1 NONLINEAR H_∞ CONTROL: A STOCHASTIC PERSPECTIVE*

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Abstract: The last few years have seen a number of exciting developments in the area known as nonlinear H_∞ control. The aim of work in this area is to design robust controllers for nonlinear systems using generalizations of the highly successful H_∞ methods used in linear systems theory. An interesting and important feature of the H_∞ problem is that it has close connections with differential games, risk-sensitive stochastic control, and dissipative systems, and as a consequence these subjects have enjoyed renewed interest in recent years. Indeed, these topics have a lot to contribute to the understanding of robust control.

In this talk I will discuss some recent developments in robust control from a "stochastic" perspective. Results to be presented include (i) singular information states and stability, (ii) semigroups and generators, and (iii) a robustness interpretation of the risk-sensitive performance index.

*Research supported in part by the Australian Research Council.

1.1 INTRODUCTION

Consider the following standard generalized regulator arrangement with nonlinear plant G and nonlinear controller K shown in Figure 1.1.1. The problem is to find a controller $K : y(\mathrm{cot}) \mapsto u(\cdot)$ for which the resulting closed-loop system (G, K) enjoys the following two properties:

(i) *Dissipation.* Given gain $\gamma > 0$ dissipation means that there exists a non-negative function β with $\beta(0) = 0$ for which the *dissipation inequality* holds:

$$\begin{cases} \frac{1}{2}\int_0^T |z(s)|^2\,ds \;\leq\; \gamma^2 \frac{1}{2}\int_0^T |w(s)|^2\,ds + \beta(x_0) \\ \text{for all} \;\; w \in L_{2,T} \text{ and all } T \geq 0. \end{cases} \tag{1.1.1}$$

(ii) *Stability.* By stability of the closed-loop system we mean that if G is initialized at any x_0, then if $w(\cdot) \in L_2[0, \infty)$, then in the closed-loop defined by $u = K(y)$ the signals $u(\cdot), y(\cdot), z(\cdot)$ belong to L_2 and the plant state $x(t)$ converges to 0 as $t \to \infty$.

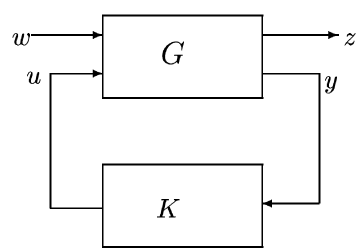

Figure 1.1.1 Closed-loop system (G, K)

The problem just specified is a generalization of basic problems from classical control engineering, and may at first sight have little to do with stochastic control, nor is any stochastic-like interpretation immediately apparent. Indeed, no probability was used in the problem definition. Background to a consideration of stochastic interpretations are the following facts.

(i) The H_∞ control problem was originally formulated in the frequency domain for linear systems. The problem can also be interpreted in the time domain (as above) as one involving L_2 gains using the *dissipative systems* framework developed by [31], [15].

(ii) It was shown in [12] that the linear H_∞ problem is equivalent to a *stochastic risk-sensitive* optimal control problem, in the sense that the solutions are the same (via the same Riccati equation). Further, it has been shown

[16] that the risk-sensitive solution is equivalent to a *deterministic differential game* (see also [29], [3]).

(iii) Except for the frequency domain formulation, all of the above control concepts and optimization problems have natural interpretations for nonlinear systems. Investigation of these concepts has been the subject of a considerable amount of research activity over the last decade (see, e.g. [28] and the references contained therein). In particular, the H_∞ control problem posed above can be solved by viewing it as a *minimax* game problem, where the controller K attempts to minimize a worst-case cost function defined by maximizing over the disturbance w, [2].

(iv) The so-called *max-plus* algebra provides an elegant framework for treating deterministic optimal control problems, see [1], [24], [26], etc. In the max-plus algebra, ordinary addition is replaced by the maximum binary operator, and so optimal control problems involving maximization over a control variable have natural stochastic analogs where the maximization is interpreted as an integration. There are also very close connections with the theory of *large deviations*, see [6], [22].

