Lectures on Stochastic Systems

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0 Introduction
1 Probability Theory
2 Stochastic Processes
3 State Space Models
4 Filtering
5 Dynamic Programming
0. Introduction

What is a stochastic system?

A dynamical system whose behaviour is subject to random influences, and therefore not perfectly predictable.

We will want to control such systems in to influence their behaviour to achieve some goal.
This is achieved by:

- using observed information to better predict system behaviour
  
  - state estimation (e.g. filtering)
  
  - system identification (e.g. parameter estimation)

- quantify the performance of the system under the influence of various control policies and choose the "best"
  
  - stochastic optimal control (e.g. LQG)
Text

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Outline

1. Probability Theory
2. Stochastic Processes
3. State Space Models
4. Filtering
5. Dynamic Programming
1. **Probability Theory**

**Random Variables and Probability**

\[ X \sim \text{Gaussian (or normal)} \quad \text{with mean } \mu \text{ and variance } \sigma^2, \]

\[ X \sim N(\mu, \sigma^2) \quad \text{(continuous r.v.)} \]

**Density**

\[ p(x) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \]

**Distribution**

\[ F(x) = P(X \leq x) = \int_{-\infty}^{x} p(x) \, dx \]
Suppose \( \mu < \mu \). 

Events:

\[ E_1 = \{ X \leq \mu \} \quad E_2 = \{ X = 0 \} \]

\[ E_3 = \{ 0 \leq X \leq \mu \} \quad E_4 = \{ X < 0 \text{ or } X > \mu \} \]

Probability of events:

\[ P(E_1) = P(X \leq \mu) = F(\mu) \]

\[ = \int_{-\infty}^{\mu} p(\xi) \, d\xi \]

\[ = \frac{1}{2} \]

\[ P(E_2) = P(X \leq 0) = F(0) \]

\[ = \int_{-\infty}^{0} p(\xi) \, d\xi \]

\[ = 1, \quad \text{and} \quad 0 < a < 1/2 \]
\[ P(\Xi_3) = P(0 \leq X \leq \mu) \]
\[ = P(0 \leq X \text{ and } X \leq \mu) \]
\[ = P(\Xi_2 \cap \Xi_1) \]
\[ = P(\Xi_1) - P(\Xi_2) \quad \text{since } \Xi_2 \subset \Xi_1 \]
\[ = \frac{1}{2} - \lambda \]
\[ = \int_{\alpha}^{\beta} p(\xi) \, d\xi \]

\[ P(\Xi_4) = P(X \leq 0 \text{ or } X > \mu) \]
\[ = P(\Xi_2) + P(\Xi_4^c) \]
\[ = \lambda + (1 - \frac{1}{2}) \]
\[ = \lambda + \frac{1}{2} \]
\[ = \int_{-\infty}^{\alpha} p(\xi) \, d\xi + \int_{\beta}^{\infty} p(\xi) \, d\xi \]

Note that \( \Xi_1, \Xi_2, \Xi_3, \Xi_4 \) are subsets of \( \mathbb{R} \).
In general, if $E$ is a (Borel) subset of $\mathbb{R}$, we define its probability of occurrence to be

$$P(E) = \int_E p(x) \, dx$$

(Lebesgue integral of $p$ over $E$)

If, say, $E = \{ x \mid x \leq 1 \}$, then

$$P(E) = \int_{\{x \leq 1\}} p(x) \, dx = \int_{-\infty}^{1} p(x) \, dx$$

If, $E = \{ x \mid x = 1 \}$, then

$$P(E) = \int_{\{x = 1\}} p(x) \, dx$$

$$= 0,$$

Thus some events have zero probability.
\[ X \sim \text{Poisson}, \text{ parameter } \lambda > 0 \]

distribution (discrete r.v.)

