Advanced Fundamental-solution-based Computational Methods for Thermal Analysis of Heterogeneous Materials

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Abstract

Heterogeneous materials such as composites and functionally graded materials (FGMs) have been widely used in space engineering, and thus their efficient numerical analysis becomes important and popular in practical design. This chapter presents an overview on the development of fundamental-solution-based numerical methods including the meshless method of fundamental solutions (MFS) and the hybrid finite-element method (FEM) and their applications in thermal analysis of FGMs and fiber-reinforced composites. Basic concepts of the two methods are briefly described, such as fundamental solution, meshless collocations, modified variational functional, intraelement fields, frame fields, and special polygonal elements. Related formulations are derived for thermal analysis of such heterogeneous materials, and some typical numerical solutions are discussed to show the application of the MFS and the fundamental-solution-based hybrid FEM. Finally, a brief summary of these two approaches are presented, and future trends in the analysis of heterogeneous materials with the fundamental-solution-based computational methods are identified.

Keywords: Functionally graded materials, fiber-reinforced composites, fundamental solutions, method of fundamental solutions, hybrid finite-element method, special elements

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11.1 Introduction

Heterogeneous materials such as functionally graded materials (FGMs) [1, 2] and fiber-reinforced composites [3–6] have been widely used in the space engineering to achieve some special purposes. For instance, FGMs, which may be characterized by a spatial gradation variation in composition and structure, were first considered in 1984 by material scientists in the Sendai area in Japan during a space plane project for the special purpose of preparing materials with high thermal barrier capability [7]. Fiber-reinforced composite materials represent a radical approach to designing new aerospace structural materials with desirable physical and chemical properties including high stiffness and strength and lightweight when compared with traditional materials such as metals and ceramics [5, 8]. It is noted that numerical methods have been extensively employed for solving thermal and mechanical problems in these heterogeneous materials [9–26], because the related analytical solutions are difficult to obtain.

Among available numerical methods, the methods related to fundamental solutions of the problem under consideration have been considerably developed and have shown to be a highly efficient computational tool for the solution of complex boundary value problems, including the method of fundamental solutions (MFS) [27–31], the boundary element method (BEM) [12, 32–34], and the fundamental-solution-based hybrid finite-element method (HFS-FEM) [35–38]. Different from the BEM, both the MFS and the HFS-FEM use the fundamental solutions of the problem of interest as trial functions for approximating the essential field (usually, the displacement or temperature fields), and the fictitious sources are placed outside the elemental domain to avoid the singularity induced when the field point approaches the source point. The MFS can be viewed as a global approximating method, whereas the HFS-FEM is a local approximating method. In this chapter, particular attention is paid to the meshless MFS and the HFS-FEM and an overview of them is made.

In contrast to the conventional finite element method (FEM) [39], the MFS is a truly meshless method, in which the sought fields are globally approximated by the linear combination of fundamental solutions of the problem of interest in the full domain by placing all singularities in the approximation of the sought fields outside the domain so that they satisfy the governing partial differential equations of the problem at any point within the domain. As a result, only boundary collocation discretization is required and the satisfaction of the boundary conditions at boundary collocations can form the final solving system of equations, whose unknown variables have usually no physical meaning and the system equation has...
non-symmetric coefficient matrix. The first attempt of the MFS was made by Kupradze for solving some second order elliptical equations in 1964 [40], and later the question of the uniqueness of the solution in Kupradze’s method is addressed by Christiansen [41]. Since then, the concept of the MFS has become increasingly popular, attracting a growing number of researchers into this field [28, 42–57]. More recently, the MFS has been applied to heat conduction and elastic problems of heterogeneous materials. For instance, Berger et al. applied the MFS to heat conduction problems and elastic problems in two-layered composite materials using special fundamental solution that satisfies the interface continuity conditions [58, 59]. Dong et al. developed the MFS formulation for analyzing the inverse heat conduction problems in anisotropic medium [60]. Marine and Lesnic established the MFS formulation for nonlinear FGMs [61]. Poullikkas et al. solved the inhomogeneous problem by the MFS [62].

As noted in literature above, the procedure of the MFS is very simple and the accuracy is high, especially for those with domains having simple geometry. However, the MFS has some inherent drawbacks [44, 63]: (1) the coefficient matrix of the resulting linear equation system is usually full and unsymmetric, (2) the coefficient matrix becomes ill-conditioned as the number of singularities increases or when the boundary shape is complicated, (3) it is inefficient for multi-domain problems, and (4) the fundamental solutions of some physical problems may not be available. The first three shortcomings can be overcome by introducing the weak-formed HFS-FEM stated below, whereas the last shortcoming can be partially handled by means of the analog equation method (AEM) [64, 65] and the radial basis functions (RBFs) [66–70].

As one of the hybrid FEMs [71–77], the HFS-FEM was developed to inherit the advantages of the MFS and BEM and overcome some disadvantages of the MFS and BEM in multi-domain problems. The method includes the use of auxiliary inter-element frame fields to link the internal fields between elements, and the local internal fields are approximated by a linear combination of fundamental solutions of the problem inside the local element domain. In the formulation of HFS-FEM based on local weak form, the inter-element continuity is enforced by using a modified variational principle together with the independent frame fields defined on each element boundary, and the element formulation, during which the internal parameters are eliminated at the element level, leads to the standard force–displacement relationship with a symmetric positive definite stiffness matrix. So far, the HFS-FEM has been well established for heat/bioheat conduction [35, 78–80], discontinuous loads [81], isotropic elasticity with holes [82, 83], anisotropic elasticity [36], elastic cracks [84], and
composites [85, 86]. More recently, the special polygonal fiber/matrix elements and interphase/fiber elements have been developed for heat transfer and elasticity in fiber-reinforced composites [14, 15, 87–90]. Besides, specially purposed graded elements (SPGEs) have been formulated for simulating thermal and elastic behaviors in FGMs [91–93].