So since the nonlinear H_∞ control problem can be formulated as a minimax game, it lends itself to interpretation using the max-plus algbebra, with it's strong stochastic analogs. Moreover, the minimax formulation is directly related to risk-sensitive stochastic optimal control via small noise limits (large deviations), see [30], [9], [19], [17], etc.

The *information state* approach to solving partially observed stochastic optimal control problems has been well known since at least the 1960's, [27], [23]. Recently this approach has proved very fruitful for solving output feedback risk-sensitive optimal control problems and minimax differential games, [17], and from this an information state theory for nonlinear H_∞ control is being developed [20], [14]. Interestingly, the "stochastic" concept of information state is the key to a general solution of the nonlinear H_∞ control problem.

Development of the information state approach is nontrivial; indeed new and deep mathematical questions have arisen. Key issues include:

(i) *Dynamic programming PDE.* The dynamic progrogramming PDE is defined on an infinite dimensional space. There is little mathematical theory available, and questions concerning the correct definition of solution, uniqueness theorems, etc, are unanswered, [21].

(ii) *Stability.* The stability or asymptotic behavior of the information state dynamical system is only just beginning to be understood, [13], [14]. The max-plus framework appears to be essential for interpreting information state convergence in general.

In this paper we will discuss aspects of these and related issues. Section 1.2 reviews the basic information state solution (see [20], [14] for full details). Then is section 1.3 singular information states are introduced, and stability of the information state system is considered in a max-plus framework. A semigroup associated with the nonlinear H_∞ control problem is defined in section 1.4, and its generator is calculated on a class of test functions.

1.2 INFORMATION STATE SOLUTION

We consider nonlinear plants G of the form

$$G : \begin{cases} \dot{x} &= A(x) + B_1(x)w + B_2(x)u \\ z &= C_1(x) + D_{12}(x)u \\ y &= C_2(x) + D_{21}(x)w. \end{cases} \tag{1.2.1}$$

Here, $x(t) \in \mathbf{R}^n$ denotes the state of the system, and is not in general directly measurable; instead an output $y(t) \in \mathbf{R}^p$ is observed. The additional output quantity $z(t) \in \mathbf{R}^m$ is a performance measure, depending on the particular problem at hand. The control input is $u(t) \in \mathbf{R}^m$, while $w(t) \in \mathbf{R}^p$ is regarded as an opposing disturbance input. We assume that all of the functions appearing in (1.2.1) are as smooth and bounded as necessary, and that zero is an equilibrium: $A(0) = 0$, $C_1(0) = 0$ and $C_2(0) = 0$. In order to simplify the notation as much as possible, we take $D_{12}(x) = I_m$ and $D_{21}(x) = I_p$; this is known as the "one block problem"(this includes some important problems from classical control, such as the mixed sensitivity problem).

The following signal spaces will be used: $L_2 = L_2([0,\infty), \mathbf{R}^q)$, $L_{2,T} = L_2([0,T], \mathbf{R}^q)$, and $L_{2,loc} = \cup_{T>0} L_{2,T}$, where the dimension of the range space will not be stated explicity but infered from the context.

A *controller* K is a causal mapping $K : L_{2,loc} \to L_{2,loc}$ taking outputs y to inputs u. Such a controller will be termed *admissible* if the closed loop equations for G (1.2.1) with $u = K(y)$ are well defined in the sense that unique solutions exist in $L_{2,loc}$. We will assume $K(0) = 0$.

For $p : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ (more will be said about p later) and a controller K define the cost functional

$$J_{p_0}(K) = \sup_{T \geq 0} \sup_{w \in L_{2,T}} \sup_{x_0 \in \mathbf{R}^n} \left\{ p_0(x(0)) + \tfrac{1}{2} \int_0^T \left[|z(s)|^2 - |w(s)|^2 \right] ds \right\}.$$

This can be interpreted as a requirement that a max-plus expectation be negative. Further, this captures the notion of dissipation, since (G, K) is dissipative if and only if

$$J_{-\beta}(K) \leq 0, \text{ some } \beta \geq 0, \ \beta(0) = 0. \tag{1.2.2}$$

