# $L^\infty$-BOUNDED ROBUSTNESS: STATE FEEDBACK ANALYSIS AND SYNTHESIS

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Abstract: In this paper we generalize the Hill-Moylan-Willems framework for dissipative systems to accommodate $L^\infty$ criteria, and a mixture of $L^\infty$ and integral criteria. The generalized dissipation property is completely characterized in terms of a partial differential inequality, interpreted in the viscosity sense. These results are then applied to derive a state feedback synthesis procedure using a minimax version of the PDI. This gives a framework for mixed $L^\infty$-bounded/integral robust control design, and in particular, for $L^\infty$-bounded robust control design. *Copyright © 2001 IFAC*

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## 1. INTRODUCTION

Techniques for the design of robust control systems and indeed for optimal control in general have primarily made use of integral-type performance criteria. These criteria are sometimes referred to as *soft criteria*, since a bound on the performance integral need not guarantee that an output quantity meets absolute bounds or constraints. In some applications it is important for outputs to meet hard constraints, such as absolute regulation error always less an a specified amount. Such situations can be formulated in terms of $L^\infty$-type criteria, which might be called *hard criteria*. Techniques for analysis and design using such criteria have begun to emerge, including Barron and Ishii (1989), Barron (1990), Dahleh and Diaz-Bobillo (1995), Fialho and Georgiou (1999).

In this paper we generalize the Hill-Moylan-Willems framework for dissipative systems to accommodate $L^\infty$ criteria, and a mixture of $L^\infty$ and integral criteria. This generalization builds on the $L^\infty$ work of Fialho and Georgiou (1999). The generalized dissipation property is completely characterized in terms of a partial differential inequality (PDI), interpreted in the viscosity sense. These results are then applied to derive a state feedback synthesis procedure using a minimax version of the PDI. This gives a framework for mixed $L^\infty$-bounded/integral robust control design, and in particular, for $L^\infty$-bounded robust control design. The measurement feedback problem is considered in the full paper James (2000).

## 2. ANALYSIS

Consider the following dynamical system

[1] Partially supported by ARC and EPSRC.

$$\begin{aligned}\dot{\xi} &= f(\xi, w)\\ z_1 &= g_1(\xi, w)\\ z_2 &= g_2(\xi, w)\end{aligned} \tag{2.1}$$

with “disturbance” input $w(\cdot)$ and real-valued performance output quantities $z_1(\cdot)$, $z_1(\cdot)$. Here, $\xi(\cdot)$, is the system state trajectory taking values in $\mathbf{R}^n$, and the input $w(\cdot)$ takes values in $\mathbf{W} \subset \mathbf{R}^s$, $\mathbf{W}$ compact. Let $\mathcal{W}_{t_0,t_1} = \{$ measurable functions $w : [t_0, t_1] \to \mathbf{W}\}$ denote the class of admissible inputs. The functions $f : \mathbf{R}^n \times \mathbf{W} \to \mathbf{R}^n$ and $g_i : \mathbf{R}^n \times \mathbf{W} \to \mathbf{R}$ $(i = 1, 2)$ are assumed bounded and Lipschitz continuous.

### 2.1 *$L^\infty$-Bounded/Integral Dissipation*

The following definition is motivated by the disturbance problem studied in Fialho and Georgiou (1999), the $\ell_1$ performance specification formulated in Dahleh and Diaz-Bobillo (1995), and the cost functions in Barron and Ishii (1989). It is one possible definition of a mixture of dissipation-like properties with $L^\infty$ and integral criteria. The dissipative systems framework was introduced by Willems (1972).

The system (2.1) is *$L^\infty$-bounded/integral (LIB/I) dissipative* if there exists a finite real-valued function $\beta$ (called a *bias*) such that

$$\int_0^t z_1(r)\,dr + z_2(t) \;\leq\; \beta(x_0), \quad \text{a.e. } t \geq 0, \tag{2.2}$$

for all $w \in \mathcal{W}_{0,\infty}$, where the performance quantities are evaluated along the corresponding trajectory of (2.1) with initial condition $\xi(0) = x_0$. This could be viewed as an input-output $(w \mapsto (z_1, z_2))$ property of the system with initialization $x_0$. Special cases of this type of dissipation property will be discussed in §2.3 below.

