) = \overline{u}^{t+\Delta t} \text{ on } \Gamma_1
\]

(60)

\[
q^{t+\Delta t} (x) = \overline{q}^{t+\Delta t} \text{ on } \Gamma_2
\]

(61)

\[
q^{t+\Delta t} (x) = h_u (u^{t+\Delta t} - u_w) \text{ on } \Gamma_3
\]

(62)

Using Eqs (52) - (57), satisfaction of the governing equation (59) at \( M \) interpolation points inside \( \Omega \) and the boundary conditions (60)-(62) at \( N \) nodal points on the physical boundary provides \( N + M \) equations to determine unknowns \( \alpha_j^\prime \) and \( \phi_j^\prime \):

\[
\begin{align*}
\sum_{j=1}^{N} \phi_j^{t+\Delta t} \left( k \nabla^2 u^* + \nabla k \cdot \nabla u^* - \frac{\partial c}{\partial t} u^* \right) + \sum_{j=1}^{M} \alpha_j^{t+\Delta t} \left( k \nabla^2 \hat{u}_j + \nabla k \cdot \nabla \hat{u}_j - \frac{\partial c}{\partial t} \hat{u}_j \right) \\
= -f^{t+\Delta t} - \frac{\partial c}{\partial t} u'
\end{align*}
\]

(63)

\[
\sum_{j=1}^{N} \phi_j^{t+\Delta t} u^* + \sum_{j=1}^{M} \alpha_j^{t+\Delta t} \hat{u}_j = \overline{u}^{t+\Delta t}
\]

\[
\sum_{j=1}^{N} \phi_j^{t+\Delta t} \left( -k \frac{\partial \hat{u}}{\partial x} n_m \right) + \sum_{j=1}^{M} \alpha_j^{t+\Delta t} \left( -k \frac{\partial \hat{u}_j}{\partial x} n_m \right) = \overline{q}^{t+\Delta t}
\]

\[
\sum_{j=1}^{N} \phi_j^{t+\Delta t} \left( h_u u^* + k \frac{\partial \hat{u}}{\partial x} n_m \right) + \sum_{j=1}^{M} \alpha_j^{t+\Delta t} \left( h_u \hat{u}_j + k \frac{\partial \hat{u}_j}{\partial x} n_m \right) = h_u u_w
\]

The unknown coefficients \( \alpha_j^{t+\Delta t} \) and \( \phi_j^{t+\Delta t} \) can thus be determined by solving the linear algebraic system (63) and using the initial condition Eq. (42). Once these unknown coefficients
are determined, the solution $u^{\xi+\Delta}$ and its normal derivative at any field point $x$ in the solution domain or on its boundary can be calculated using Eqs (52) and (53).

### 4.4. Numerical Assessment

The performance of the formulation presented in this section is assessed through the numerical example below.

**Example 4:** Consider a functionally graded finite strip with a unidirectional variation of thermal conductivity [19]. Zero initial temperature is assumed. The exponential spatial variation for thermal conductivity is written as

$$k(x) = k_0 e^{\lambda x}$$

where $k_0 = 17 \text{W/m}\cdot\text{°C}$. The scale product of density and specific heat is given by $\rho \cdot c = 10^3 \text{J/m}^3\cdot\text{°C}$. Two different exponential parameters $\lambda = 0.2$ and $0.5 \text{ cm}^{-1}$ [19] are assumed in numerical calculation. On the sides parallel to the y-axis two different temperatures are prescribed. One side is kept at zero temperature and the other has the Heaviside function of time, i.e., $u = T \cdot H(t)$ with $T = 1 \text{ °C}$. On the lateral sides of the strip the heat flux vanishes.

In the numerical calculation, a square with side length $L = 0.04 \text{m}$ is considered (see Figure 13).

![Figure 13. Geometry of a functionally graded finite square strip and boundary conditions.](image)

The special case with an exponential parameter $\lambda = 0$ is considered first. In this case the analytical solution is written as

$$u(x,t) = T \frac{x}{L} + \frac{T}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \sin \frac{n\pi x}{L} \exp \left( -\frac{n^2 \pi^2 t}{L^2} \right)$$
which can be used to check the accuracy of the present numerical method. Numerical results are obtained using 36 fictitious source points, 169 interpolation points, similarity ratio = 3.0 and, time step $\Delta t = 1s$. The following computation is carried out using the first-order interpolation function $1 + r$ only. Figure 10 shows the temperature field at the three points ($x = 0.01m$, $0.02m$ and $0.03m$). A good agreement between numerical and analytical results is observed from Figure 14.

