Piezoelectric or, more generally, electroelastic materials, exhibit electromechanical coupling. They experience mechanical deformations when placed in an electric field and become electrically polarized under mechanical loads. These materials have been used to make various electromechanical devices. Examples include transducers for converting electric energy to mechanical energy or vice versa, resonators and filters for telecommunication and time-keeping, and sensors for information collection.

Piezoelectricity has been a steadily growing field for more than a century, progressed mainly by researchers from applied physics, acoustics, materials science and engineering, and electrical engineering. After World War II, piezoelectricity research has gradually concentrated in the IEEE Society of Ultrasonics, Ferroelectrics, and Frequency Control. The two major research focuses have always been the development of new piezoelectric materials and devices. All piezoelectric devices for applications in the electronics industry require two phases of design. One aspect is the device operation principle and optimal operation which can usually be established from linear analyses; the other is the device operation stability against environmental effects such as a temperature change or stress, which is usually involved with non-linearity. Both facets of design usually present complicated electromechanical problems.

Due to the application of piezoelectric sensors and actuators in civil, mechanical, and aerospace engineering structures for control purposes, piezoelectricity has also become a topic for mechanics researchers. Mechanics can provide effective tools for piezoelectric device and material modeling. For example, the finite element and boundary element methods for numerical analysis and the one- and two-dimensional theories of piezoelectric beams, plates, and shells are effective tools for the design and optimization of piezoelectric devices. Mechanics theories of composites are useful for predicting material behaviors.

In spite of the wide and growing applications of piezoelectric devices, books published on the topic of piezoelectricity are relatively few. Following the
editor’s previous book, *An Introduction to the Theory of Piezoelectricity*, Springer ©2005, this book addresses more advanced topics that require a collective effort. Each self-contained chapter has been written by a group of international experts and includes quite a few advanced topics in the theory of piezoelectricity. Each chapter attempts to present a basic picture of the subject area addressed.

Piezoelectricity is a broad field and, practically speaking, this volume can only cover a fraction of the many relatively advanced topics. Following a brief summary of the three-dimensional theory of linear piezoelectricity, Chapters 2 through 5 discuss selected topics within the linear theory. The linear theory of piezoelectricity assumes a reference state free of deformations and fields. When initial deformations and/or fields are present, the theory for small incremental fields superimposed on a bias is needed, which is the subject of Chapter 6. The theory for incremental fields needs to be obtained from the fully nonlinear theory by linearization about an initial state, and, therefore, is a subject that is inherently nonlinear. Chapter 7 covers the fully dynamic effects due to electromagnetic coupling. Chapter 8 addresses nonlocal and gradient effects of electric field variables.

I would like to take this opportunity to thank all chapter contributors. My thanks also go to Patricia A. Worster and Ziguang Chen of the College of Engineering at the University of Nebraska-Lincoln for their editing assistance on Chapters 1, 7, and 8.

*Jiashi Yang*

*January 2009*
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Chapter 5
Boundary Element Method

Qing-Hua Qin

5.1 Introduction

In Chapter 2 Green's functions in piezoelectric materials were described. Applications of these Green's functions to the boundary element method (BEM) are discussed in this chapter. In contrast to the finite element method (FEM), BEM involves only discretization of the boundary of the structure due to the governing differential equation being satisfied exactly inside the domain leading to a relatively smaller system size with sufficient accuracy. This is an important advantage over domain-type solutions such as FEM or the finite difference method. During the past two decades several BEM techniques have been successfully developed for analyzing structure performance with piezoelectric materials [1–4]. Lee and Jiang [1] derived the boundary integral equation of piezoelectric media by the method of weighted residuals for plane piezoelectricity. Lu and Mahrenholtz [5] presented a variational boundary integral equation for the same problem. Ding, Wang, and Chen [6] developed a boundary integral formulation that is efficient for analyzing crack problems in piezoelectric material. Rajapakse [7] discussed three boundary element methods (direct boundary method, indirect boundary element method, and fictitious stress-electric charge method) in coupled electroelastic problems. Xu and Rajapakse [8] and Rajapakse and Xu [9] extended the formulations in [7, 8] to the case of piezoelectric solids with various defects (cavities, inclusions, cracks, etc.). Liu and Fan [10] established a boundary integral equation in a rigorous way and addressed the question of degeneration for problems of cracks and thin shellike structures. Pan [11] derived a single-domain BE formulation for 2D static crack problems. Denda and Lua [12] developed a BEM formulation using Stroh's formalism to derive
the fundamental solution but did not show any numerical results. Davi and Milazzo [13] used the known subdomain method to formulate a multidomain BEM, well suited for crack problems, by modeling crack faces as boundaries of the different subdomains. Groh and Kuna [14] developed a direct collocation boundary element code with a subdomain technique for analyzing crack problems and calculating stress intensity factors. Khutoryansky, Sosa, and Zu [15] introduced a BE formulation for time-dependent problems of linear piezoelectricity. A brief review of this field can be found in [4].

5.2 Boundary Integral Formulations

5.2.1 Governing Equations

In this section, the theory of piezoelectricity presented in Chapter 1 is briefly summarized for deriving the corresponding boundary integral equation. Under the condition of a static deformation, the governing equations for a linear and generally anisotropic piezoelectric solid consist of

(i) Equilibrium equations

\[ T_{ij,j} + \tilde{f}_i = 0, \quad D_{i,i} = \rho_e, \quad (5.1) \]

where \( \tilde{f}_i = \rho_0 f_i \);

(ii) Constitutive relations

\[ T_{ij} = c_{ijkl} S_{kl} - \epsilon_{mi} E_{m}, \quad D_k = \epsilon_{kj} S_{ij} + \varepsilon_{mk} E_{m}; \quad (5.2) \]

(iii) Elastic strain-displacement and electric field-potential relations

\[ S_{ij} = \frac{(u_{i,j} + u_{j,i})}{2}, \quad E_i = -\phi_{,i}; \quad (5.3) \]

(iv) Boundary conditions

\[ u_i = \bar{u}_i \quad \text{on} \quad S_u \quad \left\{ \begin{array}{l} \phi = \bar{\phi} \quad \text{on} \quad S_\phi \end{array} \right\}, \quad t_i = T_{ij} n_j = \bar{t}_i \quad \text{on} \quad S_T, \quad D_n = D_i n_i = -\bar{\sigma} \quad \text{on} \quad S_D. \quad (5.4) \]

5.2.2 Boundary Integral Equation

Several approaches have been used in the literature to establish boundary integral equations of piezoelectric materials, such as the weighted residual approach [1], the variational approach [5], and Betti’s reciprocity
A brief discussion of Betti's reciprocity theorem which is used in later sections of this chapter is given here.

With the reciprocity theorem, we consider two electroelastic states, namely 

\[ \begin{align*} 
\text{State 1: } & \quad \{ \boldsymbol{U}^{(1)} \}^T = \{ u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, \phi^{(1)} \}^T; \\
& \quad \{ \tilde{f}^{(1)} \}^T = \{ \tilde{f}_1^{(1)}, \tilde{f}_2^{(1)}, \tilde{f}_3^{(1)}, \rho_e^{(1)} \}^T; \\
& \quad \{ \tilde{t}^{(1)} \}^T = \{ t_1^{(1)}, t_2^{(1)}, t_3^{(1)}, D_n^{(1)} \}^T \\
\text{State 2: } & \quad \{ \boldsymbol{U}^{(2)} \}^T = \{ u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \phi^{(2)} \}^T; \\
& \quad \{ \tilde{f}^{(2)} \}^T = \{ \tilde{f}_1^{(2)}, \tilde{f}_2^{(2)}, \tilde{f}_3^{(2)}, -\rho_e^{(2)} \}^T; \\
& \quad \{ \tilde{t}^{(2)} \}^T = \{ t_1^{(2)}, t_2^{(2)}, t_3^{(2)}, D_n^{(2)} \}^T. 
\end{align*} \] (5.5)

The first state represents the solution to piezoelectric problems with finite domains and general loading conditions; the second state is of an artificial nature and represents the fundamental solution to the case of a fictitious infinite body subjected to a point force or a point charge. Furthermore, we introduce the compatible field \( \{ \hat{S}_{ij}, \hat{u}_i, \hat{E}_i, \hat{\phi} \} \) that satisfies Equation (5.3).

