Chapter 2

MULTI-FIELD BONE REMODELING UNDER AXIAL AND TRANSVERSE LOADS

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ABSTRACT

Internal and surface bone remodelling of inhomogeneous long cylindrical bone is studied both theoretically and numerically in this chapter. Based on the theory of adaptive elasticity, two solutions for thermoelectroelastic problems of bone remodelling, are presented to study effects of mechanical, thermal and electric loads on bone remodelling process. The external loads may be of coupled axial force, external lateral pressure, electric, magnetic and thermal loads or the load as a result of a force-fitted medullary pin. The analytical solution is used for investigating bone remodelling process on the basis of assuming a homogeneous bone material. Though all bone is heterogeneous, assumption of homogeneity is made to entail a smoothing over features such as osteons, lamellae, fibres, and other structural elements, so that a continuum representation can be obtained. In contrast, the semi-analytical solution is developed for analysing bone materials that are assumed to be radially inhomogeneous. These two solutions are then extended to include magnetic effect. Numerical results are presented to verify the proposed formulation and to show the effects of mechanical, thermal and electric loads on bone remodelling process.

Keywords: Bone remodeling, piezoelectricity, adaptive elastic theory, magnetoelectroelastic medium

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1. INTRODUCTION

Bone is a kind of dynamically adaptable material. Like any other living system, it has mechanisms for repair and growth or remodelling, and mechanisms to feed its constituent parts and ensure that any materials needed for structural work are supplied to the correct area as and when required. These bone functions are performed via three types of bone cell: osteoblast, osteoclast, and osteocyte. Osteoblasts are cells that form new bone and are typically found lining bone surfaces that are undergoing extensive remodelling. Osteoclasts are large, multinucleated, bone-removing cells. Their function is to break down and remove bone material that is no longer needed or that has been damaged in some way. The third cell type is the osteocyte. Osteocytes, called the bone “sensor cells”, are responsible for sensing the physical environment to which the skeleton is subjected. Osteocytes are characterized by many protoplasmic processes, or dendrites, emanating from the cell body. These cell dendrites form a communication network with surrounding cells, other osteocytes, osteoblasts, and possibly osteoclasts, that passes the signals from the osteocytes that control the action of osteoblasts and osteoclasts. The activities of these three cell populations, and numerous other biological and biochemical factors, are coordinated in a continuous process throughout our lives to maintain a strong, healthy skeleton system.

It should be noted that applications of the multi-field theory to bone remodelling have been the subject of fruitful scientific attention by many distinguished researchers (e.g. Fukada and Yasuda [1,2], Kryszewski [3], Robiony [4], Qin and Ye [5] and others). Early in the 1950s, Fukada and Yasuda [1,2] found that some living bone and collagen exhibit piezoelectric behaviour. Later, Gjelsvik [6] presented a physical description of the remodelling of bone tissue, in terms of a very simplified form of linear theory of piezoelectricity. Williams and Breger [7] explored the applicability of stress gradient theory for explaining the experimental data for a cantilever bone beam subjected to constant end load, showing that the approximate gradient theory was in good agreement with the experimental data. Guzelsu [8] presented a piezoelectric model for analysing a cantilever dry bone beam subjected to a vertical end load. Johnson et al. [9] further addressed the problem of a dry bone beam by presenting some theoretical expressions for the piezoelectric response to cantilever bending of the beam. Demiray [10] provided a theoretical description of electromechanical remodelling models of bones. Aschero et al. [11] investigated the converse piezoelectric effect of fresh bone using a highly sensitive dilatometer. They further investigated the piezoelectric properties of bone and presented a set of repeated measurements of coefficient $d_{23}$ on 25 cow bone samples [12]. Fotiadis et al. [13] studied wave propagation in a long cortical piezoelectric bone with arbitrary cross-section. El-Naggar and Abd-Alla [14] and Ahmed and Abd-Alla [15] further obtained an analytical solution for wave propagation in long cylindrical bones with and without cavity. Silva et al. [16] explored the physicochemical, dielectric and piezoelectric properties of anionic collagen and collagen-hydroxyapatite composites. Recently, Qin and Ye [5] and Qin et al. [17] presented a thermoelastic solution for internal and surface bone remodelling, respectively. Later, Qu et al. [18] extended the results in [5,17] to include magnetic effects. Accounts of most of the developments in this area can also be found in [3,19]. In this chapter, however, we restrict our discussion to the findings presented in [5, 17-19].
2. THERMOELECTROELASTIC INTERNAL BONE REMODELLING

2.1. Linear Theory of Thermoelectroelastic Bone

Consider a hollow circular cylinder composed of linearly thermopiezoelectric bone material subjected to axisymmetric loading. The axial, circumferential and normal to the middle-surface coordinate length parameters are denoted by $z$, $\theta$ and $r$, respectively. Using the cylindrical coordinate system, the corresponding constitutive equations can be written in the form [5,20]

$$
\begin{align*}
\sigma_{rr} &= c_{11} \varepsilon_{rr} + c_{12} \varepsilon_{\theta\theta} + c_{13} \varepsilon_{zz} - e_{31} E_z - \lambda_1 T \\
\sigma_{\theta\theta} &= c_{12} \varepsilon_{rr} + c_{11} \varepsilon_{\theta\theta} + c_{13} \varepsilon_{zz} - e_{31} E_z - \lambda_1 T \\
\sigma_{zz} &= c_{13} \varepsilon_{rr} + c_{13} \varepsilon_{\theta\theta} + c_{33} \varepsilon_{zz} - e_{33} E_z - \lambda_3 T \\
\sigma_{z\theta} &= c_{44} \varepsilon_{rr} - e_{13} E_r, \quad D_r = e_{15} \varepsilon_{rr} + \kappa_1 E_r, \\
D_z &= d_i (e_{rr} + e_{\theta\theta}) + e_{33} \varepsilon_{zz} + \kappa_3 E_z - \chi_3 T \\
h_r &= k_i H_r, \quad h_z = k_z H_z
\end{align*}
$$

(1)

where $\sigma$, $D$, $\varepsilon$, $E$, denote components of stress, electric displacement, strain, and electric field intensity, respectively; $c_{ij}$ are elastic stiffness; $e_{ij}$ are piezoelectric constants; $\kappa_i$ are dielectric permittivities; $T$ denotes temperature change; $\chi_3$ is pyroelectric constant; $\lambda_i$ are stress-temperature coefficients; $h_i$ are heat flow; $H_i$ are heat intensity; and $k_i$ are heat conduction coefficients. The associated strains, electric fields, and heat intensities are respectively related to displacements $u_i$, electric potential $\phi$, and temperature change $T$ as

$$
\begin{align*}
\varepsilon_{rr} &= u_{r,r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{zz} = u_{z,z}, \quad \varepsilon_{z\theta} = u_{z,r} + u_{r,z} \\
E_r &= -\phi_r, \quad E_z = -\phi_z, \quad H_r = -T_r, \quad H_z = -T_z
\end{align*}
$$

(2)

For quasi-stationary behaviour, in the absence of a heat source, free electric charge and body forces, the set of equations for thermopiezoelectric theory of bones is completed by adding the following equations of equilibrium for heat flow, stress and electric displacements to eqs. (1) and (2):
2.2. Adaptive Elastic Theory

Adaptive theory is used to model the normal adaptive processes that occur in bone remodelling as strain controlled mass deposition or resorption processes which modify the porosity of the porous bone material [21]. In the adaptive elastic constitutive equation presented in [21], the authors introduced an independent variable which is a measure of the volume fraction of the matrix structure. Let \( \xi \) denote the volume fraction of the matrix material in an unstrained reference state and assume that the density of the material composing the matrix is constant. Thus the conservation of mass will give the equation governing \( \xi \). Then an important constitutive assumption was made [21] that, at constant temperature and zero body force, there exists a unique zero-strain reference state for all values of \( \xi \). Thus \( \xi \) may change without changing the reference state for strain. One might imagine a block of porous elastic material with the four points, the vertices of a tetrahedron, marked on the block for the purpose of measuring the strain. When the porosity changes, material is added or taken away from the pores, but if the material is unstrained it remains so and the distance between the four vertices marked on the block do not change. Thus \( \xi \) can change while the zero-strain reference state remains the same. Keeping this in mind, a formal definition of the remodelling rate \( \dot{e} \) (the rate at which mass per unit volume is added to or removed from the porous matrix structure) and free energy \( \Psi \) can be given [21]:

\[
\dot{e} = \dot{e}(\xi, F), \quad \Psi = \Psi(\xi, F)
\]

where \( F \) stands for deformation gradient. More detailed discussion of this formulation (4) is found in [21]. Considering the adaptive property discussed above, the traditional elastic stress-strain relationship becomes [21]

\[
\sigma_{ij} = (\xi_0 + e)C_{ijkl}(e)e_{kl}, \quad \dot{e} = A'(e) + A_y(e)e_{ij}
\]

where \( \xi_0 \) is a reference volume fraction of bone matrix material, \( e \) is a change in the volume fraction of bone matrix material from its reference value \( \xi_0 \), \( C_{ijkl}(e) \) is the stiffness matrix dependent upon the volume fraction change \( e \), and \( A'(e) \) and \( A_y(e) \) are material constants also dependent upon the volume fraction change \( e \). Equation (5) is deduced from mass balance considerations. When \( e \) is very small, eq. (5) can be approximated by a simple form

\[
\dot{e} = C_0 + C_1 e + C_2 e^2 + \left( A^b_{ij} + eA^r_{ij} \right) e_{ij}
\]
where $C_0, C_1, C_2, A_{ij}^0$, and $A_{ij}$ are material constants. When $\xi_0$ is one and $e$ is zero the stress-strain relation (5) is reduced to Hooke’s law for a solid elastic material. In this situation all the pores of the bone matrix would be completely filled with bone material.

The bone remodelling equation (5) can be extended to include the effect of thermal and electric fields by introducing some new terms as [5]:

$$\dot{\varepsilon} = A^e(e) + A_{rr}^E(e)E_r + A_{rz}^E(e)E_z + A_{r\theta}^e(e)e_{r\theta} + A_{r\phi}^e(e)e_{r\phi} + A_{\phi\phi}^e(e)e_{\phi\phi}$$

where $A_{rr}^E(e)$ and $A_{rz}^e(e)$ are material coefficients dependent upon the volume fraction $e$.

Equations (1)-(3) together with eq. (7) form the basic set of equations for the adaptive theory of internal piezoelectric bone remodelling.