![Figure 14. Time variation of temperature in a finite square strip at three different positions with $\lambda = 0$.](image)

The discussion above concerns heat conduction in homogeneous materials only since there are related exact solutions to be used. To illustrate the application of the proposed algorithm to the FGM, consider now the FGM with $\lambda = 0.2$, and $0.5cm^{-1}$, respectively. The variation of temperature with time for three $\lambda$-values and at position $x = 0.02m$ is presented in Figure 15. Figure 16 shows the distribution of temperature along the x-axis at $t = 30s$. As expected, it is found from Figure 15 that the temperature increases along with an increase in $\lambda$-values (or equivalently in thermal conductivity), and the temperature approaches a steady state when $t > 20s$. It is observed from Figure 16 that the temperature increases along with an increase in $\lambda$-values again.

For final steady state an analytical solution can be obtained as

$$u(x) = T \frac{e^{-\lambda x}}{e^{-\beta a} - 1} (u \to T \frac{x}{a} \text{ with } \lambda \to 0)$$
Analytical and numerical results computed at time $t = 70s$ corresponding to stationary or static loading conditions are presented in Figure 17. The numerical results are in good agreement with the analytical results for the steady state case.

Figure 15. Time variation of temperature at position $x = 0.02\text{m}$ of functionally graded finite square strip.

Figure 16. Distribution of temperature at $t = 30s$ along x-axis of functionally graded finite square strip.
Figure 17. Distribution of temperature along x-axis for a functionally graded finite square strip under steady-state loading conditions.

5. THERMO-MECHANICAL ANALYSIS OF FGMS

5.1. Governing Equations for FGMS

Let us consider an isotropic and linear elastic domain $\Omega$ bounded by the boundary $\Gamma$. The static equilibrium requires

$$\sigma_{ij,j} + b_i = 0 \quad \text{in} \quad \Omega \quad (64)$$

where $\sigma_{ij}$ denotes the components of Cauchy stress tensor and $b_i$ the components of body force per unit volume.

For an isotropic elastic material, the constitutive equation related to stresses and strains is stated in the form

$$\sigma_{ij} = \tilde{\lambda}\tilde{\delta}_{ij} \varepsilon_{ik} + 2\tilde{\mu}\tilde{\epsilon}_{ij} - \tilde{m}\tilde{\delta}_{ij} T \quad (65)$$

where $\tilde{\lambda} = \frac{2\bar{\nu}}{1-2\bar{\nu}} \tilde{\mu}$, $\tilde{\mu} = \frac{E}{2(1+\nu)}$, $\tilde{m} = \frac{\tilde{\alpha}\tilde{E}}{1-2\bar{\nu}}$, $\tilde{E}$, $\bar{\nu}$, $\tilde{\alpha}$ have different values for plane stress and plane strain states such as

$$\left\{ \begin{array}{l}
\tilde{E} = E, \quad \bar{\nu} = \nu, \quad \tilde{\alpha} = \alpha \quad \text{for plane strain} \\
\tilde{E} = \frac{1+2\nu}{(1+\nu)^2} E, \quad \bar{\nu} = \frac{\nu}{1+\nu}, \quad \tilde{\alpha} = \frac{1+\nu}{1+2\nu}\alpha \quad \text{for plane stress}
\end{array} \right. \quad (66)$$
and parameters $E(\mathbf{x})$, $\nu(\mathbf{x})$ and $\alpha(\mathbf{x})$ are functions of space coordinates $\mathbf{x}$ and represent elastic modulus, Poisson ratio, and linear coefficient of thermal expansion, respectively.

If the displacements are small enough that the square and product of its derivatives are negligible, then the relation of Cauchy strains and displacements can be written as

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (66)$$

The BVP defined by Eqs (64)-(66) is completed by adding the following displacement and surface traction boundary conditions:

$$u_i = \overrightarrow{u}_i \quad \text{on } \Gamma_u$$
$$t_i = \sigma_y^i n_j = \overrightarrow{t}_i \quad \text{on } \Gamma_t \quad (67)$$

where $\overrightarrow{u}_i$ is the prescribed displacements on $\Gamma_u$ and $\overrightarrow{t}_i$ the given tractions on $\Gamma_t$. $\Gamma_u$ and $\Gamma_t$ are complementary parts of the boundary $\Gamma$.