The principle of virtual work is given by

\[ \int_V \tilde{f}^{(1)}_i \tilde{U}_i dV + \int_S \tilde{t}^{(1)}_i \tilde{U}_i dS = \int_V (T^{(1)}_{ij} \hat{S}_{ij} - D_i^{(1)} \hat{E}_i) dV. \] (5.7)

On the other hand, for a linear piezoelectric solid, we can show that the following reciprocal property of Betti type holds:

\[ T^{(1)}_{ij} S^{(2)}_{ij} - D_i^{(1)} E_i^{(2)} = T^{(2)}_{ij} S^{(1)}_{ij} - D_i^{(2)} E_i^{(1)}. \] (5.8)

By substituting Equation (5.8) into (5.7), the following reciprocal relation can be obtained,

\[ \int_V \tilde{f}^{(1)}_i U_i^{(2)} dV + \int_S \tilde{t}^{(1)}_i U_i^{(2)} dS = \int_V \tilde{f}^{(2)}_i U_i^{(1)} dV + \int_S \tilde{t}^{(2)}_i U_i^{(1)} dS. \] (5.9)

To convert Equation (5.9) into a boundary integral equation, assume that State 1 is the actual solution for a body \( V \) with the boundary \( S \) and State 2 is the solution for a fictitious infinite body subjected to a point force at \( \hat{X} \) in the \( x_m \) direction with no bulk charge distribution (i.e., \( \rho_c^{(2)} = 0 \)), namely

\[ \tilde{f}_m^{(2)}(\mathbf{x}) = \delta(\mathbf{x} - \hat{\mathbf{x}}) \delta_{im}, \quad (m = 1, 2, 3). \] (5.10)

The displacement, electric potential, stress, and electric displacement induced by the above-mentioned point force were discussed in Chapter 2. Using the solution given in Chapter 2, the variables \( u_i^{(2)}, \phi^{(2)}, T_{ij}^{(2)} \), and \( D_i^{(2)} \)
can be written in the form

\[ u^{(2)}_i = u^*_i(x, \hat{x})e_j, \quad \phi^{(2)} = \phi^*_j(x, \hat{x})e_j = u^*_j(x, \hat{x})e_j, \]

\[ T^{(2)}_{ij} = \sum_{jm} T^*_{ijm}(x, \hat{x})e_m = [\epsilon_{ijkl}u^*_{km,l}(x, \hat{x}) - \epsilon_{ijl}u^*_{km,l}(x, \hat{x})]e_m, \]

\[ D^{(2)}_i = D^*_{im}(x, \hat{x})e_m = [\epsilon_{iit}u^*_{km,l}(x, \hat{x}) + \kappa_{it}u^*_{km,l}(x, \hat{x})]e_m, \quad (5.11) \]

where \( u^*_j \) represents the displacement in the \( j \)th direction at field point \( x \) due to a point force in the \( j \)th direction applied at the source point \( \hat{x} \) interior to \( S \), and \( u^*_4 \) denotes the \( i \)th displacement at \( x \) due to a point electric charge at \( \hat{x} \). Making use of Equation (5.4), the traction and surface charge can be obtained as

\[ t^{(2)}_i = t^*_{im}(x, \hat{x})e_m = \sum_{jm} t^*_j(x, \hat{x})u^*_j(x) e_m = D^*_m(x, \hat{x})n_i e_m. \quad (5.12) \]

Substituting all the above quantities associated with State 2 into Equation (5.9) yields

\[ u_i(\hat{x}) = \int_V u^*_j(x, \hat{x})\tilde{f}_j(x)dV(x) - \int_S t^*_j(x, \hat{x})u_J(x)dS(x) + \int_S u^*_j(x, \hat{x})t_J(x)dS(x), \quad (5.13) \]

where \( i = 1, 2, 3 \), \( J = 1 - 4 \), \( u_4 = -\phi \), \( \tilde{f}_4 = \rho_e \), and \( t_4 = D_n \).

Next, assume that State 2 of the fictitious infinite body is subjected to a point charge at \( \hat{x} \) with no body force distribution; that is,

\[ -p^{(2)}_e(x) = \delta(x - \hat{x}), \quad \tilde{f}^{(2)}_i(x) = 0 \quad (i = 1, 2, 3) \quad (5.14) \]

The resulting displacement and electric potential induced by this point charge are given by

\[ u^{(2)}_i = u^*_i(x, \hat{x}), \quad \phi^{(2)} = \phi^*_i(x, \hat{x}) = u^*_i(x, \hat{x}), \quad (5.15) \]

where \( u^*_i \) and \( u^*_4 \) denote, respectively, the \( i \)th displacement and electric potential at \( x \) due to a point electric charge at \( \hat{x} \).

Substituting the solution in Equation (5.15) into (5.2) and (5.4), we have

\[ T^{(2)}_{ij} = \sum_{jm} T^*_j(x, \hat{x})u^*_j(x, \hat{x}) = \epsilon_{ijkl}u^*_k(x, \hat{x}) - \epsilon_{ijl}u^*_k(x, \hat{x}), \]

\[ D^{(2)}_i = D^*_j(x, \hat{x}) = \epsilon_{ijkl}u^*_k(x, \hat{x}) + \kappa_{it}u^*_k(x, \hat{x}), \]

\[ t^{(2)}_i = t^*_4(x, \hat{x}) = \sum_{jm} t^*_4(x, \hat{x})u^*_j(x) + D^{(2)}_m(x, \hat{x})n_i. \quad (5.16) \]

where \( t^{(2)}_i \) are related to \( u^{(2)}_i \) by Equations (5.2) and (5.4).
Substituting Equations (5.14) – (5.16) into (5.9), we obtain

\[
\phi(\hat{x}) = \int_V u^*_J(x, \hat{x}) f_J(x) dV(x) - \int_S t^*_J(x, \hat{x}) t_J(x) dS(x)
+ \int_S u^*_J(x, \hat{x}) t_J(x) dS(x).
\]

(5.17)

Combining Equation (5.13) with (5.17), we have

\[
u_I(\hat{x}) = \int_V u^*_J(x, \hat{x}) f_J(x) dV(x) - \int_S t^*_J(x, \hat{x}) u_J(x) dS(x)
+ \int_S u^*_J(x, \hat{x}) t_J(x) dS(x),
\]

(5.18)

where \(I, J = 1 - 4\).

Making use of Equations (5.2) and (5.18), the corresponding stresses and electric displacements are expressed as

\[
\Pi_{iJ}(\hat{x}) = \int_V D^*_K(x, \hat{x}) f_K(x) dV(x) - \int_S S^*_K(x, \hat{x}) u_K(x) dS(x)
+ \int_S D^*_K(x, \hat{x}) t_K(x) dS(x),
\]

(5.19)

where

\[
\Pi_{iJ} = \begin{cases} 
\sigma_{ij}, & J \leq 3, \\
D_i, & J = 4
\end{cases}
\]

(5.20)

\[S^*_{K,ij}(x, \hat{x}) = E_{iJ}M_n \frac{\partial u^*_M(x, \hat{x})}{\partial x_n}
\]

(5.21)

\[D^*_{K,ij}(x, \hat{x}) = E_{iJ}M_n \frac{\partial u^*_M(x, \hat{x})}{\partial x_n}
\]

(5.22)

with

\[
E_{iJ}M_n = \begin{cases} 
e_{ijmn}, & J, M \leq 3, \\
\epsilon_{ij}, & J \leq 3, M = 4, \\
\epsilon_{imn}, & J = 4, M \leq 3, \\
-\epsilon_{in}, & J = 4, M = 4.
\end{cases}
\]

(5.23)

The integral representation formula for the generalized traction components can be obtained from Equations (5.4) and (5.19) as

\[
t_J(\hat{x}) = \int_V V^*_J(x, \hat{x}) f_J(x) dV(x) - \int_S W^*_J(x, \hat{x}) u_J(x) dS(x)
+ \int_S V^*_J(x, \hat{x}) t_J(x) dS(x),
\]

(5.24)
where
\[
V_{ij}^*(x, \mathbf{x}) = D_{ikj}^*(x, \mathbf{x})n_k(\mathbf{x}), \quad W_{ij}^*(x, \mathbf{x}) = S_{ikj}^*(x, \mathbf{x})n_k(\mathbf{x}).
\] (5.25)