Generally, the approximated fields are chosen so as to \textit{a priori} satisfy the governing partial differential equations of the problem in both the MFS and the HFS-FEM so that the computation involved is only along the domain boundary in the MFS or along the element boundary in the HFS-FEM. This is one of advantages of the MFS and HFS-FEM over the conventional finite-element (FE) formulation. Besides, different from the conventional FE, the HFS-FEM can generate arbitrary polygonal or even curve-sided elements for computation because the formulation calls for integration along the element boundary only. From this point, it may be considered as a special, symmetric, substructure-oriented boundary solution approach and thus possessed the advantages of the BEM. In contrast to the conventional boundary element formulation, however, the HFS-FEM model avoids the introduction of singular integral equations, because all source points are placed outside the element, as done in the MFS. Moreover, the HFS-FEM model offers the attractive possibility of developing special hole, inclusion, or crack elements to achieve the balance of accuracy and computing efficiency with fewer elements than the conventional FEM, simply by using appropriate known special fundamental solutions as the trial functions of internal fields.

Following this introduction, the present review consists of six sections. Basic formulations of the MFS and the HFS-FEM are described in Sections 2 and 3, respectively. Sections 4 and 5 focus, respectively, on the application of the MFS and the HFS-FEM to FGMs and fiber-reinforced composite materials. Finally, a brief summary of the developments of the MFS and the HFS-FEM is provided, and areas that need further research are identified.

11.2 Basic Formulation of MFS

11.2.1 Standard MFS

To illustrate procedures of the MFS, the following homogeneous elliptic partial differential equation is considered:

\[
Lu(x) = 0, \quad x \in \Omega
\]

where \(L\) is a linear elliptic partial differential operator, \(\Omega\) is a bounded domain in two-dimensional space, \(u\) is the sought field, and \(x = (x_1, x_2)\) is
an arbitrary point in the domain Ω under consideration, as displayed in Figure 11.1.

The basic ideas for the formulation of the MFS were first proposed by Kupradze [40]. In the MFS, the approximate solution of the sought field $u$ is expressed as a linear combination of fundamental solutions

$$u(x) = \sum_{n=1}^{N} c_n G^*(x, y_n), \quad x \in \Omega, y_n \notin \Omega \quad (11.2)$$

where the source points $\{y_n\}_{n=1}^{N}$ are normally placed outside the domain $\Omega$, as shown in Figure 11.1, to avoid singular interference of fundamental solutions and $\{c_n\}_{n=1}^{N}$ are unknown coefficients. $G^*(x, y_n)$ is the fundamental solution of the differential equation (11.1) and is defined as the solution of the equation of the form in the infinite space

$$L G^*(x, y_n) + \delta(x, y_n) = 0 \quad (11.3)$$

where $\delta(x, y_n)$ is the Dirac delta function.

It is obvious that the solution $G^*(x, y_n)$ is defined everywhere except that $x = y_n$, where it is singular. Thus $y_n$ is called the singularity (or source point) of the fundamental solution. The locations of the singularities are either preassigned or are to be determined. If the latter strategy is employed, the resulting problem is nonlinear and can be solved using available software with optimization algorithm.

Moreover, it is found that the approximate solution defined in Eq. (11.2) satisfies exactly the governing equation (11.1) of the problem. Making use of the fundamental solution defined in Eq. (11.3), the unknown coefficients
in the approximate solution (11.2) can be determined by satisfying the boundary conditions only. For example, if the boundary condition of the problem has the form

$$Bu(x) = f(x), \quad x \in \partial \Omega$$  \hspace{1cm} (11.4)

the discrete linear system of equations after putting $M(M \geq N)$ collocations along the boundary can be written as

$$Bu(x_m) = \sum_{n=1}^{N} c_n BG^*(x_m, y_n) = f(x_m), \quad x_m \in \partial \Omega, y_n \notin \Omega, m = 1 \rightarrow M$$  \hspace{1cm} (11.5)

or in matrix form

$$
\begin{bmatrix}
BG^*(x_1, y_1) & BG^*(x_1, y_2) & \cdots & BG^*(x_1, y_N) \\
BG^*(x_2, y_1) & BG^*(x_2, y_2) & \cdots & BG^*(x_2, y_N) \\
\vdots & \vdots & \ddots & \vdots \\
BG^*(x_M, y_1) & BG^*(x_M, y_2) & \cdots & BG^*(x_M, y_N)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_N
\end{bmatrix}_{M \times N}
= 
\begin{bmatrix}
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_M)
\end{bmatrix}_{M \times 1}
\hspace{1cm} (11.6)
$$

where $B$ stands for a boundary operator, $f(x)$ is a specified function along the boundary, and $M(M \geq N)$ is the number of boundary collocations. If $M \neq N$, Eq. (11.5) or (11.6) can be solved by least squares fit of the boundary data, else it can be directly solved by the standard Gaussian elimination.

From the procedure of the MFS described above, it is found that the MFS formulation is very simple and has been demonstrated with higher efficiency than most of the numerical methods such as the FEM and the BEM [28]. However, the MFS has some inherent drawbacks as mentioned in the introduction.

**11.2.2 Modified MFS**

As discussed above, when the fundamental solution of a given problem is unavailable, the AEM and the RBF can be employed to make the MFS workable. In the following, cases that the partial differential equation (11.1) has not explicit expression of fundamental solution are considered.
To solve this problem with MFS, the AEM is firstly introduced to convert the original governing partial differential equation (11.1) into the simpler Poisson equation whose fundamental solution is available. To this end, applying the Laplace operator to the sought field \( u(x) \) yields
\[
\nabla^2 u(x) = b(x), \quad x \in \Omega
\]
(11.7)
where \( b(x) \) is unknown fictitious source distribution.