A system $\tilde{G}$ is obtained from G by reversing the w and y arrows; the equations defining $\tilde{G}$ are as follows:

$$\tilde{G}: \begin{cases} \dot{\xi} &= A(\xi) + B_1(\xi)(y - C_2(\xi)) + B_2(\xi)u \\ z &= C_1(\xi) + u \\ w &= -C_2(\xi) + y. \end{cases} \tag{1.2.3}$$

For fixed $u, y \in L_{2,loc}$, the *information state* p_t is defined by

$$p_t(x) = p_0(\xi(0)) + \tfrac{1}{2}\int_0^t [|C_1(\xi(s)) + u(s)|^2 - |y(s) - C_2(\xi(s))|^2]\, ds, \tag{1.2.4}$$

where $\xi(\cdot)$ is the solution of (1.2.3) with *terminal* condition $\xi(t) = x$. Using the information state, the dissipative property can be characterized as:

$$J_{p_0}(K) = \sup_{T \geq 0} \sup_{y \in L_{2,T}} \{\langle p_t \rangle\} \leq 0, \tag{1.2.5}$$

for some p_0, where $\langle p \rangle = \langle p + 0 \rangle$ and

$$\langle p + q \rangle \stackrel{\triangle}{=} \sup_{x \in \mathbf{R}^n} \{p(x) + q(x)\}$$

is the *max-plus inner product* (called "sup-pairing"in [17]).

The dynamics for p_t when smooth is a partial differential equation: for fixed $u \in L_{2,loc}$ and $y \in L_{2,loc}$ we have

$$\dot{p}_t = F(p_t, u(t), y(t)), \tag{1.2.6}$$

where $F(p, u, y)$ is the differential operator

$$\begin{aligned} F(p, u, y) &= -\nabla_x p \cdot (A + B_1(y - C_2) + B_2 u) \\ &\quad + \tfrac{1}{2}|C_1 + u| - \tfrac{1}{2}|y - C_2|^2. \end{aligned} \tag{1.2.7}$$

The central controller is obtained by finding the controller which minimizes $J(K)$. If the H_∞ problem for G is solvable, then the function

$$W(p) = \inf_K J_p(K) \tag{1.2.8}$$

is finite on a certain domain dom$W$. By dynamic programming, W formally satisfies the (infinite dimensional) PDE

$$\inf_u \sup_y \{\nabla_p W(p)[F(p, u, y)]\} = 0. \tag{1.2.9}$$

Let $\mathbf{u}^*(p)$ and $\mathbf{y}^*(p)$ denote the values of u and y which attain the infimum and supremum in (1.2.9). Indeed, a direct calculation gives

$$\begin{aligned} \mathbf{u}^*(p) &= \nabla_p W(p)[-C_1 + B_2' \nabla_x p] \\ \mathbf{y}^*(p) &= \nabla_p W(p)[C_2 - B_1' \nabla_x p] \end{aligned} \tag{1.2.10}$$

These formulas define the *optimal information state controller* K^* by

$$K^*[y](t) = \mathbf{u}^*(p_t[y]), \tag{1.2.11}$$

where p_0 is suitably chosen in dom$W$ (the particular choice of p_0 is a very important issue.) In [20] it is shown for discrete-time problems that under suitable hypotheses, K^* *solves the* H_∞ *control problem for* G *if and only if the* H_∞ *control problem for* G *is solvable.* These and more detailed results have been obtained for continuous time problems in [14].

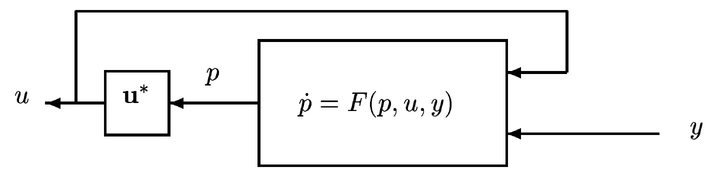

Figure 1.2.1 Optimal information state controller K^*.

Regarding this solution, we make the following remarks.