Substituting Eqs (65) and (66) into Eq (64) yields the second-order partial differential equation (PDE) in terms of displacement components

$$
\left( \tilde{\lambda} + \tilde{\mu} \right) u_{k,i,i} + \tilde{\mu} u_{k,i,k,k} + \tilde{\lambda} u_{k,k,i} + \tilde{\mu} \left( u_{k,k,i} + u_{k,i} \right) - \tilde{m} T_j - \tilde{m}_j T + b_i = 0 \quad (68)
$$

### 5.2. Graded Types of FGM

Material properties of FGMs are usually defined by the variation in the volume fractions. In the literature, there are two common descriptions to the variation of volume fractions: the power-law assumption and exponential assumption [20, 21]. In order to clearly show the variation of material properties with different assumptions, for example, we consider the material shown in Figure 18 graded through the length $L$ along the $x$ direction. If the material properties of two constituents are $P_1$ and $P_2$, respectively, then the general material property $P$ of the FGM is given in Table 2 for two different distributions.

<table>
<thead>
<tr>
<th>Table 2. Classic power-law and exponential distributions of material property in FGMs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Power-law distribution</strong></td>
</tr>
<tr>
<td>Volume fraction</td>
</tr>
<tr>
<td>Material property</td>
</tr>
</tbody>
</table>
5.3. RBF Approximation

Making use of Analog equation method, Eq. (68) can be rewritten in the form

\[
\left( \lambda + \bar{\mu} \right) u_{k,ki} + \bar{\mu} u_{,kk} + \bar{b}_i = 0
\]  

(69)

where \( \lambda, \bar{\mu} \) are elastic constants of a fictitious isotropic homogeneous solid and \( \bar{b}_i \) the ‘body force’ induced by the displacement distributions sought.

In the following, we derive the general solutions of Eq. (69) by means of RBF approximation and MFS in the new equivalent system.

As was done in the previous sections, we first divide the displacements into two parts:

\[
u_i = u_{li} + u_{pi}
\]  

(70)

where the particular parts \( u_{pi} \) satisfy

\[
\left( \lambda + \bar{\mu} \right) u_{pk,ki} + \bar{\mu} u_{,pkk} + \bar{b}_i = 0
\]  

(71)

in the infinite domain, while the complementary parts, that is, the homogeneous parts, satisfy

\[
\left( \lambda + \bar{\mu} \right) u_{,lki} + \bar{\mu} u_{,lkk} = 0
\]  

(72)

Obviously, the particular solutions and homogeneous solutions satisfying Eqs. (71) and (72) respectively are not unique, without considering the constraints of boundary conditions.

Next, we use MFS and RBF to obtain the two parts discussed above.

With the RBF approximation, the body forces in Eq. (71) are approximated by

\[
\tilde{b}_i(x) = \sum_{m=1}^{M} \phi^m(x) \alpha^m_i = \sum_{m=1}^{M} \delta_{ij} \phi^m(x) \alpha^m_i
\]  

(73)
Similarly, the particular solution \( u_{pi} \) is also approximated by means of the same coefficient set

\[
u_{pi}(x) = \sum_{m=1}^{M} \Phi_{hl}^{m}(x) \alpha_{l}^{m}
\]  

(74)

where \( \Phi_{hl}^{m} \) is a corresponding set of approximate particular solutions. Because the particular solution \( u_{pi} \) satisfies Eq (71), the precondition to this process is that such relations as

\[
(\lambda + \mu) \Phi_{hl,ki}^{m}(x) + \mu \delta_{hl,kl}^{m}(x) = -\delta_{hl,kl} \rho^{m}(x)
\]  

(75)

hold.

For the piecewise smooth power spline (PS), also known as conical spline, \( \phi = r^{2n-1} \), and thin plate spline (TPS), also called Duchon spline, \( \phi = r^{2n} \ln r \), the corresponding set of particular solutions and its first and second order differentials, respectively, are given in the appendix for plane strain states [22]:

5.4. Method of Fundamental Solutions

To obtain approximated solutions of homogeneous equation (72), \( N \) fictitious source points \( y_{n} \) \( (n = 1, 2, \cdots, N) \) located on the virtual boundary outside the domain are selected. Moreover, assume that at each source point there is a pair of fictitious point loads \( \phi_{1}, \phi_{2} \) along \( x \) and \( y \) direction, respectively. According to the main construct of MFS, the approximated displacement fields at arbitrary points \( x \) in the domain or on the boundary can be expressed as a linear combination of fundamental solutions in terms of fictitious sources outside the domain of interest, that is,

\[
u_{hi}(x) = \sum_{n=1}^{N} u_{hi}^{*}(x, y_{n}) \rho_{l}^{n}
\]  

(76)

in which the displacement fundamental solution \( u_{hi}^{*}(x, y) \) denoting the induced displacement distribution along the \( i \) direction at the field point \( x \) due to the unit concentrated load acting in the \( l \) direction at source point \( y \) satisfies the Navier equation

\[
(\lambda + \mu) u_{hi,ki}(x, y) + \mu u_{hi,kl}(x, y) = -\delta_{xy} e_{li}
\]  

(77)
such that \( \delta \) is the Dirac delta function concentrated at the source point \( y \) and \( e_{\alpha} \) are the components of the \( 2 \times 2 \) identity matrix.