2.3. Analytical Solution of a Homogeneous Hollow Circular Cylindrical Bone

We now consider a hollow circular cylinder of bone subjected to an external temperature change $T_0$, a quasi-static axial pressure load $P$, an external pressure $p$ and an electric load $\phi_e$ (or/and $\phi_a$). The boundary conditions are

$$T = 0, \quad \sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0, \quad \phi = \phi_a \quad \text{at} \quad r = a$$

$$T = T_0, \quad \sigma_{rr} = -p, \quad \sigma_{r\theta} = \sigma_{rz} = 0, \quad \phi = \phi_b, \quad \text{at} \quad r = b$$

and

$$\int_S \sigma_{zz} dS = -P$$

where $a$ and $b$ denote, respectively, the inner and outer radii of the bone, and $S$ is the cross-sectional area. For a long bone, it is assumed that all displacements, temperature and electrical potential except the axial displacement $u_z$ are independent of the $z$ coordinate and that $u_z$ may have linear dependence on $z$. Using eqs. (1) and (2), differential equations (3) can be written as

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) T = 0, \quad c_{11} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) u_r = \lambda_1 \frac{\partial T}{\partial r}$$

$$c_{44} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z + c_{15} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \phi = 0$$
The solution to the heat conduction equation (10) satisfying boundary conditions (8) can be written as

\[ T = \frac{\ln(r/a)}{\ln(b/a)} T_0 \]  

(13)

It is easy to prove that eqs. (10)-(12) will be satisfied if we assume

\[ u_r = A(t)r + \frac{B(t)}{r} + \frac{\sigma r[\ln(r/a) - 1]}{c_{11}} \]  

(14)

\[ u_z = zC(t) + D(t)\ln(r/a), \quad \varphi = F(t)\ln(r/a) + \varphi_a \]  

(15)

where \( A, B, C, D \) and \( F \) are unknown variables to be determined by introducing boundary conditions, and \( \varpi = \frac{\lambda_1 T_0}{2\ln(b/a)} \). Substituting eqs. (14) and (15) into eq. (2), and later into eq. (1), we obtain

\[ \sigma_{rr} = A(t)(c_{11} + c_{12}) - \frac{B(t)}{r^2}(c_{11} - c_{12}) + c_{13}C(t) + \varpi \left[ \frac{c_{12}}{c_{11}} \ln \frac{r}{a} - 1 - \ln \frac{r}{a} \right] \]  

(16)

\[ \sigma_{\theta\theta} = A(t)(c_{11} + c_{12}) + \frac{B(t)}{r^2}(c_{11} - c_{12}) + c_{13}C(t) + \varpi \left[ \frac{c_{12}}{c_{11}} \ln \frac{r}{a} - \ln \frac{r}{a} - 1 \right] \]  

(17)

\[ \sigma_{zz} = 2A(t)c_{13} + c_{33}C(t) + \varpi \left[ \frac{c_{13}}{c_{11}} \left[ 2\ln(r/a) - 1 \right] - \chi_3 T_0 \frac{\ln(r/a)}{\ln(b/a)} \right] \]  

(18)

\[ \sigma_{xz} = \frac{1}{r} \left[ e_{14}^2 D(t) + e_{15} F(t) \right], \quad D_r = \frac{1}{r} \left[ e_{15} D(t) - \chi_3 F(t) \right] \]  

(19)

\[ D_z = 2A(t)e_{31} + C(t)e_{33} + \varpi \left[ e_{31} \left[ 2\ln(r/a) - 1 \right] - \chi_3 T_0 \frac{\ln(r/a)}{\ln(b/a)} \right] \]  

(20)

The boundary conditions (8) and (9) of stresses and electric potential require that

\[ c_{44} D(t) + d_4 F(t) = 0, \quad \varphi_b = F(t)\ln(b/a) + \varphi_a \]  

(21)
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\[ A(t)(c_{11} + c_{12}) - \frac{B(t)}{a^2} (c_{11} - c_{12}) + c_{13} C(t) - \frac{c_{12}}{c_{11}} \omega = 0 \]  
\[ A(t)(c_{11} + c_{12}) - \frac{B(t)}{b^2} (c_{11} - c_{12}) + c_{13} C(t) + \omega \left[ \frac{c_{12}}{c_{11}} (\ln \frac{b}{a} - 1) - \ln \frac{b}{a} \right] = -p \]

\[ \pi(b^2 - a^2) [2 A(t)c_{13} + C(t)c_{13} - F_1^* T_0] + F_2^* T_0 = -P \] 

where

\[ F_1^* = \frac{1}{\ln(b/a)} \left( \frac{c_{13} \lambda_1}{c_{11}} - \frac{\lambda_3}{2} \right), \quad F_2^* = \pi b^2 \left( \frac{c_{13} \lambda_1}{c_{11}} - \lambda_3 \right) \] 

The unknown functions \( A(t), B(t), C(t), D(t) \) and \( F(t) \) are readily found from eqs. (21)-(24) as

\[ A(t) = \frac{1}{F_3^*} \left( c_{33} \beta_1^* [\beta_2^* T_0 + p(t)] + \omega \frac{c_{33} c_{12}}{c_{11}} + \frac{F_2^* T_0 + P(t)}{\pi (b^2 - a^2)} c_{13} - F_1^* T_0 c_{13} \right) \] 
\[ B(t) = \frac{a^2 \beta_1^* [\beta_2^* T_0 + p(t)]}{c_{11} - c_{12}} \] 
\[ C(t) = \frac{1}{F_3^*} \left[ F_1^* T_0 - \frac{F_2^* T_0 + P(t)}{\pi (b^2 - a^2)} \right] (c_{11} + c_{12}) - 2 c_{13} \beta_1^* [\beta_2^* T_0 + p(t)] - \frac{2 c_{13} c_{12} \omega}{c_{11}} \] 
\[ D(t) = -\frac{e_{15} (\varphi_b - \varphi_a)}{c_{44} \ln(b/a)} \] 
\[ F(t) = \frac{\varphi_b - \varphi_a}{\ln(b/a)} \] 

where

\[ F_3^* = c_{33} (c_{11} + c_{12}) - 2 c_{13}^2, \quad \beta_1^* = \frac{b^2}{(a^2 - b^2)}, \quad \beta_2^* = \frac{\lambda_1}{2} \left( \frac{c_{12}}{c_{11}} - 1 \right) \]
Using expressions (26)-(30), the displacements \( u_r, u_z \) and electrical potential \( \varphi \) are given by

\[
\begin{align*}
  u_r &= \frac{r}{F_3} \left( c_{33} \beta_1^*[\beta_2^*T_0 + p(t)] + \omega \frac{c_{33}c_{12}}{c_{11}} + \frac{F_2^*T_0 + P(t)}{\pi(b^2 - a^2)} c_{13} - F_1^*T_0 c_{13} \right) \\
  &\quad + \frac{a^2 \beta_1^*[\beta_2^*T_0 + p(t)]}{r(c_{11} - c_{12})} + \frac{\omega r[\ln(r/a) - 1]}{c_{11}} \\
  u_z &= \frac{z}{F_3} \left[ \left( F_1^*T_0 - \frac{F_2^*T_0 + P(t)}{\pi(b^2 - a^2)} \right)(c_{11} + c_{12}) - 2c_{13}\beta_1^*[\beta_2^*T_0 + p(t)] \\
  &\quad - \frac{2c_{13}c_{12}\sigma}{c_{11}} - \frac{e_{15}(\varphi_b - \varphi_a)\ln(r/a)}{c_{44}\ln(b/a)} \right) \\
  \varphi &= \frac{\ln(r/a)}{\ln(b/a)}(\varphi_b - \varphi_a) + \varphi_a 
\end{align*}
\]

The strains and electric field intensity appearing in eq. (7) can be found by substituting eqs. (13) and (32)-(34) into eq. (2). They are, respectively,

\[
\begin{align*}
  \varepsilon_{rr} &= \frac{1}{F_3} \left( c_{33} \beta_1^*[\beta_2^*T_0 + p(t)] + \omega \frac{c_{33}c_{12}}{c_{11}} + \frac{F_2^*T_0 + P(t)}{\pi(b^2 - a^2)} c_{13} - F_1^*T_0 c_{13} \right) \\
  &\quad - \frac{a^2 \beta_1^*[\beta_2^*T_0 + p(t)]}{r^2(c_{11} - c_{12})} + \frac{\omega \ln(r/a)}{c_{11}} \\
  \varepsilon_{\theta\theta} &= \frac{1}{F_3} \left( c_{33} \beta_1^*[\beta_2^*T_0 + p(t)] + \omega \frac{c_{33}c_{12}}{c_{11}} + \frac{F_2^*T_0 + P(t)}{\pi(b^2 - a^2)} c_{13} - F_1^*T_0 c_{13} \right) \\
  &\quad + \frac{a^2 \beta_1^*[\beta_2^*T_0 + p(t)]}{r^2(c_{11} - c_{12})} + \frac{\omega[\ln(r/a) - 1]}{c_{11}} \\
  \varepsilon_{zz} &= \frac{1}{F_3} \left[ \left( F_1^*T_0 - \frac{F_2^*T_0 + P(t)}{\pi(b^2 - a^2)} \right)(c_{11} + c_{12}) - 2c_{13}\beta_1^*[\beta_2^*T_0 + p(t)] \\
  &\quad - \frac{2c_{13}c_{12}\sigma}{c_{11}} \right) 
\end{align*}
\]
\[ e_{\varepsilon} = - \frac{e_{15}(\varphi_b - \varphi_a)}{r_c_{44} \ln(b/a)} \] (38)

\[ E_r = - \frac{(\varphi_b - \varphi_a)}{r \ln(b/a)} \] (39)

Then substituting the solutions (35)-(39) into eq. (7) yields

\[
\dot{e} = A^0 (e) + \frac{2A^e}{F_3} \left( c_{33}^{*} [\beta_1^{*} T_0 + p(t)] + \omega \frac{c_{13}^{*} c_{12}^{*}}{c_{11}^{*}} + \frac{F_2^{*} T_0 + P(t)}{\pi (b^2 - a^2)} c_{13} - F_1^{*} c_{13} \right) \\
+ \frac{A^e}{F_3} \left[ F_1^{*} T_0 - \frac{F_2^{*} T_0 + P(t)}{\pi (b^2 - a^2)} \right] (c_{11} + c_{12}) \\
- 2c_{13}^{*} \beta_1^{*} [\beta_2^{*} T_0 + p(t)] - \frac{2c_{13} c_{12} \sigma}{c_{11}} \right) - \frac{\varphi_b - \varphi_a}{r \ln(b/a)} \left( A^E + \frac{e_{15}^0}{c_{44}} A^E \right) (40)
\]

Since we do not know the exact expressions of the material functions \( A^0 (e), \ A^E (e), A^y (e), c_{ij}, d, \lambda_j, \kappa_j, \) and \( \chi_3, \) the following approximate forms of them, as proposed by Cowin and van Buskirk [22] for small values of \( e, \) are used here

\[ A^0 (e) = C_a + C_1 e + C_2 e^2, \ A^E (e) = A_{\varepsilon}^{E0} + e A_{\varepsilon}^{E1}, \ A^y (e) = A_{\varepsilon}^{y0} + e A_{\varepsilon}^{y1} \] (41)

and

\[ c_{ij} (e) = c_{ij}^0 + \frac{e}{\xi_0} (c_{ij}^1 - c_{ij}^0), \quad e_{ij} (e) = e_{ij}^0 + \frac{e}{\xi_0} (e_{ij}^1 - e_{ij}^0), \]

\[ \lambda_i (e) = \lambda_i^0 + \frac{e}{\xi_0} (\lambda_i^1 - \lambda_i^0), \quad \kappa_i (e) = \kappa_i^0 + \frac{e}{\xi_0} (\kappa_i^1 - \kappa_i^0), \]

\[ \chi_3 (e) = \chi_3^0 + \frac{e}{\xi_0} (\chi_3^1 - \chi_3^0) \] (42)

where \( C_a, \ C_1, \ C_2, \ A_{\varepsilon}^{E0}, \ A_{\varepsilon}^{E1}, \ A_{\varepsilon}^{y0}, \ A_{\varepsilon}^{y1}, \ c_{ij}^0, \ e_{ij}^0, \ e_{ij}^1, \ \lambda_i^0, \ \lambda_i^1, \ \kappa_i^0, \ \kappa_i^1, \ \chi_3^0, \ \chi_3^1, \ \chi_3^2, \) and \( \chi_3^3 \) are material constants. Using these approximations the remodelling rate equation (40) can be simplified as
\[ \dot{e} = \alpha(e^2 - 2\beta e + \gamma) \]  
\[ (43) \]

by neglecting terms of \( e^3 \) and the higher orders of \( e \), where \( \alpha, \beta \) and \( \gamma \) are constants. The solution to eq. (43) is straightforward and has been discussed by Hegedus and Cowin [23]. For the reader’s benefit, the solution process is briefly described here. Let \( e_1 \) and \( e_2 \) denote solutions to \( e^2 - 2\beta e + \gamma = 0 \), i.e.

\[ e_{1,2} = \beta \pm (\beta^2 - \gamma)^{1/2} \]  
\[ (44) \]

When \( \beta^2 < \gamma \), \( e_1 \) and \( e_2 \) are a pair of complex conjugate, the solution of eq. (43) is

\[ e(t) = \beta + \sqrt{(\gamma - \beta^2)} \tan \left( \frac{\alpha t \sqrt{(\gamma - \beta^2)}}{\beta - e_0} + \arctan \frac{\sqrt{(\gamma - \beta^2)}}{\beta - e_0} \right) \]  
\[ (45) \]

where \( e=e_0 \) is initial condition. When \( \beta^2 = \gamma \), the solution is

\[ e(t) = e_1 - \frac{e_1 - e_0}{1 + \alpha(e_1 - e_0)t} \]  
\[ (46) \]

Finally, when \( \beta^2 > \gamma \), we have

\[ e(t) = \frac{e_1(e_0 - e_2) + e_2(e_1 - e_0) \exp(\alpha(e_1 - e_2)t)}{(e_0 - e_2) + (e_1 - e_0) \exp(\alpha(e_1 - e_2)t)} \]  
\[ (47) \]

Since it has been proved that both solutions (45) and (46) are physically unlikely [22], we will use the solution (47) in our numerical analysis.