It can be seen from Equations (5.13), (5.17), and (5.24) that to obtain the fields at internal points, the boundary data of traction, displacement, electric potential, and the normal component of electric displacement need to be known throughout the boundary \(S\). For this purpose, we can examine the limiting forms of Equations (5.13), (5.17), and (5.24) as \(\mathbf{x}\) approaches the boundary. To properly circumvent the singular behavior when \(x\) approaches \(\mathbf{x}\), Chen and Lin [16] assumed a singular point \(\mathbf{x}\) on the boundary surrounded by a small hemispherical surface of radius \(\varepsilon\), say \(S_\varepsilon\), centered at the point \(\mathbf{x}\) with \(\varepsilon \to 0\). Because the asymptotic behavior of Green’s function in piezoelectric solids at \(r = |x - \mathbf{x}| \to 0\) is mathematically similar to that of uncoupled elasticity, Equations (5.13), (5.17), and (5.24) can be rewritten as [16]

\[
c_{kj}u_k(\mathbf{x}) = \int_V u_{j1}^*(x, \mathbf{x})f_j(x)dV(x) - \int_S t_{j1}^*(x, \mathbf{x})u_J(x)dS(x)
+ \int_S u_{j4}^*(x, \mathbf{x})t_{j4}(x)dS(x) \tag{5.26}
\]

\[
b\phi(\mathbf{x}) = \int_V u_{j4}^*(x, \mathbf{x})f_j(x)dV(x) - \int_S t_{j4}^*(x, \mathbf{x})u_J(x)dS(x)
+ \int_S u_{j4}^*(x, \mathbf{x})t_{j4}(x)dS(x) \tag{5.27}
\]

\[
c_{kj}t_k(\mathbf{x}) = \int_V V_{j1}^*(x, \mathbf{x})f_j(x)dV(x) - \int_S W_{j1}^*(x, \mathbf{x})u_J(x)dS(x)
+ \int_S V_{j4}^*(x, \mathbf{x})t_{j4}(x)dS(x) \tag{5.28}
\]

\[
bD_n(\mathbf{x}) = \int_V V_{j4}^*(x, \mathbf{x})f_j(x)dV(x) - \int_S W_{j4}^*(x, \mathbf{x})u_J(x)dS(x)
+ \int_S V_{j4}^*(x, \mathbf{x})t_{j4}(x)dS(x), \tag{5.29}
\]

where \(\mathbf{x} \in S\), and the coefficient \(c_{kj}\) and \(b\) are defined as

\[
c_{kj}(\mathbf{x}) = \delta_{kj} + \lim_{\varepsilon \to 0} \int_{S_\varepsilon} t_{ki}^*(x, \mathbf{x})dS(x) \tag{5.30}
\]

\[
b(\mathbf{x}) = 1 + \lim_{\varepsilon \to 0} \int_{S_\varepsilon} t_{44}^*(x, \mathbf{x})dS(x). \tag{5.31}
\]

In the field of BEM, the coefficients \(c_{kj}\) and \(b\) are usually known as boundary shape coefficients: \(c_{ii}(\mathbf{x}) = b(\mathbf{x}) = 1\) if \(\mathbf{x} \in \Omega\), \(c_{ii}(\mathbf{x}) = b(\mathbf{x}) = 1/2\) if \(\mathbf{x}\) is on the smooth boundary [17]. Using the concept of boundary shape
coefficients, Equation (5.18) can be rewritten as

\[ c_{K1uK}(\hat{x}) = \int_V u^*_J(x, \hat{x}) f_J(x)dV(x) - \int_S t^*_J(x, \hat{x}) u_J(x)dS(x) + \int_S u^*_J(x, \hat{x}) t_J(x)dS(x), \]

(5.32)

where \( \hat{x} \in \Gamma \), and \( c_{K4} = c_{4K} = b\delta_{K4} \).

It is more convenient to work with matrices rather than continue with indicial notation. To this effect the generalized displacement \( \mathbf{U} \), traction \( \mathbf{T} \), body force \( \mathbf{b} \), and boundary shape coefficients \( \mathbf{C} \) are defined as \([6]\)

\[
\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}, \quad \tilde{f} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 \\ c_{21} & c_{22} & c_{23} & 0 \\ c_{31} & c_{32} & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{bmatrix}^T,
\]

(5.33)

where \( c_{44} = b \).

Similarly, the fundamental solution coefficients can be defined in matrix form as

\[
\mathbf{U}^* = \begin{bmatrix} u^*_{11} & u^*_{12} & u^*_{13} & u^*_{14} \\ u^*_{21} & u^*_{22} & u^*_{23} & u^*_{24} \\ u^*_{31} & u^*_{32} & u^*_{33} & u^*_{34} \\ u^*_{41} & u^*_{42} & u^*_{43} & u^*_{44} \end{bmatrix}^T, \quad \mathbf{T}^* = \begin{bmatrix} t^*_{11} & t^*_{12} & t^*_{13} & t^*_{14} \\ t^*_{21} & t^*_{22} & t^*_{23} & t^*_{24} \\ t^*_{31} & t^*_{32} & t^*_{33} & t^*_{34} \\ t^*_{41} & t^*_{42} & t^*_{43} & t^*_{44} \end{bmatrix}^T,
\]

(5.34)

where \( t^*_{ij} (i = 1, 2, 3) \) represents the traction in the \( j \)th direction at a field point \( x \) due to a unit point load acting at the source point \( \hat{x} \), and \( t^*_{4j} \) denotes the \( j \)th traction at \( x \) due to a unit electric charge at \( \hat{x} \). Using the matrix notation defined in Equations (5.33) and (5.34), the integral equation (5.32) can be written in matrix form as

\[
\mathbf{C} \mathbf{U} = \int_V \mathbf{U}^* \tilde{f} dV - \int_S \mathbf{T}^* \mathbf{U} dS + \int_S \mathbf{U}^* \mathbf{T} dS.
\]

(5.35)

Using Equation (5.35), the generalized displacement at a point, say point \( \hat{x}_i \), can be obtained by enforcing a point load at the same point. In this case Equation (5.35) becomes

\[
\mathbf{C} (\hat{x}_i) \mathbf{U} (\hat{x}_i) = \int_V \mathbf{U}^* (\hat{x}_i, x) \tilde{f}(x)dV(x) - \int_S \mathbf{T}^* (\hat{x}_i, x) \mathbf{U}(x)dS(x) + \int_S \mathbf{U}^* (\hat{x}_i, x) \mathbf{T}(x)dS(x).
\]

(5.36)
5.2.3 Boundary Element Equation

To obtain a weak solution of Equation (5.36), as in the usual BEM, the boundary $S$ and the domain $V$ in Equation (5.36) are divided into $K$ boundary elements and $M$ internal cells, respectively. Boundary displacements, tractions, electric potential, and electric displacement are written in terms of their values at a series of nodal points. After discretization is performed by use of various kinds of BE (e.g., constant elements, linear elements, or higher-order elements), the boundary integral equation (5.36) becomes a set of linear algebraic equations including boundary variables $U$ and $T$. Once the boundary conditions are applied the linear algebraic equations can be solved to obtain all the unknown values. For a particular boundary element $e$, the variables $U$ and $T$ can be approximated in terms of the shape function $N$ in the form [17]

$$U = \sum_{i=1}^{Q} N_i U^e_i = NU^e, \quad T = \sum_{i=1}^{Q} N_i T^e_i = NT^e,$$  \hspace{1cm} (5.37)

where $N_i$ is the shape function associated with node $i$ and is discussed in the next two sections, $U^e_i$ and $T^e_i$ are, respectively, the values of $U$ and $T$ at node $i$ of the element, and $Q$ is the number of nodes of the element. The interpolation function $N$ is a $4 \times Q$ array of shape functions; that is,

$$N = \begin{bmatrix} N_1 & 0 & 0 & 0 & N_2 & \cdots & N_Q & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 & 0 & \cdots & 0 & N_Q & 0 & 0 \\ 0 & 0 & N_1 & 0 & 0 & \cdots & 0 & 0 & N_Q & 0 \\ 0 & 0 & 0 & N_1 & 0 & \cdots & 0 & 0 & 0 & N_Q \end{bmatrix}$$  \hspace{1cm} (5.38)

for a three-dimensional electroelastic problem. Substituting Equation (5.37) into (5.36) yields

$$C^i U^i + \sum_{j=1}^{K} \left\{ \int_{S_j} T^{*(i)} N dS \right\} U^j = \sum_{j=1}^{K} \left\{ \int_{S_j} U^{*(i)} N dS \right\} T^j$$

$$+ \sum_{s=1}^{M} \left\{ \int_{V_s} U^{*(i)} \tilde{f} dV \right\},$$  \hspace{1cm} (5.39)

where $K$ is the number of boundary elements, $C^i = C(x_i)$, $U^i = U(x_i)$, $U^{*(i)} = U^*(x_i, x)$, $T^{*(i)} = T^*(x_i, x)$, and $S_j$ is the surface of a “$j$” element.