It is apparent that Eq (76) completely satisfies Eq (72) in the domain based on the definition of the fundamental solutions, that is Eq (77), and the fact that source point \( y_n \) and field point \( x \) are different.

The related expressions of fundamental solutions and the derivatives for the plane strain state can be found in the Appendix.

### 3.4. Final Complete Solutions

According to Eq (70), the complete solutions of displacement components are written as the sum of the particular and homogeneous solutions, thus

\[
 u_i(x) = \sum_{n=1}^{N} u_{i,n}^*(x, y_n) \phi_i^n + \sum_{m=1}^{M} \Phi_{i,m}^n(x) \alpha_i^m \tag{78}
\]

Differentiating Eq (78) yields

\[
 u_{i,j}(x) = \sum_{n=1}^{N} u_{i,j,n}^*(x, y_n) \phi_i^n + \sum_{m=1}^{M} \alpha_i^m \Phi_{i,j,m}^n(x) \tag{79}
\]

\[
 u_{i,j,k}(x) = \sum_{n=1}^{N} u_{i,j,k,n}^*(x, y_n) \phi_i^n + \sum_{m=1}^{M} \alpha_i^m \Phi_{i,j,k,m}^n(x) \tag{80}
\]

Consequently, the stress components can be expressed by substituting Eqs (79) and (80) into Eqs (65) and (66) as

\[
 \sigma_{ij}(x) = \sum_{n=1}^{N} \sigma_{ij,n}^*(x, y_n) \phi_i^n + \sum_{m=1}^{M} S_{ij,m}^n(x) \alpha_i^m - \tilde{m} \delta_{ij} T \tag{81}
\]

where

\[
 \sigma_{ij,n}^* = \tilde{\lambda} \delta_{ij} u_{i,k,n}^* + \tilde{\mu} \left( u_{i,j,n}^* + u_{j,i,n}^* \right) \tag{82}
\]

\[
 S_{ij,n}^m = \tilde{\lambda} \delta_{ij} \Phi_{i,k,n}^m + \tilde{\mu} \left( \Phi_{i,j,n}^m + \Phi_{j,i,n}^m \right)
\]

Furthermore, the traction components can be derived as

\[
 t_i = \sigma_{ij} n_j = \sum_{n=1}^{N} t_i^n \phi_i^n + \sum_{m=1}^{M} P_{i,m}^n \alpha_i^m - \bar{m} n_i T \tag{83}
\]

where \( t_i^* = \sigma_{ij}^* n_j \) and \( P_{i,m}^n = S_{ij,n}^m n_j \).
Finally, making Eqs. (79) and (80) satisfy the governing Eq. (68) at \( M \) interpolation points and substituting Eq. (78) and (83) into boundary conditions (67) at \( N \) boundary nodes produce a set of linear algebraic equations in matrix form for the determination of unknown coefficients

\[
\begin{bmatrix} H \end{bmatrix} \{ A \} = \{ B \}
\]

(84)

where

\[
\text{vector } \{ A \} = \{ \varphi_1, \varphi_2, \ldots, \varphi_N, \alpha_1, \alpha_2, \ldots, \alpha_M, \alpha_M^N \}^T.
\]

Figure 19. Configuration of hollow circular plate under internal pressure.

The first and second order derivatives of kernel functions \( u_{li}^* \) and \( \Phi_{li}^m \) used in above process are given in the Appendix.

**Example 5: Symmetrical thermoelastic problem in a long cylinder**

Consider a thick hollow cylinder with geometries and mechanical boundary conditions given in Figure 19. The same power-law assumptions are used to define the elastic modulus

\[
E(r) = E_0 \left( \frac{r}{a} \right)^\eta
\]

and coefficient of thermal expansion, that is, \( \alpha(r) = \alpha_0 \left( \frac{r}{a} \right)^\eta \). The temperature change in the entire domain is given in a closed form

\[
T = \begin{cases} 
\frac{T_a \left( b^{-\eta} - r^{-\eta} \right) + T_b \left( r^{-\eta} - a^{-\eta} \right)}{b^{-\eta} - a^{-\eta}} & \text{for } \eta \neq 0 \\
\frac{T_a \ln \frac{b}{r} + T_b \ln \frac{r}{a}}{\ln \frac{b}{a}} & \text{for } \eta = 0 
\end{cases}
\]

