RECENT DEVELOPMENTS IN NONLINEAR H_∞ CONTROL*

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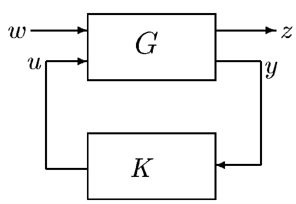
Abstract. The last few years have seen a number of exciting developments in the area known as nonlinear H_∞ control. The aim of work in this area is to design robust controllers for nonlinear systems using generalizations of the highly successful H_∞ methods used in linear systems theory. In this paper we give an overview of the problems and methods of nonlinear H_∞ control, including the H_∞ norm (L_2 gain), stability, information state methods, and factorization. The development of this area has involved important contributions from a number of researchers using a variety of techniques. In addition, H_∞ control has spurred renewed interest in some traditional areas such as differential games, dissipative systems, and stochastic (risk-sensitive) control.

Key Words. Nonlinear control systems, robust H_∞ control, risk-sensitive stochastic control, differential games, dissipative systems.

1. INTRODUCTION

Consider the closed loop system $(G, K) : w \mapsto z$ depicted in the following diagram.



Here, G is the plant to be controlled, K is the controller, w is a disturbance input, u is the controlled input, z is a performance output and y is the observed or measured outputs.

The basic problem of H_∞ control is the following. Given a plant G find a controller K such that the closed loop system (G, K) is

(i) internally stable, and
(ii) γ-dissipative, i.e.,

$$\begin{aligned} &\tfrac{1}{2}\int_0^T |z(s)|^2\, ds \\ &\quad \le \gamma^2 \tfrac{1}{2}\int_0^T |w(s)|^2\, ds + \beta(x_0) \qquad (1.1) \end{aligned}$$

for all $w \in L_{2,T}$ and all $T \ge 0$.

for some $\beta(x) \ge 0$ with $\beta(0) = 0$ (the function β depends on K, and is a bias term accounting for the initial states x_0 of G).

The H_∞ problem originated in the work of (Zames, 1981) on robust control, which focused on frequency domain ideas (this is where the H_∞ terminology comes from). This work sparked a lot of interest, and eventually an elegant and fundamentally important state space solution was obtained by (Doyle *et al*, 1989). Let us recall the solution to a particular linear H_∞ problem, see (Petersen *et al*, 1991), (Green & Limebeer, 1994). The plant G is described by the model

$$G: \quad \begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + u \\ y &= C_2 x + w. \end{aligned} \qquad (1.2)$$

Under standard assumptions, the H_∞ control problem for G is solvable if and only if

(i) there exists $X \ge 0$ solving

$$\begin{aligned} &(A - B_2 C_1)' X + X (A - B_2 C_1) \\ &\quad + X(\tfrac{1}{\gamma^2} B_1 B_1' - B_2 B_2') X = 0 \end{aligned} \qquad (1.3)$$

with $A - B_2 C_1 + (\frac{1}{\gamma^2} B_1 B_1' - B_2 B_2')X$ stable,

(ii) there exists $Y \ge 0$ solving

$$\begin{aligned} &(A - B_1 C_2) Y + Y (A - B_1 C_2)' \\ &\quad + Y(\tfrac{1}{\gamma^2} C_1' C_1 - C_2' C_2) Y = 0 \end{aligned} \qquad (1.4)$$

with $A - B_1C_2 + Y(\frac{1}{\gamma^2}C_1'C_1 - C_2'C_2)$ stable, and

(iii) $XY < \gamma^2 I$.

The (central) controller is given by

$$\begin{aligned} \dot{\hat{x}} &= (A - B_1C_2 + Y(\tfrac{1}{\gamma^2}C_1'C_1 - C_2'C_2))\hat{x} + \\ &\quad (B_1 + YC_2')y + (B_2 + \tfrac{1}{\gamma^2}YC_1')u \\ u &= -(C_1 + (B_2 + \tfrac{1}{\gamma^2}YC_1)'X(1 - \tfrac{1}{\gamma^2}YX)^{-1})\hat{x}. \end{aligned}$$

In this paper we are interested in the H_∞ problem for nonlinear systems, a problem which has attracted much interest in recent years, and it is our purpose to review some of the main developments. For illustration, we consider the following particular nonlinear system:

$$G: \quad \begin{aligned} \dot{x} &= A(x) + B_1(x)w + B_2(x)u \\ z &= C_1(x) + u \\ y &= C_2(x) + w. \end{aligned} \tag{1.5}$$

We assume $A(0) = 0$, $C_1(0) = 0$, $C_2(0) = 0$. The controller K is a causal map

$$K : y \mapsto u, \tag{1.6}$$

the structure of which is in general unspecified. The key task is to obtain necessary and sufficient conditions from appropriate nonlinear generalizations of the Riccati equations (1.3), (1.4). As we shall see, these involve partial differential equations and inequalities (even infinite dimensional ones).

An interesting and important feature of the H_∞ problem is that it has close connections with differential games, risk-sensitive stochastic control, and dissipative systems, and as a consequence these subjects have enjoyed renewed interest. Indeed, they have a lot to contribute to the understanding of robust control.

We begin in §2 with a discussion of the H_∞ norm and dissipative systems, and present some of the main partial differential inequalities and equations. In §3 we describe the risk-sensitive index and its relationship to the H_∞ and H_2 norms. The state feedback H_∞ control problem is discussed in §4, while the output feedback problem is treated in §5. This paper is written in a heuristic, tutorial style, and mathematical technicalities are kept to a minimum. Also, we confine our attention to the so-called one block problem. For full details, readers should consult the references. The book (Basar & Bernhard, 1991) presents the game theoretic approach in detail, while the book (van der Schaft, 1996) covers a number of aspects of the theory; and the forthcoming book (Helton & James, 1997) presents the information state approach in detail.

2. THE H_∞ NORM

Consider a nonlinear system Σ with input w and output z:

$$\begin{aligned} \dot{x} &= f(x) + g(x)w \\ z &= h(x). \end{aligned} \tag{2.1}$$

We assume that $f(0) = 0$ and $h(0) = 0$.

One measure of the size of Σ, i.e., of the amount of influence of w on z, is the "H_∞ norm", or more properly, the L_2 gain. By *definition*, let us say that $\| \Sigma \|_{H_\infty} \leq \gamma$ if and only if we have

$$\tfrac{1}{2}\int_0^T |z(t)|^2 dt \leq \gamma^2 \tfrac{1}{2}\int_0^T |w(t)|^2 dt + \beta(x_0) \tag{2.2}$$

for all $w \in L_{2,T}$, $T \geq 0$, for some $\beta \geq 0$, $\beta(0) = 0$. Here, $L_{2,T}$ is the space of square integrable functions on $[0, T]$. Then $\| \Sigma \|_{H_\infty}$ is the smallest number γ for which $\| \Sigma \|_{H_\infty} \leq \gamma$.