### 2.4. Semi-Analytical Solution for Inhomogeneous Cylindrical Bone Layers

The solution obtained in the previous section is suitable for analyzing bone cylinders if they are assumed to be homogeneous [22]. It can be useful if explicit expressions and a simple analysis are required. It is a fact, however, that all bone materials exhibit inhomogeneity. In particular, for a hollow bone cylinder, the volume fraction of bone matrix materials varies from the inner to the outer surface. To solve this problem we present here a semi-analytical model.

Considering eqs. (1), (2) and (3) and assuming a constant longitudinal strain, the following first-order differential equations can be obtained [5]
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\[
\frac{\partial}{\partial r} \begin{bmatrix} u_r \\ \sigma_r \end{bmatrix} = \begin{bmatrix} -\frac{c_{12}}{c_{11}R} & \frac{1}{r} \\ \psi & (c_{12}/c_{11} - 1) \end{bmatrix} \begin{bmatrix} u_r \\ \sigma_r \end{bmatrix} + \begin{bmatrix} -\frac{c_{13}}{c_{11}R} \\ \frac{c_{11} (1 - c_{12}/c_{11})}{r} \end{bmatrix} \varepsilon_r + \begin{bmatrix} \lambda_1 \\ \bar{\lambda}_1 \end{bmatrix} \end{bmatrix}^T \tag{48}
\]

where \( \psi = c_{11} - c_{12}^2/c_{11} \). In the above equation, the effect of electrical potential is absent. This is because it is independent of \( u_r \) and \( \sigma_r \). The contribution of electrical field can be calculated separately as described in the previous section and then included in the remodeling rate equation.

Assuming that a bone layer is sufficiently thin, we can replace \( r \) with its mean value \( R \), and let \( r = a + s \), where \( 0 \leq s \leq h \), \( a \) and \( h \) are the inner radius and the thickness of the thin bone layer, respectively. Thus, eq. (48) is reduced to

\[
\frac{\partial}{\partial S} \begin{bmatrix} u_r \\ \sigma_r \end{bmatrix} = \begin{bmatrix} -\frac{c_{12}}{c_{11}R} & \frac{1}{R^2} \\ \psi & (c_{12}/c_{11} - 1) \end{bmatrix} \begin{bmatrix} u_r \\ \sigma_r \end{bmatrix} + \begin{bmatrix} -\frac{c_{13}}{c_{11}R} \\ \frac{c_{11} (1 - c_{12}/c_{11})}{R} \end{bmatrix} \varepsilon_r + \begin{bmatrix} \lambda_1 \\ \bar{\lambda}_1 \end{bmatrix} \end{bmatrix}^T \tag{49}
\]

The above equation can be written symbolically as

\[
\frac{\partial}{\partial S} \{ \mathbf{F} \} = [\mathbf{G}] \{ \mathbf{F} \} + \{ \mathbf{H}_L \} + \{ \mathbf{H}_T \} \tag{50}
\]

where \([\mathbf{G}]\), \(\{ \mathbf{H}_L \}\) and \(\{ \mathbf{H}_T \}\) are all constant matrices.

Equation (50) can be solved analytically and the solution is \[24\]

\[
\begin{bmatrix} u_r(s) \\ \sigma_r(s) \end{bmatrix} = e^{[\mathbf{G}]S} \begin{bmatrix} u_r(0) \\ \sigma_r(0) \end{bmatrix} + \int_0^S e^{[\mathbf{G}]\tau} \{ \mathbf{H}_L \} d\tau + \int_0^S e^{[\mathbf{G}]\tau} \{ \mathbf{H}_T \} d\tau \tag{51}
\]

where \( u_r(0) \) and \( \sigma_r(0) \) are, respectively, the displacement and stress at the bottom surface of the layer. Rewrite eq. (51) as

\[
\{ \mathbf{F}(s) \} = [\mathbf{D}(s)] \{ \mathbf{F}(0) \} + \{ \mathbf{D}_L \} + \{ \mathbf{D}_T \} \tag{52}
\]

The exponential matrix can be calculated as follows

\[
[\mathbf{D}(s)] = e^{[\mathbf{G}]s} = \alpha_0(s) \mathbf{I} + \alpha_1(s) [\mathbf{G}] \tag{53}
\]

where \( \alpha_0(s) \) and \( \alpha_1(s) \) can be solved from
\[ \alpha_0(s) + \alpha_1(s)\beta_1 = e^{\beta_1s} \]
\[ \alpha_0(s) + \alpha_1(s)\beta_2 = e^{\beta_2s} \]  

(54)

In eq. (54) \( \beta_1 \) and \( \beta_2 \) are two eigenvalues of \([G]\), which are given by

\[
\begin{bmatrix}
\beta_1 \\
\beta_2
\end{bmatrix} = \frac{1}{2R} \pm \frac{1}{2R} \sqrt{5 - 4 \frac{c_{12}}{c_{11}}}
\]  

(55)

Considering now \( s = h \), i.e. the external surface of the bone layer, we obtain

\[
\{F(h)\} = [D(h)]\{F(0)\} + \{D_L\} + \{D_T\}
\]  

(56)

The axial stress applied at the end of the bone can be found as

\[
\sigma_z = c_{13}(1 - \frac{c_{12}}{c_{11}}) \frac{u}{R} + (c_{33} - \frac{c_{13}^2}{c_{11}})\varepsilon_z + \frac{c_{13}}{c_{11}} \sigma_r + (\frac{c_{13}}{c_{11}} \lambda_1 - \lambda_3)T
\]  

(57)

The stress problem (57) can be solved by introducing the boundary conditions described on the top and bottom surfaces into eq. (56) and

\[
\int_S \{c_{13}(1 - \frac{c_{12}}{c_{11}}) \frac{u}{R} + (c_{33} - \frac{c_{13}^2}{c_{11}})\varepsilon_z + \frac{c_{13}}{c_{11}} \sigma_r + (\frac{c_{13}}{c_{11}} \lambda_1 - \lambda_3)T\} dS = -P(t)
\]  

(58)

For a thick-walled bone section or a section with variable volume fraction in the radial direction, we can divide the bone into a number of sub-layers, each of which is sufficiently thin and is assumed to be composed of a homogeneous material. Within a layer we take the mean value of volume fraction of the layer as the layer’s volume fraction. As a consequence, the analysis described above for a thin and homogeneous bone can be applied here for the sub-layer in a straightforward manner. For instance, for the \( j \)-th layer, eq.(56) becomes

\[
\{F^{(j)}(h_j)\} = [D^{(j)}(h_j)]\{F^{(j)}(0)\}_{j} + \{D_L^{(j)}\} + \{D_T^{(j)}\}
\]  

(59)

where \( h_j \) denotes thickness of the \( j \)-th sub-layer.

Considering the continuity of displacements and transverse stresses across the interfaces between these fictitious sub-layers, we have

\[
\{F^{(j)}(h_j)\} = \{F^{(j+1)}(0)\}
\]  

(60)
After establishing eq. (59) for all sub-layers, the following equation can be obtained by using eqs. (59) and (60) recursively.

\[
\{\mathbf{F}(h_N)\} = \left[ \mathbf{D}^{(N)}(h_N) \right] \{\mathbf{F}(h_{N-1})\} + \{\mathbf{D}^{(N)}_L\} + \{\mathbf{D}^{(N)}_T\}
\]

\[
= [\mathbf{D}^{(N)}(h_N)]([\mathbf{D}^{(N-1)}(h_{N-1})]\{\mathbf{F}(h_{N-2})\} + \{\mathbf{D}^{(N-1)}_L\} + \{\mathbf{D}^{(N-1)}_T\}) + \{\mathbf{D}^{(N)}_L\} + \{\mathbf{D}^{(N)}_T\}
\]

\[
= [\mathbf{D}^{(N)}(h_N)]([\mathbf{D}^{(N-1)}(h_{N-1})]\{\mathbf{F}(h_{N-2})\}) + \{\mathbf{D}^{(N-1)}_L\} + \{\mathbf{D}^{(N-1)}_T\} + \{\mathbf{D}^{(N)}_L\} + \{\mathbf{D}^{(N)}_T\}
\]

\[
= \ldots \ldots
\]

\[
= [\mathbf{D}^{(N)}(h_N)]([\mathbf{D}^{(N-1)}(h_{N-1})][\mathbf{D}^{(N-2)}(h_{N-2})][\mathbf{D}^{(N-3)}(h_{N-3})][\mathbf{D}^{(N-j)}(h_{N-j})] \{\mathbf{F}(h_{N-j-1})\}) + \{\mathbf{D}^{(N-j)}_L\} + \{\mathbf{D}^{(N-j)}_T\}) + \ldots
\]

\[
= \ldots \ldots
\]

\[
= \{\Psi\}\{\mathbf{F}(0)\} + \{\Omega\} \tag{61}
\]

where

\[
\{\Psi\} = \prod_{j=N}^{1} [\mathbf{D}^{(j)}(h_j)]
\]

\[
\{\Omega\} = \sum_{i=2}^{N} \left( \prod_{j=N}^{i} [\mathbf{D}^{(j)}(h_j)] \right) \{\{\mathbf{D}^{(N-1)}_L\} + \{\mathbf{D}^{(N-1)}_T\} + \{\mathbf{D}^{(N)}_L\} + \{\mathbf{D}^{(N)}_T\} \}
\]

It can be seen that eq. (61) has the same structure and dimension as those of eq. (56). After introducing the boundary condition imposed on the two transverse surfaces and considering eq. (58), the surface displacements and/or stresses can be obtained. Introducing these solutions back into the equations at sub-layer level, the displacements, stresses and then strains within each sub-layer can be further calculated.

### 2.5. Internal Surface Pressure Induced by a Medullar Pin

Prosthetic devices often employ metallic pins fitted into the medulla of a long bone as a means of attachment. These medullar pins will cause the bone in the vicinity of the pin to change its internal structure and external shape. In this section we introduce the model presented in [17,18,21] for external changes in bone shape. The theory is applied here to the problem of determining the changes in external bone shape that result from a pin force-fitted
into the medulla. The diaphysial region of a long bone is modelled here as a hollow circular cylinder, and external changes in shape are changes in the external and internal radii of the hollow circular cylinder.