Eq. (11.7) indicates that the solution of Eq. (11.1) can be established by solving this linear equation under the same boundary condition (11.4). The solution procedure can be implemented through the RBF interpolation, as detailed below.

Since Eq. (11.7) is a linear partial differential equation, its solution can be written as a sum of the homogeneous solution and the particular solution as
\[
u(x) = u_h(x) + u_p(x)
\]
(11.8)
where \( u_h(x) \) is the homogeneous solution, which satisfies the following homogeneous partial differential equation
\[
\nabla^2 u_h(x) = 0
\]
(11.9)
and \( u_p(x) \) is the particular solution, which satisfies the following inhomogeneous partial differential equation:
\[
\nabla^2 u_p(x) = b(x)
\]
(11.10)

### 11.2.2.1 RBF Interpolation for the Particular Solution

Making use of the properties of RBFs, the unknown fictitious source distribution \( b(x) \) is firstly approximated by
\[
b(x) = \sum_{i=1}^{I} a_i \varphi(x, x_i), \quad x, x_i \in \Omega
\]
(11.11)
where \( I \) is the total number of interpolation points inside the domain and on the boundary (see Figure 11.2), \( a_i \) are the coefficients to be determined, and \( \varphi(x, x_i) = \varphi(r) \) are a set of approximating RBFs, which are typically in terms of the radial distance \( r \)
\[
r = |x - x_i|
\]
(11.12)
Furthermore, the particular solution can be expressed as

\[ u_p(x) = \sum_{i=1}^{l} a_i \psi(x, x_i) \]  

(11.13)

where \( \psi(x, x) = \psi(r) \) is the particular solution kernel (PSK) which has the relation to the RBF \( \varphi(r) \)

\[ \nabla^2 \psi(r) = \varphi(r) \]  

(11.14)

In the RBF interpolation, the effectiveness and accuracy of the interpolation depend on the type of the RBFs. Besides the *ad hoc* RBF, which is used almost exclusively and uncritically in the engineering literature, the thin plate splines (TPS) and multiquadrics (MQ) are also commonly used in meshless formulation. Here, the three types of RBFs and the corresponding PSKs are listed in Table 11.1 for reference.

### 11.2.2.2 MFS for the Homogeneous Solution

To obtain a weak homogeneous solution of the Laplace equation (11.9), the standard MFS can be employed. Provided that \( N \) singularities \( \{y_n\}_{n=1}^N \) are located outside the domain \( \Omega \) as illustrated in Figure 11.2, we have

\[ u_h(x) = \sum_{n=1}^{N} c_n G^*(x, y_n), \quad x \in \Omega, y_n \not\in \Omega \]  

(11.15)

where \( \{c_n\}_{n=1}^N \) are unknown coefficients and \( G^*(x, y_n) \) is the fundamental solution of the Laplace partial differential equation, i.e. for the case of two dimension, \( G^*(x, y_n) \) can be written as

**Figure 11.2** Schematic of the modified MFS procedure.
with

\[ r = |x - y_n| \]  

(11.17)

### 11.2.2.3 Complete Solution

Based on the discussion above, the complete solution of Eq. (11.7) can be expressed in terms of the homogeneous solution and the particular solution as

\[ u(x) = \sum_{n=1}^{N} c_n G^*(x, y_n) + \sum_{i=1}^{N} a_i \psi(r) \]  

(11.18)

To determine all unknowns in Eq. (11.18), \( M(M \geq N) \) collocations are placed along the physical boundary, and Eq. (11.18) is required to satisfy the boundary condition (11.4) at \( M \) collocations and satisfy the original governing equation (11.1) at \( I \) interpolation points, i.e. for the system consisting of Eqs (11.1) and (11.4), the final discrete linear equations can be written as

\[
\begin{align*}
\sum_{n=1}^{N} c_n L G^*(x_j, y_n) + \sum_{i=1}^{I} a_i L \psi(x_j, x_i) &= 0 & x_i \in \Omega, x_j \in \Omega, y_n \notin \Omega \\
\sum_{n=1}^{N} c_n B G^*(x_m, y_n) + \sum_{i=1}^{I} a_i B \psi(x_m, x_i) &= f(x_m) & x_i \in \Omega, x_m \in \partial \Omega, y_n \notin \Omega
\end{align*}
\]  

(11.19)

### Table 11.1 Three types of RBFs and the corresponding PSKs.

<table>
<thead>
<tr>
<th>Radial basis function ( \phi(r) )</th>
<th>Particular solution kernel ( \psi(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 + r 2^{n-1} ) ((n \geq 1))</td>
<td>( \frac{r^2}{4} + \frac{r^{2n+1}}{(2n+1)^2} )</td>
</tr>
<tr>
<td>( r^2 \ln r )</td>
<td>( \frac{1}{16} r^4 \ln r - \frac{1}{32} r^4 )</td>
</tr>
<tr>
<td>( \sqrt{r^2 + c^2} )</td>
<td>( \frac{c^2 \ln(c^2 + r^2)}{3} + \frac{(r^2 + 4c^2) \phi}{9} )</td>
</tr>
</tbody>
</table>

\[ G^*(x, y_n) = -\frac{1}{2\pi} \ln r \]  

(11.16)
11.3 Basic Formulation of HFS-FEM

To solve heat transfer problem of heterogeneous materials, for example, the fiber-reinforced composites, the MFS usually needs to establish separate equation for each material subdomain and then generate the final linear equation system by coupling them with additional equations satisfying the interfacial conditions of adjacent material subdomains. The complication of the MFS increases as multiple material subdomains are involved. To bypass this problem, the fundamental-solution-based hybrid FEM has been developed for thermal analysis in heterogeneous materials [14, 15, 78, 92, 93]. More interestingly, various special $n$-sided polygonal elements can be developed to simplify the analysis and reduce the quantity of elements.