The property (2.2) (a generalization of a time domain analog of the frequency domain definition of H_∞ norm for linear systems) can be treated efficiently with the theory of dissipative systems, see (Willems, 1972), (Hill & Moylan, 1976). Indeed, we say that Σ is γ-*dissipative* (with respect to the *supply rate* $\gamma^2|w|^2 - |z|^2$) iff there exists a nonnegative function $V(x)$ (called a *storage function*) such that $V(0) = 0$ and

$$\begin{aligned} V(x) \geq \sup\nolimits_{T \geq 0, w \in L_{2,T}} &\{V(x(T)) \\ &- \tfrac{1}{2}\textstyle\int_0^T [\gamma^2|w(r)|^2 - |z(r)|^2]dr : x(0) = x\}. \end{aligned} \tag{2.3}$$

Now $\| \Sigma \|_{H_\infty} \leq \gamma$ iff Σ is γ-dissipative. To see this, if $\| \Sigma \|_{H_\infty} \leq \gamma$ then define (the *available storage*)

$$\begin{aligned} V_a(x) = \sup\nolimits_{T \geq 0, w \in L_{2,T}} \\ \{-\textstyle\int_0^T [\gamma^2|w(r)|^2 - |z(r)|^2]dr : x(0) = x\}. \end{aligned} \tag{2.4}$$

In view of (2.2) and the definition, it follows that $0 \leq V_a(x) \leq \beta(x)$, $V_a(0) = 0$, and by dynamic programming, V_a satisfies (2.3); hence Σ is γ-dissipative. Conversely, if Σ is γ-dissipative, then (2.3) implies (2.2) with $\beta = V$.

Storage functions can be characterized by a PDI (an infinitesimal version of (2.3)), see (Willems, 1972), (Hill & Moylan, 1976), (James, 1993):

$$\sup_w\{\nabla_x V\cdot(f(x)+g(x)w) \\ -\tfrac{1}{2}[\gamma^2|w|^2-|h(x)|^2]\}\leq 0 \tag{2.5}$$

A version of the Bounded Real Lemma then says that Σ is γ-dissipative iff $\parallel \Sigma \parallel_{H_\infty}\leq\gamma$ iff there exists a non-negative solution $V(x)$ of (2.5) with $V(0)=0$. In general, V need not be globally differentiable, but (2.5) can be interpreted in the viscosity sense, see (James, 1993), (Ball & Helton, 1993), (Soravia, 1994), (Fleming & Soner, 1993).

The utility of the PDI (2.5) is that it can be used to determine or compute storage functions V, and thereby evaluate the H_∞ norm of a system. Indeed, if the PDI (2.5) is solvable, then $\parallel \Sigma \parallel_{H_\infty}\leq\gamma$; conversely, if the PDI is not solvable, then $\parallel \Sigma \parallel_{H_\infty}>\gamma$. This leads to the well known iterative search for $\gamma^*=\parallel\Sigma\parallel_{H_\infty}$.

In general, (2.5) cannot be solved explicitly, and so one is forced to use approximations, see (James & Yuliar, 1995). The finite difference discretization

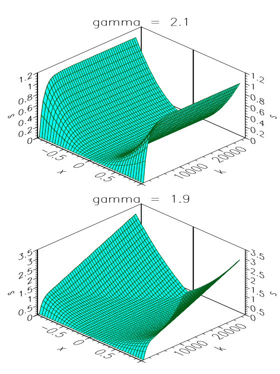
$$\sup_w\{\textstyle\sum_{x'}p^\delta(x,x';w)V^\delta(x') \\ -\Delta t(x)(\gamma^2|w|^2-|h(x)|^2)\}\leq V^\delta(x)$$

is an approximation to (2.5) as the discretization parameter $\delta\to 0$ (for suitable $p^\delta(x,x';w)$, $\Delta t(x)$; see (James & Yuliar, 1995)). For example, consider the one dimensional linear system, with transfer function $\Sigma(s)=1/(s+0.5)$, and minimal realization

$$\dot{x}=-0.5x+u$$

$$y=x$$

Now $\parallel\Sigma\parallel_\infty=2$, and the numerical scheme converges for $\gamma>2$ and diverges for $\gamma<2$, as shown in the graphs.



In the case of linear systems the PDI can be solved in terms of a matrix $X\geq 0$ satisfying a matrix inequality. Indeed, if $f(x)=Ax$, $g(x)=B$, and $h(x)=Cx$, then $V(x)=\frac{1}{2}x'Xx$, where

$$A'X+XA+\tfrac{1}{\gamma^2}XBB'X+C'C\leq 0. \tag{2.6}$$

A γ-dissipative system is often also asymptotically stable, with Lyapunov function $V(x)$. Indeed, if Σ is, say, zero-state observable (meaning that if $w(t)=0$ and $z(t)=0$ for all $t\geq 0$ then $x(0)=0$), then $V(x)>0$ if $x\neq 0$ and Σ is locally asymptotically stable; and if also V has compact level sets (proper), then Σ is globally asymptotically stable. This is because the PDI (2.5) implies that

$$\tfrac{d}{dt}V(x(t))=\nabla_x V(x(t))\cdot f(x(t)) \\ \leq -\tfrac{1}{2}|h(x(t))|^2\leq 0.$$

Geometrical interpretation. An interesting geometrical interpretation of the PDI (2.5) is given in (van der Schaft, 1991), (van der Schaft, 1992), (van der Schaft, 1993), (van der Schaft, 1996), which, together with the local linearization of Σ at $x=0$, $\Sigma_{lin}=(A,B,C)=(\frac{df}{dx}(0),\frac{dg}{dx}(0),\frac{dh}{dx}(0))$, can be used to prove local results. In particular, storage functions can be produced which are smooth at least in a neighborhood of $x=0$.