The solution of this problem can be obtained by decomposing the problem into two separate sub-problems: the problem of the remodelling of a hollow circular cylinder of adaptive bone material subjected to external loads, and the problem of an isotropic solid elastic cylinder subjected to an external pressure. These two problems are illustrated in Fig. 1.

![Fig. 1. Decomposition of the medullar pin problem into two separate sub-problems.](image)

For an isotropic solid elastic cylinder subjected to an external pressure \( p(t) \), the displacement in the radial direction is given by

\[
u = -\frac{(2\mu + \lambda)p(t)r}{2\mu(3\lambda + 2\mu)}
\]

where \( \lambda \) and \( \mu \) are Lamé’s constants for an isotropic solid elastic cylinder.

In this problem we calculate the pressure of interaction \( p(t) \) which occurs when an isotropic solid cylinder of radius \( a_0 + \frac{\delta}{2} \) is forced into a hollow adaptive bone cylinder of radius \( a_0 \).

Let \( a \) and \( b \) denote the inner and outer radii, respectively, of the hollow bone cylinder at the instant after the solid isotropic cylinder has been forced into the hollow cylinder. Although the radii of the hollow cylinder will actually change during the adaptation process, the deviation of these quantities from \( a \) and \( b \) will be a small quantity negligible in small strain theory.

At an arbitrary time instant after the two cylinders have been forced together the pressure of the interaction is \( p_i(t) \). The radial displacement of the solid cylinder at its surface is
Using the expression (32), the radial displacement of the bone at its inner surface is obtained as

\[
   u_2 = \frac{a}{F^*} \left( c_{33} B_1^{*} [B_2^{*} T_0 - p_1(t) + p(t)] + \frac{c_{31} c_{12}}{c_{11}} F_2^{*} T_0 + P(t) \right) + \frac{\alpha a}{(c_{11} - c_{12})} \left( c_{13} - F_1^{*} T_0 c_{13} \right)
\]

(65)

Since it is assumed that the two surfaces have perfect contact, the two displacements have the following relationship

\[
   a_0 + \frac{\delta}{2} + u_1 = a_0 + u_2
\]

(66)

Hence we find

\[
   \delta = 2(u_2 - u_1)
\]

(67)

Substituting eqs. (64) and (65) into eq. (66), and then solving eq. (66) for \( p_1(t) \) we obtain

\[
   p_1(t) = -\frac{1}{H} \left[ \frac{\delta}{a} - \left( H_1 \frac{b^2}{b^2 - a^2} + H_2 \frac{1}{b^2 - a^2} + H_3 \frac{1}{\ln b^2/a} \right) \right]
\]

(68)

where

\[
   H = \frac{2\mu + \lambda}{\mu(3\lambda + 2\mu)} - 2 \left( \frac{c_{33}}{F_3^{*}} + \frac{1}{c_{11} - c_{12}} \right) \frac{b^2}{b^2 - a^2}
\]

(69)

\[
   H_1 = \frac{2}{F_3^{*}} \left[ c_{13} \left( \frac{c_{12} \lambda_1 - \lambda_3}{c_{11}} \right) T_0 \right] - 2 \left( \frac{c_{33}}{F_3^{*}} + \frac{1}{c_{11} - c_{12}} \right) \left[ B_2^{*} T_0 + p(t) \right]
\]

(70)

\[
   H_2 = \frac{2}{F_3^{*}} \frac{c_{13} P(t)}{\pi}
\]

(71)
\[ H_3 = \frac{c_{33} c_{12} \lambda_4 T_0}{F_3 c_{11}} - \frac{2}{F_3} \left( \frac{c_{13} \lambda_4 - \lambda_3}{2c_{11}} \right) c_{13} T_0 - \frac{\lambda_3 T_0}{c_{11}} \]  

Equation (68) is the solution of the internal surface pressure induced by an inserting medullar pin.

2.6. Numerical Examples

As numerical illustration of the proposed analytical and semi-analytical solutions, we consider a femur with \( a = 25 \text{mm} \) and \( b = 35 \text{mm} \). The material properties assumed for the bone are

\[
c_{11} = 15(1 + e) \text{GPa}, \quad c_{12} = c_{13} = 6.6(1 + e) \text{GPa}, \quad c_{33} = 12(1 + e) \text{GPa},
\]

\[
c_{44} = 4.4(1 + e) \text{GPa}, \quad \lambda_4 = 0.621(1 + e) \times 10^5 \text{NK}^{-1} \text{m}^{-2},
\]

\[
\lambda_3 = 0.551(1 + e) \times 10^5 \text{NK}^{-1} \text{m}^{-2}, \quad \chi_3 = 0.0133(1 + e) \text{CK}^{-1} \text{m}^{-2},
\]

\[
e_{31} = -0.435(1 + e) \text{C/m}^2, \quad e_{33} = 1.75(1 + e) \text{C/m}^2,
\]

\[
e_{15} = 1.14(1 + e) \text{C/m}^2, \quad \kappa_1 = 111.5(1 + e) \kappa_0,
\]

\[
\kappa_3 = 126(1 + e) \kappa_0, \quad \kappa_0 = 8.85 \times 10^{-12} \text{C}^2/\text{Nm}^2 = \text{permittivity of free space}
\]

The remodelling rate coefficients are assumed to be

\[
C_0 = 3.09 \times 10^{-9} \text{sec}^{-1}, \quad C_1 = 2 \times 10^{-7} \text{sec}^{-1}, \quad C_2 = 10^{-6} \text{sec}^{-1},
\]

and

\[
A_{rr}^{x0} = A_{mm}^{x1} = A_{zz}^{x0} = A_{zz}^{x1} = A_{rz}^{x0} = A_{rz}^{x1} = 10^{-5} \text{sec}^{-1},
\]

\[
A_r^{E0} = A_r^{E1} = 10^{-15} \text{V}^{-1} \text{m/sec} = 10^{-15} \text{N}^{-1} \text{C/sec}
\]

The initial inner and outer radii are assumed to be

\[
a_0 = 25 \text{mm}, \quad b_0 = 35 \text{mm}
\]
and $e_0=0$ is assumed. In the calculation, $u_r(t) << a_0$ has been assumed for the sake of simplicity, i.e., $a(t)$ and $b(t)$ may be approximated by $a_0$ and $b_0$.

### 2.6.1. A Hollow, Homogeneous Circular Cylindrical Bone Subjected to Various External Loads

To analyse remodelling behaviour affected by various loading cases we distinguish the following five loading cases:

(i) $P(t) = n \times 2\text{MPa}$ ($n=1, 2, 3$ and $4$), $P(t)=1500\text{N}$, with no other types of load applied

Table 1 lists the results at some typical time instances obtained by both the analytical and semi-analytical solutions. The semi-analytical solution is obtained by dividing the bone into $N (=10, 20, 40)$ sub-layers. It is evident from the table, and also from other extensive comparisons that are not shown here, that the solutions have excellent agreement on the change rate of porosity $e$. Hence, for the numerical results presented below, no references are given regarding which method is used to obtain the solution, unless otherwise stated. It is also evident from the table that the numerical results will gradually converge to the exact value as the layer number $N$ increases.

| Table 1: Comparison of porosity $e$ obtained by analytical and semi-analytical solution ($P=1500\text{N}, p=2\text{Mpa}$). |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|
| Time (sec)                      | 500000          | 1000000         | 1500000         | 2000000         |
| Semi-analytical                 |                 |                 |                 |                 |
| $N=10$                          | $7.283 \times 10^{-5}$ | $1.533 \times 10^{-4}$ | $2.423 \times 10^{-4}$ | $3.406 \times 10^{-4}$ |
| $N=20$                          | $7.294 \times 10^{-5}$ | $1.536 \times 10^{-4}$ | $2.427 \times 10^{-4}$ | $3.412 \times 10^{-4}$ |
| $N=40$                          | $7.297 \times 10^{-5}$ | $1.536 \times 10^{-4}$ | $2.428 \times 10^{-4}$ | $3.413 \times 10^{-4}$ |
| Analytical                      | $7.298 \times 10^{-5}$ | $1.536 \times 10^{-4}$ | $2.428 \times 10^{-4}$ | $3.414 \times 10^{-4}$ |

The extended results for this loading case are shown in Fig. 2 to demonstrate the effect of external pressure on the bone remodelling process. It is evident that there is a critical value $p_{r_0}$, above which the porosity of the femur will be reduced. The critical value $p_{r_0}$ in this problem is approximately $2.95\text{MPa}$. It is also evident that the porosity of the femur increases along with the increase of external pressure $p$.

(ii) $P=1500\text{N}$ and internal pressure is produced by inserting a rigid pin whose radius $a^*$ is greater than $a$.
Fig. 2. Variation of $e$ with time $(\varphi_0 - \varphi_a = T_0 = 0$ and $P = 1500 N)$.

Fig. 3. Variation of $e$ with time induced by a solid pin.

The values of $e$ as a function of $t$ for $a'-a=0.01\text{mm}$, $0.03\text{mm}$, and $0.05\text{mm}$ are shown in Fig. 3. It is interesting to note that for the three cases, the bone structure at the pin-bone interface adapts itself initially to become less porous and then to a state with even less porosity. This is followed by a quick recovery of porosity, indicated by a sharply decreased value of $e$. As time approaches infinity, the bone structure stabilizes itself at a moderately reduced porosity. Although dramatic change of the remodelling constant is observed during
the remodelling process, it is believed that the effect of the change on bone structures is limited by the fact that duration of the change is very short compared to the entire remodelling process. This result coincides with Cowin and van Buskirk’s [22] theoretical observation which showed that a bone structure might tend to a physiologically impossible bone structure in finite time. Both of these have been observed clinically and classified as osteoporosis (excess density with the maximum value of $e$) and osteopetrosis (excess porosity with the minimum value of $e$), respectively. Figure 3 also shows the variation of $e$ against the tightness of fit. It is evident that the tightness of fit has significant effects on the remodelling process, especially during the time period when the abrupt change of porosity occurs. It must be mentioned here is that the remodelling rate for this period can only serve as an indication of the modelling process, since eq. (43) is only valid for predicting a low remodelling rate. Thus, detailed analysis of the equation will not provide any further reliable information. More sophisticated and advanced remodelling models are apparently needed. Nevertheless, the prediction does suggest that the possibility exists of loss of grip on the pin, or of high level tensile stresses in the bone layer surrounding the pin that may induce cracks.

(iii) $T_0(t) = 10^\circ C, 20^\circ C, 30^\circ C, 40^\circ C$

Figure 4 shows the effects of temperature change on bone remodelling rate at $r = b_0$ when $\phi_b - \phi_a = p(t) = P(t) = 0$. In general, low temperature induces more porous bone structures, while a warmer environment may improve the remodelling process with a less porous bone structure. After considering all other factors, it is expected that there is a preferred temperature under which an ideal remodelling rate may be achieved.

(iv) $\phi_b - \phi_a = -60V, -30V, 30V, and 60V$, $r = b_0$, and $T_0 = P = P = 0$

Figure 5 shows the variation of $e$ with time $t$ for various values of electric potential difference with $T_0 = P = P = 0$. It can be observed from Fig. 5 that there are no significant differences between the remodelling rates when the external electric potential difference $\phi_b - \phi_a$ changes from -60V to 60V, though it is observed that the remodelling rate increases as the electric potential difference decreases. However, the result does suggest that the remodelling process may be improved by exposing a bone to an electric field. Further theoretical and experimental studies are needed to investigate the implication of this in medical practice.
Qing-Hua Qin

Fig. 4. Variation of \( e \) with time \( t \) for several temperatures \((\varphi_b - \varphi_a = p = P = 0)\).

Fig. 5. Variation of \( e \) with time \( t \) for several potential differences \((T_0 = p = P = 0)\).