Following Willems (1972), we call a function $V : \mathbf{R}^n \to \mathbf{R}$ a *storage function* for the system (2.1) if it is finite valued and

$$\begin{aligned} V(x) \;\geq \max\{&V(\xi(t)) + \int_0^t z_1(r)\,dr,\\ &\text{ess.sup}_{s\in[0,t]}\{\int_0^s z_1(r)\,dr + z_2(s)\}\} \end{aligned} \tag{2.3}$$

for all $w \in \mathcal{W}_{0,t}$, and all $t \geq 0$, where $\xi(\cdot)$ is the state trajectory of (2.1) with initial state $\xi(0) = x$; i.e.

$$\begin{aligned} V(x) \geq &\\ &\sup_{t\geq 0}\sup_{w\in\mathcal{W}_{0,t}} \max\{V(\xi(t)) + \int_0^t z_1(r)\,dr,\\ &\quad \text{ess.sup}_{s\in[0,t]}\{\int_0^s z_1(r)\,dr + z_2(s)\}\}. \end{aligned} \tag{2.4}$$

Inequality (2.3) (or (2.4)) is called the *LIB/I dissipation inequality.*

In the general theory of dissipative systems, two particular storage functons are of special interest, viz. the available storage and the required supply. In our present context, the *available storage* is defined by

$$\begin{aligned} V_a(x) \;\stackrel{\triangle}{=} &\sup_{t\geq 0}\sup_{w\in\mathcal{W}_{0,t}}\\ &\quad \text{ess.sup}_{s\in[0,t]}\{\int_0^s z_1(r)\,dr + z_2(s)\}. \end{aligned} \tag{2.5}$$

The available storage $V_a$ is lower semicontinuous (l.s.c.) and is a storage function for the system (2.1); in fact, it is the minimal storage function; i.e. $V \geq V_a$ for any storage function.

*Proposition 2.1.* The system (2.1) is LIB/I dissipative iff there exists a storage function.

Proof. If $V$ is a storage function for the system (2.1), then it follows directly from the definition (2.3) that

$$V(x) \geq \int_0^t z_1(r)\,dr + z_2(t) \quad \text{a.e. } t \geq 0, \tag{2.6}$$

which says that the output quantity on the RHS is essentially bounded by the initial stored “energy” $V(x)$. We can take $\beta = V$, and conclude that (2.1) is LIB/I dissipative.

On the other hand, if (2.1) is LIB/I dissipative the inequality (2.2) implies the finiteness of the available storage $V_a$ (defined by (2.5)). □

*Proposition 2.2.* If a locally bounded function $V$ satisfies the dissipation inequality (2.4), then so does its lower semicontinuous envelope $V_*$ defined by

$$V_*(x) = \liminf_{x'\to x} V(x'); \tag{2.7}$$

i.e.

$$V_*(x) \geq \sup_{t\geq 0} \sup_{w\in\mathcal{W}_{0,t}} \max\{V_*(\xi(t)) + \int_0^t z_1(r)\,dr, \quad \text{ess.sup}_{s\in[0,t]}\{\int_0^s z_1(r)\,dr + z_2(s)\}\} \tag{2.8}$$

PROOF. The proof is similar to that of (James, 1993, Proposition 2.3). □

## 2.2 *Partial Differential Inequality*

In the general theory of dissipative systems, storage functions are characterized infinitesimally by a *partial differential inequality* (PDI) (Bounded Real Lemma), and storage functions can be obtained by solving the PDI, e.g. van der Schaft (1996). In Fialho and Georgiou (1999), the available storage $V_a$ in the context of $L_\infty$-bounded gain was shown to be a viscosity solution of a partial differential equation (PDE) of the variational inequality type (VI). In this section we extend this by giving a complete characterization of the LIB/I dissipation property using a PDI, following James (1993); actually, there are at least two possible ways of expressing it, following Barron and Ishii (1989), Barron (1990), Fialho and Georgiou (1999). The definition of viscosity solutions is given in Appendix 4.