The *Hamiltonian* associated with the PDI (2.5) is

$$\begin{aligned}&H^\gamma(x,\lambda)\\ &=\sup_w\{\lambda\cdot f(x)+g(x)w)-\tfrac{1}{2}[\gamma^2|w|^2-|h(x)|^2]\}\\ &=\lambda\cdot f(x)+\tfrac{1}{2\gamma^2}\lambda g(x)g(x)'\lambda'+\tfrac{1}{2}h(x)'h(x),\end{aligned}$$

and the corresponding Hamiltonian vector field X_{H^γ} is given by the equations

$$\begin{aligned}\dot{x}&=\tfrac{\partial}{\partial\lambda}H^\gamma(x,\lambda)\\ \dot{\lambda}&=-\tfrac{\partial}{\partial x}H^\gamma(x,\lambda).\end{aligned} \tag{2.7}$$

(These are the characteristic equations for (2.5).) X_{H^γ} is called *hyperbolic* iff the Jacobian

$$DX^\gamma(0,0)=\begin{bmatrix}\frac{\partial^2}{\partial x\partial\lambda}H^\gamma(0,0) & \frac{\partial^2}{\partial\lambda^2}H^\gamma(0,0)\\ -\frac{\partial^2}{\partial x^2}H^\gamma(0,0) & -\frac{\partial^2}{\partial\lambda\partial x}H^\gamma(0,0)\end{bmatrix}$$

$$=\begin{bmatrix}A & \frac{1}{\gamma^2}BB'\\ -C'C & -A'\end{bmatrix}$$

has no purely imaginary eigenvalues. Now if X_{H^γ} is hyperbolic, there exists (at least locally) a stable invariant submanifold N^- through $(0,0)$ consisting of trajectories of X_{H^γ} attracted to $(0,0)$. It turns out that if A is asymptotically stable and X_{H^γ} is hyperbolic, then there exists a neighborhood U of $x=0$ and a smooth non-negative function $V^-(x)$ solving the PDE

$$H^\gamma(x, \nabla_x V) = 0 \qquad (2.8)$$

(i.e. the PDI (2.5) with equality) in U with $V^-(0) = 0$, $f + \frac{1}{\gamma^2} gg'(\nabla_x V^-)'$ asymptotically stable in U, and that locally N^- is determined by V^-:

$$N^- = \{(x, \lambda) : \lambda = \nabla_x V^-(x), x \in U\}. \qquad (2.9)$$

Also, V^- equals the available storage V_a in U, under appropriate stability assumptions.

These results depend on the corresponding results for the linear system Σ_{lin} relating to the existence of a solution $X \geq 0$ of (2.6) with equality and $A + \frac{1}{\gamma^2} BB'X$ asymptotically stable.

If $\| \Sigma \|_{H_\infty} \leq \gamma$, then $\| \Sigma_{lin} \|_{H_\infty} \leq \gamma$. Conversely, if $\| \Sigma_{lin} \|_{H_\infty} \leq \gamma$, then (2.2) holds for all w for which the corresponding trajectory stays near $x = 0$. For further details, see (van der Schaft, 1991), (van der Schaft, 1992), (van der Schaft, 1993).

Summarizing, we see that while storage functions need not be smooth globally, there often exist storage functions which are smooth locally and can be used to study the local H_∞ behavior, and this is tied to an explicit geometrical structure of a Hamiltonian system. To make global statements regarding storage functions and the PDI (2.5), one can appeal to the theory of viscosity solutions which does not require smoothness. Of course, much stronger results are available when the storage functions are in fact globally smooth, as in the linear case.

We will see generalizations and variants of the PDI (2.5) and the corresponding PDE (2.8) in the sequel—they are the key mathematical equations needed for H_∞ control.

3. THE RISK-SENSITIVE INDEX

The paper (Jacobson, 1973) introduced a new type of performance index for stochastic optimal control. This index is the average-of-exponential type. When a parameter multiplying the cost in the exponential is positive, this index is commonly called *risk-sensitive*. Jacobson solved this problem for linear systems, and showed that the solution also solved a deterministic differential game. Later, (Glover & Doyle, 1988) established a link between risk-sensitive control and H_∞ control. Interestingly, risk-sensitive control, which is stochastic, is closely related to game theory and H_∞ control, which is deterministic. Recently, this relationship has been explored in the nonlinear context by a number of authors, and further insights have been obtained; see (Fleming & McEneaney, 1991), (Fleming & McEneaney, 1993), (Fleming & James, 1995), (James, 1992), (Runolfsson, 1991), (Whittle, 1990a). In particular, using asymptotic methods, it was shown that the risk-sensitive index includes both H_2 and H_∞ terms.

Consider once more the system Σ (2.1), but this time replace w by scaled white noise: $\sqrt{\varepsilon} v$ (here, $\varepsilon > 0$ is the noise variance and v is the formal derivative of a standard Wiener process). The *risk-sensitive index* for Σ is a real number $I^{\gamma,\varepsilon} \geq 0$ (Fleming & McEneaney, 1993), (Fleming & James, 1995) defined by

$$I^{\gamma,\varepsilon} = \limsup_{T \to \infty} \frac{\gamma^2 \varepsilon}{T} \log \mathbf{E}_x[\exp \tfrac{1}{2\gamma^2 \varepsilon} \int_0^T |z(t)|^2 dt] \qquad (3.1)$$

This is a particular case of a large deviations formula, (Donsker & Varadhan, 1975). The number $\mu = 1/\gamma^2 > 0$ is called the risk-sensitive parameter.

Associated with this index is a function $V^{\gamma,\varepsilon}$ solving the PDE

$$I^{\gamma,\varepsilon} = \frac{\varepsilon}{2} \mathrm{tr}(gg' D_x^2 V^{\gamma,\varepsilon}) + H^\gamma(x, \nabla_x V^{\gamma,\varepsilon}) \qquad (3.2)$$

and $V^{\gamma,\varepsilon}(0) = 0$. This PDE arises as follows (Fleming & McEneaney, 1993). If $S(x,T)$ denotes the expectation in (3.1), then $S(x,T)$ satisfies a parabolic PDE by the Feynman-Kac formula. If, for large T,

$$S(x,T) \asymp \exp[\tfrac{1}{\gamma^2 \varepsilon}(I^{\gamma,\varepsilon} T + V^{\gamma,\varepsilon}(x))], \qquad (3.3)$$

then one is led to (3.2).

Interestingly, and importantly, the PDE (3.2) has an interpretation as the HJB equation for an ergodic stochastic control problem, giving a different interpretation for $I^{\gamma,\varepsilon}$:

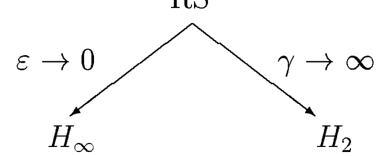
$$I^{\gamma,\varepsilon} = \sup_w \limsup_{T \to \infty} \tfrac{1}{2T} \mathbf{E}_x[\int_0^T (|h(\xi((t))|^2 - \gamma^2 |w(t)|) dt] \qquad (3.4)$$

where ξ is the solution of the auxiliary stochastic differential equation

$$\dot{\xi} = f(\xi) + g(\xi) w + \sqrt{\varepsilon} g(\xi) v,$$

where w is the control and $\sqrt{\varepsilon} v$ is the noise.