11.3.1 Problem Statement

For a 2D thermal conduction problem of heterogeneous materials without internal heat sources, as shown in Figure 11.3, the related governing equations include the following:

1. Energy conservation (heat balance) equation

\[
\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0 \quad (11.20)
\]

2. Heat transfer constitutive law (Fourier law)

\[
q_i = -k_{ij} \frac{\partial u}{\partial x_j} \quad (i, j = 1, 2) \quad (11.21)
\]

where $q_i (i = 1, 2)$ are the heat flux components along $i$-direction, $u$ is the temperature field, and $k_{ij}$ represents anisotropic thermal conductivity coefficient of material constituents, which should obey Onsager's reciprocity relation $k_{ij} = k_{ji}$ and positive determinant

\[
\Lambda = k_{11}k_{22} - k_{12}^2 > 0 \quad (11.22)
\]

Generally, for heterogeneous or inhomogeneous materials, the thermal conductivity may be expressed as function in terms of spatial variable $x = (x_1, x_2)$, while for homogeneous materials, the thermal conductivity is a constant.
Substituting Eq. (11.21) into Eq. (11.20) yields
\[
\frac{\partial}{\partial x_1} \left( k_{11} \frac{\partial u}{\partial x_1} + k_{12} \frac{\partial u}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( k_{21} \frac{\partial u}{\partial x_1} + k_{22} \frac{\partial u}{\partial x_2} \right) = 0 \quad (11.23)
\]
Specially, when
\[
k_{11} = k_{22} \neq 0, \quad k_{12} = k_{21} = 0 \quad (11.24)
\]
the problem becomes isotropic and Eq. (11.23) reduces to
\[
\frac{\partial}{\partial x_1} \left( k_{11} \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( k_{11} \frac{\partial u}{\partial x_2} \right) = 0 \quad (11.25)
\]
Besides, the boundary conditions for heat transfer usually include
- Dirichlet condition: \( u = \bar{u} \) on \( \Gamma_u \)
- Neumann condition: \( q = \bar{q} \) on \( \Gamma_q \) \quad (11.26)

where the normal heat flux \( q \) is given by
\[
q = q_i n_i = -k_{ij} \frac{\partial u}{\partial x_j} n_i \quad (11.27)
\]
and \( n_i \) is the direction cosine of the unit outward normal vector \( \mathbf{n} \) to the boundary \( \Gamma \).

### 11.3.2 Implementation of the HFS-FEM

To study the thermal behavior of heterogeneous materials mentioned above, it is necessary to solve the corresponding boundary value problem described by the heat transfer in the bounded domain with specific boundary conditions, as illustrated in Figure 11.4a, in which the computing domain \( \Omega \) is discretized with finite hybrid FEs. Then, the hybrid FE formulation in weak form can be established in a particular \( n \)-sided element domain \( \Omega_e \) with the boundary \( \Gamma_e \) shown in Figure 11.4b, for heat transfer analysis by means of a modified variational functional [15, 35, 80, 94, 95]

\[
\Pi = -\frac{1}{2} \int_{\Omega_e} k_e \mu H \mu d\Omega_e - \int_{\Gamma} q(\tilde{u} - u)d\Gamma_e \tag{11.28}
\]

where \( u \) is the intraelement temperature field satisfying, \textit{a priori}, the governing equation of heat transfer and \( \tilde{u} \) is the element frame field which is introduced to enforce conformity on the field variable between adjacent elements. \( q \) and \( \tilde{q} \) are the normal heat flux and specified value on the boundary \( \Gamma_q = \Gamma_q \cap \Gamma_e \), respectively. \( n_i (i = 1, 2) \) are components of outward unit normal to the boundary \( \Gamma_e \).

Applying the divergence theorem

\[
\int_{\Omega} h_i d\Omega = \int_{\Gamma} h n_i d\Gamma \tag{11.29}
\]

to the functional (11.28) yields

\[
\Pi_e = -\frac{1}{2} \int_{\Gamma} q_n \mu d\Gamma - \int_{\Gamma} \tilde{q} \tilde{u} d\Gamma + \int_{\Gamma} q_n \tilde{u} d\Gamma \tag{11.30}
\]

In Eq. (11.30), the intraelement temperature \( u \) satisfying automatically the governing equation of heat transfer is approximated by the linear combination of the fundamental solutions of the problem, as done in the standard MFS,

\[
u = \sum_{j=1}^{n} G^*(\mathbf{x}, \mathbf{x_j}) c_j = \mathbf{N} c \tag{11.31}
\]
where $G^*(x, x'_f)$ is the related fundamental solution, $x(x_1, x_2)$ and $x_f(x'_{i,j}, x'_{2,j})$ are the field point and source point, respectively. It has to be noted that the so-called source points are placed outside the element domain to avoid computing singular integrals. Such a technique has proved to be efficient [35–38, 78–83].

Furthermore, differentiating the temperature field (11.31) to the coordinate variable and then substituting it into Eq. (11.27) yield the following normal derivative of temperature

$$q = -k \frac{\partial T}{\partial n} = Qc$$

(11.32)
with

\[ Q = -\{n_1 n_2\} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial N}{\partial x_1} \\ \frac{\partial N}{\partial x_2} \end{bmatrix} \] (11.33)

Meanwhile, the independent element frame field \( \tilde{u} \) is defined over the element boundary and can be expressed in terms of the nodal temperature vector \( d \)

\[ \tilde{u} = \tilde{N}d \] (11.34)

where \( \tilde{N} \) represents the conventional interpolating shape functions for line element in the FEM or BEM.