(i) In the general case, with $D_{21} \neq I$, the differential operator $F(p, u, y)$ is nonlinear, and so the information state p_t is not smooth in general. So the PDE (1.2.6) governing the information state dynamics needs to be interpreted in a weak (e.g. viscosity) sense.

(ii) Information states need not be everywhere finite (these are called *singular information states* in [14]).

(iii) A framework is needed which will permit interpretation of the dynamic programming PDE (1.2.9), and the formulas (1.2.10) for the optimal control and observation as well as for convergence of information states (and hence stability).

These points are discussed in detail in [14].

1.3 SINGULAR INFORMATION STATES AND STABILITY

A crucial issue in the information state solution is initialization of the controller $K^*_{p_0}$, and this in turn is very closely related to equilibrium solutions p_e of the information state equation:

$$0 = F(p_e, 0, 0). \tag{1.3.1}$$

By stability of the information state system (1.2.6) we will mean convergence, in a sense to be described, to a particular stable equilibrium (modulo additive constants):

$$p_t \to p_e + c \text{ as } t \to \infty,$$

where $c \in \mathbf{R}$ and the equilibrium p_e is defined by

$$p_e(x) = \lim_{T\to\infty} \{ \tfrac{1}{2} \int_{-T}^{0} [|C_1(\xi(s))|^2 - |C_2(\xi(s))|^2]\, ds \}, \tag{1.3.2}$$

where $\xi(\cdot)$ is the solution of (1.2.3) with $u = 0$, $y = 0$, and $\xi(0) = x$:

$$\dot{\xi} = A^\times(\xi), \tag{1.2.3$'$}$$

where the vector field $A^\times$ is defined by

$$A^\times \triangleq A - B_1 C_2. \tag{1.3.3}$$

The nature of the equilibrium p_e depends on the stability properties of $A^\times$. If $A^\times$ is L_2 exponentially stable, i.e.

$$|\Phi^v_{t,s}(x)| \le C|x|e^{-c(t-s)/2} + C(\int_s^t e^{-c(t-r)}|v(r)|^2 dr)^{\frac{1}{2}}, \tag{1.3.4}$$

where $v = (u, y)$ is input to (1.2.3), then

$$p_e = \delta_0, \tag{1.3.5}$$

where δ_0 is a *singular* information state of the form

$$\delta_M(x) = \begin{cases} 0 & \text{if } x \in M, \\ -\infty & \text{if } x \notin M, \end{cases} \tag{1.3.6}$$

where $M \subset \mathbf{R}^n$. On the other hand, if $-A^\times$ is L_2 exponentially stable, then p_e is a function which is everywhere finite. If $A^\times$ is hyperbolic, with stable and antistable directions, then

$$p_e(x) = \delta_{M_{as}} + \breve{p}, \tag{1.3.7}$$

where M_{as} is the antistable manifold for $A^\times$, and $\breve{p}$ is a finite function on M_{as}.

In [14], it is shown that the choice $p_0 = p_e$ is a natural and correct initialization of the information state controller. Thus it is important to make use of singular information states in the theory of nonlinear H_∞ control.

Let's consider the convergence

$$p_t \to \delta_0 + c$$

in the case where $A^\times$ is L_2 exponentially stable, with initialization $p_0 = \delta_0$ and inputs $u, y \in L_2$. By [14], p_t is given by

$$p_t = \delta_{\xi(t)} + c(t),$$

where $\xi(\cdot)$ is the trajectory of the system (1.2.3) with initial condition $\xi(0) = 0$ and inputs u, y, and $c(t)$ is the integral accummulated long this trajectory:

$$c(t) = \tfrac{1}{2}\int_0^t \left[|C_1(\xi(s)) + u(s)|^2 - |y(s) - C_2(\xi(s))|^2\right] ds.$$

The stability assumption ensures that

$$\xi(t) \to 0 \quad \text{and} \quad c(t) \to c(u,y) \quad \text{as } t \to \infty,$$

with $c(u,y) \in \mathbf{R}$. However,

$$\delta_{\xi(t)} \quad \text{does not converge } \textbf{pointwise} \text{ to} \quad \delta_0 \quad \text{as} \quad t \to \infty.$$