*Theorem 2.3.* If the system (2.1) is LIB/I dissipative with locally bounded storage function $V$, then $V$ is a viscosity solution of the PDI

$$\sup_{w\in\mathbf{W}} \max\{g_2(x,w) - V(x), \quad \nabla V(x) f(x,w) + g_1(x,w)\} \leq 0 \quad \text{in} \quad \mathbf{R}^n. \tag{2.9}$$

Conversely, if a locally bounded function $V$ is a viscosity solution of the PDI (2.9), then the system (2.1) is LIB/I dissipative with storage function $V_*$.

PROOF. We give the proof for the case $g_1 = 0$ only, without loss of generality (write $g = 0$, $g = g_2$, etc).

1. Let $\phi \in C^1(\mathbf{R}^n)$ and suppose that $V_* - \phi$ attains a local minimum at $x_0 \in \mathbf{R}^n$. Then

$$V_*(x_0) - \phi(x_0) \leq V_*(x) - \phi(x) \tag{2.10}$$

for all $|x - x_0| < r$, for some $r > 0$.

Fix $w \in \mathbf{W}$ and define a constant control $w_0(t) = w$ for all $t \geq 0$, and let $\xi_0(\cdot)$ denote the corresponding state trajectory of (2.1) with $\xi_0(0) = x_0$. Then from the dissipation inequality (2.8) we have

$$V_*(x_0) \geq \max\{V_*(\xi_0(t)), \text{ess.sup}_{s\in[0,t]}\{g(\xi_0(s), w)\}\},$$

and in particular,

$$V_*(x_0) \geq V_*(\xi_0(t)), \tag{2.11}$$

and

$$V_*(x_0) \geq \text{ess.sup}_{s\in[0,t]}\{g(\xi_0(s), w)\}. \tag{2.12}$$

Now for all $t > 0$ sufficiently small, we have $|\xi_0(t) - x_0| < r$, and so combining (2.10) and (2.11) we get

$$\frac{\phi(\xi_0(t)) - \phi(x_0)}{t} \leq 0,$$

and sending $t \downarrow 0$ we obtain

$$\nabla\phi(x_0) f(x_0, w) \leq 0. \tag{2.13}$$

Next, since $s \mapsto g(\xi_0(s), w)$ is continuous, (2.12) implies

$$V_*(x_0) \geq (\xi_0(s), w) \quad \forall \quad 0 \leq s \leq t.$$

Send $t \downarrow 0$ to get

$$V_*(x_0) \geq g(x_0, w). \tag{2.14}$$

This proves that $V$ is a viscosity solution of (2.9).

2. Suppose $V$ is a locally bounded viscosity solution of (2.9). We must prove that $V_*$ solves the dissipation inequality.

To this end, let $\{\psi_i\}_{i=1}^\infty \subset C(\mathbf{R}^n)$ be such that $\psi_i \leq V_*$ and $\psi_i \uparrow V_*$ as $i \to \infty$. Fix $t > 0$ and define for $s \in [0,t]$ the two functions

$$Z_1^i(x,s) = \sup_{w\in\mathcal{W}_{s,t}} \{\psi_i(\xi(t))\}, \tag{2.15}$$

and

$$Z_2^i(x,s) = \sup_{w\in\mathcal{W}_{s,t}} \left\{\text{ess.sup}_{\tau\in[s,t]}\{z(\tau)\}\right\}, \tag{2.16}$$

where $\xi(\cdot)$ is the trajectory of (2.1) on $[s,t]$ with initial state $\xi(s) = x$.