(85)

with \( T_a = T(a) \) and \( T_b = T(b) \).
The two-phase aluminum/ceramic FGM is examined here. The metal aluminum constituent is arranged on the inner surface, while the ceramic constituent is on the outer surface. The related material properties are $E_{Al} = 70$ GPa, $\alpha_{Al} = 1.2 \times 10^{-6} / ^\circ C$, $E_{ceramic} = 151$ GPa, $\alpha_{ceramic} = 2.59 \times 10^{-6} / ^\circ C$. Poisson’s ratio is taken to be $\nu = 0.3$. The inner and outer boundary temperature changes, respectively, are $T_a = 10^\circ C$ and $T_b = 0^\circ C$.

Analytical solutions of displacements and stresses for the case of plane strain state are provided by Jabbari et al. [23]. However, it is necessary to point out that there are some important written errors in the work of Jabbari et al. The results in Figure 20 show good agreement between the analytical solutions and the numerical results in FGM and homogeneous material, which corresponds to $\eta = 0$. Furthermore, we again find that after graded treatment, the maximum value of hoop stress decreases from 82.6 MPa to 53 MPa. Additionally, the radial displacement in FGM also decreases, compared to the response in homogeneous media. Since the value of radial displacement is very small, radial deformation can be neglected in practical analysis.

Figure 20. Stresses and radial displacement distributions in FGM and homogeneous material with $N = 32$, $M = 220$.

6. THIN PLATE BENDING

6.1. Basic Equations of Thin Plate Bending

Consider a thin plate under an arbitrary transverse loads as shown in Figure 21. It is assumed that the thickness $h$ of the thin plate is in the range of $1/20 \sim 1/100$ of its span approximately.
Under the assumption above the Kirchhoff thin plate bending theory can be employed. The governing equation of thin plate on an elastic foundation under arbitrary transverse load \( p(\mathbf{x}) \) is, thus, written as

\[
D
\nabla^4 w(\mathbf{x}) + k_\nu w(\mathbf{x}) = p(\mathbf{x})
\]

where \( w(\mathbf{x}) \) denotes the lateral deflection of interest at the point \( \mathbf{x} = (x_1, x_2) \in \Omega \subset \mathbb{R}^2 \), \( D \) is the flexural rigidity defined by

\[
D = \frac{Eh^3}{12(1-\nu^2)}
\]

where \( E \) is Young’s modulus, \( \nu \) Poisson’s ratio, \( h \) plate thickness, \( k_\nu \) the parameter of Winkler foundation, and \( \nabla^4 \) is the biharmonic differential operator defined by

\[
\nabla^4 = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}
\]

What follows is to establish a linear equation system of thin plate bending for determining the unknown deflection \( w(\mathbf{x}) \) which satisfies Eq (86) and boundary conditions listed in Table 3. The boundary conditions in Table 3 are described by two displacement components \((w, \theta_n)\) and two internal forces \((M_n, V_n)\).

![Figure 21](image.png)

Figure 21. Configuration of thin plate bending on elastic foundation under transverse distributed loads.
### Table 3. Common boundary conditions in thin plate bending

<table>
<thead>
<tr>
<th>Types of support</th>
<th>Mathematical expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple support</td>
<td>( w = 0, M_n = 0 )</td>
</tr>
<tr>
<td>Fixed edge</td>
<td>( w = 0, \theta_n = 0 )</td>
</tr>
<tr>
<td>Free edge</td>
<td>( V_n = 0, M_n = 0 )</td>
</tr>
</tbody>
</table>

The variables \( \theta_n, M_n \), and \( V_n \) are, respectively, outward normal derivative of deflection, bending moment, and Kirchhoff’s equivalent shear force. They can be expressed in terms of deflection \( w(\mathbf{x}) \) as

\[
\theta_n = w_j n_i, \\
M_n = -D\left[v w_{,ji} + (1-v) w_{,i} n_j n_i \right], \\
V_n = Q_n + \frac{\partial M_n}{\partial s} = -D\left[w_{,iji} n_i + (1-v) w_{,ijk} n_j t_k t_i \right],
\]

where \( \mathbf{n} = [n_1, n_2] \) and \( \mathbf{t} = [-n_2, n_1] \) are the outward unit normal vector and tangential vector on the boundary, respectively, \( s \) is the arc length along the boundary measured from a certain boundary point, and

\[
M_{,ij} = -D(1-v) w_{,ij} n_i, \quad Q_n = -D w_{,iji} n_i
\]