As mentioned above, the risk-sensitive index contains information about both the H_∞ and H_2 norms. This is indicated in the following diagram.



Fix $\gamma > 0$ and send $\varepsilon \to 0$, so that (at least formally) $I^{\gamma,\varepsilon} \to I^\gamma$, $V^{\gamma,\varepsilon} \to V^\gamma$, and

$$I^\gamma = H^\gamma(x, \nabla_x V^\gamma). \tag{3.5}$$

If $I^\gamma = 0$, we expect that V^γ will be a storage function (given stability), so that Σ will be γ-dissipative, and conversely. Thus the H_∞ norm is determined from the small noise limit of the risk-sensitive index. The H_2 norm is obtained by fixing $\varepsilon > 0$ and sending $\gamma \to \infty$, so that $I^{\gamma,\varepsilon} \to I^\varepsilon$, $V^{\gamma,\varepsilon} \to V^\varepsilon$, and

$$\begin{aligned} I^\varepsilon &= \limsup_{T\to\infty} \tfrac{1}{2T}\mathbf{E}_x[\int_0^T |z(t)|^2 dt] \\ &\triangleq \| \Sigma \|_{H_2} . \end{aligned} \tag{3.6}$$

In the linear case, where $f(x) = Ax$, $g(x) = B$, and $h(x) = Cx$, we have (provided A is stable and $\| \Sigma \|_{H_\infty} \leq \gamma$)

$$I^{\gamma,\varepsilon} = \frac{\varepsilon}{2}\mathrm{tr}(B'X^\gamma B), \quad V^{\gamma,\varepsilon}(x) = \tfrac{1}{2}x'X^\gamma x,$$

where $X^\gamma \geq 0$ is a solution of (2.6) with equality, and

$$I^\gamma = 0, \quad V^\gamma(x) = \tfrac{1}{2}x'X^\gamma x.$$

Also,

$$I^\varepsilon = \frac{\varepsilon}{2}\mathrm{tr}(B'XB), \quad V^\varepsilon(x) = \tfrac{1}{2}x'Xx,$$

where $X \geq 0$ is solution of the Lyapunov equation

$$A'X + XA + C'C = 0.$$

4. STATE FEEDBACK H_∞ CONTROL

It is much easier to solve control problems if complete state information is available. This is because the state of a system summarises the past history of the system and together with the inputs the future evolution can be determined. Accordingly, we consider first the state feedback H_∞ control problem, where the controller K is a memoryless function of the state x, so that $u = K(x)$; see (van der Schaft, 1991), (van der Schaft, 1992), (van der Schaft, 1993), (van der Schaft, 1996), (Ball & Helton, 1989), (Ball & Helton, 1992), (James & Baras, 1995), (McEneaney, 1995), (Soravia, 1994).

The closed loop system (G, K) is therefore

$$\begin{aligned} \dot{x} &= A(x) + B_1(x)w + B_2(x)K(x) \\ z &= C_1(x) + K(x). \end{aligned} \tag{4.1}$$

If we assume that (G, K) is γ-dissipative, then from §2 we know that there exists a storage function $V(x) \geq 0$, with $V(0) = 0$, and solving the PDI

$$\begin{aligned} &\sup_w \{\nabla_x V \cdot (A(x) + B_1(x)w + B_2(x)K(x)) \\ &\qquad -\tfrac{1}{2}[\gamma^2|w|^2 - |C_1(x) + K(x)|^2]\} \leq 0. \end{aligned}$$

Taking the min over $u = K(x)$ gives

$$\begin{aligned} &\inf_u \sup_w \{\nabla_x V \cdot (A(x) + B_1(x)w + B_2(x)u) \\ &\qquad -\tfrac{1}{2}[\gamma^2|w|^2 - |C_1(x) + u|^2]\} \leq 0. \end{aligned} \tag{4.2}$$

This is the key PDI for state feedback H_∞ control. Evaluating the min and max gives

$$\begin{aligned} &\nabla_x V \cdot (A - B_2 C_1) \\ &\qquad + \tfrac{1}{2}\nabla_x V(\tfrac{1}{\gamma^2}B_1 B_1' - B_2 B_2')\nabla_x V' \leq 0, \end{aligned} \tag{4.3}$$

with the optimizing control and disturbance (if they exist) given by

$$u^*(x) = -C_1(x) - B_2(x)'\nabla_x V(x)'$$

$$w^*(x) = \tfrac{1}{\gamma^2}B_1(x)'\nabla_x V(x)'.$$

In the linear case, $V(x) = \frac{1}{2}x'Xx$, where $X \geq 0$ is a solution of (1.3) with inequality.

Conversely, if there exists a *smooth* solution $V(x) \geq 0$, $V(0) = 0$ of the PDI (4.2), then the controller

$$K^*(x) \triangleq -C_1(x) - B_2(x)'\nabla_x V(x)' \tag{4.4}$$

renders the closed loop system γ-dissipative. Stability follows if $(A - B_2K^*, C_1 + K^*)$ is zero state observable. Notice that the controller depends explicitly on the gradient of the storage function, and consequently the existence of such controllers depends critically on the smoothness of the storage function. As discussed in §2, it follows that it is too much to expect that such controllers will exists globally.

These results can be refined to obtain a PDE, viz.:

$$\begin{aligned} &\nabla_x V \cdot (A - B_2 C_1) \\ &\qquad + \tfrac{1}{2}\nabla_x V(\tfrac{1}{\gamma^2}B_1 B_1' - B_2 B_2')\nabla_x V' = 0. \end{aligned} \tag{4.5}$$

Additionally, geometrical and local results can be obtained. Indeed, if the H_∞ control problem for the linearized system is solvable, then the H_∞ problem for the nonlinear system is solvable locally near $x = 0$; see (van der Schaft, 1991), (van der Schaft, 1992), (van der Schaft, 1993), (van der Schaft, 1996). In particular, a controller of the form (4.4) can be constructed near $x = 0$ using a storage function which is locally smooth.

Differential games. The PDE (4.5) is a Hamilton-Jacobi-Isaacs (HJI) equation corresponding to a min-max differential game, viz.

$$\inf_K J_x(K),$$

where

$$J_x(K) = \sup_{T \geq 0, w \in L_{2,T}}$$

$$\{\tfrac{1}{2}\int_0^T [|z(t)|^2 - \gamma^2 |w(t)|^2] dt : x(0) = x\}.$$

The two players are the controller $u = K(x)$ (minimizing), and the competing disturbance w (maximizing). Note that (G, K) is γ-dissipative iff

$$J_x(K) \leq \beta(x),$$

for some $\beta \geq 0$, $\beta(0) = 0$. Also, the value function

$$V(x) = \inf_K J_x(K)$$

solves the PDE (4.5).