By Lemma 4.1, $Z_1^i$ is the unique continuous viscosity solution of the Cauchy problem

$$\frac{\partial}{\partial s} Z(x,s) + \sup_{w\in\mathbf{W}} \{\nabla Z(x,s) f(x,w)\} = 0 \tag{2.17}$$

in $\mathbf{R}^n \times (0,t)$, $Z(x,t) = \psi(x)$ (with $\psi = \psi_i$), and $Z_2^i$ unique continuous viscosity solution of

$$\sup_{w\in\mathbf{W}} \max\{g(x,w) - Z(x,s), \quad \frac{\partial}{\partial s} Z(x,s) + \nabla Z(x,s) f(x,w)\} = 0 \tag{2.18}$$

in $\mathbf{R}^n \times (0,t)$, $Z(x,t) = \max_{w\in\mathbf{W}}\{g(x,w)\}$ (with $\psi = \psi_i$).

Now $V_*$ is a l.s.c. supersolution of both (2.17) and (2.18), and again by Lemma 4.1, we have

$$V_*(x) \geq \max\{Z_1^i(x,s),\, Z_2^i(x,s)\}. \tag{2.19}$$

However,

$$\begin{aligned}&\max\{Z_1^i(x,s),\, Z_2^i(x,s)\} = \\ &\sup_{w\in\mathcal{W}_{s,t}} \max\left\{\psi_i(\xi(t)),\, \text{ess.sup}_{\tau\in[s,t]}\{z(\tau)\}\right\},\end{aligned} \tag{2.20}$$

and sending $i \to \infty$ we obtain the dissipation inequality. □

### 2.3 *Special Cases*

**Standard Hill-Moylan-Willems Dissipation.** Consider the case where $g_1(x,w) = S(g(x,w),w)$, $g_2(x,w) = 0\ \forall\, w \in \mathbf{W}$, and the system (2.1) has an equilibrium $x = 0$. This corresponds to the standard Hill-Moylan-Willems framework with *supply rate* $S(z,w)$, which of course includes $L^p$ ($p < +\infty$) gain and passivity, see Willems (1972), van der Schaft (1996). The LIB/I dissipation inequality (2.3) becomes

$$\begin{aligned}V(x) \geq &\sup_{t\geq 0}\sup_{w\in\mathcal{W}_{0,t}} \{V(\xi(t)) \\ &+ \int_0^t S(g(\xi(r),w(r)),w(r))\,dr\},\end{aligned} \tag{2.21}$$

and if, say, $S(z,0) = 0$, this implies $V \geq 0$. Hence the mixed dissipation definition (2.3) is completely equivalent to the standard definition in this case.

**$L^\infty$-Bounded Dissipation.** The $L^\infty$-bounded case corresponds to $g_1(x,w) = 0$, $g_2(x,w) = |g(x,w)|$, $\forall\ w \in \mathbf{W}$. We may take, e.g., $\mathbf{W} = \{w \in \mathbf{R}^s \ :\ |w| \leq \alpha\}$ for some $\alpha > 0$. The LIB/I dissipation inequality (2.4) reads in this case

$$\begin{aligned}V(x)\ \geq &\sup_{t\geq 0}\sup_{w\in\mathcal{W}_{0,t}} \max\{V(\xi(t)), \\ &\text{ess.sup}_{s\in[0,t]} z_2(s)\}.\end{aligned} \tag{2.22}$$

which we call the *$L_\infty$-bounded (LIB) dissipation inequality.* This abstracts the approach to worst case analysis in Fialho and Georgiou (1999); the definition (2.5) of available storage corresponds to the function defined by equations (2) and (3) in Fialho and Georgiou (1999). Storage functions for LIB dissipative systems can be used can be used to analyse $L_\infty$-bounded gain functions and induced $L_\infty$ gains over bounded signals Fialho and Georgiou (1999).