\[ p(x) = P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!} \quad x = 0, 1, 2, \ldots \]

\[ F(x) = \sum_{\tilde{x} = 0}^{x} p(\tilde{x}) \]

\[ P(x) \]

\[ F(x) \]

\[ \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
\end{array} \]

\[ \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
\end{array} \]
events

\[ E_1 = \{ X = 3 \} \]

\[ E_2 = \{ X = 2 \} \]

\[ E_3 = \{ X = 7 \} \text{ or } X = 8 \]

probability of event

\[ P(E_1) = P(X = 3) = e^{-\lambda} \frac{\lambda^3}{3!} \]

\[ P(E_2) = P(X = 2) \]

\[ = P(X = 0, 1, \text{ or } 2) \]

\[ = e^{-\lambda} \frac{\lambda^0}{0!} + e^{-\lambda} \frac{\lambda^1}{1!} + e^{-\lambda} \frac{\lambda^2}{2!} \]

\[ = e^{-\lambda} \left( 1 + \lambda + \frac{1}{2} \lambda^2 \right) \]

\[ P(E_3) = P(X = 7 \text{ or } X = 8) \]

\[ = P(X = 7) + P(X = 8) \]

\[ = e^{-\lambda} \frac{\lambda^7}{7!} + e^{-\lambda} \frac{\lambda^8}{8!} \]
In general, if \( E \) is a subset of \( \mathbb{N} = \{1, 2, \ldots \} \), define

\[
P(E) = \sum_{\mathbf{x} \in E} p(\mathbf{x})
\]

\[
= \int_E p(d\mathbf{x})
\]

(Cauchy integral of \( p \) over \( E \))

If \( E = E_1 = \{3\} \subset \mathbb{N} \),

\[
P(E) = \int_{\{3\}} p(d\mathbf{x})
\]

\[
= \sum_{\mathbf{x} \in \{3\}} p(\mathbf{x})
\]

\[
= e^{-3} \frac{3!}{3!}
\]

If \( E = E_3 = \{7, 8\} \subset \mathbb{N} \),

\[
P(E) = \int_{\{7, 8\}} p(d\mathbf{x})
\]

\[
= \sum_{\mathbf{x} \in \{7, 8\}} p(\mathbf{x})
\]

\[
= e^{-1} \frac{7!}{7!} + e^{-1} \frac{8!}{8!}
\]
A probability space is a triple
\[(\Omega, \mathcal{F}, P)\]

where
* \(\Omega\) is a set of points (or outcomes of an experiment) called the sample space.
* \(\mathcal{F}\) is a collection of subsets of \(\Omega\) (or event space).
  \(E \in \mathcal{F}\) is an event.
We assume \(\mathcal{F}\) is a \(\sigma\)-algebra, (or \(\sigma\)-field).
* \(P\) is a probability measure, which assigns to each event \(E \in \mathcal{F}\) the probability of occurrence \(P(E)\) \(\in [0,1]\).

\[\dagger\] Defined below.
Eq. 1 \quad X \sim N(\mu, \sigma^2)

\mathcal{L} = \mathbb{R} \quad \mathcal{F} = \text{all sets of } \mathbb{R}

\rho : \mathcal{F} \to [0, 1] \quad \text{defined by}

\rho(E) = \int_E (2\pi \sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2} x^2\right) \, dx

(= \int_E \rho(dx))

\rho(dx) = (2\pi \sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2} x^2\right) \, dx

Eq. 2 \quad X \sim \text{Poisson } \lambda

\mathcal{L} = \mathbb{N} \quad \mathcal{F} = \text{all subsets of } \mathbb{N}

\rho : \mathcal{F} \to [0, 1] \quad \text{defined by}

\rho(E) = \sum_{x \in E} e^{-\lambda} \frac{\lambda^x}{x!}

Def. The pair \((\mathcal{L}, \mathcal{F})\) is called a **measurable space**
Dfn. A collection of subsets \( \mathcal{F} \) of \( \mathbb{R} \) is called a \( \mathcal{F} \) -algebra (or \( \mathcal{F} \) -field) if:

1. \( \emptyset \in \mathcal{F} \), \( \mathbb{R} \in \mathcal{F} \)