Then, substituting Eqs (11.31) and (11.34) into the functional (11.30) produces

\[ \Pi_e = -\frac{1}{2} c^T H c - d^T g + c^T G d \] (11.35)

in which

\[ H = \int_{\Gamma_e} Q^T N d\Gamma \quad G = \int_{\Gamma_e} Q^T \tilde{N} d\Gamma \quad g = \int_{\Gamma_e} \tilde{N}^T \tilde{q} d\Gamma \]

Minimization of the functional \( \Pi_e \) with respect to \( c \) and \( d \), respectively, yields

\[ \frac{\partial \Pi_e}{\partial c^T} = -H c + G d = 0 \]

\[ \frac{\partial \Pi_e}{\partial d^T} = G^T c - g = 0 \] (11.36)

from which the optional relationship between \( c \) and \( d \)

\[ c = H^{-1} G d \] (11.37)

and the element stiffness equation

\[ K d = g \] (11.38)
are established. In Eq. (11.38), the symmetric stiffness matrix can be expressed by

$$K = G^T H^{-1} G$$

(11.39)

Assembling the element stiffness equation (11.38) element by element yields the final global stiffness equation, which can be used to determine the nodal temperature vector \(d\), and furthermore, the unknown interpolation coefficients \(c\) can be determined using Eq. (11.37). However, we notice that the rigid-body motion mode should be recovered in the assumed internal temperature field (11.31), because the fundamental solution used does not include any rigid-body motion [73, 96, 97].

### 11.3.4 Recovery of Rigid-body Motion

For the scale temperature field under consideration, the missing rigid-body mode can be recovered by adding a constant \(c_0\) to the assumed internal field (11.31) in each element, that is,

$$u = Nc + c_0$$

(11.40)

Then, the least square matching of \(u\) and \(\tilde{u}\) at element nodes

$$\sum_{i=1}^{n} (Nc + c_0 - \tilde{u})^2 |_{node\ i} = \min$$

(11.41)

can be used to determine the constant \(c_0\)

$$c_0 = \frac{1}{n} \sum_{i=1}^{n} (\tilde{u} - Nc) |_{node\ i}$$

(11.42)

where \(n\) is the total number of element nodes.

### 11.4 Applications in Functionally Graded Materials

#### 11.4.1 Basic Equations in Functionally Graded Materials

FGMs are a class of relatively new and promising heterogeneous materials that have optimized material properties by combining different material components following a predetermined law. The gradual change in terms
of spatial variables gives place to a gradient of properties and performances [2]. Generally, the thermal conductivities of the FGMs can be written as

\[ k_{ij} = k_{ij}(x) \quad (11.43) \]

Usually, the following three variations are assumed in the practical engineering:

- **Exponential form**
  \[ k_{ij}(x) = k_{ij}^0 e^{2(\beta_1 x_1 + \beta_2 x_2)} \quad (11.44) \]

- **Quadratic form**
  \[ k_{ij}(x) = k_{ij}^0(a_0 + a_1 x_1 + a_2 x_2)^2 \quad (11.45) \]

- **Trigonometric form**
  \[ k_{ij}(x) = k_{ij}^0[a_{1} \cos(\beta_1 x_1 + \beta_2 x_2) + a_{2} \sin(\beta_1 x_1 + \beta_2 x_2)]^2 \quad (11.46) \]

where \( k_{ij}^0, a_0, a_1, a_2, \beta_1, \beta_2 \) are material constants.

### 11.4.2 MFS for Functionally Graded Materials

Theoretically, the thermal conductivities can change in arbitrary form in terms of the spatial variable \( x \), so it is difficult to derive a general form of fundamental solution for such material. To address this issue, we can apply the modified MFS to solve this problem. Here, the transient heat conduction in isotropic FGMs is studied [54]. For this case, the governing equation can be written as

\[ \nabla \cdot \nabla [k(x)\nabla u(x, t)] = k(x)\nabla^2 u(x, t) + \nabla k(x) \cdot \nabla u(x, t) = \rho c \frac{\partial u(x, t)}{\partial t} \quad (11.47) \]

with the following boundary condition

\[ Bu(x, t) = f(x, t), \quad x \in \partial \Omega \quad (11.48) \]
and the initial condition

\[ u(x, t = 0) = g(x) \]  

(11.49)

In order to obtain the temperature field and its flux at any time, the time domain is divided into elements and a simple backward time stepping scheme is used, i.e.

\[
\frac{\partial u}{\partial t} \bigg|_{t+\Delta t} = \frac{u^{t+\Delta t} - u^t}{\Delta t}
\]  

(11.50)

where \( \Delta t \) is the time step, and \( u^{t+\Delta t} = u(x, t + \Delta t) \), \( u^t = u(x, t) \).

Substituting Eq. (11.50) into Eqs (11.47) and (11.48), we have

\[
k(x) \nabla^2 u^{t+\Delta t}(x) + \nabla k(x) \cdot \nabla u^{t+\Delta t}(x) - \frac{\rho c}{\Delta t} u^{t+\Delta t}(x) = -\frac{\rho c}{\Delta t} u^t(x)
\]  

(11.51)

and

\[
Bu^{t+\Delta t}(x) = f^{t+\Delta t}(x), \quad x \in \partial \Omega
\]  

(11.52)

Thus, the operator \( L \) in Eq. (11.1) can be written as

\[
Lu^{t+\Delta t}(x) = k(x) \nabla^2 u^{t+\Delta t}(x) + \nabla k(x) \cdot \nabla u^{t+\Delta t}(x) - \frac{\rho c}{\Delta t} u^{t+\Delta t}(x)
\]  

(11.53)

Applying the modified MFS to the system consisting of Eqs (11.51) and (11.52), we finally have

\[
\begin{align*}
\sum_{n=1}^{N} c_n L G^*(x_j, y_n) + \sum_{i=1}^{l} a_i \psi(x_j, x_i) &= -\frac{\rho c}{\Delta t} u^t(x_j), \quad x_i \in \Omega, x_j \in \Omega, y_n \notin \Omega \\
\sum_{n=1}^{N} c_n B G^*(x_m, y_n) + \sum_{i=1}^{l} a_i \psi(x_m, x_i) &= f(x_m), \quad x_i \in \Omega, x_j \in \partial \Omega, y_n \notin \Omega
\end{align*}
\]  

(11.54)

which can be solved for all unknowns at each time step.
Next, let us consider a square functionally graded finite strip, as displayed in Figure 11.5, in which the thermal conductivity changes in a unidirectional exponential form

\[ k(x) = k_0 e^{2\beta_1 x_1} \]  

(11.55)

with \( k_0 = 17 \text{W/(m \cdot K)} \) and \( \rho c = 10^{-6} \text{J/(m}^3 \cdot \text{K)} \). \( H(t) \) is the Heaviside step function. Usually, the constant coefficient \( \beta_1 \) is called as the graded parameter.