To see this, consider the generic situation so that $\xi(t) \neq 0$ for $t > 0$ (recall $\xi(0) = 0$). Then

$$\delta_{\xi(t)}(0) = -\infty < \delta_0(0) = 0 \text{ for all } t > 0$$

implies that

$$\lim_{t\to\infty} \delta_{\xi(t)}(0) = -\infty \neq \delta_0(0) = 0.$$

Therefore it is necessary to relax the mode of convergence. We do this by making use of weak convergence in the max-plus sense. Here, the information states are interpreted as *max-plus measures.*

Let $\mathcal{X}_e$ denote the space of u.s.c. functions $p : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ which are bounded above: $\langle p \rangle < +\infty$. This is the natural state space for the information state system. Given a subset $A \subset \mathbf{R}^n$, the *max-plus p-measure of A*, denoted $p(A)$, is defined by

$$p(A) = \sup_{x \in A} p(x). \tag{1.3.8}$$

This p-measure has many properies analogous to the usual measures. The *max-plus integral of f with respect to p* is defined by

$$p(f) = \sup_{x \in \mathbf{R}^n} \{p(x) + f(x)\} = \langle p + f \rangle. \tag{1.3.9}$$

Note that $p(\delta_A) = p(A)$, just as in usual measure theory.

A sequence $\{p_n\} \subset \mathcal{X}_e$ is said to *converge weakly* to $p_\infty \in \mathcal{X}_e$ if

(i) $\liminf_{n\to\infty} p_n(G) \geq p_\infty(G)$ for all open subsets $G \subset \mathbf{R}^n$, and

(ii) $\limsup_{n\to\infty} p_n(F) \leq p_\infty(F)$ for all closed subsets $F \subset \mathbf{R}^n$.

Weak convergence is denoted $p_n \Longrightarrow p_\infty$.

Theorem 1 *Let $p_n, p_\infty \in \mathcal{X}_e$. Then $p_n \Longrightarrow p_\infty$ if and only if*

$$p_n(f) \to p_\infty(f)$$

in $\mathbf{R}_e$ for all $f \in C_b(\mathbf{R}^n)$ (continuous, bounded).

For a proof of this and related theorems, see [22], [6], [14].

In the next theorem we apply this concept of weak convergence to obtain a stability result for the information state.

Theorem 2 *Assume $A^\times$ is L_2 exponentially stable. Let $u, y \in L_2[0,\infty)$ and $p_0 = p_e$ given by (1.3.5). Then*

$$p_t \Longrightarrow p_e + c(u,y) \text{ as } t \to \infty \tag{1.3.10}$$

where $c(u,y)$ is a real number depending on u, y.

PROOF. Given the above discussion, it remains to prove $\delta_{\xi(t)} \Longrightarrow \delta_0$ as $t \to \infty$. Select a test function $f \in C_b(\mathbf{R}^n)$, as in Theorem 1. Then for any $x \in \mathbf{R}^n$,

$$\delta_x(f) = \langle \delta_x + f \rangle = f(x),$$

and so

$$\delta_{\xi(t)}(f) = f(\xi(t)) \to f(0) = \delta_0(f) \text{ as } t \to \infty.$$

□

In summary, we see that information states have natural and useful interpretation as max-plus measures, and using max-plus weak convergence, stability of the information state system can be addressed.

1.4 SEMIGROUPS AND GENERATORS

We turn now to the dynamic programming PDE

$$\inf_u \sup_y \{\nabla_p W(p)[F(p,u,y)] = 0, \tag{1.2.9}$$

and the formula for the optimal control

$$\mathbf{u}^*(p) = \nabla_p W(p)[-C_1 + B_2' \nabla_x p]. \qquad (1.2.10)'$$

Both of these expressions present serious difficulties since one would like to interpret them for p in the space $\mathcal{X}_e$. The main difficulties are the interpretation of the gradient $\nabla_p W$ (since $\mathcal{X}_e$ is not a standard space of functional analysis such as a Banach or Hilbert space) if we are fortunate enough for a derivative to exist, the definition of weak solution since in general W will not be differentiable in a classical sense, and interpretation of the terms in (1.2.9) and (1.2.10) involving $\nabla_x p$ when p is not differentiable, and in particular, when p is singular.