## 3. STATE FEEDBACK SYNTHESIS

In this section we consider the problem of finding a controller (determining the signal $u$) which results in a mixed dissipative closed loop (relative to $w \mapsto z_1, z_2$) for the system

$$\begin{aligned}\dot{\xi}\ &= f(\xi,w,u)\\ z_1\ &= g_1(\xi,w,u)\\ z_2\ &= g_2(\xi,w,u)\end{aligned} \tag{3.1}$$

This is similar in many ways to the problem of synthesizing $H_\infty$ or $L_2$-gain controllers, see van der Schaft (1996), Helton and James (1999). These problems can be addressed using methods from game theory and/or dissipative systems theory. Here we use a simplified approach using the LIB/I dissipation inequality and static state feedback controllers.

In (3.1), $\xi \in \mathbf{R}^n$, $w \in \mathbf{W} \subset \mathbf{R}^s$, $\mathbf{W}$ compact, $w \in \mathbf{U} \subset \mathbf{R}^m$, $\mathbf{U}$ compact, $z_1 \in \mathbf{R}$, and $z_2 \in \mathbf{R}$. Write $\mathcal{U}_{t_0,t_1} = \{$ measurable functions $u\ :\ [t_0,t_1] \to \mathbf{U}\}$. The functions $f : \mathbf{R}^n \times \mathbf{W} \times \mathbf{U} \to \mathbf{R}^n$, $g_i : \mathbf{R}^n \times \mathbf{W} \times \mathbf{U} \to \mathbf{R}$ ($i = 1,2$) are assumed bounded and Lipschitz continuous.

We consider static state feedback controllers

$$\mathcal{U}_{\text{state}} = \{\mathbf{u} : \mathbf{R}^n \to \mathbf{U} \mid \mathbf{u} \text{ admissible}\}$$

where *admissible* means here that $\mathbf{u}$ is Lipschitz continuous. For any $\mathbf{u} \in \mathcal{U}_{\text{state}}$, $w \in \mathcal{W}_{0,\infty}$ and initial condition $x_0$ there exists unique solution $\xi(\cdot)$ of (3.1) and the control trajectory $t \mapsto \mathbf{u}(\xi(t))$ belongs to $\mathcal{U}_{0,\infty}$, and the closed loop system obtain from (3.1) is given explicity by

$$\begin{aligned}\dot{\xi}\ &= f(\xi,w,\mathbf{u}(\xi))\\ z_1\ &= g_1(\xi,w,\mathbf{u}(\xi))\\ z_2\ &= g_2(\xi,w,\mathbf{u}(\xi)).\end{aligned} \tag{3.2}$$

*Theorem 3.1.* Assume there exists a controller $\mathbf{u}_0 \in \mathcal{U}_{\text{state}}$ for which the closed loop system (3.2) (with $\mathbf{u} = \mathbf{u}_0$) is LIB/I dissipative with locally bounded storage function $V_0$. Then there exists a locally bounded viscosity solution of the PDI

$$\begin{aligned}\inf_{u\in\mathbf{U}}\sup_{w\in\mathbf{W}} &\max\{g_2(x,w,u) - V(x), \\ &\nabla V(x) f(x,w,u) + g_1(x,w,u)\} \leq 0.\end{aligned} \tag{3.3}$$

Proof. By hypothesis and Theorem 2.3, the function $V_0$ is a viscosity solution of the PDI (2.9), i.e.

$$\sup_{w\in\mathbf{W}} \max\{g_2(x,w,\mathbf{u}_0(x)) - V_0(x),$$
$$\nabla V_0(x) f(x,w,\mathbf{u}_0(x)) + g_1(x,w,\mathbf{u}_0(x))\} \leq 0.$$