2. \( A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \)  
   (closure under complements)

3. \( \{ A_i \}_{i=1}^{\infty} \subseteq \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \)  
   (closure under countable unions)

\[ \text{eg:} \mathcal{F} = \mathbb{R} \]

\[ \mathcal{F} = \mathcal{B}(\mathbb{R}), \text{ the Borel } \mathcal{F} \text{-algebra} \]

= smallest \( \sigma \)-algebra containing

all intervals \((a, b]\).
Let $\mathcal{F}$ be a $\sigma$-algebra in $\Omega$. A probability measure is a function

$$P : \mathcal{F} \rightarrow [0, 1]$$

such that

1. $0 \leq P(E) \leq 1 \quad \forall \; E \in \mathcal{F}$
2. $P(\Omega) = 1$
3. If $A_1, A_2, A_3, \ldots$ are pairwise disjoint ($A_i \cap A_j = \emptyset$ if $i \neq j$), then

$$P\left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i)$$

(co-absolute additivity)

\textit{eg. 3.} \quad \Omega = \mathbb{R}, \mathcal{F} = \mathcal{B}(\mathbb{R})$

$$P = \lambda, \quad \text{Lebesgue measure}$$

$$\lambda((a, b]) = b - a \quad \text{(length of interval)}$$

If $E = \bigcup_{i=1}^{\infty} (a_i, b_i]$ disjoint, then

$$A(E) = \sum_{i=1}^{\infty} (b_i - a_i).$$

For instance, $E = (0, \frac{1}{2})$. 
Then \( C = \bigcup_{n=1}^{\infty} \left( \frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \)

\[
\chi(C) = \sum_{n=1}^{\infty} \chi \left( \left( \frac{1}{2^n}, \frac{1}{2^{n-1}} \right] \right) \\
= \sum_{n=1}^{\infty} \frac{1}{2^n} \\
= \frac{1}{2} \quad (\text{Geometric series})
\]

Some properties

1. \( \mathcal{P}(A^c) = 1 - \mathcal{P}(A) \)
2. \( \mathcal{P}(\emptyset) = 0 \)
3. \( A \subseteq B \quad \Rightarrow \quad \mathcal{P}(A) \leq \mathcal{P}(B) \)
4. \( \mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \cap B) \)
Distributions and Densities

Let \( \mathbb{R} = \mathbb{R}^d \) (or \( \mathbb{E}_0, \mathbb{I} \))

\( Y = \mathbb{R} \), and \( \mathbb{P} \) a prob. measure

on \( (\mathbb{R}, \Sigma, \mathbb{P}) \). The function

\[ F(x) \triangleq \mathbb{P} \left( \langle -\infty, x \rangle \right) \]

is called the (cumulative) distribution function associated with \( (\mathbb{R}, \Sigma, \mathbb{P}) \).

If \( F(\cdot) \) is absolutely continuous,
then there exists a function \( p(x) \),
called a density such that

\[ F(x) = \int_{-\infty}^{x} p(\xi) \, d\xi \]

(or more generally,

\[ \mathbb{P}(\xi) = \int_{\xi}^{\infty} p(\xi) \, d\xi \quad \forall \xi \in \mathbb{E} \])

Notation

\[ \text{d}F(x) = p(x) \, dx \]

\[ \text{P}(dx) = p(x) \, dx \]
Note: Densities do not always exist:

\[ F(x) \]

\[ \Rightarrow x \]

Random Variables

Let \((\Omega_1, \mathcal{F}_1), (\Omega_2, \mathcal{F}_2)\) be two measurable spaces.

Define a function

\[ X : \Omega_1 \rightarrow \Omega_2 \]

is called measurable if for every \( A \in \mathcal{F}_2 \), the set

\[ X^{-1}(A) = \{ w \in \Omega_1 : X(w) \in A \} \]

belongs to \( \mathcal{F}_1 \).