(v) \( \varphi_b - \varphi_a = -60V, -30V, 30V, \text{ and } 60V, \) \( p(t) = 2\text{MPa}, P(t) = 1500\text{N}, \) and \( T_0 = 0 \)

This loading case is considered in order to study the coupling effect of electric and mechanical loads on bone remodelling rate. Figure 6 shows the numerical results of volume fraction change against different values of electric potential difference \( \varphi_b - \varphi_a \) when \( T_0 = 0 \),
$P(t)=1500\text{N}$ and $p(t)=2\text{MPa}$. As observed in Fig. 5, it can again be seen from Fig. 6 that the bone remodelling rate increases along with the decrease of the potential difference $\varphi_b - \varphi_a$. The combination of electrical and mechanical loads results in significantly different values of the remodelling rate when different electrical fields are applied.

![Graph showing variation of strain with time for coupling loads](image1)

**Fig. 6.** Variation of $\epsilon$ with time $t$ for coupling loads ($p = 2\text{MPa}, P = 1500\text{N},$ and $T_0=0$).

![Graph showing variation of potential difference with time for an inhomogeneous bone](image2)

**Figure 7.** Variation of $\epsilon$ with time $t$ for an inhomogeneous bone subjected to coupling loads ($p = 4\text{MPa}, P = 1500\text{N}, T_0=40$ and $\varphi_b - \varphi_a = 30V$).
2.6.2. A Hollow, Inhomogeneous Circular Cylindrical Bone Subjected to External Loads

The geometrical and material parameters of this problem are the same as those used in the above cases except that all material constants in eq. (73) are now modified by a multiplier \[1-(1-\xi)(b-r)/(b-a)\], where \(0 \leq \xi \leq 1\) and represents a percentage reduction of stiffness at the inner surface of the bone. It is worth mentioning that by using the semi-analytical approach, the form of stiffness variation in the radial direction can be arbitrary. Figure 7 shows the results of \(\xi\) at the outside surface of the bone for \(\xi=1, 0.8, 0.6\) and \(0.4\) The external loads are \(p=4\text{MPa}, P=1500\text{N}, T=40\text{, and } \varphi_b - \varphi_a = 30\text{V}\). In general, the remodelling rate declines as the initial stiffness of inner bone surface decreases. When time approaches infinity, it is observed that the stiffness reduction in the radial direction has an insignificant effect on the remodelling rate of the outside bone surface. This observation suggests that ignoring stiffness reduction in the radial direction can yield satisfactory prediction of the remodelling process occurring at the outside layer of the bone.

3. Thermoelectroelastic Surface Bone Remodelling

3.1. Equation for Surface Bone Remodelling

The elastic model for surface remodelling described here is based on the work of Cowin and Buskirk [25]. They presented a hypothesis that the speed of the remodelling surface is linearly proportional to the strain tensor under the assumption of small strain:

\[
U(n, Q, t) = C_y(n, Q)[\varepsilon_y(Q, t) - \varepsilon_y^0(Q, t)]
\]  

(74)

where \(U(n, Q, t)\) denote the speed of the remodelling surface normal to the surface at the surface point \(Q\). It is assumed the velocity of the surface in any direction in the tangent plane is zero because the surface is not moving tangentially with respect to the body. \(n\) is the normal to the bone surface at the point \(Q\), \(\varepsilon_y^0(Q, t)\) is a reference value of strain where no remodelling occurs, and \(C_y(n, Q)\) are surface remodelling rate coefficients which are, in general, dependent upon the point \(Q\) and the normal \(n\) to the surface at \(Q\). Equation (74) gives the normal velocity of the surface at the point \(Q\) as a function of the existing strain state at \(Q\). If the strain state at \(Q\), \(\varepsilon_y(Q, t)\), is equal to the reference strain state, \(\varepsilon_y^0(Q, t)\), then the velocity of the surface is zero and no remodelling occurs. If the right side of eq. (74) is positive, the surface is growing by deposition of material. If, on the other hand, the right side of eq. (74) is negative, the surface is resorbing.

Equation (74) can be extended to include piezoelectric effects by adding some new terms as below [17,18]
\[
U = C_y(n, Q) \left[ e_y(Q, t) - e_y^0(Q, t) \right] + C_z(n, Q) \left[ e_z(Q, t) - e_z^0(Q, t) \right] \\
= C_{r_z} e_{r_z} + C_{r_y} e_{r_y} + C_{r_z} e_{r_z} + C_{E_z} E_z + C_{E_r} E_r - C_0
\]

(75)

where \( C_0 = C_{r_z} e_{r_z} + C_{r_y} e_{r_y} + C_{r_z} e_{r_z} + C_{E_z} E_z + C_{E_r} E_r \), \( C_i \) are surface remodelling coefficients.

### 3.2. Differential Field Equation for Surface Remodelling Rate

We now consider again the hollow circular cylinder of bone used in Section 2. The bone cylinder is subjected to the same external load and boundary conditions as those in Section 2.

Substituting eqs. (35)-(39) into eq. (75) yields

\[
\begin{align*}
U_e &= N_1^e \frac{b^2}{b^2 - a^2} + N_2^e \frac{1}{\ln \left( \frac{b}{a} \right)} + N_3^e \frac{1}{b^2 - a^2} + N_4^e \frac{1}{a \ln \left( \frac{b}{a} \right)} - C_0^e \\
U_p &= N_1^p \frac{b^2}{b^2 - a^2} + N_2^p \frac{a^2}{b^2 - a^2} + N_3^p \frac{1}{b \ln \left( \frac{b}{a} \right)} + N_4^p \frac{1}{b^2 - a^2} + C_0^p
\end{align*}
\]

(76)

where

\[
N_1^e = \frac{1}{F_3} \left[ c_{13} \left( \frac{c_{13}}{c_{11}} \lambda_1 - \lambda_3 \right) T_0 - c_{33} \left( \frac{c_{13}}{c_{11}} \lambda_1 - \lambda_3 \right) T_0 + p(t) \right] \left( C_{r_z}^e + C_{r_y}^e \right) \\
+ \frac{1}{F_3} \left[ 2c_{13} C_{r_z}^e \frac{c_{13}}{c_{11}} \left( \frac{c_{12}}{c_{11}} - 1 \right) T_0 + p(t) \right] - (c_{11} + c_{12}) C_{r_z}^e \left( \frac{c_{13}}{c_{11}} \lambda_1 - \lambda_3 \right) T_0 \right] \\
(\frac{C_{r_z}^e - C_{r_y}^e}{c_{11} - c_{12}}) \left( \frac{c_{13}}{c_{11}} \lambda_1 - \lambda_3 \right) T_0 + p(t)
\]

(77)
\[ N_2^e = \frac{1}{F_3^e} \left[ \frac{c_{12} c_{33}}{2c_{11}} \lambda_1 T_0 - \left( \frac{c_{13}}{c_{11}} \lambda_1 - \frac{\lambda_3}{2} \right) c_{13} T_0 \right] \left( C_{rr}^e + C_{\theta\theta}^e \right) \]

\[ + \frac{C_{zz}^e}{F_3^e} \left[ (c_{11} + c_{12}) \left( \frac{c_{13}}{c_{11}} \lambda_1 - \frac{\lambda_3}{2} \right) T_0 - \frac{c_{12} c_{13}}{c_{11}} \lambda_1 T_0 \right] - \frac{C_{\theta\theta}^e \lambda_1 T_0}{2c_{11}} \]  

\[ (78) \]

\[ N_3^e = \frac{1}{F_3^e} \left[ c_{13} \left( C_{rr}^e + C_{\theta\theta}^e \right) - (c_{11} + c_{12}) C_{zz}^e \right] \frac{P(t)}{\pi} \]  

\[ (79) \]

\[ N_4^e = -\left( \frac{e_{15}}{e_{44}} C_{rr}^e + C_r \right) (\varphi_b - \varphi_a) \]  

\[ (80) \]

\[ N_1^p = \frac{1}{F_3^p} \left[ c_{13} \left( \frac{c_{13}}{c_{11}} \lambda_1 - \lambda_3 \right) T_0 - c_{33} \left[ \frac{c_{12}}{c_{11}} \left( \frac{c_{12}}{c_{11}} - 1 \right) T_0 + p(t) \right] \right] \left( C_{rr}^p + C_{\theta\theta}^p \right) \]

\[ + \frac{1}{F_3^p} \left[ 2c_{13} C_{\theta\theta} \left[ \frac{\lambda_1}{2} \left( \frac{c_{12}}{c_{11}} - 1 \right) T_0 + p(t) \right] - (c_{11} + c_{12}) C_{zz}^p \left[ \frac{c_{12}}{c_{11}} \lambda_1 - \lambda_3 \right] T_0 \right] \]  

\[ (81) \]

\[ N_2^p = \frac{1}{F_3^p} \left[ \frac{c_{12} c_{33}}{2c_{11}} \lambda_1 T_0 - \left( \frac{c_{13}}{c_{11}} \lambda_1 - \frac{\lambda_3}{2} \right) c_{13} T_0 \right] \left( C_{rr}^p + C_{\theta\theta}^p \right) \]

\[ + \frac{C_{zz}^p}{F_3^p} \left[ (c_{11} + c_{12}) \left( \frac{c_{13}}{c_{11}} \lambda_1 - \frac{\lambda_3}{2} \right) T_0 - \frac{c_{12} c_{13}}{c_{11}} \lambda_1 T_0 \right] - \frac{C_{\theta\theta}^p \lambda_1 T_0}{2c_{11}} \]  

\[ (83) \]

\[ N_3^p = \frac{1}{F_3^p} \left[ c_{13} \left( C_{rr}^p + C_{\theta\theta}^p \right) - (c_{11} + c_{12}) C_{zz}^p \right] \frac{P(t)}{\pi} \]  

\[ (84) \]

\[ N_4^p = -\left( \frac{e_{15}}{e_{44}} C_{rr}^p + C_r \right) (\varphi_b - \varphi_a) \]  

\[ (85) \]

\[ N_3' = \frac{(C_{\theta\theta}^p + C_{rr}^p) \lambda_1 T_0}{2c_{11}} \]  

\[ (86) \]
and the subscripts \( p \) and \( e \) refer to periosteal and endosteal, respectively. Since \( U_e \) and \( U_p \) are the velocities normal to the inner and outer surfaces of the cylinders, respectively, they are calculated as

\[
U_e = -\frac{da}{dt}, \quad U_p = \frac{db}{dt} \tag{87}
\]

where the minus sign appearing in the expression for \( U_e \) denotes that the outward normal of the endosteal surface is in the negative coordinate direction. Substituting eq. (87) into eq. (76) yields

\[
\begin{aligned}
\frac{da}{dt} &= N_i^e \frac{b^2}{b^2-a^2} + N_2^e \frac{1}{\ln \left( \frac{b}{a} \right)} + N_3^e \frac{1}{a \ln \left( \frac{b}{a} \right)} - C_0' \\
\frac{db}{dt} &= N_i^p \frac{b^2}{b^2-a^2} + N_2^p \frac{a^2}{b^2-a^2} + N_3^p \frac{1}{\ln \left( \frac{b}{a} \right)} + N_4^p \frac{1}{b \ln \left( \frac{b}{a} \right)} - C_0^{p'} \\
&\quad + N_4^e \frac{1}{b \ln \left( \frac{b}{a} \right)}
\end{aligned} \tag{88}
\]

where \( C_0' = C_0^p - N_3^e \).