The game theoretic approach to H_∞ control is discussed in detail in (Basar & Bernhard, 1991); see also (James & Baras, 1995).

Risk-sensitive stochastic control. Consider the following risk-sensitive control problem with stochastic dynamics

$$\dot{x} = A(x) + \sqrt{\varepsilon} B_1(x) v + B_2(x) K(x), \qquad (4.6)$$

and cost

$$I^{\gamma,\varepsilon}(K) = \limsup_{T\to\infty} \frac{\gamma^2 \varepsilon}{T} \log \mathbf{E}_x[\exp \tfrac{1}{2\gamma^2 \varepsilon} \int_0^T |z(t)|^2 dt],$$

which is to be minimised (see (Fleming & McEneaney, 1993)):

$$I^{*\gamma,\varepsilon} = \inf_K I^{\gamma,\varepsilon}(K).$$

We won't go through the details of the solution to this problem, but rather discuss its connection with game theory, H_∞, and H_2 control. In fact, by exploiting the parameters γ and ε, all of these problems can be regarded as perturbations of a simple deterministic optimal control problem, revealing stochastic and deterministic methods for modelling disturbances.

Using the alternate representation (3.4), we see that the risk-sensitive stochastic control problem is equivalent to a *stochastic* differential game:

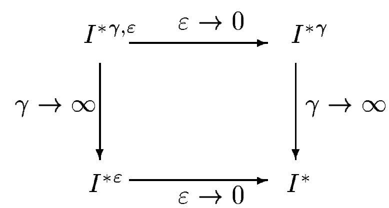
$$I^{*\gamma,\varepsilon} = \inf_K \sup_w \limsup_{T\to\infty} \tfrac{1}{2T} \mathbf{E}_x[\int_0^T (|z(t)|^2 - \gamma^2 |w(t)|) dt].$$

Here, $z(t) = C_1(\xi(t)) + K(\xi(t))$ where ξ is the solution of the auxiliary stochastic differential equation

$$\dot{\xi} = A(\xi) + B_1(\xi) w + \sqrt{\varepsilon} B_1(\xi) v + B_2(\xi) K(\xi).$$

Note that here there is a controller $u = K(x)$ (minimizing), a competing disturbance input w (maximizing), and a white noise input $\sqrt{\varepsilon} v$ (averaging).

Sending $\varepsilon \to 0$ and $\gamma \to \infty$ yields the relationships depicted in the following diagram.



In this diagram, the performance indices have the following interpretations.

Deterministic differential game and H_∞ control:

$$I^{*\gamma} = \inf_K \sup_w \limsup_{T\to\infty} \tfrac{1}{2T} \int_0^T (|z(t)|^2 - \gamma^2 |w(t)|) dt$$

with dynamics (1.5).

H_2 or risk-neutral optimal control:

$$I^{*\varepsilon} = \inf_K \limsup_{T\to\infty} \tfrac{1}{2T} \mathbf{E}_x[\int_0^T |z(t)|^2 dt]$$

with dynamics (4.6).

Deterministic optimal control:

$$I^{*} = \inf_K \limsup_{T\to\infty} \tfrac{1}{2T} \int_0^T |z(t)|^2 dt$$

with dynamics

$$\dot{x} = A(x) + B_2(x) K(x). \qquad (4.7)$$

Formally, the optimal risk-sensitive controller has an expansion (for large γ and small ε)

$$K^{*\gamma,\varepsilon}(x) = K^*(x) + \tfrac{1}{\gamma^2} K_{H_\infty}(x) + \varepsilon K_{H_2}(x) + \ldots,$$

where $K^*(x)$ is the optimal (stabilizing) controller for the limit deterministic optimal control problem

$$K^*(x) = -C_1(x) - B_2(x)'\nabla_x V(x)'$$

where $V(x)$ solves the PDE

$$I^* = \nabla_x V \cdot (A - B_2 C_1) - \tfrac{1}{2}\nabla_x V B_2 B_2' \nabla_x V',$$

and

$$K_{H_\infty}(x) = -B_2(x)'\nabla_x V_{H_\infty}(x)',$$
$$K_{H_2}(x) = -B_2(x)'\nabla_x V_{H_2}(x)'.$$

Here, $V^{\gamma,\varepsilon} = V + \frac{1}{\gamma^2}V_{H_\infty} + \varepsilon V_{H_2} + \ldots$, where

$$V_{H_\infty}(x) =$$

$$\tfrac{1}{2}\int_0^\infty \nabla_x V(x(t))B_1(x(t))B_1(x(t))'\nabla_x V(x(t))'dt,$$

and

$$V_{H_2}(x) =$$

$$\tfrac{1}{2}\int_0^\infty [\mathrm{tr}(B_1(x(t))B_1(x(t))'D^2 V(x(t)))$$
$$-\mathrm{tr}(B_1(0)B_1(0)'D^2 V(0))]dt,$$

where $x(t)$ is the optimal (asymptotically stable, with $K(x) = K^*(x)$) trajectory of (4.7).

These asymptotic calculations are only formal; see (Campi & James, 1996), (James, 1992), (Fleming & McEneaney, 1993), (Fleming & James, 1995) for precise theorems. The significance of these formulas lies in the detail they reveal regarding the structure of the risk-sensitive controller, and its H_∞ and H_2 components. The H_∞ component $K_{H_\infty}(x)$ depends on the disturbance energy, whereas the H_2 component $K_{H_2}(x)$ depends on the diffusion effects of the noise.

In the linear case, we have

$$V(x) = \tfrac{1}{2}x'Xx, I^* = 0,$$

where $X \geq 0$ is a solution of the Riccati equation

$$0 = (A - B_2C_1)'X + X(A - B_2C_1) - XB_2B_2'X,$$

and

$$V_{H_\infty}(x) = \tfrac{1}{2}x'Px, V_{H_2}(x) = 0,$$

where $P \geq 0$ is given by

$$P = \int_0^\infty e^{A^{*\prime}t}XB_1B_1'Xe^{A^*t}dt,$$

and $A^* = A - B_2(C_1 + B_2'X)$. Thus

$$K^{\gamma,\varepsilon}(x) = -(C_1 + B_2'X + \tfrac{1}{\gamma^2}P)x + \ldots$$

It is interesting to note that in the linear case the H_2 term in the risk-sensitive controller is zero!