### 6.2. Fundamental Solution And Determination of Source Points

The meshless formulation for thin plate bending with an elastic foundation is, here, presented using AEM, MFS, and RBF approximation. To convert Eq (86) into a standard plate bending equation, the application of AEM to Eq (86) gives

\[
D \nabla^4 w(\mathbf{x}) = \tilde{p}(\mathbf{x})
\]

where \( \tilde{p}(\mathbf{x}) \) is fictitious transverse load including the term with the unknown deflection. The equation above is a plate bending equation without elastic foundation and its fundamental solution is available in the literature. The fictitious transverse load \( \tilde{p}(\mathbf{x}) \) can be expressed in terms of RBFs.
The solution to Eq. (91) is firstly divided into two parts as was done before: homogeneous solution and particular solution, which satisfy the following equations, respectively,

\[
\begin{align*}
D^4 w_h(x) &= 0 \\
D^4 w_p(x) &= \tilde{p}(x)
\end{align*}
\]  
(92)

Specially, for the case of thin plate bending problems without elastic foundation (\(k_w = 0\)) we have \(\tilde{p}(x) = p(x)\). The procedure of AEM is unnecessary in this case.

For a well-posed thin plate bending problem, there are two known and two unknown quantities at each point on the boundary. Therefore, we need two equations to determine the two unknowns at each point. Considering this feature the corresponding MFS is constructed based the following two foundational solutions.

It’s well known that the general solution of a biharmonic equation can be expressed in the following form

\[ w_h(x) = A + r^2 B \]  
(93)

where \(A\) and \(B\) are two independent functions satisfying the Laplace equation, respectively,

\[ \nabla^2 A = 0, \quad \nabla^2 B = 0 \]  
(94)

So, we can combine the fundamental solutions of biharmonic operator and Laplace operator to fulfill the character of boundary conditions mentioned above, that is

\[ w_h(x) = \sum_{i=1}^{N_s} \left[ \phi_{h,i} w_1^*(x,y_i) + \phi_{h,i} w_2^*(x,y_i) \right] \quad x \in \Omega, \quad y_i \not\in \Omega \]  
(95)

where \(N_s\) are source points outside the domain, \(w_1^*(x,y)\) and \(w_2^*(x,y)\) are fundamental solutions of biharmonic operator and Laplace operator, respectively, which can be written as

\[ w_1^*(x,y) = -\frac{1}{8\pi D} r^2 \ln r \]

\[ w_2^*(x,y) = -\frac{1}{2\pi D} \ln r \]

with \(r = \|x - y\|\).
Unlike in the approaches in [7] constructing fundamental solutions to adapt the requirement of boundary conditions in thin plate bending, the proposed approach is simpler and more convenient in practical applications.

It is easy to verify that Eq (95) satisfies the first equation in Eq (92).

The proper location of the source points is an important issue in the MFS with respect to the accuracy of numerical solutions. Here the position of the source points can be evaluated by means of the following equation [24]:

\[ y = x_b + \gamma \left( x_b - x_c \right) \]  

where \( y \) is the spatial coordinates of a particular source point, \( x_b \) the spatial coordinates of related boundary points, and \( x_c \) the central coordinates of the solution domain. \( \gamma \) is a non-dimensionless real parameter, which is positive for the case of external boundary and negative for the case of internal boundary (see Figure 22).

![Figure 22. Configuration of source points.](image)

### 6.3. Radial Basis Function (RBF)

In order to obtain the particular solution corresponding to the fictitious transverse load \( \tilde{p}(x) \), the radial basis function approximation of \( \tilde{p}(x) \) is written in the form [25].

\[ \tilde{p}(x) = \sum_{j=1}^{N} \alpha_j \phi_j (x) \]  

where the set of radial basis functions \( \phi_j (x) \) is taken as \( \phi_j (r_j) \) where \( r_j = \| x - x_j \| \). \( \phi(r_j) \) is defined in Table 4.
Similarly, the particular solution \( w_p(x) \) is expressed by the linear combination of approximated particular solutions \( \Phi_j(x) = \Phi(x_j) \), that is

\[
w_p(x) = \sum_{j=1}^{N_x} \alpha_j \Phi_j(x)
\]

(98)

The satisfaction of the relation of \( w_p(x) \) and \( \tilde{p}(x) \) in Eq (92) requires

\[
D\nabla^4 \Phi(x_j) = \phi(x_j)
\]

(99)

Therefore, once the expression of radial basis function \( \phi(x_j) \) is given, the approximated particular solutions \( \Phi(x_j) \) can be determined from Eq (99).