### 3.3. Approximation for Small Changes in Radii

It is apparent that eq. (88) is non-linear and cannot, in general, be solved analytically. However, the equations can be approximately linearized when they are applied to solve problems with small changes in radii. In the bone surface remodelling process, we can assume that the radii of the inner and outer surface of the bone change very little compared to their original values. This means that the changes in \( a(t) \) and \( b(t) \) are small. This is a reasonable assumption from the viewpoint of physics of the problem. To introduce the approximation two non-dimensional parameters

\[
\varepsilon = \frac{a}{a_0} - 1, \quad \eta = \frac{b}{b_0} - 1 \tag{89}
\]

are adopted in the following calculations. As a result, \( a(t) \) and \( b(t) \) can be written as
\[ a(t) = (1 + \varepsilon(t))a_0, \quad b(t) = (1 + \eta(t))b_0, \quad (\varepsilon, \eta << 1) \quad (90) \]

Since both \( \varepsilon \) and \( \eta \) are far smaller than one, their squares can be ignored from the equations. Consequently, we can have the following approximations

\[
\frac{b^2}{b^2 - a^2} \approx L_0 + 2L_0^2 \frac{a_0^2}{b_0^2} (\varepsilon - \eta) \quad (91)
\]

\[
\frac{a^2}{b^2 - a^2} \approx L_0' + 2L_0'^2 \frac{b_0^2}{a_0^2} (\varepsilon - \eta) \quad (92)
\]

\[
\frac{1}{b^2 - a^2} \approx L_2 + 2L_2 (a_0^2 \varepsilon - b_0^2 \eta) \quad (93)
\]

\[
\frac{1}{\ln\left(\frac{b}{a}\right)} \approx L_1 + L_1^2 (\varepsilon - \eta) \quad (94)
\]

\[
\frac{1}{a \ln\left(\frac{b}{a}\right)} \approx \frac{1}{a_0} L_1 (1 - \varepsilon) + \frac{1}{a_0} L_1^2 (\varepsilon - \eta) \quad (95)
\]

\[
\frac{1}{b \ln\left(\frac{b}{a}\right)} \approx \frac{1}{b_0} L_1 (1 - \eta) + \frac{1}{b_0} L_1^2 (\varepsilon - \eta) \quad (96)
\]

where

\[
L_0 = \frac{b_0^2}{b_0^2 - a_0^2}, \quad (97)
\]

\[
L_0' = \frac{a_0^2}{b_0^2 - a_0^2}, \quad (98)
\]

\[
L_1 = \frac{1}{\ln\left(\frac{b_0}{a_0}\right)}. \quad (99)
\]
\[ L_2 = \frac{1}{b_0^2 - a_0^2}, \]  
(100)

Thus, eq. (88) can be approximately represented in terms of \( \varepsilon \) and \( \eta \), as follows

\[
\begin{align*}
\frac{d\varepsilon}{dt} &= B_1 \varepsilon + B_2 \eta + B_3 \\
\frac{d\eta}{dt} &= B_4 \varepsilon + B_5 \eta + B_6
\end{align*}
\]  
(101)

where

\[
B_1 = -\frac{1}{a_0} \left( 2L_0^2 a_0^2 N_1^\varepsilon + L_1^2 N_2^\varepsilon - \frac{L_4 N_4^\varepsilon}{a_0} + \frac{L_2^2 N_3^\varepsilon}{a_0} + 2L_2^2 a_0^2 N_3^\varepsilon \right) 
\]  
(102)

\[
B_2 = \frac{1}{a_0} \left( 2L_0^2 a_0^2 N_1^\varepsilon + L_1^2 N_2^\varepsilon + \frac{L_4^2 N_4^\varepsilon}{a_0} + 2L_2^2 a_0^2 N_3^\varepsilon \right) 
\]  
(103)

\[
B_3 = -\frac{1}{a_0} \left( L_0 N_1^\varepsilon + L_1 N_2^\varepsilon + L_2 N_3^\varepsilon + \frac{L_4 N_4^\varepsilon}{a_0} - C_0^\varepsilon \right) 
\]  
(104)

\[
B_1' = \frac{1}{b_0} \left( 2L_0^2 a_0^2 N_1^\rho + L_1^2 N_2^\rho + \frac{L_4^2 N_4^\rho}{b_0} + 2L_2^2 a_0^2 N_3^\rho + 2L_0^2 \frac{b_0^2}{a_0^2} N_1^\rho \right) 
\]  
(105)

\[
B_2' = -\frac{1}{b_0} \left( 2L_0^2 a_0^2 N_1^\rho + L_1^2 N_2^\rho - \frac{L_4 N_4^\rho}{b_0} + \frac{L_2^2 a_0^2 N_3^\rho}{b_0} + 2L_2 b_0^2 N_3^\rho + 2L_0 \frac{b_0^2}{a_0^2} N_1^\rho \right) 
\]  
(106)

\[
B_3' = \frac{1}{b_0} \left( L_0 N_1^\rho + L_1 N_2^\rho + L_2 N_3^\rho + \frac{L_4 N_4^\rho}{b_0} + L_0' N_1^\rho - C_0^\rho \right) 
\]  
(107)

3.4. Analytical Solution of Surface Remodelling

An analytical solution of eq. (101) can be obtained if smeared homogeneous property is assumed for bone material. In such a case, the inhomogeneous linear differential equations system (101) can be converted into the following homogeneous one.
by introducing two new variables such that

\[
\begin{align*}
\varepsilon' &= \varepsilon - \varepsilon_\infty \\
\eta' &= \eta - \eta_\infty
\end{align*}
\]  

(109)

\[
\begin{align*}
\varepsilon_\infty &= \frac{1}{\det \mathbf{M}} (B_3' B_2' - B_2' B_3') \\
\eta_\infty &= \frac{1}{\det \mathbf{M}} (B_3 B_1' - B_1 B_3')
\end{align*}
\]  

(110)

\[
\mathbf{M} = \begin{pmatrix} B_1' & B_2' \\ B_1 & B_2 \end{pmatrix}
\]  

(111)

\[
\det \mathbf{M} = B_1 B_2' - B_1' B_2
\]  

(112)

The solution of eq. (108), subject to the initial conditions that \( \varepsilon(0) = 0 \) and \( \eta(0) = 0 \), can be expressed in four possible forms that fulfil the physics of the problem, i.e., when \( t \rightarrow \infty \), \( \varepsilon \) and \( \eta \) must be limited quantities, \( a < b \) and the solution must be stable. The form of the solution depends on the roots of the following quadratic equation

\[
s^2 - \text{tr} \mathbf{M} s + \det \mathbf{M} = 0
\]  

(113)

where

\[
\text{tr} \mathbf{M} = B_1 + B_2'
\]  

(114)

All the theoretically possible solutions are shown as follows:

**Case A:** when \( \left( B_1 - B_2' \right)^2 + 4 B_2 B_1' > 0 \), \( B_1 + B_2' < 0 \) and \( B_1 B_2' - B_2 B_1' > 0 \), eq. (113) has two different roots, \( s_1 \) and \( s_2 \), both of which are real and distinct. Then the solutions of the equations are
\[
\begin{align*}
\varepsilon' &= \frac{1}{s_1 - s_2} \left[ (s_2 \varepsilon_\infty - B_3) e^{-s_2 t} + (B_3 - s_1 \varepsilon_\infty) e^{-s_1 t} \right] \\
\eta' &= \frac{1}{s_1 - s_2} \left[ (s_2 \eta_\infty - B_3') e^{-s_2 t} + (B_3' - s_1 \eta_\infty) e^{-s_1 t} \right]
\end{align*}
\] (115)

which can also be written as

\[
\begin{align*}
\varepsilon(t) &= \varepsilon_\infty + \frac{1}{s_1 - s_2} \left[ (s_2 \varepsilon_\infty - B_3) e^{-s_2 t} + (B_3 - s_1 \varepsilon_\infty) e^{-s_1 t} \right] \\
\eta(t) &= \eta_\infty + \frac{1}{s_1 - s_2} \left[ (s_2 \eta_\infty - B_3') e^{-s_2 t} + (B_3' - s_1 \eta_\infty) e^{-s_1 t} \right]
\end{align*}
\] (116)

The formulae for the variation of the radii, i.e., \( a(t) \) and \( b(t) \), with time can be obtained by substituting eq. (116) into eq. (89). Thus

\[
\begin{align*}
a(t) &= a_0 + a_0 \varepsilon_\infty + \frac{a_0}{s_1 - s_2} \left[ (s_2 \varepsilon_\infty - B_3) e^{-s_2 t} + (B_3 - s_1 \varepsilon_\infty) e^{-s_1 t} \right] \\
b(t) &= b_0 + b_0 \eta_\infty + \frac{b_0}{s_1 - s_2} \left[ (s_2 \eta_\infty - B_3') e^{-s_2 t} + (B_3' - s_1 \eta_\infty) e^{-s_1 t} \right]
\end{align*}
\] (117)

The final radii of the cylinder are then

\[
\begin{align*}
a_\infty &= \lim_{t \to \infty} a(t) = a_0 (1 + \varepsilon_\infty) \\
b_\infty &= \lim_{t \to \infty} b(t) = b_0 (1 + \eta_\infty)
\end{align*}
\] (118)

**Case B:** when \( (B_1 - B_2')^2 + 4B_2B_1' = 0 \), \( B_1 \neq B_2' \) and \( B_1 + B_2' < 0 \), eq. (113) has two equal roots, \( B_2' + B_1 \). The solutions of the equations are

\[
\begin{align*}
\varepsilon' &= \left\{ \varepsilon_\infty + \left[ \frac{B_1 - B_2'}{2} \varepsilon_\infty + B_2 \eta_\infty \right] t \right\} e^{\frac{B_1 + B_2'}{2}} e^{-s_2 t} \\
\eta' &= \left\{ \eta_\infty - \left[ \frac{(B_2' - B_1)^2}{4B_2} \varepsilon_\infty - \frac{B_2' - B_1}{2} \eta_\infty \right] t \right\} e^{\frac{B_1 + B_2'}{2}} e^{-s_1 t}
\end{align*}
\] (119)
which can also be written as

\[
\varepsilon(t) = \varepsilon_\infty - \left\{ \varepsilon_\infty + \left[ \frac{B_1 - B_2'}{2} \varepsilon_\infty + B_2' \eta_\infty \right] t \right\}_e^{\frac{b_i + b_i}{2}} \\
\eta(t) = \eta_\infty - \left\{ \eta_\infty - \left[ \frac{(B_2' - B_1)^2}{4B_2} \varepsilon_\infty - \frac{B_2' - B_1}{2} \eta_\infty \right] t \right\}_e^{\frac{b_i + b_i}{2}}
\]

(120)

The formulae for the variation of \( a(t) \) and \( b(t) \) with time can be obtained by substituting eq. (120) into eq. (89) as

\[
a(t) = a_0 + a_0 \varepsilon_\infty - a_0 \left\{ \varepsilon_\infty + \left[ \frac{B_1 - B_2'}{2} \varepsilon_\infty + B_2' \eta_\infty \right] t \right\}_e^{\frac{b_i + b_i}{2}} \\
b(t) = b_0 + b_0 \eta_\infty - b_0 \left\{ \eta_\infty - \left[ \frac{(B_2' - B_1)^2}{4B_2} \varepsilon_\infty - \frac{B_2' - B_1}{2} \eta_\infty \right] t \right\}_e^{\frac{b_i + b_i}{2}}
\]

(121)

The final radii of the cylinder are then

\[
\begin{align*}
a_\infty &= \lim_{t \to \infty} a(t) = a_0(1 + \varepsilon_\infty) \\
b_\infty &= \lim_{t \to \infty} b(t) = b_0(1 + \eta_\infty)
\end{align*}
\]

(122)

**Case C:** when \( B_1 = B_2' < 0 \) and \( B_2 = 0 \), the solutions of the equations are

\[
\begin{align*}
\varepsilon' &= -\varepsilon_\infty e^{B_1 t} \\
\eta' &= \left( B_1' \varepsilon_\infty t + \eta_\infty \right) e^{B_1 t}
\end{align*}
\]