In the numerical analysis, the side length of the square is assumed to be 0.4 m, and zero initial temperature is considered. The graded parameter is assumed to have three different values: \( \beta_1 = 0.1, 0.1, \) and 0.25 cm\(^{-1} \) for comparison. Numerical results are obtained using 36 fictitious source points, 169 interpolation points, and the time step \( \Delta t = 1 \text{s} \). Figure 11.6 displays the time variation of temperature in the FGM square strip at different positions for the case of homogeneous case, that is, \( \beta_1 = 0.0 \). Figure 11.7 shows the time variation of temperature at position \( x_1 = 0.02 \text{ m} \) of the FGM square strip for various graded parameters. Finally, the steady-state temperature distributions are illustrated in Figure 11.8 for comparison when the graded parameter has different values.

---

**Figure 11.5** Geometry of a functionally graded finite square strip and boundary conditions.
Figure 11.6  Time variation of temperature in the FGM square strip at different positions with $\beta_1 = 0$.

Figure 11.7  Time variation of temperature at position $x_1 = 0.02$ m of the FGM square strip.

11.4.3  HFS-FEM for Functionally Graded Materials

For simplicity, we assume the thermal conductivity varies exponentially with position vector, as defined in Eq. (11.44). Substituting Eq. (11.44) into the governing equation (11.23) yields

$$ k_{ij}^0 u_j(x) + 2 \beta_i k_{ii}^0 u_i(x) = 0 $$

(11.56)
whose fundamental solution satisfies following equation

\[ k_{ij}^{0} u_{ij}(x) + 2 \beta k_{ij}^{0} u_{ij}(x) + \delta(x, x_s) = 0 \]  (11.57)

where \( x \) and \( x_s \) denote arbitrary field point and source point in the infinite domain, respectively.

The closed-form solution to Eq. (11.57) in two dimensions can be expressed as [98]

\[ G^*(x, x_s) = -\frac{K_0(\kappa R)}{2\pi \sqrt{\Lambda^0}} \exp\{-\beta \cdot (x + x_s)\} \]  (11.58)

where

\[ \Lambda^0 = k_{11}^{0} k_{22}^{0} - (k_{12}^{0})^2 > 0 \]  (11.59)

\[ \kappa = \sqrt{\beta \cdot \kappa^0 \beta} \]  (11.60)

\( R \) is the geodesic distance defined as

\[ R = R(x, x_s) = \sqrt{r \cdot (k^{0})^{-1} r} \]  (11.61)

with \( r = r(x, x_s) = x - x_s \). \( K_0 \) is the modified Bessel function of the second kind of zero order.

Figure 11.8 Steady-state distribution of temperature along \( x_1 \)-axis for the FGM square strip.
By means of the fundamental solutions corresponding to the gradation variation of the thermal conductivities, we can construct the so-called specially purposed graded element (SPGE), in which the thermal conductivities allow to change as defined in Eq. (11.44). As shown in Figure 11.9, the constructed SPGE is evidently different from the constant FE in which constant material definition is given, and the existing approximately isoparametric-graded element in which the shape function interpolation is employed to approximate the practical variation of the thermal conductivities [99].

Once the fundamental solution of the problem is known, following the procedure described in Section 3, one can obtain the temperature distribution in the solution domain. Next, an example is considered to demonstrate the applicability and efficiency of the proposed SPGE. Figure 11.10 shows the square FG plate with specified boundary conditions. We assume that the plate is isotropic and the thermal conductivity graded along the \( x_2 \)-direction. The side length of the plate to be \( a = 0.04 \) m. The analytical solution of the temperature distribution is given by

\[
\begin{align*}
    u = e^{-2\beta_2 x_2} - 1 \\
    e^{-2\beta_2 a} - 1
\end{align*}
\]  

In the analysis, the graded parameter \( \beta_2 = 10, 25, 50 \) and material constant \( k_0 = 17 \) are used. One 16-node polygonal SPGE is used to model the solution domain, as shown in Figure 11.11a. For comparison, the solution domain is also modelled by the 16 standard quadratic isoparametric FEs with stepwise constant material definition, as shown in Figure 11.11b. Figure 11.12 shows the temperature distribution along direction \( x_2 \), which is the graded direction for various graded parameter \( \beta_2 \). It is found that a higher level of temperature is obtained at the same position for a bigger graded parameter \( \beta_2 \). We also observe that an excellent agreement between

Figure 11.9 Schematic of the conventional FE and SPGE.
Figure 11.10 Square FG plate under specified boundary conditions.

Figure 11.11 The polygonal hybrid graded FE mesh in the proposed HFS-FEM (a) and the conventional FE mesh in the FEM (b).