Once the integrals in Equation (5.39) have been carried out, they can be further written as

$$C^i U^i + \sum_{j=1}^{K} \overline{H}_{ij} U^j = \sum_{j=1}^{K} G_{ij} T^j + \sum_{s=1}^{M} B_{is},$$  \hspace{1cm} (5.40)
where the inference matrices $\mathbf{H}_{ij}$ and $\mathbf{G}_{ij}$ are evaluated by

$$
\mathbf{H}_{ij} = \sum_t \int_{S_t} T^*(\hat{x}_i, x) N_q(x) dS(x),
$$

$$
\mathbf{G}_{ij} = \sum_t \int_{S_t} U^*(\hat{x}_i, x) N_q(x) dS(x),
$$

(5.41)

where the summation extends to all the elements to which node $j$ belongs and $q$ is the number of the order of node $j$ within element $t$. The pseudo-loading component $\mathbf{B}_{is}$ is defined as

$$
\mathbf{B}_{is} = \int_{V_s} U^*(\hat{x}_i, x) \tilde{f}(x) dV(x).
$$

(5.42)

If we define $\mathbf{H}_{ij} = \mathbf{H}_{ij} + \delta_{ij} \mathbf{C}(\hat{x}_i)$ and apply Equation (5.40) to the $K$ nodes "j" the following system of equations is obtained,

$$
\mathbf{H} \mathbf{U} = \mathbf{G} \mathbf{T} + \mathbf{B} .
$$

(5.43)

The vectors $\mathbf{U}$ and $\mathbf{T}$ represent all the values of generalized displacements and tractions before applying boundary conditions. These boundary conditions can be introduced by collecting the unknown terms to the left-hand side and the known terms to the right-hand side of Equation (5.43). This gives the final system of equations; that is,

$$
\mathbf{E} \mathbf{X} = \mathbf{R} .
$$

(5.44)

By solving the above system the vector $\mathbf{X}$ of boundary variables is fully determined.

5.3 One-Dimensional Elements

5.3.1 Shape Functions

One-dimensional elements are used to model the boundary of the plane domain of a problem. Within a particular element $e$ (see Figure 5.1), field variables can be interpolated using constant, linear, and higher-order polynomials. In this section, however, we focus on BEM formulations with linear elements.

Consider a linear element as shown in Figure 5.2, in which $\xi$ is a dimensionless coordinate whose value equals zero at the center and $\pm 1$ at the ends.
It is easy to verify that the Cartesian coordinates of a point on the
element shown in Figure 5.2 are related to the dimensional coordinate \( \xi \)
by the relation

\[
x(\xi) = \frac{1 - \xi}{2} x_i + \frac{1 + \xi}{2} x_j = N_1(\xi)x_i + N_2(\xi)x_j
\]
\[
y(\xi) = \frac{1 - \xi}{2} y_i + \frac{1 + \xi}{2} y_j = N_1(\xi)y_i + N_2(\xi)y_j,
\]

(5.45)

where \( N_1 \) and \( N_2 \) are the two shape functions of the element. Similarly, the
field variables \( U \) and \( T \) of this element can be approximated in terms of the
same shape functions as

\[
U(\xi) = \frac{1 - \xi}{2} U_i + \frac{1 + \xi}{2} U_j = N_1(\xi)U_i + N_2(\xi)U_j
\]
\[
T(\xi) = \frac{1 - \xi}{2} T_i + \frac{1 + \xi}{2} T_j = N_1(\xi)T_i + N_2(\xi)T_j.
\]

(5.46)
Substituting Equation (5.46) into (5.41) and noting the node $i$ shown in Figure 5.1, we have

$$
\mathbf{H}_{ki} = \int_{S_e^{-1}} \mathbf{T}^*(\tilde{x}_k, \mathbf{x}) \mathbf{N}_2(\mathbf{x}) dS(\mathbf{x}) + \int_{S_e} \mathbf{T}^*(\tilde{x}_k, \mathbf{x}) \mathbf{N}_1(\mathbf{x}) dS(\mathbf{x})
$$

$$
\mathbf{G}_{ki} = \int_{S_e^{-1}} \mathbf{U}^*(\tilde{x}_k, \mathbf{x}) \mathbf{N}_2(\mathbf{x}) dS(\mathbf{x}) + \int_{S_e} \mathbf{U}^*(\tilde{x}_k, \mathbf{x}) \mathbf{N}_1(\mathbf{x}) dS(\mathbf{x}) .
$$

(5.47)

### 5.3.2 Differential Geometry

In evaluating Equation (5.47), the information on the unit vector normal to a line element is required. The best way to determine the unit vector is by using vector algebra. To this end, consider a one-dimensional element as shown in Figure 5.3 [18]. The tangential vector in the direction of $\xi$ can be obtained by the differentiation of Equation (5.45) as

$$
\mathbf{v}_\xi = \nu_x \mathbf{i} + \nu_y \mathbf{j} = \frac{d}{d\xi} x(\xi) \mathbf{i} + \frac{d}{d\xi} y(\xi) \mathbf{j} ,
$$

(5.48)

where $\mathbf{i}$ and $\mathbf{j}$ are, respectively, the unit vectors in the $x$- and $y$-directions.

A vector normal to the line element $e$ in Figure 5.3, $\mathbf{v}_n$, may then be obtained by taking the cross-product of $\mathbf{v}_\xi$ with a unit vector in the $z$-direction ($\mathbf{v}_\xi = \{0, 0, 1\}^T$):

$$
\mathbf{v}_n = \mathbf{v}_\xi \times \mathbf{v}_z = \begin{vmatrix}
\frac{dx}{d\xi} & 0 & -\frac{dy}{d\xi} \\
\frac{dy}{d\xi} & 0 & \frac{dx}{d\xi} \\
0 & 1 & 0
\end{vmatrix} .
$$

(5.49)

Fig. 5.3 Normal and tangential vectors for one-dimensional element [18].
The length of the vector $v_n$ is equal to

$$v_n = |v_n| = \sqrt{\left(\frac{dy}{d\xi}\right)^2 + \left(-\frac{dx}{d\xi}\right)^2}$$

which is also the length of $d\xi$ and hence the Jacobian of the coordinate transformation from the Cartesian coordinate system to the intrinsic coordinate $\xi$. Therefore the unit vector normal to the line element is given by

$$n = \frac{v_n}{|v_n|} = \frac{1}{\sqrt{\left(\frac{dy}{d\xi}\right)^2 + \left(-\frac{dx}{d\xi}\right)^2}} \begin{pmatrix} \frac{dy}{d\xi} \\ -\frac{dx}{d\xi} \\ 0 \end{pmatrix}.$$  

5.4 Two-Dimensional Elements

5.4.1 Shape Functions

For modeling the boundary of three-dimensional problems, two-dimensional elements are used. For illustration we take an 8-node isoparametric element shown in Figure 5.4 as an example. The shape functions $N_i (i = 1–8)$ for the 8-node isoparametric element take the form [18]

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi - \eta - 1),$$

$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)(\xi - \eta - 1),$$

$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)(\xi + \eta - 1),$$

$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi + \eta - 1),$$

$$N_5 = \frac{1}{2}(1 - \xi^2)(1 - \eta), \quad N_6 = \frac{1}{2}(1 + \xi)(1 - \eta^2),$$

$$N_7 = \frac{1}{2}(1 - \xi^2)(1 + \eta), \quad N_8 = \frac{1}{2}(1 - \xi)(1 - \eta^2).$$

These shape functions have the property that $N_i$ is equal to unity at node $i$ and zero at all other nodes. In the isoparametric formulation, both the geometry and the variable fields are approximated with the same shape
functions as

\[ x(\xi, \eta) = \sum_{i=1}^{8} N_i(\xi, \eta)x^i \]  
\[ U(\xi, \eta) = \sum_{i=1}^{8} N_i(\xi, \eta)U^i \]  
\[ T(\xi, \eta) = \sum_{i=1}^{8} N_i(\xi, \eta)T^i. \]

To illustrate how to evaluate Equation (5.41) using Equations (5.57) and (5.58), consider node 11 in Figure 5.5. Substituting Equations (5.57) and (5.58) into (5.41), we obtain

\[ H_{ki} = \int_{S_e} T^*(\tilde{x}_k, x)N_3(x)dS(x) + \int_{S_e} T^*(\tilde{x}_k, x)N_4(x)dS(x) 
+ \int_{S_e} T^*(\tilde{x}_k, x)N_1(x)dS(x) + \int_{S_e} T^*(\tilde{x}_k, x)N_2(x)dS(x) \]  
\[ G_{ki} = \int_{S_e} U^*(\tilde{x}_k, x)N_3(x)dS(x) + \int_{S_e} U^*(\tilde{x}_k, x)N_4(x)dS(x) 
+ \int_{S_e} U^*(\tilde{x}_k, x)N_1(x)dS(x) + \int_{S_e} U^*(\tilde{x}_k, x)N_2(x)dS(x). \]
5.4.2 Differential Geometry

The unit vector \( \mathbf{n} \) normal to a surface can be obtained in a way similar to that in the one-dimensional element. Consider a two-dimensional element as shown in Figure 5.6. The tangential vectors \( \mathbf{v}_\xi \) in the \( \xi \)-direction and \( \mathbf{v}_\eta \) in the \( \eta \)-direction are obtained by differentiating Equation (5.56) [18]:

\[
\mathbf{v}_\xi = \frac{\partial \mathbf{x}}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} \mathbf{x}_i,
\]

\[
\mathbf{v}_\eta = \frac{\partial \mathbf{x}}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} \mathbf{x}_i.
\]