5. OUTPUT FEEDBACK H_∞ CONTROL

We turn now to the output feedback H_∞ control problem, where the controller K is a causal map $y \mapsto u$. A number of authors (Ball & Helton, 1989), (Ball *et al*, 1993), (Isidori & Astolfi, 1992), (van der Schaft, 1991), (van der Schaft, 1992), (van der Schaft, 1993), (van der Schaft, 1996) assume that K has a specified form, namely a finite dimensional realization whose state ζ has the same dimension as the state x of the plant G. In (James & Baras, 1995) the structure of K was left unspecified, and the information state approach was used to determine the general nature of K (an *infinite* dimensional system). Indeed, it is likely that generically the H_∞ problem is solvable only by an infinite dimensional controller (this is well known to be the case in stochastic control).

Specified controller structure. Let's begin by assuming that K has the finite dimensional realization:

$$K: \begin{array}{rcl} \dot{\zeta} & = & a(\zeta, u, y) \\ u & = & b(\zeta, y), \end{array} \qquad (5.1)$$

where $\dim \zeta = \dim x$, for some functions $a(\cdot)$, $b(\cdot)$ to be determined. The closed loop system (G, K) (with state (x, ζ)) is

$$\begin{array}{rcl} \dot{x} & = & A(x) + B_1(x)w + B_2(x)b(\zeta, y) \\ \dot{\zeta} & = & a(\zeta, b(\zeta, y), y) \\ z & = & C_1(x) + b(\zeta, y) \end{array}$$

If (G, K) is γ-dissipative, then there exists a storage function $e(x, \zeta) \geq 0$, with $e(0,0) = 0$ satisfying the PDI

$$\begin{array}{l} \sup_w \{\nabla_x e \cdot (A(x) + B_1(x)w \\ \qquad + B_2(x)b(\zeta, C_2(x) + w)) \\ + \nabla_\zeta e \cdot (a(\zeta, b(\zeta, y), C_2(x) + w)) \\ \quad - \frac{1}{2}[\gamma^2|w|^2 - |C_1(x) + b(\zeta, C_2(x) + w)|^2] \\ \quad \leq 0. \end{array} \qquad (5.2)$$

The goal is to find functions $a(\cdot)$, $b(\cdot)$ for which there does exist a storage function satisfying (5.2). In (Ball *et al*, 1993), the authors minimize the LHS of (5.2) over $a(\cdot)$, $b(\cdot)$ and obtain some interesting formulas for the resulting optimal $a^*(\cdot)$, $b^*(\cdot)$. This gives a controller K^* of the form (5.1) solving the H_∞ control problem under certain conditions. A different approach was used in (Isidori & Astolfi, 1992), where K is obtained by assuming the existence of an observer-type gain, together with other conditions; thus the dynamical part of K is essentially an observer.

In (van der Schaft, 1993) (and also in (Ball *et al*, 1993)), nonlinear analogs of the Riccati equations (1.3), (1.4) were obtained. To see this, suppose $a(\zeta,u,y)=k(\zeta)+\ell(\zeta)y$ and $b(\zeta,y)=m(\zeta)$. Substitution of these expressions in (5.2) and evaluating the max over w gives the PDI

$$\begin{pmatrix} \nabla_x e \\ \nabla_\zeta e \end{pmatrix}' \begin{pmatrix} A(x)+B_2(x)m(\zeta) \\ k(\zeta)+\ell(\zeta)C_2(x) \end{pmatrix} +$$

$$\tfrac{1}{2\gamma^2}\begin{pmatrix} \nabla_x e \\ \nabla_\zeta e \end{pmatrix}' \begin{pmatrix} B_1(x) \\ \ell(\zeta) \end{pmatrix} \begin{pmatrix} B_1(x) \\ \ell(\zeta) \end{pmatrix}' \begin{pmatrix} \nabla_x e \\ \nabla_\zeta e \end{pmatrix}$$

$$+\tfrac{1}{2}|C_1(x)+m(\zeta)|^2 \le 0$$

Now define

$$V(x)=e(x,\phi(x)),$$

where ϕ is a smooth function such that $\phi(0)=0$ and $\nabla_\zeta e(x,\phi(x))=0$. Then substituting this into the PDI with $\zeta=\phi(x)$ and minimizing over $m(\zeta)$ shows that V solves the state feedback PDI (4.3). Next, define

$$R(x)=V(x,0).$$

Then substituting this into the PDI with $\zeta=0$ and minimizing over $\ell(0)$ shows that R solves the (so-called *dual*) PDI

$$\nabla_x R\cdot(A-B_1C_2) + \tfrac{1}{2}(C_1'C_1-\gamma^2C_2'C_2)\le 0. \tag{5.3}$$

It also follows that (at least locally) $V(x)\le R(x)$, akin to the linear coupling condition, and it is possible to use these two storage functions to obtain output feedback controllers, under certain conditions.

In the linear case, $V(x)=\frac{1}{2}x'Xx$ and $R(x)=\frac{\gamma^2}{2}x'Y^{-1}x$, where $X\ge 0$ and $Y>0$ correspond to subsolutions of (1.3) and (1.4) respectively. Also, $e(x,\zeta)=\frac{1}{2}[\gamma^2(x-\zeta)'Y^{-1}(x-\zeta)+\zeta'X(1-\frac{1}{\gamma^2}YX)^{-1}\zeta]$.

Information state solution. We now relax the above restriction on the structure of the controller K, and simply let K be any causal map $y\mapsto u$. We employ the information state approach used in (James *et al*, 1994), (James *et al*, 1993), (James & Baras, 1995), (James & Baras, 1996), (Bernhard, 1994b), (Helton & James, 1997) for output feedback dynamic games and H_∞ control. It is interesting to note that this approach was motivated by considering a related risk-sensitive stochastic control problem; in stochastic control theory, the idea of information state (or sufficient statistic) is well known, see (Kumar & Varaiya, 1986).

Define the min-max cost functional

$$J_{p_0}(K)=\sup_{T\ge0}\sup_{w\in L_{2,T}}\sup_{x_0\in\mathbf{R}^n} \left\{p_0(x(0))+\tfrac{1}{2}\int_0^T\left[|z(s)|^2-\gamma^2|w(s)|^2\right]ds\right\}.$$

Here, $p_0(x_0)$ is a penalty term for the initial condition x_0 of the plant G, much like an a prior density in stochastic control. Clearly, (G,K) is dissipative if and only if

$$J_{-\beta}(K)\le 0, \tag{5.4}$$

for some $\beta\ge 0$ with $\beta(0)=0$. The min-max game is to minimize this cost function over the class of all causal maps K. To solve the game, we transform it into an equivalent full state information game by introducing an appropriate state, viz., the information state.