Numerical and analytical results is achieved for different graded parameters. Besides, Table 11.2 listed the numerical results from the present HFS-FEM using SPGE at some specific locations, in which the analytical solutions and the numerical results from the conventional FEM using the mesh shown in Figure 11.11b are also provided for comparison. It can be seen from Table 11.2 that the proposed SPGE is more efficient than the conventional FE, because it can achieve more accurate results with much less elements.
11.5 Applications in Composite Materials

11.5.1 Basic Equations of Composite Materials

The heat conduction in composites that usually include two or more material constituents is more complicated than that in homogeneous materials. Here, for the sake of convenience, the heat conduction in fiber-reinforced composites is reviewed, as shown in Figure 11.13. Assume that $x = (x_1, x_2)$ denotes the spatial coordinates, $\Omega_m$ and $\Omega_f$ are the matrix and fiber domains, respectively, and $u_m$ and $u_f$ are the temperature fields in the corresponding domains. If both the matrix and the fiber are isotropic, then the heat transfer equations in fiber-reinforced composites can be written as follows.

1. The governing equations for the fiber and matrix material phases:

$$\frac{\partial^2 u_m}{\partial x_1^2} + \frac{\partial^2 u_m}{\partial x_2^2} = 0, \quad x \in \Omega_m$$

$$\frac{\partial^2 u_f}{\partial x_1^2} + \frac{\partial^2 u_f}{\partial x_2^2} = 0, \quad x \in \Omega_f$$ (11.63)
Table 11.2  Comparison of temperatures along $x_2$ for different methods.

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>0.000</th>
<th>0.005</th>
<th>0.010</th>
<th>0.015</th>
<th>0.020</th>
<th>0.025</th>
<th>0.03</th>
<th>0.035</th>
<th>0.040</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>0.0000</td>
<td>0.2558</td>
<td>0.4551</td>
<td>0.6102</td>
<td>0.7311</td>
<td>0.8252</td>
<td>0.8985</td>
<td>0.9555</td>
<td>1.0000</td>
</tr>
<tr>
<td>HFS-FEM</td>
<td>0.0000</td>
<td>0.2556</td>
<td>0.4553</td>
<td>0.6102</td>
<td>0.7295</td>
<td>0.8252</td>
<td>0.8982</td>
<td>0.9556</td>
<td>1.0000</td>
</tr>
<tr>
<td>FEM</td>
<td>0.0000</td>
<td>0.2275</td>
<td>0.4551</td>
<td>0.5931</td>
<td>0.7311</td>
<td>0.8148</td>
<td>0.8984</td>
<td>0.94923</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
2. The continuity conditions at the interface $\Gamma_{mf}$ between the fiber and the matrix

\[ u_m = u_f, \quad k_m \frac{\partial u_m}{\partial n} = k_f \frac{\partial u_f}{\partial n} \]  \hspace{1cm} (11.64)

where $n$ is the unit direction normal to the fiber/matrix interface, and $k_m$ and $k_f$ are the thermal conductivities of the matrix and fiber materials, respectively.

Besides, according to Fourier’s law of heat transfer in isotropic media, one has the following relationship between the temperature variable $u$ and the heat flux component $q_i$

\[ q_{mi} = -k_m \frac{\partial u_m}{\partial x_i} \quad (i = 1, 2) \]  \hspace{1cm} (11.65)

\[ q_{fi} = -k_f \frac{\partial u_f}{\partial x_i} \quad (i = 1, 2) \]

Finally, the boundary conditions of the problem defined along the outer boundary of the matrix domain are given by

\[ B u_m (x) = f(x), \quad x \in \partial \Omega_{m1} \]  \hspace{1cm} (11.66)
where \( B \) stands for a boundary operator and \( f(x) \) is a specified function along the boundary. \( \partial \Omega_{m1} = \partial \Omega_m / \Gamma_{m1} \) is the outer boundary of the matrix domain.

**11.5.2 MFS for Composite Materials**

For the heat conduction problem in the composite shown in Figure 11.13, the procedure of MFS is relatively complicated [100, 101]. The MFS needs to establish discrete equation for each material constituent and then link them by the interfacial conditions (11.64). For simplification, let us consider the simplest case that only one fiber is embedded in the matrix.

**11.5.2.1 MFS for the Matrix Domain**

Following the procedure of the MFS described above, the approximation of temperature field in the matrix domain shown in Figure 11.14 has the form

\[
 u_m(x) = \sum_{n=1}^{N_m} c_n^m G_m^*(x, y_n^m), \quad x \in \Omega_m, y_n^m \notin \Omega_m \quad (11.67)
\]

where \( N_m \) is the number of unknown singularities (sources) for the matrix domain, \( c_n^m \) are unknown real coefficients, and \( G_m^*(x, y_n^m) \) is the fundamental solution related to the matrix material

\[
 G_m^*(x, y_n^m) = -\frac{1}{2\pi k_m} \ln r \quad (11.68)
\]

with

\[
 r = |x - y_n^m| \quad (11.69)
\]

**11.5.2.2 MFS for the Fiber Domain**

Similarly, the MFS approximation of temperature field in the fiber domain shown in Figure 11.15 has the form

\[
 u_f(x) = \sum_{n=1}^{N_f} c_n^f G_f^*(x, y_n^f), \quad x \in \Omega_f, y_n^f \notin \Omega_f \quad (11.70)
\]

where \( N_f \) is the number of unknown singularities (sources) for the fiber domain, \( c_n^f \) are unknown real coefficients, and \( G_f^*(x, y_n^f) \) is the fundamental solution related to the matrix material
\[ G_f^*(x, y_n^f) = -\frac{1}{2\pi k_f} \ln r \]  
\[ r = |x - y_n^f| \]  
\( \text{(11.71)} \)  
\( \text{(11.72)} \)

**11.5.2.3 Complete Linear Equation System**

In Eq. (11.67), there are \( N_m \) unknowns, while in Eq. (11.70) there are \( N_f \) unknowns. These unknown coefficients can be determined by collocating the boundary conditions and the interfacial conditions at \( M_1 \) distinct points on the outer boundary \( \partial\Omega_m = \partial\Omega_m \setminus \Gamma_m^f \) and \( M_2 \) distinct points on the interface \( \Gamma_m^f \).
Substituting Eq. (11.67) into the boundary condition (11.66), we have

$$Bu_m(x_i) = \sum_{n=1}^{N_m} c_n^m B G^*_m(x_i, y_n^m) = f(x_i), \quad x_i \in \partial \Omega_{m_1}, i = 1 \rightarrow M_1 \quad (11.73)$$