An initial attempt at formulating a definition of viscosity solution for the PDE (1.2.9) was made in [21]. However, no uniqueness results have been obtained and it is likely that this definition will be eventually superceded. Even if a satisfactory viscosity theory is developed, there remains the basic issue of synthesis of optimal controls.

We now present a theory of semigroups and generators for the nonlinear H_∞ control problem that has recently been developed in [14], [18]. This theory was inspired by the Nisio semigroup as studied in the case of partially observed stochastic control in [8]. The idea is to define the semigroup associated with the H_∞ problem and describe it generator on a class of test functions. As we shall see, this leads to an interpretation of $\nabla_p \psi(p)[F(p,u,y)]$ which is well defined for non-differentiable and even singular p, for a certain class of test functions ψ.

Let $S_t^{u,y}$ denote the operator which maps an initial information state $p_0 = p$ to the information state at time t determined by the inputs u, y. By dynamic programming, $S_t^{u,y}$ satisfies the semigroup property. The semigroup $S_t^{u,y}$ induces a semigroup $\mathcal{S}_t^{u,y}$ in the space

$$\mathbf{F}(\mathcal{X}_e) = \{p \mid p : \mathcal{X}_e \to \mathbf{R}\}$$

of real valued functions defined on $\mathcal{X}_e$. For $\psi \in \mathbf{F}(\mathcal{X}_e)$, write

$$\mathcal{S}_t^{u,y}\psi(p) = \psi(S_t^{u,y}p) \tag{1.4.1}$$

whenever the RHS is defined. In general there will be a domain of functions ψ and points p for which the RHS is defined. Indeed, if $\psi(p)$ is defined for all $p \in \mathcal{X}_e$, then $\mathcal{S}_t^{u,y}\psi(p)$ is defined for all $p \in P_t^{u,y}$, where

$$P_t^{u,y} = \{p \in \mathcal{X}_e \ : \ S_t^{u,y}p \in \mathcal{X}_e\}.$$

Using this semigroup, we can express the value function W defined by (1.2.8) as

$$W(p) = \inf_K \sup_{T \geq 0, y \in L_{2,T}} \mathcal{S}_T^{K(y),y}\langle \cdot \rangle (p)$$

(the semigroup is applied to the function $\psi(p) = \langle p \rangle$).

We shall say that an operator $\mathcal{L}^{u,y}$ is a *"generator"* for the transition operator $\mathcal{S}^{u,y}$ if there exists a nonempty set $\mathrm{dom}\mathcal{L}^{u,y} \subset \mathbf{F}(\mathcal{X}_e)$ such that for $\psi \in \mathrm{dom}\mathcal{L}^{u,y}$, and each constant pair $(u,y) \in \mathbf{R}^m \times \mathbf{R}^p$ the limit

$$\mathcal{L}^{u,y}\psi(p) = \lim_{t \to 0} \frac{\mathcal{S}_t^{u,y}\psi(p) - \psi(p)}{t} \tag{1.4.2}$$

exists for p belonging to a nonempty set $\mathrm{dom}\mathcal{L}^{u,y}\psi \subset \mathcal{X}_e$.

In general it will be difficult or impossible to evaluate $\mathcal{L}^{u,y}\psi$ for arbitrary ψ. However, it is possible to evaluate $\mathcal{L}^{u,y}$ for certain types of functions ψ. Let us define the following class of test functions:

$$\hat{\mathcal{G}}_b = \{\psi \in \mathcal{G}_b \ : \ \psi(p) = \ g(\langle p + f_1 \rangle, \ldots, \langle p + f_k \rangle),$$
$$\text{for some } k \geq 1, \ g \in C_b^1(\mathbf{R}^k), \ f_1, \ldots, f_k \in C_b^1(\mathbf{R}^n)\}$$