The vector normal to the surface in Figure 5.6, \( \mathbf{v}_n \) can then be determined by taking the cross-product of \( \mathbf{v}_\xi \) and \( \mathbf{v}_\eta \):

\[
\mathbf{v}_n = \mathbf{v}_\xi \times \mathbf{v}_\eta = \begin{pmatrix} v_{nx} \\ v_{ny} \\ v_{nz} \end{pmatrix} = \begin{pmatrix} \frac{dx}{d\xi} \\ \frac{du}{d\xi} \\ \frac{dz}{d\xi} \end{pmatrix} \times \begin{pmatrix} \frac{dx}{d\eta} \\ \frac{du}{d\eta} \\ \frac{dz}{d\eta} \end{pmatrix}.
\]

The unit normal vector \( \mathbf{n} \) is then given by

\[
\mathbf{n} = \frac{\mathbf{v}_n}{|\mathbf{v}_n|},
\]

where

\[
|\mathbf{v}_n| = \sqrt{(v_{nx})^2 + (v_{ny})^2 + (v_{nz})^2}
\]

is the Jacobian of the transformation from the Cartesian coordinate system to the intrinsic coordinate system \( (\xi, \eta) \).
5.5 Numerical Integration over Elements

Generally, the analytical solution to the integrals in Equation (5.41) is very difficult, and numerical integration over the element is thus required. Because Gaussian integration formulae are popular, simple, and very accurate for a given number of points, they are adopted in this chapter.

5.5.1 One-Dimensional Elements

In the numerical treatment of one-dimensional elements the integral to be evaluated is written as

$$I_1 = \int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i, \quad (5.65)$$

where $\xi_i$ is the coordinate of the $i$th integration point, $w_i$ is the associated weighting factor, and $n$ is the total number of integration points, which can be found from the BEM textbooks [19]. Notice that $\xi_i$ values are symmetric with respect to $\xi = 0$, $w_i$ being the same for the two symmetric values.

When we apply the Gaussian integration formulae (5.65) to the expression (5.41), the limitations of the integrals need to be converted from $S_t$ to $[-1,1]$ and the relationship between $dS$ and the increment of intrinsic coordinate $d\xi$. Noting Equation (5.50), this relationship is given by

$$dS = v_n d\xi = J d\xi = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2} d\xi, \quad (5.66)$$
where \( J \) is the Jacobian of the intrinsic transformation. Substituting Equations (5.65) and (5.66) into (5.41) and using the relation (5.45) yields

\[
\mathbf{H}_{ij} = \sum_t \left[ \int_{-1}^{+1} T^*(\hat{x}_i, x(\xi))N_q(x(\xi))J_t(\xi) d\xi \right] \\
\approx \sum_t \left[ \sum_k^{n} w_k T^*(\hat{x}_i, x(\xi_k))N_q(x(\xi_k))J_t(\xi_k) \right] 
\tag{5.67}
\]

\[
\mathbf{G}_{ij} = \sum_t \left[ \int_{-1}^{+1} U^*(\hat{x}_i, x(\xi))N_q(x(\xi))J_t(\xi) d\xi \right] \\
\approx \sum_t \left[ \sum_k^{n} w_k U^*(\hat{x}_i, x(\xi_k))N_q(x(\xi_k))J_t(\xi_k) \right] , 
\tag{5.68}
\]

where \( n \) is the number of Gaussian sampling points employed in the Gaussian numerical integration, and \( U^*(\hat{x}_i, x(\xi)) \) and \( T^*(\hat{x}_i, x(\xi)) \) are the fundamental solutions at \( x(\xi) \) for a source at point \( \hat{x}_i \).

### 5.5.2 Two-Dimensional Elements

Two-dimensional integration formulation is obtained by combining expression (5.65) in the form

\[
I_2 = \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) d\xi d\eta \approx \sum_{j=1}^{n} \sum_{i=1}^{n} f(\xi_i, \eta_j) w_i w_j . 
\tag{5.69}
\]

The relationship between \( dS \) and the increment of intrinsic coordinates \( d\xi d\eta \) is given by

\[
dS = v_n d\xi d\eta = J(\xi, \eta) d\xi d\eta = \sqrt{\nu_{nx}^2 + \nu_{ny}^2 + \nu_{nz}^2} d\xi d\eta . 
\tag{5.70}
\]

Substituting Equations (5.69) and (5.70) into (5.41) and using the relation (5.56), we have

\[
\mathbf{H}_{ij} = \sum_t \left[ \int_{-1}^{+1} T^*(\hat{x}_i, x(\xi, \eta))N_q(x(\xi, \eta))J_t(\xi, \eta) d\xi d\eta \right] \\
\approx \sum_t \left[ \sum_{k=1}^{n} \sum_{l=1}^{n} w_k w_l T^*(\hat{x}_i, x(\xi_k, \eta_l))N_q(x(\xi_k, \eta_l))J_t(\xi_k, \eta_l) \right] 
\tag{5.71}
\]
\[ G_{ij} = \sum_{t} \left[ \int_{-1}^{+1} U^*(\hat{x}_i, x(\xi, \eta))N_q(x(\xi, \eta))J_t(\xi, \eta) \, d\xi \, d\eta \right] \]
\[ \approx \sum_{t} \left[ \sum_{i=1}^{n} \sum_{k=1}^{n} w_i w_k U^*(\hat{x}_i, x(\xi_k, \eta_l))N_q(x(\xi_k, \eta_l))J_t(\xi_k, \eta_l) \right] \] (5.72)

5.6 Treatment of Singular Integrals

The accuracy of BEM for piezoelectric problems is critically dependent upon proper evaluation of the boundary integrals. The integrals (5.41) and (5.42) present a singular behavior of the order \( O(1/r) \) and \( O(1/r^2) \) for the generalized displacement and traction fundamental solutions, where \( r \) is the distance from a source point to the element under evaluation. The discussion which follows illustrates the basic procedure in treating singular integrals, taken from the results in [20, 21].

5.6.1 Nearly Singular Integrals [20]

Nearly singular integrals are evaluated using the standard Gauss quadrature formulae (5.69) in a special element subdivision scheme [22]. In this scheme, the element is subdivided into \( M \) and \( N \) subelements in the directions \( \xi \) and \( \eta \), respectively. The Gauss quadrature formulae are applied in each element, obtaining the value of the integral as the sum of all individual contributions from the subdivisions. The criteria for choosing the number of subdivisions \( M \) and \( N \) are based on the size of the element of integration and its relative distance to the collocation point. These criteria are formulated in such a way that, for a given element, the closer the source point is to the element, the greater is the number of subdivisions.

5.6.2 Weakly Singular Integrals

Integrals of the kernels \( U^* \) in Equation (5.41) show a weak singularity of the type \( O[\ln(z_K - z_{K0})] \) when the source point and the field point are either coincident or a short distance apart in comparison with the size of the element, which can be dealt with by a method of transformation in which the Jacobian of the transformation cancels out the singularity or weakens its effect, allowing a regular quadrature formula to give accurate results. For example, Rizzo and Shippy [23] used a Cartesian to polar transformation to integrate weak singular kernels. A polar coordinate system \((r, \theta)\) is
introduced, originating at the singular node, such that the element of area $d\xi d\eta$ in the Cartesian system $(\xi, \eta)$ becomes $r dr d\theta$ in the polar coordinate system. The additional $r$ in the integrand cancels the $O(1/r)$ singularity.

5.6.3 Nonhypersingular Integrals [21]

The kernels $T^*$ appearing in Equation (5.41) show a strong singularity of $O[1/(z_K - \hat{z}_K)]$ as $\hat{x} \to x$, where $(z_K - \hat{z}_K)$ is defined in Equation (5.75) below. Integration of such kernels over the element $S_j$ that contains the source point $\hat{x}$ can be achieved as follows.

It is obvious that integrals of the type

$$\int_{S_j} T^*(\hat{x}_j, x)N_q(x)dS(x) \quad (5.73)$$

contain the basic integral as

$$I_K = \int_{F_j} [p_K n_1(x) - n_2(x)] \frac{1}{z_K - \hat{z}_K} N_q(x)dS(x) \quad (K = 1-4) \quad (5.74)$$

where $n_1, n_2$ are the components of the external unit normal to the boundary at the observation point $x$ (see Figure 5.7) and $p_K$ is the material’s eigenvalues [2]. Define

$$r_K = z_K - \hat{z}_K = (x_1 - \hat{x}_1) + p_K(x_2 - \hat{x}_2). \quad (5.75)$$

It follows that

$$\frac{dr_K}{dS} = \frac{dr_K}{dx_1} \frac{dx_1}{dS} + \frac{dr_K}{dx_2} \frac{dx_2}{dS} = -n_2 + p_K n_1. \quad (5.76)$$

![Fig. 5.7 Outward unit normal at boundary point $x$.](image)
Equation (5.76) is the key for all the transformations proposed below, and this illustrates that the Jacobian $dr_K/dS$ of the coordinate transformation that maps the geometry of the boundary element $S_j$ onto the complex plane $r_K$ is included in the fundamental solution itself for the piezoelectric case.