If \( \mu \) is a probability measure on \((\Omega_1, \mathcal{F}_1)\), then we say that \( X \) is a random variable.
Note. A random variable

\[ X : (\mathbb{R}_1, \mathcal{F}_1, P_1) \to (\mathbb{R}_2, \mathcal{F}_2) \]

defines a probability measure \( P_X \) on

\[ (\mathbb{R}_2, \mathcal{F}_2) \]

via

\[ P_X(A) = P_1(X^{-1}(A)) \quad \forall A \in \mathcal{F}_2, \]

where \( P_1 \) is the probability measure induced by \( X \).

An \underline{real valued} random variable is a measurable function

\[ X : (\mathbb{R}, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}) \]

on an abstract underlying probability space.

By

A \underline{Gaussian} random variable is a \underline{real valued} r.v. for which

\[ P_X(A) = \int_A (2\pi \sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right) \, dx \]

\[ \forall A \in \mathcal{B}. \]
The distribution function $F_X$ of a real-valued $X$ is the distribution function of $P_X$.

$$F_X(x) = P_X((-\infty, x]) = P\left(\{\omega : X(\omega) \leq x\}\right)$$

$$= P(X = x)$$

**Example:** If $X$ is Gaussian, then

$$F_X(x) = \int_{(-\infty, x]} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(\xi - \mu)^2\right) d\xi$$

$$= \int_{-\infty}^{x} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(\xi - \mu)^2\right) d\xi$$

**Theorem:** If $F_X(x)$ is absolutely continuous, then the density $P_X(x)$ of $X$ exists and is given by

$$P_X(x) = F_X'(x)$$

**Example:** If $X$ is Gaussian,

$$P_X(x) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right)$$
**Expectation** (Integration)

Let $X$ be a real valued r.v., defined on a probability space $(\Omega, \mathcal{F}, P)$. The expectation of $E$ is the real number

$$E[X] = \int_{\Omega} X(\omega) \, dP(\omega)$$

where the RHS denotes the integral of $X$ with respect to the probability measure $P$. (This will be defined shortly.)

E.g., $X \sim \text{Gaussian} \sim N(\mu, \sigma^2)$

$$E[X] = \int_{\mathbb{R}} x \, dP$$

$$= \int_{\mathbb{R}} x \, P_X(dx)$$

$$= \int_{-\infty}^{\infty} x \, p(x) \, dx \quad \text{(Gaussian density)}$$

$$= \mu$$
The definition of the (Lebesgue) integral involves a number of steps.

We are given a probability space $(\Omega, \mathcal{F}, P)$ and a real-valued r.v. $X$. We wish to define $E(X) = \int X \, dP$.

a. Let $A \in \mathcal{F}$ be an event.

The random variable

$$Y = I_A,$$

called the \textit{indicator function} of $A$. 

$$E[X] = \int_\mathbb{R} X(\omega) \, d\omega$$

$$= \sum_{\omega \in \mathbb{R}} X(\omega) \mathbb{P}(\{\omega\})$$

$$= \sum_{x=0} e^{-x} \frac{x^x}{x!}$$

$$= \lambda$$
is defined by

\[ I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} \]

Define

\[ E[Y] = E\left[ I_A \right] = \int I_A(\omega) \, P(d\omega) \]

\[ \Delta P(A), \]

b. Let \( A_1, A_2, A_3, \ldots \) be a sequence of disjoint subsets in \( \Omega \) (let \( x_1, x_2, \ldots \) the real numbers).

\[ Y = \sum_{i=1}^{\infty} x_i \cdot I_{A_i} \]

is called a simple function.
The integral of \( Y \) is defined by

\[
E[Y] = \int Y(w) \, P(\, dw) = \sum_{i=1}^{\infty} x_i \cdot P(A_i)
\]

Note \( Y(w) \) is finite for each \( w \in \mathbb{R} \).

\( E[Y] \) may be \( \infty \) too.

We say \( Y \) is integrable if

\[
E[Y] < \infty.
\]

c. Now suppose that \( X \) is positive:

\( X(w) \geq 0 \) for all \( w \in \mathbb{R} \).