The next theorem provides explicit evaluation of $\mathcal{L}^{u,y}\psi$ for $\psi \in \hat{\mathcal{G}}_b$. Write

$$[[p+f]] = \operatorname*{argmax}_{x\in\mathbf{R}^n}\{p(x)+f(x)\}.$$

Theorem 3 *Let $\psi \in \hat{G}_b$ and let $p \in \mathcal{X}_e$ be tight. Then for each constant pair $(u,y) \in \mathbf{R}^m \times \mathbf{R}^p$*

$$\begin{aligned}\mathcal{L}^{u,y}\psi(p) &= \textstyle\sum_{i=1}^k \partial_i g(\langle p+f_1\rangle,\ldots,\langle p+f_k\rangle)\cdot \\ &\qquad \sup_{\bar{x}\in[[p+f_i]]}\{\nabla_x f_i(\bar{x})\cdot(A^\times(\bar{x})+B_1(\bar{x})y+B_2(\bar{x})u) \\ &\qquad\qquad +\tfrac{1}{2}|C_1(\bar{x})+D_{12}u|^2-\gamma^2\tfrac{1}{2}|y-C_2(\bar{x})|^2]\} \\ &= \textstyle\sum_{i=1}^k \partial_i g(\langle p+f_1\rangle,\ldots,\langle p+f_k\rangle)\cdot \sup_{\bar{x}\in[[p+f_i]]}\{F(-f_i,u,y)(\bar{x})\}\end{aligned} \tag{1.4.3}$$

Thus $\mathcal{L}^{u,y}$ is a "generator" for the semigroup $\mathcal{S}^{u,y}$ in the above sense.

A remarkable feature of the expression (1.4.3) is that it does not involve any derivatives of p; the derivatives have been "transfered" to the test functions f_i. Thus (1.4.3) provides a "weak-sense" interpretation of the formal expression $\nabla_p\psi(p)[F(p,u,y)]$ (recall from the definition (1.2.7) that $F(p,u,y)$ involves the gradient $\nabla_x p$, which may not exist, whereas $F(-f_i,u,y)$ involves $\nabla_x f_i$, which does exist).

PROOF. (SKETCH). The complete proof is given in [14], and we give here the main idea.

Assume $k=1$, so that $\psi(p) = g(\langle p+f\rangle)$. Due to the smoothness of g, it is enough to show that

$$\begin{aligned}\lim_{t\to 0}\frac{p_t(f)-p(f)}{t} &= \sup_{\bar{x}\in[[p+f]]}\{\nabla_x f(\bar{x})\cdot(A^\times(\bar{x})+B_1(\bar{x})y+B_2(\bar{x})u) \\ &\qquad +\tfrac{1}{2}|C_1(\bar{x})+D_{12}u|^2-\gamma^2\tfrac{1}{2}|y-C_2(\bar{x})|^2\}\end{aligned} \tag{1.4.4}$$

Fix $(u,y)\in\mathbf{R}^m\times\mathbf{R}^p$ and write $\xi(s)=\Phi_{s,t}(x)=\Phi^{u,y}_{s,t}(x)$ for the transition operator of (1.2.3) with end point $\xi(t)=x$. So the solution at time s is $\xi(s)=\Phi^{u,y}_{s,t}(x)$. By the definition (1.2.4) we have

$$p_t(f)=\sup_x\{p(\Phi_{0,t}(x))+\int_0^t[\tfrac{1}{2}|C_1(\xi(s))+D_{12}u|^2-\gamma^2\tfrac{1}{2}|y-C_2(\xi(s))|^2]\,ds+f(x)\},$$

where $\xi(t) = x$, for any $t > 0$. Next, subtract $p(f) = \langle p + f \rangle$, divide by t, and add-subtract $f(\Phi_{0,t}(x))/t$, to get

$$\begin{aligned}\frac{p_t(f) - p(f)}{t} &= \sup_x \{ \frac{f(x) - f(\Phi_{0,t}(x))}{t} \\ &\quad + \frac{1}{t}\int_0^t [\tfrac{1}{2}|C_1(\xi(s)) + D_{12}u|^2 - \gamma^2 \tfrac{1}{2}|y - C_2(\xi(s))|^2]\, ds \\ &\quad + \frac{-p(f) + p(\Phi_{0,t}(x)) + f(\Phi_{0,t}(x))}{t} \}\end{aligned} \tag{1.4.5}$$