Making use of Equations (5.76) and (5.74) can be rewritten as

$$I_K = \int_{S_j} \frac{1}{r_K} N_q(x) dr_K$$  \hspace{1cm} (5.77)

which can be decomposed into the sum of a regular integral plus a singular integral with a known analytical solution

$$I_K = \int_{\Gamma_j} \frac{1}{r_K} (N_q(x) - 1) dr_K + \int_{\Gamma_j} \frac{1}{r_K} dr_K.$$  \hspace{1cm} (5.78)

Integration of the kernels $D^{*}_{K:J}$ and $V^{*}_{IJ}$ in Equations (5.19) and (5.24) can be achieved in a similar way as for $T^{*}$ kernels because they contain singularities of the same type when $\hat{x} \rightarrow x$. From Equation (5.22), we have the singular integral of the type [21]

$$I''_K = \int_{\Gamma_j} \left[ p_K n_1(x) - n_2(x) \right] \frac{1}{r_K} N_q(x) d\Gamma(x) = \int_{\Gamma_j} \frac{1}{r_K} N_q(x) dr_K,$$  \hspace{1cm} (5.80)

The first integral in Equation (5.80) is regular and the second integral can be easily evaluated.

### 5.6.4 Hypersingular Integrals [21]

Note that the integration of $W^{*}_{IJ}$ in Equation (5.24) has a hypersingularity of the order $O(1/r^2)$ as $x \rightarrow \hat{x}$. From Equation (5.21) it follows that the hypersingular integral in Equation (5.24) is of the form

$$I''_K = \int_{\Gamma_j} \left[ p_K n_1(x) - n_2(x) \right] \frac{1}{r_K^2} N_q(x) d\Gamma(x) = \int_{\Gamma_j} \frac{1}{r_K} N_q(x) dr_K \hspace{1cm} (K = 1-4).$$  \hspace{1cm} (5.81)
As indicated in [21], the integral (5.81) can be again decomposed into the sum of a regular integral plus singular integrals with known analytical solutions by using Equation (5.76) and the first two terms of the series expansion of the shape function $N_q$ at $\hat{x}$, considered as a function of the complex space variable $r_K$

$$N_q(r_K) = N_q|_{r_K=0} + \frac{dN_q}{dr_K}|_{r_K=0} r_K + 0(r_K^2) \approx N_q^0 + N_q'^0 r_K . \quad (5.82)$$

Thus, $I''_K$ can be written as

$$I''_K = \int_{\Gamma_j} \frac{1}{r_K} N_q(x) dr_K = \int_{\Gamma_j} \frac{1}{r_K} [N_q(x) - N_q^0 - N_q'^0 r_K] dr_K$$

$$+ N_q^0 \int_{\Gamma_j} \frac{1}{r_K} N_q(x) dr_K + N_q'^0 \int_{\Gamma_j} \frac{1}{r_K} N_q(x) dr_K . \quad (5.83)$$

The first integral in (5.83) is regular and the other two can be easily evaluated analytically.

### 5.7 Evaluation of Domain Integrals

It is noted that the boundary integral equation (5.36) contains a domain integral due to the existence of body forces and body charges. The simplest way of computing the domain integral in Equation (5.36) is by subdividing the region into a series of internal cells, on each of which a numerical integration scheme such as Gauss quadrature can be applied. The use of cells to evaluate domain integrals implies an internal discretization, which considerably increases the amount of data needed to run the corresponding program. Hence the method is an obvious disadvantage and diminishes the elegance and computational efficiency of BEM which relies on the transformation of domain integrals into boundary ones. This can be overcome using the dual reciprocity method [24, 25]. The method may, theoretically, be used with any type of fundamental solution and does not need internal cells. The discussion in this section follows the developments in [17, 26].

#### 5.7.1 Dual Reciprocity Formulation

The dual reciprocity formulation is derived by weighting the inhomogeneous differential equation [26],

$$\Xi_{JK} U_{KN}^{l} + b_{JN}^{l} = 0 , \quad (5.84)$$
with the fundamental solution $u_{MJ}^*$ leading to
\[
\int_V u_{MJ}^* b_{JN}^l dV = C_{MJ} \hat{U}_{JN}^l + \int_S (u_{MJ}^* \hat{U}_{JN}^l - u_{MJ}^* T_{JN}^l) dS, \tag{5.85}
\]
where $\Xi_{JK} = E_{iJKm} \partial_i \partial_m$ is the elliptic operator of piezoelectricity, $C_{MJ}$ is defined in Equation (5.33), $u_{MJ}^*$ and $t_{MJ}^*$ are given in Equation (5.34), and $\hat{T}_{JN}^l = E_{iJKq} \hat{U}_{KN}^l n_i$ is the corresponding traction field. Consequently, the dual reciprocity method requires the use of a series of particular solutions $\hat{U}_{KM}^l$, where the number of $\hat{U}_{KM}^l$ is equal to the total number of nodes in the problem with collocation points. If there are $W$ boundary nodes and $L$ internal nodes, there will be $W + L$ values of $\hat{U}_{KM}^l$. Now the source term $\tilde{f}_J$ is approximated by a series of tensor functions $b_{JN}^l$ and unknown coefficients $\alpha_N^l$,
\[
\tilde{f}_J \approx \sum_{l=1}^{W+L} b_{JN}^l \alpha_N^l, \tag{5.86}
\]
where the approximation functions $b_{JN}^l$ are linked to the particular $\hat{U}_{KM}^l$ through the relationship (5.84). Substituting the approximation (5.86) into Equation (5.36) and making use of Equation (5.85), the following equation is obtained
\[
C_i U_i + K \sum_{j=1}^{W+L} H_{ij} U_j = \sum_{j=1}^{W+L} G_{ij} T_j + K \sum_{k=1}^{W+L} \left( C_i \hat{U}_{jk}^l + \sum_{j=1}^{K} \sum_{k=1}^{K} \hat{H}_{ij} \hat{U}_{jk}^l - \sum_{j=1}^{K} \sum_{k=1}^{K} G_{ij} \hat{T}_{jk}^l \right) \alpha_k, \tag{5.87}
\]
After discretization and integrating over each element using interpolation equation (5.37), Equation (5.87) becomes
\[
C^i U_i + \sum_{j=1}^{K} \hat{H}_{ij} U_j = \sum_{j=1}^{K} G_{ij} T_j + \sum_{k=1}^{W+L} \left( C^i \hat{U}_{jk}^l + \sum_{j=1}^{K} \sum_{k=1}^{K} \hat{H}_{ij} \hat{U}_{jk}^l - \sum_{j=1}^{K} \sum_{k=1}^{K} G_{ij} \hat{T}_{jk}^l \right) \alpha_k, \tag{5.88}
\]
where $\hat{H}_{ij}$ are $G_{ij}$ defined in Equation (5.41), $i$ is a source node, $j$ a boundary element, and $k$ the collocation points of the dual reciprocity scheme.

The contribution for all “$i$” nodes can be written together in matrix form as
\[
H U = G T + (H \hat{U} - G \hat{T}) \alpha. \tag{5.89}
\]
5.7.2 Coefficients $\alpha$

The coefficients $\alpha$ can be calculated by point allocation, that is, by forcing approximation (5.86) to be exact at $W + L$ collocation points. This leads to the following system equations:

$$
\begin{bmatrix}
    b_1(x_1) & b_2(x_1) & \cdots & b_P(x_1) \\
    b_1(x_2) & b_2(x_2) & \cdots & b_P(x_2) \\
    \vdots & \vdots & \ddots & \vdots \\
    b_1(x_P) & b_2(x_P) & \cdots & b_P(x_P)
\end{bmatrix}
\begin{bmatrix}
    \alpha_1 \\
    \alpha_2 \\
    \vdots \\
    \alpha_P
\end{bmatrix}
= 
\begin{bmatrix}
    \tilde{f}_1 \\
    \tilde{f}_2 \\
    \vdots \\
    \tilde{f}_P
\end{bmatrix},
$$