For fixed $u(\cdot),y(\cdot)$, the *information state* p_t is defined by

$$p_t(x)=p_0(\xi(0))+\tfrac{1}{2}\int_0^t[|C_1(\xi(s))+u(s)|^2 -\gamma^2|y(s)-C_2(\xi(s))|^2]\,ds, \tag{5.5}$$

where $\xi(\cdot)$ is the solution of

$$\dot\xi=A(\xi)+B_1(\xi)(y-C_2(\xi))+B_2(\xi)u \tag{5.6}$$

with *terminal* condition $\xi(t)=x$. This quantity describes the worst-case performance up to time t using the control u which is consistent with the observed output y and the constraint $x(t)=x$; in (Krener, 1994), this quantity is called a *conditional storage function.* Using the information state, the dissipative property can be represented in terms of the new completely observed information state, viz.

$$J_{p_0}(K)=\sup_{T\ge0}\sup_{y\in L_{2,T}}\{(p_T,0):p_0\text{ given }\}, \tag{5.7}$$

where $(p,q)\overset{\triangle}{=}\sup_x(p(x)+q(x))$ is the "sup-pairing" (James *et al*, 1994). Thus (G,K) is dissipative if and only if

$$\sup_{T\ge0}\sup_{y\in L_{2,T}}\{(p_T,0):p_0\text{ given }\}\le 0, \tag{5.8}$$

where $p_0=-\beta$ for some $\beta\ge0$, $\beta(0)=0$. Note that y is the disturbance for the transformed problem!

The dynamics for p_t is a partial differential equation: for fixed $u(\cdot),y(\cdot)$ we have

$$\dot p_t=F(p_t,u(t),y(t)), \tag{5.9}$$

where $F(p,u,y)$ is the differential operator

$$F(p,u,y) = -\nabla_x p \cdot (A + B_1(y - C_2) + B_2 u) + \tfrac{1}{2}[|C_1 + u|^2 - \gamma^2 |y - C_2|^2]. \quad (5.10)$$

This is the nonlinear analog of the Riccati equation (1.4). The dual PDI (5.3) (with equality) is just a stationary version the the information state dynamics (5.9), (with $u = 0$, $y = 0$).

The central controller is obtained by finding the controller which minimizes $J_p(K)$. If the H_∞ problem for G is solvable, then the function

$$W(p) = \inf_K J_p(K) \quad (5.11)$$

is finite on a domain dom$W$. By dynamic programming, W satisfies the (infinite dimensional) PDE

$$\begin{aligned} &\inf\nolimits_u \sup\nolimits_y \langle \nabla_p W(p), F(p,u,y)\rangle = 0, \\ &(p, 0) \le W(p), \quad W(-\beta) = 0. \end{aligned} \quad (5.12)$$

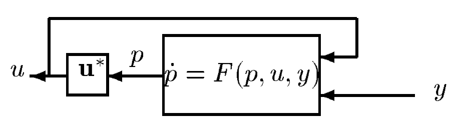
(For the purpose of this paper, we treat this equation in a formal sense; for more details, see (James & Baras, 1996), (Helton & James, 1997).) This is the fundamental PDE for output feedback H_∞ control (rather than (4.3)). Let $\mathbf{u}^*(p)$ and $\mathbf{y}^*(p)$ denote the values of u and y which attain the infimum and supremum in (5.12). Indeed, a direct calculation gives (when they exist)

$$\begin{aligned} \mathbf{u}^*(p) &= \langle \nabla_p W(p), -C_1 + B_2' \nabla_x p\rangle \\ \mathbf{y}^*(p) &= \langle \nabla_p W(p), C_2 - \tfrac{1}{\gamma^2} B_1' \nabla_x p\rangle. \end{aligned} \quad (5.13)$$

These formulas define the central controller K^* by

$$K^*[y](t) = \mathbf{u}^*(p_t[y]), \quad (5.14)$$

where p_0 is suitably chosen in dom$W$. Of course, the existence of the central controller depends on the smoothness of $W(p)$, in an approriate sense. The particular choice of p_0 is a very important issue (see (Helton & James, 1994), (Helton & James, 1997)). In (James & Baras, 1995) it is shown (in discrete-time) that under suitable hypotheses, K^* solves the H_∞ control problem for G if and only if the H_∞ control problem for G is solvable. Stability depends on observability conditions. A block diagram of the central controller is as follows.



This is the general structure of the output feedback controller.

The PDE (5.12) is perhaps unfamiliar in the context of H_∞ control. However, a storage function for the closed loop system (G, K) can be derived from it, (Helton & James, 1994), (Helton & James, 1997). Indeed, the non-negative function $e(x,p)$ defined by the simple formula

$$e(x,p) = -p(x) + W(p) \quad (5.15)$$

is a storage function. This follows because $e(x,p) \ge 0$, $e(0,\beta) = 0$, $\nabla_x e(x,p) = -\nabla_x p(x)$ and $\nabla_p e(x,p) = -E_x + \nabla_p W(p)$ (where E_x is the evaluation operator $\langle E_x, f\rangle = f(x)$), which impies that $e(x,p)$ satisfies the dissipation PDE

$$\begin{aligned} &\inf_u \sup_y \{\nabla_x e \cdot (A + B_1(y - C_2) + B_2 u) \\ &\quad + \langle \nabla_p e, F(p,u,y)\rangle \\ &\quad + \tfrac{1}{2}[|C_1 + u|^2 - \gamma^2 |y - C_2|^2]\} = 0, \end{aligned} \quad (5.16)$$

where the infinum and supremum are attained at $u = \mathbf{u}^*(p)$ and $y = \mathbf{y}^*(p)$ respectively. The optimal disturbance is $w^*(x,p) = -C_2(x) + \mathbf{y}^*(p)$. Integrating the PDE (5.16) yields the dissipation inequality

$$\begin{aligned} &e(x(t), p(t)) + \tfrac{1}{2}\int_0^t |z(s)|^2\, ds \\ &\quad \le e(x_0, p_0) + \tfrac{1}{2}\gamma^2 \int_0^t |w(s)|^2\, ds \end{aligned} \quad (5.17)$$

for any $w \in L_{2,t}$, for all $t \ge 0$.

In general, the information state solution to the (nonlinear) H_∞ control problem is *infinite dimensional*, since in general it is not possible to compute $p_t(x)$ using a finite set of ODE's. Thus in general, p_t evolves in the infinite dimensional space. Moreover, computing $\mathbf{u}^*(p)$ requires the solution of an infinite dimensional PDE—a very difficult task in general! However, the information state is finite dimensional in some cases, see (James & Yuliar, 1995), (Teolis *et al*, 1994).