We define

\[
E[X] = \int X(w) \, P(\, dw)
\]

\[
\sup \{ E[Y] : 0 \leq Y \leq X, \ Y \text{ simple} \}
\]

Say \( X \) is integrable if \( E[X] < \infty \).
d. General case. Write

\[ X = X^+ - X^- \]

where

\[ X^+(\omega) = \begin{cases} X(\omega) & \text{if } X(\omega) \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ = X(\omega) \uparrow M_{\max} \]

\[ X^-(\omega) = -(X(\omega) \downarrow m_{\min}) \]

Note: \[ |X| = X^+ + X^- \]
Define

\[ E[X] = E[X^+] - E[X^-] \]

whenever \( X \) is integrable, i.e., whenever

\[ E[|X|] < +\infty. \]

(This means that both \( E[X^+] \) and \( E[X^-] \) must be finite.)

(Note \( \int X(w) P(dw) = \int X^+(w) P(dw) \)

- \( \int X^-(w) P(dw) \)

Some Properties

1. \( E[aX_1 + bX_2] = aE[X_1] + bE[X_2] \) (Linearity)

2. If \( X = Y \) a.s., then

\( E[X] = E[Y] \) \( \text{almost surely} \)

\[ (X = Y \text{ a.s.} \iff P(X \neq Y) = 0) \]
3. If $X \leq Y$ a.s., then 
\[ E[X] \leq E[Y] \]

4. If $A \in \mathcal{F}$, we define 
\[ \int_A X \, dP = \int \mathbf{I}_A(w) X(w) \, f(dw) = \mathbb{E} \left( \mathbf{I}_A X \right) \]

5. Numerous convergence theorems (see Assignment 1)

---

Above we have defined expectation $\mathbb{E}[X]$ of $X$ in terms of $(\Omega, \mathcal{F}, P)$:
\[ \mathbb{E}[X] = \int X \, dP \]

This can also be expressed in terms of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$, where $P_X$ is the probability measure induced by $X$.

\[ \mathbb{E}[X] = \int_{\mathbb{R}} x \, P_X(dx) \]
If \( X \sim N(\mu, \sigma^2) \), then

\[
E[X] = \int_\mathbb{R} x \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right) dx
\]

\[
= \mu.
\]

More generally, if \( g : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is measurable, we define

\[
E[g(X)] = E[g(X) - X]
\]

\[
= \int_\mathbb{R} (g(X) - X)(\omega) \, p_X(\omega) dx
\]

\[
= \int_\mathbb{R} g(x) \, p_X(\omega) dx
\]

Moments and correlation, etc. are obtained by appropriate choice of \( g \), etc.

1. \( k \)-th moment of \( X \)

\[
E[X^k]
\]

2. Variance of \( X \)

\[
E[ (X - E[X])^2 ]
\]
3. Correlation of $X - Y$

$E[XY]$ (a constant, if $E[XY] = 0$)

4. Covariance of $X - Y$

$E[(X - E[X])(Y - E[Y])]$

$L_p$ spaces and some inequalities

Given a probability space $(\Omega, \mathcal{F}, P)$, $0 < p < \infty$.

$L_p = L_p(\Omega, \mathcal{F}, P)$ is the vector space of real valued random variables $X$ such that

$E[|X|^p] < +\infty$.

In fact, $L_p$ is a Banach space, with norm

$||X||_p = E[|X|^p]^{1/p}$.