(5.90)

where $P = W + L$, $b_i(x_j) = b_j(x_i) = b(|x_i - x_j|)$. Then, the coefficient vector $\alpha$ can be expressed in terms of nodal values of the generalized body force vector $\tilde{f}$ as

$$
\begin{bmatrix}
    \alpha_1 \\
    \alpha_2 \\
    \vdots \\
    \alpha_P
\end{bmatrix}
= 
\begin{bmatrix}
    b_1(x_1) & b_2(x_1) & \cdots & b_P(x_1) \\
    b_1(x_2) & b_2(x_2) & \cdots & b_P(x_2) \\
    \vdots & \vdots & \ddots & \vdots \\
    b_1(x_P) & b_2(x_P) & \cdots & b_P(x_P)
\end{bmatrix}^{-1}
\begin{bmatrix}
    \tilde{f}_1 \\
    \tilde{f}_2 \\
    \vdots \\
    \tilde{f}_P
\end{bmatrix}.
$$

(5.91)

5.7.3 Particular Solutions $\hat{U}$ and Approximation Functions $b$

Due to the complexity of the governing differential equation it is very difficult to derive particular solutions in closed form. Several methods for approximate calculation of particular solutions have been presented in the literature [27–29]. Using a simple approach as presented in [17], instead of choosing radial basis functions for the approximation functions $b$ and solving the differential equation (5.84) to obtain the particular solutions, we can simply choose $\hat{U}_{KN} = \hat{U}_{KN}^l(r_l)$ and calculate the corresponding body force term $b_{jN}$ using Equation (5.84). Using this approach, Kogl and Gaul [26] selected the following particular solution,

$$
\hat{U}_{KN}^l = \delta_{KN}(r_l^2 + r_l^3),
$$

(5.92)

which yields the derivatives

$$
\hat{U}_{KN,q}^l = \delta_{KN}(2r_l + 3r_l^2)r_{q,l} \\
\hat{U}_{KN,qi}^l = \delta_{KN}[2 + 3r_l]\delta_{qi} + 3r_lr_{q,i}.
$$

(5.93) (5.94)

Then $b_{jN}$ can be obtained through use of Equations (5.84) and (5.94).
5.8 Multidomain Problems

The discussion in the previous sections of this chapter is suitable for problems with a single solution domain only, as the fundamental solution used assumes that material properties do not change inside the domain being analyzed. If the solution domain is made up piecewise of different materials the problem can be solved by multidomain BEM [13, 18]. The basic idea is to consider a number of regions that are connected to each other, much like pieces of a puzzle. Each region is treated in the same way as discussed previously, but can now be assigned different material properties. Because at the interfaces between the regions both $U$ and $T$ are not known, the number of unknowns is increased and additional equations are required to solve the problem. These equations can be obtained from the conditions of equilibrium and compatibility at the region interfaces. The multidomain approach presented in [13] can be adopted in implementation of the method. It is based on the division of the origin domain into homogeneous subregions (see Figure 5.8) so that Equation (5.43) still holds for each single subdomain, and we can write

$$H^{(i)}U^{(i)} - G^{(i)}T^{(i)} = B^{(i)}, \quad (i = 1, 2, \ldots, J), \quad (5.95)$$

where $J$ is the number of subregions and the superscript $(i)$ indicates quantities associated with the $i$th subregion. To obtain the solution it is necessary to restore domain unity by enforcing generalized displacement and traction continuity conditions along the interfaces between contiguous subdomains. We now introduce a partition of the linear algebraic system

![Fig. 5.8 Multidomain configuration.](image-url)
given by Equation (5.95) in such a way that the generic vector can be written as [13]

\[ y^{(i)} = \left\{ y_{S_{ij}}^{(i)} \cdots y_{S_{ij}}^{(i)} \right\}^T, \quad (5.96) \]

where the vector \( y_{S_{ij}}^{(i)} \) collects the components of \( y^{(i)} \) associated with the nodes belonging to the interface \( S_{ij} \) between the \( i \)th and \( j \)th subdomain, with the convention that \( S_{ii} \) stands for the external boundary of the \( i \)th subdomain (see Figure 5.8). Based on this arrangement, the interface compatibility and equilibrium conditions are given by [13]

\[ U_{S_{ij}}^{(i)} = U_{S_{ij}}^{(j)}, \quad T_{S_{ij}}^{(i)} = -T_{S_{ij}}^{(j)}, \quad (i = 1, \ldots, J - 1; j = i + 1, \ldots, J). \quad (5.97) \]

It should be noted that if the \( i \)th and \( j \)th subdomain have no common boundary, \( y_{S_{ij}}^{(i)} \) is a zero-order vector, and Equation (5.97) is no longer valid. The system of Equation (5.95) and the interface continuity conditions (5.97) provide a set of relationships that, together with the external boundary conditions, allows derivation of the electroelastic solution in terms of generalized displacement and traction on the boundary of each subdomain. It should be mentioned that the multidomain approach described here is suitable for modeling general fracture problems in piezoelectric media [13].

### 5.9 Numerical Examples

To illustrate the application of the element model described above, three examples are presented. The first example deals with a piezoelectric column subjected to tension at its two ends; the second treats an infinite piezoelectric solid with a horizontal finite crack; and the third illustrates the behavior of crack-tip fields in a skew-cracked rectangular panel.

#### 5.9.1 A Piezoelectric Column Under Uniaxial Tension [6]

In this example, a piezoelectric prism under simple extension is considered (see Figure 5.9). The size of the prism is \( 2a \times 2a \times 2b \). The corresponding boundary conditions are given by

- \( T_{zz} = p, \quad T_{xz} = T_{yz} = D_z = 0, \quad \text{for} \quad z = \pm b, \)
- \( \phi = 0, \quad \text{for} \quad z = 0 \)
- \( T_{xx} = T_{xz} = D_x = 0, \quad \text{for} \quad x = \pm a \)
- \( T_{yy} = T_{yz} = D_y = 0, \quad \text{for} \quad y = \pm a. \)
In the calculation, $2a = 3\,\text{m}$, $2b = 10\,\text{m}$, and $p = 100\,\text{Nm}^2$ are assumed and 32 elements are used [6]. The material considered is PZT−4 whose material parameters are

\[
\begin{align*}
c_{11} &= 13.9 \times 10^{10}\,\text{NM}^{-2}, \\
c_{12} &= 7.78 \times 10^{10}\,\text{NM}^{-2}, \\
c_{13} &= 7.43 \times 10^{10}\,\text{NM}^{-2}, \\
c_{33} &= 11.5 \times 10^{10}\,\text{NM}^{-2}, \\
c_{44} &= 2.56 \times 10^{10}\,\text{NM}^{-2}, \\
e_{15} &= 12.7\,\text{Cm}^{-2}, \\
e_{31} &= -5.2\,\text{Cm}^{-2}, \\
e_{33} &= 15.1\,\text{Cm}^{-2}, \\
\varepsilon_{11} &= 730\varepsilon_0, \\
\varepsilon_{33} &= 635\varepsilon_0,
\end{align*}
\]

where $\varepsilon_0 = 8.854 \times 10^{-12}\,\text{C}^2/\text{Nm}^2$. Table 5.1 lists the displacements and electric potential at points A(2, 0), B(3, 0), C(0, 5), and D(0, 10) using BEM, and comparison is made with analytical results. It is evident that the BEM results are in good agreement with the analytical results even when only six boundary elements are used in the calculation [6].

<table>
<thead>
<tr>
<th>Point</th>
<th>A(2, 0)</th>
<th>B(3, 0)</th>
<th>C(0, 5)</th>
<th>D(0, 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BEM</td>
<td>$u_1 \times 10^{10}$ (m)</td>
<td>$-0.9663$</td>
<td>$-1.4550$</td>
<td>$0.4967$</td>
</tr>
<tr>
<td></td>
<td>$u_2 \times 10^{8}$ (m)</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$\phi$ (V)</td>
<td>$0$</td>
<td>$0$</td>
<td>$0.6879$</td>
</tr>
<tr>
<td>Exact [6]</td>
<td>$u_1 \times 10^{10}$ (m)</td>
<td>$-0.9672$</td>
<td>$-1.4508$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$u_2 \times 10^{8}$ (m)</td>
<td>$0$</td>
<td>$0$</td>
<td>$0.5006$</td>
</tr>
<tr>
<td></td>
<td>$\phi$ (V)</td>
<td>$0$</td>
<td>$0$</td>
<td>$0.6888$</td>
</tr>
</tbody>
</table>
5.9.2 A Horizontal Finite Crack in an Infinite Piezoelectric Medium