In the case of linear systems, the information state is given (James & Yuliar, 1995) explicitly by

$$p_t(x) = -\frac{\gamma^2}{2}(x - \hat{x}(t))' Y^{-1}(t)(x - \hat{x}(t)) + \phi(t),$$

where

$$\begin{aligned} \dot{\hat{x}} &= (A - B_1 C_2 + Y(t)(\tfrac{1}{\gamma^2} C_1' C_1 - C_2' C_2))\hat{x} \\ &\quad + (B_1 + Y C_2) y(t) + (B_2 + \tfrac{1}{\gamma^2} Y C_1) u(t), \\ \dot{Y} &= (A - B_1 C_2) Y + Y (A - B_1 C_2)' \\ &\quad + Y(\tfrac{1}{\gamma^2} C_1' C_1 - C_2' C_2) Y, \\ \dot{\phi} &= \tfrac{1}{2}[|C_1 \hat{x} + u|^2 - \gamma^2 |y - C_2 \hat{x}|^2], \end{aligned}$$

with initial data p_0 specified by $x_0 = 0$, $Y_0 \geq 0$, $\phi_0 = 0$. Consequently, the infinite dimensional state variable p can be replaced by the finite dimensional state $(\hat{x}, Y, \phi)$. Note that we are using a Riccati differential equation for $Y(t)$, where Y_0 is chosen in the domain of attraction of the stabilizing solution of the corresponding algebraic Riccati equation (1.4), and should satisfy $Y_0 X < \gamma^2 I$, where $X \geq 0$ is a stabilizing solution of the Riccati equation (1.3). The value function is:

$$W(\hat{x}, Y, \phi) = \tfrac{1}{2}\hat{x}' X(1 - \tfrac{1}{\gamma^2} Y X)^{-1} \hat{x} + \phi.$$

Note that dom$W$ contains points $(\hat{x}, Y, \phi)$ for which $YX < \gamma^2 I$ whenever $\hat{x} \neq 0$. The optimal control and observation are:

$$\mathbf{u}^*(\hat{x}, Y, \phi) = -(C_1 + (B_2 + \tfrac{1}{\gamma^2} Y C_1) X (1 - \tfrac{1}{\gamma^2} Y X)^{-1})\hat{x},$$

$$\mathbf{y}^*(\hat{x}, Y, \phi) = (C_2 + (B_1 + Y C_2) X (1 - \tfrac{1}{\gamma^2} Y X)^{-1})\hat{x}.$$

The storage function is

$$e(x, \hat{x}, Y, \phi) = \tfrac{1}{2}[\gamma^2 (x - \hat{x})' Y^{-1} (x - \hat{x}) + \hat{x}' X (I - \tfrac{1}{\gamma^2} Y X)^{-1} \hat{x}].$$

In summary, the solution to the nonlinear output feedback H_∞ control problem involves:

(i) the existence of a solution $W(p)$ to the infinite dimensional PDE (5.12),
(ii) the existence of a solution p_t to the information state dynamics (5.9), and
(iii) correct choice of the initial condition $p_0 \in$ dom$W$.

An appropriate choice for p_0 is the stabilizing equilibrium solution p_e of (5.9). For further details, see (James & Baras, 1995), (Helton & James, 1994), (Helton & James, 1997). Note that the information state framework is quite general, and applies in other contexts, see (Baras & James, 1994).

Certainty equivalence. An approach to output feedback games building on a certainty equivalence principle was been developed by (Bernhard, 1990), (Basar & Bernhard, 1991), and is similar to the framework of (Whittle, 1981). In this approach, the key assumption is that the maximum of

$$p_t(x) + V(x)$$

is attained at a unique point $\bar{x}(t)$, called the *minimum stress estimate*. When this is true (assuming adequate smoothness), the output feedback controller is

$$u^*(t) = u^*(\bar{x}(t)),$$

where $u^*(x)$ is the optimal state feedback controller (4.4). This principle is discussed further in (James, 1994), (James & Baras, 1996), where it is placed in the information state framework.

Risk-sensitive stochastic control. Historically, the output feedback risk-sensitive stochastic optimal control problem was first solved by (Whittle, 1981) (for discrete-time linear systems), by (Bensoussan & van Schuppen, 1985) (for continuous-time linear systems), and by (James *et al*, 1994) (for discrete-time nonlinear systems). Significantly, the conditional density is *not* a sufficient statistic for the risk-sensitive problem, and a new information state was needed. This observation by Whittle was fundamental to stochastic control and to H_∞ control, (Glover & Doyle, 1988). An interesting example of risk-sensitive control not directly related to H_∞ control is being investigated by (Fernandez-Gaucherland & Marcus, 1994).

As mentioned above, the definition of information state p_t for the output feedback game problem was motivated (in fact originally derived) from an information state σ_t for a related output feedback risk-sensitive problem, see (James *et al*, 1994), (James *et al*, 1993). Indeed, one has

$$\lim_{\varepsilon \to 0} \varepsilon \gamma^2 \log \sigma_t(x) = p_t(x)$$

(up to a constant). If $S(\sigma, T)$ denotes the value function for a finite horizon output feedback risk-sensitive problem (c.f. (3.3)), then

$$\lim_{\varepsilon \to 0} \varepsilon \gamma^2 \log S(e^{\frac{p}{\varepsilon \gamma^2}}, T) = W(p, T)$$

is the value function for a finite horizon output feedback game.

Factorization. The information state framework can be applied to solve the J-inner outer factorization problem for the plant $\tilde{G}$ obtined from G by reversing the w and y arrows. The problem is to find a factorization $\tilde{G} = \theta R$, where θ is J-inner (J-dissipative and lossless), and R is outer (R and R^{-1} stable). These factors can be obtained by augmenting the information state central controller. For preliminary results, see (Helton & James, 1994).

6. CONCLUSION

In this paper we have attempted to summarize some of the key recent developments in nonlinear H_∞ control theory. As we have shown, partial differential equations and inequalities of various types related to the theory of dissipative systems are the fundamental mathematical equations of H_∞ control. These equations are also related to the equations of dynamic game theory and risk-sensitive stochastic control. The main technical (and practical) difficulty concerns solving these PDI's and PDE's, which in general have solutions which are not smooth. Further, one is faced with Bellman's "curse of dimensionality" in general. In spite of this, considerable insight into the nonlinear H_∞ control problem has been attained, and in future, much work needs to be done to obtain approximate solutions, and to produce controllers which are in fact *robust*.

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