Similarly, substituting Eqs (11.67) and (11.70) into the interfacial condition (11.64), we have

$$\sum_{n=1}^{N_m} c_n^m G^*_m(x, y_n^m) = \sum_{n=1}^{N_f} c_n^f G^*_f(x_j, y_n^f), \quad x_j \in \Gamma_{mf}, j = 1 \rightarrow M_2 \quad (11.74)$$

$$k_m \sum_{n=1}^{N_m} c_n^m \frac{\partial G^*_m(x_j, y_n^m)}{\partial n} = k_f \sum_{n=1}^{N_f} c_n^f \frac{\partial G^*_f(x_j, y_n^f)}{\partial n}, \quad x_j \in \Gamma_{mf}, j = 1 \rightarrow M_2 \quad (11.75)$$

Thus, we have a total number of $N_m + N_f$ unknowns and a total of $M_1 + 2M_2$ equations to be solved. The minimization strategy can be carried to solve these equations and then obtain all knowns.

From the procedure above, we find that the MFS is not very suitable for such problem. In particular, when the number of fibers in the matrix becomes large, the computation time of the MFS increases dramatically. In contrast, the HFS-FEM discussed in Section 11.5.3 can handle this problem efficiently.

11.5.3 HFS-FEM for Composite Materials

According to the feature that arbitrarily shaped polygonal hybrid FE can be constructed in the HFS-FEM, the special $n$-sided fiber/matrix elements can be developed for such analysis, as described below.

11.5.3.1 Special Fundamental Solutions

Fundamental solutions play an important role in the derivation of the special $n$-sided fiber/matrix elements. With the proposed HFS-FEM, for two-dimensional heat conduction problems in fiber-reinforced composites, the temperature response in an infinite matrix region containing a centered circular fiber is needed under the unit heat source in the matrix.

In Figure 11.16, a unit heat source is applied at the source point $z_0$ in the infinite matrix $\Omega_{m'}$, then the temperature responses $G_m$ and $G_f$ at any field point $z$ in matrix and fiber regions are, respectively, obtained as [102]
\[
G_m = -\frac{1}{2\pi k_m} \left[ \text{Re}\ln(z - z_0) + \frac{k_m - k_f}{k_m + k_f} \text{Re}\left[ \ln\left( \frac{R^2}{z - z_0^2} \right) \right] \right] z \in \Omega_m \\
G_f = -\frac{1}{(k_m + k_f)\pi} \text{Re}\ln(z - z_0) \quad z \in \Omega_f
\]

where \( R \) is the radius of the fiber, \( z = x_1 + x_2i \) is a complex number defined in a local coordinate system \( x = (x_1, x_2) \) with its origin coincident with the fiber center, and \( i = \sqrt{-1} \) denotes the unit imaginary number.

11.5.3.2 Special n-Sided Fiber/Matrix Elements

By means of the special fundamental solutions described above, special \( n \)-sided fiber/matrix elements can be constructed. Here, one and nine fibers embedded in the unit square matrix region (see Figure 11.17) are considered to demonstrate the performance of the present special \( n \)-sided fiber/matrix elements. A uniform heat flux of 100.0 is applied on the left side of the square matrix region, while its opposite side is maintained at a temperature of 0. All other edges of the square are assumed to be insulated.

For simplicity, the thermal conductivities in consistent units are assumed for the matrix and the fiber, that is, \( k_f = 20k_m = 20 \). Figures 11.18 and 11.19 show, respectively, the temperature distribution along the middle line \( x_2 = 0.5 \) for the cases of one and nine fibers. Also, the numerical results from the conventional FEs (ABAQUS) are provided as reference values for comparison. To obtain steady results, the conventional FE mesh is chosen to be refined enough. It is found that there are good agreements between results.
Figure 11.17 Different fiber numbers in the matrix.

Figure 11.18 Temperature variation along the line $x_2 = 0.5$ for one fiber case.

Figure 11.19 Temperature variation along the line $x_2 = 0.5$ for nine-fiber case.
from the present HFS-FEM and ABAQUS for both cases. However, very few elements are employed in the present HFS-FEM, and mesh discretization inside the fiber domain is avoided.

11.6 Conclusions

On the basis of the preceding discussion, the following conclusions can be made. In contrast to the conventional FEM and the BEM, the fundamental-solution-based MFS and HFS-FEM have some advantages:

1. The formulation of MFS is very simple, and relatively few boundary collocations and singularities are required to produce results with acceptable accuracy. Besides, the MFS has good adaptivity for problems local singularities such as geometric corner and crack. For example, by adjusting the locations of boundary collocations, the corner problem meeting in the BEM is not a specific source of inaccuracy.

2. As for the HFS-FEM, all integrals are performed along the element boundary, and thus we can develop a general formulation of $n$-sided general or special polygonal element to simplify mesh division and reduce numbers of elements, which permits that different elements can have a different number of sides. In addition, it is easy to calculate fields everywhere inside the element.

In the future, there are many possible extensions and areas in need of further development by using the MFS and the HFS-FEM. Among those developments, one could list the following:

- Development of efficient formulations of the MFS and HFS-FEM to thermoelastic analysis of FGM and composites.
- Construction of more special polygonal elements for our use in the HFS-FEM.
- Extensions of the MFS and HFS-FEM to coated composite structures.
- Combination of the MFS and HFS-FEM with topology optimization for designing microstructure of materials.
- Development of efficient formulations of the MFS and HFS-FEM for interaction between fluid and structure.
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Conflict of Interest

The authors confirm that there is no conflict of interest related to the content of this article.

References