The second example is a finite horizontal crack along the $x$-direction in an infinite PZT$-4$ medium under a uniform far-field stress or electric displacement. To effectively evaluate crack-tip fields, Pan [11] constructed the following crack-tip element with its tip at $\zeta = -1$, where $\zeta$ is the intrinsic coordinate in a quarter-point element defined in Figure 5.10,

$$\Delta U = \sum_{k=1}^{3} \Phi_k \Delta U^k,$$

(5.98)

where the superscript $k(k = 1, 2, 3)$ denotes the the generalized relative crack displacement (GRCD) at nodes $\zeta = -2/3, 0, 2/3$, respectively. $\zeta$ is the boundary natural coordinate in a quarter-point element defined in Figure 5.10. The quarter-point quadratic crack-tip element is obtained by setting [21]

$$\zeta = 2 \sqrt{\frac{r}{L}} - 1,$$

(5.99)

where $r$ is the distance from a field point to the crack-tip, and $L$ the element length (see Figure 5.10). In the element, the collocation points NC1, NC2, and NC3 for the quarter-point element are located at $\zeta_1 = -3/4$, $\zeta_2 = 0$, and $\zeta_3 = 3/4$, respectively (see Figure 5.10). In such a case, the distance $r$ from the collocation nodes of the quarter-point element to the crack-tip follows from Equation (5.99)

$$r_1 = \frac{L}{64} \text{ at NC1},$$

$$r_2 = \frac{L}{4} \text{ at NC2},$$

$$r_3 = \frac{49L}{64} \text{ at NC3}.$$  

(5.100)

Fig. 5.10 Configuration of a quarter-point element.
Making use of the boundary natural coordinate $\zeta$, the shape functions $\Phi_k$ in Equation (5.98) can be defined by [11]

$$\Phi_1 = \frac{3\sqrt{3}}{8}\sqrt{\zeta + 1}[{-5 + 18(\zeta + 1) - 9(\zeta + 1)^2}],$$

$$\Phi_2 = \frac{1}{4}\sqrt{\zeta + 1}[{5 - 8(\zeta + 1) + 3(\zeta + 1)^2}],$$

$$\Phi_3 = \frac{3\sqrt{3}}{8\sqrt{5}}\sqrt{\zeta + 1}[{1 - 4(\zeta + 1) + 3(\zeta + 1)^2}].$$  \hfill (5.101)

For calculation of the generalized stress intensity factors (GSIF), Pan [11] employed the extrapolation method of the GRCD, which requires an analytical relation between the generalized displacement and the GSIF. This relation can be expressed as

$$\Delta U(r) = 2\sqrt{\frac{2r}{\pi}}(iAB^{-1})K,$$  \hfill (5.102)

where the matrices $A$ and $B$ are matrices associated with material properties [2], and $K$ is defined by [21],

$$K = \begin{cases} K_{II} \\ K_I \\ K_{IV} \end{cases} = 2\sqrt{\frac{2\pi}{L}}(\text{Re}[B])^{-1} \begin{cases} \Delta u_1^{NC1} \\ \Delta u_2^{NC1} \\ \Delta \phi^{NC1} \end{cases}.$$  \hfill (5.103)

In this example, Pan [11] used 20 discontinuous quadratic elements described above to discretize the crack surface which has a length of $2a$ (=1m). Tables 5.2 and 5.3 list the GRCD caused by a far-field stress $T_{yy}(=1 \text{ N/m}^2)$ and a far-field electric displacement $D_y(=1 \text{ C/m}^2)$, and comparison is made with analytical results. It is obvious that a far-field stress induces a nonzero

| Table 5.2 GRCD caused by a far-field $T_{yy}(=1 \text{ N/m}^2)$[11] |
|-----------------|-----------------|-----------------|
| $x$ (m) | $\Delta u_y(10^{-12} \text{ m})$ | $\Delta \phi(10^{-1} \text{ V})$ |
| BEM Analytical | BEM Analytical |
|-----------------|-----------------|-----------------|
| 0.492 | 0.032 | 0.032 | 0.040 | 0.040 |
| 0.425 | 0.094 | 0.093 | 0.116 | 0.116 |
| 0.358 | 0.124 | 0.124 | 0.154 | 0.154 |
| 0.292 | 0.144 | 0.144 | 0.179 | 0.179 |
| 0.225 | 0.158 | 0.158 | 0.197 | 0.197 |
| 0.158 | 0.168 | 0.168 | 0.210 | 0.210 |
| 0.092 | 0.174 | 0.174 | 0.217 | 0.217 |
| 0.025 | 0.177 | 0.177 | 0.221 | 0.221 |
Table 5.3 GRCD caused by a far-field $D_y (= 1 \text{ C/m}^2) [11]$

<table>
<thead>
<tr>
<th>$x$ (m)</th>
<th>$\Delta \phi (10^8 \text{ V})$</th>
<th>$\Delta u_y (10^{-1} \text{ m})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BEM Analytical</td>
<td>BEM Analytical</td>
<td></td>
</tr>
<tr>
<td>0.492</td>
<td>0.161</td>
<td>0.160</td>
</tr>
<tr>
<td>0.425</td>
<td>0.466</td>
<td>0.465</td>
</tr>
<tr>
<td>0.358</td>
<td>0.616</td>
<td>0.616</td>
</tr>
<tr>
<td>0.292</td>
<td>0.717</td>
<td>0.717</td>
</tr>
<tr>
<td>0.225</td>
<td>0.789</td>
<td>0.789</td>
</tr>
<tr>
<td>0.158</td>
<td>0.838</td>
<td>0.838</td>
</tr>
<tr>
<td>0.092</td>
<td>0.868</td>
<td>0.868</td>
</tr>
<tr>
<td>0.025</td>
<td>0.882</td>
<td>0.882</td>
</tr>
</tbody>
</table>

$\Delta \phi$ even though the corresponding $K_{IV}$ is zero. Similarly, a far-field electric displacement can induce a nonzero $\Delta u_y$.

5.9.3 A Rectangular Piezoelectric Solid with a Central Inclined Crack [13]

The third example is a rectangular piezoelectric solid with a central crack ($a = 0.1 \text{ m}$) inclined $\theta = 45^\circ$ with respect to the positive $x$-direction. The ratios of crack length to width and of height to width are $a/w = 0.2$ and $h/w = 2$, respectively (see Figure 5.11). The analysis is carried out for the

![Fig. 5.11 Finite rectangular solid with an inclined crack under uniform tension or electric displacement in the y-direction.](image-url)
Table 5.4 GSIF for cracked rectangle loaded by $\sigma_y$ or $D_y$

<table>
<thead>
<tr>
<th>Ref</th>
<th>$K_I/T_{yy}\sqrt{\pi a}$</th>
<th>$K_{II}/T_{yy}\sqrt{\pi a}$</th>
<th>$K_{IV}/D^*\sqrt{\pi a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loaded by $\sigma_y$ [11]</td>
<td>0.5303 0.5151</td>
<td>$-2.97 \times 10^{-12}$</td>
<td>$-2.79 \times 10^{-12}$</td>
</tr>
<tr>
<td>[11]</td>
<td>0.5292 0.5163</td>
<td>$-2.79 \times 10^{-12}$</td>
<td>$-2.79 \times 10^{-12}$</td>
</tr>
</tbody>
</table>

Table 5.5 GRCD for cracked rectangle caused by a far-field $T_{yy}(=1 \text{ N/m}^2)$ [13]

<table>
<thead>
<tr>
<th>$x = y \times 10^{-1} \text{ m}$</th>
<th>$\Delta u_x(10^{-13} \text{ m})$</th>
<th>$\Delta u_y(10^{-11} \text{ m})$</th>
<th>$\Delta \phi (10^{-2} \text{ V})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.477</td>
<td>0.256</td>
<td>$-0.190$</td>
<td>0.232</td>
</tr>
<tr>
<td>0.424</td>
<td>0.279</td>
<td>$-0.206$</td>
<td>0.252</td>
</tr>
<tr>
<td>0.371</td>
<td>0.299</td>
<td>$-0.219$</td>
<td>0.268</td>
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<tr>
<td>0.318</td>
<td>0.315</td>
<td>$-0.230$</td>
<td>0.281</td>
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<tr>
<td>0.265</td>
<td>0.328</td>
<td>$-0.239$</td>
<td>0.292</td>
</tr>
<tr>
<td>0.212</td>
<td>0.339</td>
<td>$-0.246$</td>
<td>0.300</td>
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<tr>
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<td>$-0.251$</td>
<td>0.307</td>
</tr>
<tr>
<td>0.106</td>
<td>0.352</td>
<td>$-0.255$</td>
<td>0.312</td>
</tr>
<tr>
<td>0.053</td>
<td>0.356</td>
<td>$-0.257$</td>
<td>0.314</td>
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Table 5.6 GRCD for cracked rectangle caused by a far-field $D_y (= 1 \text{ C/m}^2)$ [13]

<table>
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<tr>
<th>$x = y$ ($10^{-1}$ m)</th>
<th>$\Delta u_x$ ($10^{-5}$ m)</th>
<th>$\Delta u_y$ ($10^{-2}$ m)</th>
<th>$\Delta \phi$ ($10^8$ V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.477</td>
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<td>0.232</td>
<td>0.093</td>
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References