**Table 4. Particular solutions for the biharmonic equation**

<table>
<thead>
<tr>
<th></th>
<th>Power spline (PS) RBF</th>
<th>Thin plate spline (TPS) RBF</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>( r^{2n-1} )</td>
<td>( r^{2n} ) ln ( r )</td>
</tr>
<tr>
<td>( D\Phi )</td>
<td>( \frac{r^{2n+3}}{(2n+1)^2(2n+3)^2} )</td>
<td>( \frac{r^{2n+4}}{16(n+1)^2(n+2)^2} \left[ \ln r - \frac{2n+3}{(n+1)(n+2)} \right] )</td>
</tr>
</tbody>
</table>

**3.4. Solution \( w(x) \)**

The solution \( w(x) \) can be obtained by putting the obtained homogeneous and particular parts together and written as

\[
w(x) = \sum_{j=1}^{N_x} \alpha_j \Phi_j(x) + \sum_{i=1}^{N_x} \left[ \phi_i \Phi_1(x, y_i) + \phi_{2i} \Phi_2(x, y_i) \right]
\]

(100)

The unknowns \( \alpha_j \), \( \phi_{1i} \), and \( \phi_{2i} \) can be determined by substituting Eq. (100) into the original governing equation (86) at \( N_x \) interpolation points and boundary conditions (89) at \( N_s \) boundary points. For example, the substitution of Eq (100) into Eqs (86) and (89) yields following system of linear equations

\[
(D\nabla^4 + k_w)w(x)|_{x=x_i} \{A\} = p(x_i) \quad (i = 1, 2, \cdots, N_x)
\]
for a simply-supported plate,

\[
\begin{bmatrix}
\mathbf{w}(x)
\end{bmatrix}
= \begin{bmatrix}
-D(\mathbf{v}\mathbf{w}(x))_{nk} + (1 - \nu)\mathbf{w}(x)_{nk}n_i
\end{bmatrix}
\begin{bmatrix}
\{A\} = \begin{bmatrix}
\mathbf{\bar{w}}(x_i) \\
\mathbf{\bar{\theta}}(x_i)
\end{bmatrix}
\end{bmatrix}
(i = 1, 2, \ldots, N_s)
\]  

where

\[
\mathbf{w}(x)|_{k \rightarrow k} = \{\Phi_1(x_1) \Phi_2(x_1) \cdots \Phi_{N_s}(x_1) \ w_1^*(x_1, y_1) \ w_2^*(x_1, y_1) \ w_1^*(x_1, y_{N_s}) \ w_2^*(x_1, y_{N_s})\}
\]

\[
\{A\} = \begin{bmatrix}
\alpha_1 \alpha_2 \cdots \alpha_{N_s} \phi_1 \phi_2 \cdots \phi_{1N_s} \phi_{2N_s}
\end{bmatrix}^T
\]

Once all unknown coefficients are determined, the deflection \(w\), rotation \(\theta\), moment \(M\) and reaction force \(V\) can be calculated by using Eqs (89) and (100).

**Example 6:** Square plate on a Winkler elastic foundation

Consider a square plate resting on a Winkler elastic foundation and subjected to uniformly distributed load \(q_0\). All four edges of the plate are simply-supported, i.e. \(w = 0\) and \(M = 0\) along all edges.

The parameter of Winkler foundation \(k_w\) is taken to be \(4.9 \times 10^7\) N/m\(^3\). The analytical solution for this problem is

\[
w(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{a}
\]

with

\[
A_{mn} = \frac{q_{mn}}{D\pi^4 \left( \frac{m^2 + n^2}{a^2} \right)^2 + k_w}
\]

\[
q_{mn} = \frac{4q_0}{mn\pi^2} \left[ -1 + \cos(m\pi) \right] \left[ -1 + \cos(n\pi) \right]
\]

Due to the symmetry of the problem, one quarter of the solution domain is considered. The distribution of deflection and moment along \(y = 0.5\) is evaluated with \(N_s = 36\) and \(N_l = 121\) and the corresponding results are shown in Figure 23. It can be seen from Figure 23 that the proposed MFS-based method provides a very good accuracy approximation for the corresponding analytical solution.
Figure 23. Distribution of deflection and moment along $y = 0.5$.