(123)

which can also be written as

\[
\begin{align*}
\varepsilon(t) &= \varepsilon_\infty - \varepsilon_\infty e^{B_1 t} \\
\eta(t) &= \eta_\infty - \left( B_1' \varepsilon_\infty t + \eta_\infty \right) e^{B_1 t}
\end{align*}
\]

(124)
The formulae for the variation of \( a(t) \) and \( b(t) \) with time can be obtained by substituting eq. (124) into eq. (89), as follows

\[
\begin{align*}
    a(t) &= a_0 + a_0 \varepsilon_\infty \left(1 - e^{B_1t}\right) \\
    b(t) &= b_0 + b_0 \eta_\infty - b_0 \left(B_1 \varepsilon_\infty t + \eta_\infty\right) e^{B_1t}
\end{align*}
\]  

(125)

The final radii of the cylinder are then

\[
\begin{align*}
    a_\infty &= \lim_{t \to \infty} a(t) = a_0 (1 + \varepsilon_\infty) \\
    b_\infty &= \lim_{t \to \infty} b(t) = b_0 (1 + \eta_\infty)
\end{align*}
\]  

(126)

**Case D:** when \( B_1 < 0 \) and \( B_1' = 0 \), the solutions of the equations are

\[
\begin{align*}
    \varepsilon' &= -B_1 \eta_\infty t + \varepsilon_\infty e^{B_1t} \\
    \eta' &= -\eta_\infty e^{B_1t}
\end{align*}
\]  

(127)

which can also be written as

\[
\begin{align*}
    \varepsilon(t) &= \varepsilon_\infty - \left(B_1 \eta_\infty t + \varepsilon_\infty\right) e^{B_1t} \\
    \eta(t) &= \eta_\infty - \eta_\infty e^{B_1t}
\end{align*}
\]  

(128)

The formulae for the variation of the radii with time can be obtained by substituting eq. (128) into eq. (89). Thus

\[
\begin{align*}
    a(t) &= a_0 + a_0 \varepsilon_\infty - a_0 \left(B_1 \eta_\infty t + \varepsilon_\infty\right) e^{B_1t} \\
    b(t) &= b_0 + b_0 \eta_\infty \left(1 - e^{B_1t}\right)
\end{align*}
\]  

(129)

The final radii of the cylinder are then

\[
\begin{align*}
    a_\infty &= \lim_{t \to \infty} a(t) = a_0 (1 + \varepsilon_\infty) \\
    b_\infty &= \lim_{t \to \infty} b(t) = b_0 (1 + \eta_\infty)
\end{align*}
\]  

(130)

All the above solutions are theoretically valid. However, the first is the most likely solution to the problem, as it is physically possible when \( t \to \infty \) [17, 18]. Therefore it can be used to calculate the bone surface remodelling.
3.5. Application of Semi-Analytical Solution to Surface Remodelling of Inhomogeneous Bone

The semi-analytical solution presented in Section 2.4 can be used to calculate strains and stresses at any point on the bone surface. These results form the basis for surface bone remodelling analysis. This section presents applications of solution (61) to the analysis of surface remodelling behaviour in inhomogeneous bone.

It is noted that surface bone remodelling is a time-dependent process. The change in the radii (ε or η) can therefore be calculated by using the rectangular algorithm of integral (see Fig. 8). The procedure is described here. Firstly, let $T_0$ be the starting time and $T$ be the length of time to be considered, and divide the time domain $T$ into $m$ equal interval $\Delta T = T / m$. At the time $t$, calculate the strain and electric field using eqs. (35)-(39). The results are then substituted into eq. (75) to determine the normal velocity of the surface bone remodelling. Assuming that $\Delta T$ is sufficiently small, we can replace $U$ with its mean value $\overline{U}$ at each time interval $[t, t + \Delta T]$. The change in the radii (ε or η) at time $t$ can thus be determined using the results of surface velocity. Accordingly, the strain and electric field are updated by considering the change in the radii. The updated strain and electric field are in turn used to calculate the normal surface velocity at the next time interval. This process is repeated up to the last time interval $[T_0 + (m - 1)\Delta T, T_0 + T]$. Figure 8 shows the rectangular-algorithm of integral when we replace $U$ with its initial value $U_i$ (rather than its mean value $\overline{U}$) at the time interval $[t, t + \Delta T]$.

![Fig. 8. Illustration of the rectangular algorithm](image)
3.6. Surface Remodelling Equation Modified By an Inserting Medullar Pin

Substituting eq. (68) into eq. (88) yields

\[
\begin{aligned}
\frac{da}{dt} &= N_1' \frac{b^2}{b^2 - a^2} + N_2' \frac{1}{\ln\left(\sqrt{\frac{b}{a}}\right)} + N_3' \frac{1}{b^2 - a^2} + N_4' \frac{1}{a \ln\left(\sqrt{\frac{b}{a}}\right)} \\
-M_1 p_1(t) \frac{b^2}{a^2 - b^2} - C_0' \\
\frac{db}{dt} &= N_1^p \frac{b^2}{b^2 - a^2} + N_1^p' \frac{a^2}{b^2 - a^2} + N_4^p \frac{1}{\ln\left(\sqrt{\frac{b}{a}}\right)} + N_3^p \frac{1}{b^2 - a^2} \\
&- \left(M_2 \frac{b^2}{a^2 - b^2} + M_3 \frac{a^2}{a^2 - b^2}\right) p_1(t) + N_4^p \frac{1}{b \ln\left(\sqrt{\frac{b}{a}}\right)} - C_0'' \\
\end{aligned}
\]

where

\[
M_1 = \frac{(C_{rr}^e + C_{\theta\theta}^e) c_{33} - 2 c_{13} C_{zz}^e}{F_3^*} - \frac{C_{rr}^e - C_{\theta\theta}^e}{c_{11} - c_{12}}
\]

\[
M_2 = \frac{(C_{rr}^p + C_{\theta\theta}^p) c_{33} - 2 c_{13} C_{zz}^p}{F_3^*}
\]

\[
M_3 = -\frac{C_{rr}^p - C_{\theta\theta}^p}{c_{11} - c_{12}}
\]

It can be seen that eq. (131) is similar to eq. (88). It can also be simplified as

\[
\begin{aligned}
\frac{d \epsilon}{dt} &= Y_1 \epsilon + Y_2 \eta + Y_3 \\
\frac{d \eta}{dt} &= Y_1' \epsilon + Y_2' \eta + Y_3'
\end{aligned}
\]

where

\[
Y_i = B_i - M_3 \left[H_1 \left(\frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_3 \right) + H_1' \left(2 L_0^2 H_1 \frac{a_0^2}{b_0^2} + 2 L_2 H_2 a_0^2 + H_3 L_3 + \frac{\delta}{a_0}\right)\right]
\]

\[136\]
\[ Y_2 = B_2 + M_1 \left[ H_3 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \right. \]
\[ \left. + H_4 \left( 2L_0^2 H_1 \frac{a_0^2}{b_0} + 2L_2 H_2 b_0^2 + H_3 L_1 \right) \right] \]

(137)

\[ Y_3 = B_3 + M_1 H_4 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \]

(138)

\[ Y_1' = B_1' - M_2 \left[ H_4 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \right. \]
\[ \left. + H_4 \left( 2L_0^2 H_1 \frac{a_0^2}{b_0} + 2L_2 H_2 a_0^2 + H_3 L_1 + \frac{\delta}{a_0} \right) \right] \]
\[ + M_3 \left[ H_4 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) - H_4 \left( 2L_0^2 H_1 \frac{a_0^2}{b_0} + 2L_2 H_2 a_0^2 + H_3 L_1 + \frac{\delta}{a_0} \right) \right] \]

(139)

\[ Y_2' = B_2' + M_2 \left[ H_5 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \right. \]
\[ \left. + H_4 \left( 2L_0^2 H_1 \frac{a_0^2}{b_0} + 2L_2 H_2 b_0^2 + H_3 L_1 \right) \right] \]
\[ - M_3 \left[ H_4 \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) - H_4 \left( 2L_0^2 H_1 \frac{a_0^2}{b_0} + 2L_2 H_2 a_0^2 + H_3 L_1 \right) \right] \]

(140)

\[ Y_3' = B_3' + (M_2 H_4 + M_2 H_6) \left( \frac{\delta}{a_0} - H_1 L_0 - H_2 L_2 - H_3 L_1 \right) \]

(141)

\[ H_4 = \frac{1}{2 \left( \frac{c_{33}}{F^*} + \frac{1}{c_{11} - c_{12}} \right) - \frac{2 \mu + \lambda}{\mu(3\lambda + 2\mu)} \left( 1 - \frac{a_0^2}{b_0^2} \right)} \]

(142)

\[ H_5 = \frac{(4\mu + 2\lambda)a_0^2}{\mu(3\lambda + 2\mu)b_0^2} \]

(143)

\[ H_6 = \frac{1}{\frac{2 \mu + \lambda}{\mu(3\lambda + 2\mu)} + \left[ \frac{2 \left( \frac{c_{33}}{F^*} + \frac{1}{c_{11} - c_{12}} \right) - \frac{2 \mu + \lambda}{\mu(3\lambda + 2\mu)} \left( 1 - \frac{a_0^2}{b_0^2} \right) }{a_0^2} \right]} \]

(144)
Multi-Field Bone Remodeling Under Axial and Transverse Loads

\[ H_\gamma = 2 \left( \frac{c_{33}}{F_3^*} + \frac{1}{c_{11} - c_{12}} + \frac{2\mu + \lambda}{\mu(3\lambda + 2\mu)} \right) \frac{b_0^2}{a_0^2} \left( \frac{2\mu + \lambda}{\mu(3\lambda + 2\mu)} + \frac{2\mu + \lambda}{\mu(3\lambda + 2\mu)} \right) \right] \] (145)

Equation (135) is similar to eq. (108) and can thus be solved by following the solution procedure described in Section 3.4.

### 3.7. Numerical Examples

Consider again the femur used in Section 2.6. The geometrical and material coefficients of the femur are the same as those used in Section 2.6 except that the volume fraction change \( e \) is now taken to be zero here. In addition, the surface remodelling rate coefficients are assumed to be

\[
C_{rr}^e = -9.6 \text{ m/day} \quad C_{\theta\theta}^e = -7.2 \text{ m/day} \quad C_{zz}^e = -5.4 \text{ m/day} , \\
C_{zr}^e = -8.4 \text{ m/day} \quad C_{rr}^p = -12.6 \text{ m/day} \quad C_{\theta\theta}^p = -10.8 \text{ m/day} , \\
C_{zz}^p = -9.6 \text{ m/day} \quad C_{zr}^p = -12 \text{ m/day} , \quad C_r = 10^{-9} \text{ m/day}
\]

and

\[
C_0^e = 0.0008373 \text{ m/day} \quad C_0^p = 0.00015843 \text{ m/day}
\]

and \( \varepsilon_0 = 0, \eta_0 = 0 \) are assumed.

In the following, numerical results are provided to show the effect of temperature and external electric load on the surface bone remodelling process. While the results for the effects of mechanical loading, inserted pin, and material inhomogeneity on the surface remodelling behaviour are omitted here, they can be found in [17,18].