Let

$$L = \limsup_{t\to 0} \frac{p_t(f) - p(f)}{t}$$

and let t_i be a subsequence such that

$$\lim_{i\to\infty} t_i = 0, \text{ and } \lim_{i\to\infty} \frac{p_{t_i}(f) - p(f)}{t_i} = L.$$

Select $\bar{x}_i \in \operatorname{argmax}\{p_{t_i}(\cdot) + f(\cdot)\}$. By the tightness of p, and $\lim_{t\to 0} p_t = p$ weakly, we can assume (by selecting a further subsequence if necessary) $\lim_{i\to\infty} \bar{x}_i = \bar{x} \in [[p + f]] = \operatorname{argmax}_x\{p(x) + f(x)\}$.

Then since

$$-p(f) + p(\Phi_{0,t}(x)) + f(\Phi_{0,t}(x)) \le 0$$

for any x, we have, setting $x = \bar{x}_i$ in (1.4.5),

$$\begin{aligned}\frac{p_{t_i}(f) - p(f)}{t_i} &\le \frac{f(\bar{x}_i) - f(\Phi_{0,t_i}(\bar{x}_i))}{t_i} \\ &\quad + \frac{1}{t_i}\int_0^{t_i} [\tfrac{1}{2}|C_1(\xi(s)) + D_{12}u|^2 - \gamma^2 \tfrac{1}{2}|y - C_2(\xi(s))|^2]\, ds,\end{aligned}$$

where $\xi(t_i) = \bar{x}_i$. Sending $i \to \infty$ gives

$$\begin{aligned}&\limsup_{i\to\infty} \frac{p_{t_i}(f) - p(f)}{t_i} \\ &\le \nabla_x f(\bar{x}) \cdot (A^\times(\bar{x}) + B_1(\bar{x})y + B_2(\bar{x})u) \\ &\quad + \tfrac{1}{2}|C_1(\bar{x}) + D_{12}u|^2 - \gamma^2 \tfrac{1}{2}|y - C_2(\bar{x})|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{t\to 0} \frac{p_t(f) - p(f)}{t} &\le \sup_{\bar{x}' \in [[p+f]]} \{\nabla_x f(\bar{x}') \cdot (A^\times(\bar{x}') + B_1(\bar{x}')y + B_2(\bar{x}')u) \\ &\quad + \tfrac{1}{2}|C_1(\bar{x}') + D_{12}u|^2 - \gamma^2 \tfrac{1}{2}|y - C_2(\bar{x}')|^2\}.\end{aligned} \tag{1.4.6}$$

This proves the upper half of (1.4.4). The reverse inequality can also be proven. □

In the important special case of *certainty equivalence* [2], the value function turns out to be $W(p) = \langle p + V \rangle$ (for some p), where V is the state feedback H_∞ value function, and one assumes that $[[p + V]] = \bar{x}$ is unique. Then we can evaluate

$$\begin{aligned} \mathbf{u}^*(p) &= \underset{u}{\operatorname{argmin}} \sup_y \mathcal{L}^{u,y} \langle p + V \rangle \\ &= -C_1(\bar{x}) - B_2(\bar{x})' \nabla_x V(\bar{x})' \end{aligned}$$

which makes sense for singular p (provided V is smooth). The p dependence is via $\bar{x} = \bar{x}(p)$.

In terms of the operator $\mathcal{L}^{u,y}$, the dynamic programming PDE (1.2.9) takes the form

$$\inf_{u \in \mathbf{R}^m} \sup_{y \in \mathbf{R}^p} \{ \mathcal{L}^{u,y} W(p) \} = 0. \tag{1.2.9$'$}$$

This is a "weak-sense" view of the dynamic programming PDE of nonlinear H_∞ control. It is analogous to simlar equations in stochastic control.

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