Therefore

$$\inf_{u\in\mathbf{U}} \sup_{w\in\mathbf{W}} \max\{g_2(x,w,u) - V_0(x),$$
$$\nabla V_0(x) f(x,w,u) + g_1(x,w,u)\} \leq 0,$$

which implies that $V_0$ is a viscosity solution of the PDI (3.3). $\square$

*Theorem 3.2.* Assume there exists a classical solution $V \in C^1(\mathbf{R}^n)$ of the PDI (3.3), and assume there exists $\mathbf{u}^* \in \mathcal{U}_{\text{state}}$ such that

$$\begin{aligned} &\sup_{w\in\mathbf{W}} \max\{g_2(x,w,\mathbf{u}^*(x)) - V(x), \\ &\quad \nabla V(x) f(x,w,\mathbf{u}^*(x)) + g_1(x,w,\mathbf{u}^*(x))\} \\ &= \inf_{u\in\mathbf{U}} \sup_{w\in\mathbf{W}} \max\{g_2(x,w,u) - V(x), \\ &\quad \nabla V(x) f(x,w,u) + g_1(x,w,u)\} \leq 0 \end{aligned} \tag{3.4}$$

for all $x \in \mathbf{R}^n$. Then the closed loop system (3.2) (with $\mathbf{u} = \mathbf{u}^*$) is LIB/I dissipative with storage function $V$.

## 4. APPENDIX: VISCOSITY SOLUTIONS

A locally bounded function $V$ is said to satisfy the PDI (2.9) in the *viscosity* sense if for every $\phi \in C^1(\mathbf{R}^n)$ and any local minimum $x_0 \in \mathbf{R}^n$ of $V_* - \phi$ one has

$$\begin{aligned} &\sup_{w\in\mathbf{W}} \max\{g_2(x_0,w) - V_*(x_0), \\ &\quad \nabla\phi(x_0) f(x_0,w) + g_1(x_0,w)\} \leq 0. \end{aligned} \tag{4.1}$$

A locally bounded function $Z$ is said to be a *viscosity supersolution* of the PDE (2.18) if for every $\phi \in C^1(\mathbf{R}^n \times (0,t))$ and any local minimum $(x_0,s_0) \in \mathbf{R}^n \times (0,t)$ of $Z_* - \phi$ the inequality

$$\begin{aligned} &\sup_{w\in\mathbf{W}} \max\{g_2(x_0,w) - Z_*(x_0,s_0), \\ &\quad \frac{\partial}{\partial s}\phi(x_0,s_0) + \nabla\phi(x_0,s_0) f(x_0,w) \\ &\quad +g_1(x_0,w)\} \leq 0 \end{aligned} \tag{4.2}$$

A locally bounded function $Z$ is said to be a *viscosity subsolution* of the PDE (2.18) if for every $\phi \in C^1(\mathbf{R}^n \times (0,t))$ and any local maximum $(x_0,s_0) \in \mathbf{R}^n \times (0,t)$ of $Z^* - \phi$ the inequality

$$\begin{aligned} &\sup_{w\in\mathbf{W}} \max\{g_2(x_0,w) - Z^*(x_0,s_0), \\ &\quad \frac{\partial}{\partial s}\phi(x_0,s_0) + \nabla\phi(x_0,s_0) f(x_0,w) \\ &\quad +g_1(x_0,w)\} \geq 0 \end{aligned} \tag{4.3}$$

A locally bounded function $V$ is said to be a *viscosity solution* of the PDE (2.18) if it is both a supersolution and a subsolution.

*Lemma 4.1.* The function $Z = Z_1^i$ defined by (2.15) is the unique continuous viscosity solution of the PDE (2.17) (with $\psi = \psi_i \in C(\mathbf{R}^n)$), and any l.s.c. supersolution $V$ of (2.17) with $V(x,t) \geq \psi(x)$ satisfies

$$V(x,s) \geq Z(x,s) \quad \text{for all } x \in \mathbf{R}^n, s \in [0,t]. \tag{4.4}$$

Similarly, the function $Z = Z_2^i$ defined by (2.16) is the unique continuous viscosity solution of the PDE (2.18) (with $\psi = \psi_i \in C(\mathbf{R}^n)$), and any l.s.c. supersolution $V$ of (2.18) with $V(x,t) \geq \max_{w\in\mathbf{W}} g_2(x,w)$ satisfies (4.4).

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