(i) Effect of temperature change on surface bone remodelling. The temperature is assumed to change between \( 29.5^\circ C \sim 30.5^\circ C \), i.e. \( T_0(t) = 29.5^\circ C, 29.8^\circ C, 30^\circ C, 30.2^\circ C, 30.5^\circ C \), while the other external loadings are specified as: \( \varphi_h - \varphi_a = 30V, p(t) = 1 \text{ MPa}, P(t) = 1500 \text{ N} \). Figures 9 and 10 show the effects of temperature change on bone surface remodelling. In general, the radii of the bone decrease when the temperature increases and they increase when the temperature decreases. It can also be seen from Figs. 9 and 10...
that $\varepsilon$ and $\eta$ are almost the same. Since $a_0 < b_0$, the change of the outer surface radius is normally greater than that of the inner one. The area of the bone cross-section decreases as the temperature increases. This also suggests that a lower temperature is likely to induce thicker bone structures, while a warmer environment may improve the remodelling process with a less thick bone structure. This result seems to coincide with actual fact. Thicker and stronger bones maybe make a person living in Russia look stronger than one who lives in Vietnam. It should be mentioned here that how this change may affect the bone remodelling process is still an open question. As an initial investigation, the purpose of this study is to show how a bone may respond to thermal loads and to provide information for the possible use of imposed external temperature fields in medical treatment and in controlling the healing process of injured bones.

Fig. 9. Variation of $\varepsilon$ with time $t$ for several temperatures

(ii) Effect of external electrical potential on surface bone remodelling. In this case, the coupled loading is assumed as: $\phi_b - \phi_a = -60V, -30V, 30V, \text{ and } 60V$, $p(t) = 1\text{MPa}$, $P(t)=1500\text{N}$, and $T_0=0$. Figures 11 and 12 show the variation of $\varepsilon$ and $\eta$ with time $t$ for various values of electric potential difference. It can be seen that the effect of the electric potential is just the opposite to that of temperature. A decrease in the intensity of electric field results in a decrease of the inner and outer surface radii of the bone by almost the same magnitude. Theoretically, the results suggest that the remodelling process may be improved by exposing a bone to an electric field. Clearly, further theoretical and experimental studies are needed to investigate the implications of this in medical practice.
Fig. 10. Variation of $\eta$ with time $t$ for several temperatures.

Fig. 11. Variation of $\varepsilon$ with time $t$ for several potential differences.
4. Extension to Thermomagneto Electroelastic Solid

In the case of thermomagneto electroelastic solid, the constitutive relation (1) and the remodelling rate equation (75) must be augmented by some additional terms related to magnetic field as follows:

\[
\begin{align*}
\sigma_{rr} &= c_{11}s_{rr} + c_{12}s_{00} + c_{13}s_{zz} - e_{31}E_z - \alpha_{31}H_z - \lambda_I T, \\
\sigma_{00} &= c_{12}s_{rr} + c_{11}s_{00} + c_{13}s_{zz} - e_{31}E_z - \alpha_{31}H_z - \lambda_I T, \\
\sigma_{zz} &= c_{13}s_{rr} + c_{13}s_{00} + c_{33}s_{zz} - e_{33}E_z - \alpha_{33}H_z - \lambda_3 T, \\
\sigma_{r\varphi} &= c_{44}s_{z\varphi} - e_{15}E_r - \alpha_{15}H_r, \\
D_r &= e_{15}s_{z\varphi} + \kappa_1E_r + d_1H_r, \\
D_z &= e_{31}(s_{rr} + s_{00}) + e_{33}s_{zz} + \kappa_3E_z + d_3H_z - \gamma_3 T, \\
B_r &= \alpha_{15}s_{z\varphi} + d_1E_r + \mu_1H_r, \\
B_z &= \alpha_{31}(s_{rr} + s_{00}) + \alpha_{33}s_{zz} + d_3E_z + \mu_3H_z - m_3 T, \\
h_r &= k_1q_r, \\
h_z &= k_2q_z
\end{align*}
\]

(146)

\[
U = C_p(n, Q)[s_{y}(Q) - s_{y0}(Q)] + C_1[E_i(Q) - E_i^0(Q)] + G_h[H_i(Q) - H_i^0(Q)]
= C_v s_{rr} + C_{00} s_{00} + C_{zz} s_{zz} + C_{r\varphi} s_{r\varphi} + C_{r\varphi} E_r + C_\varphi E_\varphi + C_{r\varphi} E_r + C_\varphi E_\varphi + G_i H_r + G_i H_r - C_\varphi
\]

(147)

where \( B_i \) and \( H_i \) are components of magnetic induction and magnetic field respectively; \( \alpha_y \) are piezomagnetic constants; \( d_i \) are magnetoelectric constants; \( \mu_i \) are magnetic
permeabilities; \( m_3 \) is pyromagnetic constant; The associated magnetic field is related to magnetic potential \( \psi \) by

\[
H_r = -\psi_r, \quad H_z = -\psi_z
\]  

(148)

If we consider a hollow circular cylinder of bone, subjected to an external temperature change \( T_0 \), a quasi-static axial load \( P \), an external pressure \( p \), an electric potential load \( \varphi_a \) (and/or \( \varphi_b \)) and an magnetic potential load \( \psi_a \) (and/or \( \psi_b \)), the governing equations (11)-(12) and boundary condition (8) are, in this case, replaced by

\[
c_{44} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z + c_{15} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \varphi + \alpha_{15} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \psi = 0
\]  

(149)

\[
e_{15} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z - \kappa_e \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \varphi - \mu_e \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \psi = 0
\]  

(150)

\[
T = 0, \quad \sigma_{rr} = \sigma_{zr} = \sigma_{zz} = 0, \quad \varphi = \varphi_a, \psi = \psi_a \quad \text{at } r = a
\]

\[
T = T_0, \quad \sigma_{rr} = -p, \quad \sigma_{zr} = \sigma_{zz} = 0, \quad \varphi = \varphi_b, \psi = \psi_b \quad \text{at } r = b
\]  

(151)

The solution to the governing equations (10), (149)-(150) satisfying boundary conditions (9) and (151) is given by

\[
u_r = \frac{r}{F_3} \left( c_{13} \beta_1^* \left[ \beta_2^* T_0 + p(t) \right] + \alpha \frac{c_{13} c_{12}}{c_{11}} \frac{F_2^* T_0 + P(t)}{\pi (b^2 - a^2)} c_{13} - F_1^* T_0 c_{13} \right)
\]

\[
+ \frac{a^2 \beta_1^* \left[ \beta_2^* T_0 + p(t) \right]}{r(c_{11} - c_{12})} + \frac{\alpha r[\ln(r/a) - 1]}{c_{11}}
\]  

(152)

\[
u_z = \frac{z}{F_3} \left[ \left( F_1^* T_0 - \frac{F_2^* T_0 + P(t)}{\pi (b^2 - a^2)} \right) (c_{11} + c_{12}) - 2 c_{13} \beta_1^* \left[ \beta_2^* T_0 + p(t) \right] \right]
\]

\[
- \frac{2 c_{13} c_{12} \alpha}{c_{11}} - e_{15} (\varphi_b - \varphi_a) \ln(r/a) - \frac{\alpha_{15} (\psi_b - \psi_a) \ln(r/a)}{c_{44} \ln(b/a)}
\]  

(153)

\[
\varphi = \frac{\ln(r/a)}{\ln(b/a)} (\varphi_b - \varphi_a) + \varphi_a \quad \psi = \frac{\ln(r/a)}{\ln(b/a)} (\psi_b - \psi_a) + \psi_a
\]  

(154)
The strains, electric and magnetic field intensity can be found by introducing eqs. (152)-(155) into (146). They are, respectively,

\[
 s_{rr} = \frac{1}{F_3^*} \left( c_{33} \beta_1^* \left( \beta_2^* T_0 + p(t) \right) + \sigma \frac{c_{33}c_{12}}{c_{11}} + \frac{F_2^* T_0 + P(t)}{\pi (b^2 - a^2)} c_{13} - F_1^* T_0 c_{13} \right) - \frac{a^2 \beta_1^* \left( \beta_2^* T_0 + p(t) \right)}{r^2 (c_{11} - c_{12})} + \frac{\sigma \ln(r/a)}{c_{11}} \tag{156}
\]

\[
 s_{\theta \theta} = \frac{1}{F_3^*} \left( c_{33} \beta_1^* \left( \beta_2^* T_0 + p(t) \right) + \sigma \frac{c_{33}c_{12}}{c_{11}} + \frac{F_2^* T_0 + P(t)}{\pi (b^2 - a^2)} c_{13} - F_1^* T_0 c_{13} \right) + \frac{a^2 \beta_1^* \left( \beta_2^* T_0 + p(t) \right)}{r^2 (c_{11} - c_{12})} + \frac{\sigma [\ln(r/a) - 1]}{c_{11}} \tag{157}
\]

\[
 s_{zz} = \frac{1}{F_3^*} \left[ \frac{F_1^* T_0}{\pi (b^2 - a^2)} \right] (c_{11} + c_{12}) - 2 c_{13} \beta_1^* \left( \beta_2^* T_0 + p(t) \right) - \frac{2 c_{13} c_{12} \sigma}{c_{11}} \tag{158}
\]

\[
 s_{rz} = - \frac{e_{15} (\phi_b - \phi_a)}{r c_{44} \ln(b/a)} - \frac{\alpha_{15} (\psi_b - \psi_a)}{r c_{44} \ln(b/a)} \tag{159}
\]

\[
 E_r = - \frac{(\phi_b - \phi_a)}{r \ln(b/a)} \quad H_r = - \frac{(\psi_b - \psi_a)}{r \ln(b/a)} \tag{160}
\]

Substituting (156)-(160) into (147) yields eq (76), in which all \( N_j \) are the same as those in Section 3 except that

\[
 N_4^e = - \left( \frac{e_{15}}{c_{44}} C_{\sigma}^e + C_r \right) (\phi_b - \phi_a) - \left( \frac{\alpha_{15}}{c_{44}} C_{\psi}^e + G_r \right) (\psi_b - \psi_a) \tag{161}
\]

\[
 N_4^p = - \left( \frac{e_{15}}{c_{44}} C_{\sigma}^p + C_r \right) (\phi_b - \phi_a) - \left( \frac{\alpha_{15}}{c_{44}} C_{\psi}^p + G_r \right) (\psi_b - \psi_a) \tag{162}
\]
As numerical illustration of the analytical solutions in this Section, we consider again the femur used in Section 3.7. The geometrical and material constants of the femur are the same as those used in Section 3.7. In addition, The surface magnetic remodeling rate coefficients are assumed to be

\[ G_r = 10^{-10} \, \text{m/day}. \]

To study the effect of magnetic loading on bone remodelling, we consider following loading case:

\[ \psi_b - \psi_a = -2A, -1A, 1A, \text{ and } 2A, \quad p(t) = 1\text{MPa}, \quad P(t) = 1500\text{N}, \quad \psi_b - \psi_a = 30\text{V} \text{ and } T_0 = 0. \]

![Graph](image)

Fig.13. Variation of \( \varepsilon \) and \( \eta \) with time \( t \) for several magnetic potential differences.

Figure 13 shows the variation of \( \varepsilon \) and \( \eta \) with time \( t \) for various values of magnetic potential difference. The change of the outer and inner surfaces of the bone due to the
magnetic influence is similar to that in Fig 12 for the effect of electric field on bone remodelling.

5. Conclusion

The problem of thermopiezoelectric bone internal and surface remodeling has been addressed within the framework of adaptive elastic theory in this chapter. The control equations for bone materials have been derived through use of the adaptive elastic theory. By assuming a homogeneous bone material, the analytical solution can provide explicit solutions for analyzing circular cylindrical bones. Then, in the numerical analysis various load conditions were considered, including transverse pressure, electric load, magnetic load and thermal load.

The numerical results show that electric field, magnetic field, and thermal load can also affect bone remodeling process in addition to mechanical loads. This feature may be considered and utilized in controlling healing process of injured bones. It should be mentioned here that all the results are obtained on the basis of the numerical model which may have difference from those of individual bone materials. Therefore further experimental validation is required before it can apply to the clinical practice.

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References


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