7. CONCLUSION

Appendix. First and second order differentials of fundamental solutions and approximated particular solutions

A1. Fundamental solutions and their derivatives

\[ u^*_i = \frac{1}{8 \pi \mu (1 - \nu)} \left( (3 - 4\nu) \delta_i \ln r + r_j r_j \right) \]

\[ u^*_{i,j} = \frac{1}{8 \pi \mu (1 - \nu)} \left( (3 - 4\nu) \delta_i r_j + r_j \delta_j + r_j \delta_j - 2 r_j r_j r_j \right) \]

\[ u^*_{i,k} = \frac{1}{4 \pi \mu (1 - \nu)} \left( (1 - 2\nu) \right) r_j \]

\[ u^*_{i,k,k} = \frac{1}{4 \pi \mu (1 - \nu)} \left( (1 - 2\nu) \right) (2 r_j r_j - \delta_j) \]

\[ u^*_{i,k,k} = \frac{1}{4 \pi \mu (1 - \nu)} \left( (1 - 2\nu) \right) \left( \delta_i - 2 r_j r_j \right) \]

A2. Approximated particular solutions and their derivatives

A2.1 Power spline (PS) function

\[ \Phi_i = -\frac{1}{2 \mu (1 - \nu)} \frac{1}{(2n + 1)(2n + 3)} r^{2n+1} (A_i \delta_i + A_i r_j r_j) \]
\[ \Phi_{i,j} = -\frac{1}{2\mu(1-\nu)(2n+1)} \frac{1}{(2n+3)} r^{2n} \left[ B_i \delta_{i,j} + B_j \left( \delta_{i,j} + \delta_{j,i} \right) + B_j r_i r_j \right] \]

\[ \Phi_{k,\ell} = -\frac{1}{2\mu(1-\nu)(2n+1)^2} \frac{1}{(2n+3)} r^{2n-1} B_4 \left[ \delta_{k,\ell} + (2n-1) r_\ell r_\ell \right] \]

\[ \Phi_{\ell,\kappa} = -\frac{1}{2\mu(1-\nu)(2n+1)^3} \frac{1}{(2n+3)} r^{2n-1} \left( C_1 \delta_{\ell,\kappa} + C_2 r_\ell r_{\kappa} \right) \]

where

\[ A_i = (4n + 5) - 2\nu(2n + 3) \]

\[ A_2 = -(2n + 1) \]

\[ B_i = A_i (2n + 1) \]

\[ B_2 = A_2 \]

\[ B_3 = A_2 (2n - 1) \]

\[ B_4 = B_i + 3B_2 + B_3 \]

\[ C_1 = 2B_2 + B_i (2n + 1) \]

\[ C_2 = 2B_2 (2n - 1) + B_i (2n + 1) \]

A2.2 Thin plate spline (TPS) function

\[ \Phi_{\ell,j} = -\frac{1}{32\mu(1-\nu)(n+1)^3(n+2)} r^{2n+1} \left( A_i \delta_{\ell,j} + A_j r_\ell r_j \right) \]

\[ \Phi_{\ell,j} = -\frac{1}{32\mu(1-\nu)(n+1)^3(n+2)} r^{2n+1} \left[ B_i r_\ell r_j + B_j \delta_{\ell,j} + B_\ell \left( \delta_{\ell,j} + \delta_{j,\ell} \right) \right] \]

\[ \Phi_{\ell,k} = -\frac{1}{32\mu(1-\nu)(n+1)^3(n+2)} r^{2n} \left( C_1 r_\ell r_k + C_2 \delta_{\ell,k} \right) \]

\[ \Phi_{\ell,k} = -\frac{1}{32\mu(1-\nu)(n+1)^3(n+2)} r^{2n} \left( C_2 r_\ell r_k + C_3 \delta_{\ell,k} \right) \]
where
\[
A_i = -\left(8n^2 + 29n + 27\right) + 8\nu(n+2)^2 + 2(n+1)(n+2)\left[4n + 7 - 4\nu(n+2)\right]\ln r
\]
\[
A_2 = 2(n+1)\left[(2n+3) - 2(n+1)(n+2)\ln r\right]
\]
\[
B_1 = 2nA_2 - 4(n+1)^2(n+2)
\]
\[
B_2 = 2(n+1)\left[A_i + (n+2)\left[4n + 7 - 4\nu(n+2)\right]\right]
\]
\[
B_4 = B_1 + B_2 + 3B_3
\]
\[
C_1 = 2nB_3 + 8(n+1)^2(n+2)^2(1-2\nu)
\]
\[
C_2 = 2(n+1)B_1 + 4nB_2 - 8(n+1)^3(n+2)
\]
\[
C_4 = 2(n+1)B_2 + 2B_3 - 4(n+1)^2(n+2)\left[-4n + 7 - 4\nu(n+2)\right]
\]

REFERENCES

Meshless Approach and its Application in Engineering Problems