$L_2$ is a Hilbert space, with inner product

$(X, Y) = E[XY]$. 
For $p = \infty$, $L_\infty$ is the Banach space consisting of (essentially) bounded $f(t)$ with norm

$$
\|X\|_\infty = \text{ess sup}_w |X(w)|
$$

Some inequalities:

1. $\|X + Y\|_p \leq \|X\|_p + \|Y\|_p$ (Minkowski ≤) for all $X, Y \in L_p$

2. If $X \in L_p$, $Y \in L_2$, where $\frac{1}{p} + \frac{1}{2} = 1$, then

$$
E[|XY|] \leq \|X\|_p \|Y\|_2 \quad \text{(Hölder ≤)}
$$

($p = 2$, Cauchy–Schwarz ≤)

3. If $X \in L_p$, $\alpha > 0$ constant,

$$
P(|X| \geq \alpha) \leq \frac{1}{\alpha^p} \|X\|_p^p \quad \text{(Markov ≤)}$$

($p = 2$, Chebyshev ≤)

(many others ... )
Independence (of events)

Given \((\Omega, \mathcal{F}, P)\), two events
\(A, B \in \mathcal{F}\) are independent if
\[ P(A \cap B) = P(A) P(B). \]

\(N\) events \(A_1, A_2, \ldots, A_N\) are independent if for every subset \(\{k_1, k_2, \ldots, k_i\}\) of \(\{1, 2, \ldots, N\}\),
\[ P\left( \bigcap_{i=1}^{N} A_{k_i} \right) = \prod_{i=1}^{N} P(A_{k_i}) . \]

An infinite (perhaps uncountable) collection of events \(\mathcal{G} = \{A_n\}\) is independent if every finite subcollection of \(\mathcal{G}\) is independent.

Collections, or classes, of events \(A_1, A_2, \ldots, A_N\) are independent if for each choice
\(A_i \in \mathcal{G}_{k_i}\), the events \(A_1, A_2, \ldots, A_N\) are independent.

Intuition

Recall the conditional probability of \(B\) given \(A\) is defined by
\[ P(B|A) = \frac{P(B \cap A)}{P(A)} \]
whenever \(P(A) \neq 0\).
If \( A \) and \( B \) are independent,

\[
P(A \cap B) = P(A)P(B)
\]

so that

\[
P(B \mid A) = P(B).
\]

Similarly, if \( A \cap B \) are independent,

\[
P(B \mid A) = P(A)
\]

Thus two events are independent if knowledge of occurrence of either event does not affect the assessment of likelihood of the other.

**Independence of Random Variables**

Recall that a r.v. \( X \) is a measurable function

\[
X: (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}, \mathcal{B})
\]

i.e.

\[
X^{-1}(A) \in \mathcal{F} \quad \forall \ A \in \mathcal{B}
\]

A sub-\( \sigma \)-algebra \( \mathcal{G} \) of \( \mathcal{F} \) is a subset of \( \mathcal{F} \) which is a \( \sigma \)-algebra in its own right.

(c.f. subspace of a vector space.)
The σ-algebra generated by $X$, $\sigma(X)$, is the smallest sub-σ-algebra of $\mathcal{F}$ with respect to which $X$ is measurable, i.e.

$$\sigma(X) = \sigma(\{X^{-1}(A) ; A \in \mathcal{F}\})$$

In general,

$$\sigma(X) \subset \mathcal{F}$$

with strict inclusion (unless $\mathcal{F} = \sigma(X)$).

$\sigma(X)$ summarizes information about $X$.

If $X$ is a real r.v., then

$$\sigma(X) = \sigma(\{X^{-1}((-\infty, x]) ; x \in \mathbb{R}\})$$

$X : (\mathcal{L}^0, \mathcal{B}(\mathcal{L}^0), \mathcal{A}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

defined by

$$X(\omega) = \begin{cases} 0 & 0 \leq \omega < \frac{1}{2} \\ 1 & \frac{1}{2} \leq \omega \leq 1 \end{cases}$$

Then

$$\sigma(X) = \{ \{0, 1\}, \emptyset, \{0, \frac{1}{2}\}, \{\frac{1}{2}, 1\} \}.$$
Say \( x_1, x_2, \ldots, x_n \) are independent if

\[
\mathbb{P}(x_1, x_2, \ldots, x_n) = \mathbb{P}(x_1) \mathbb{P}(x_2) \cdots \mathbb{P}(x_n)
\]

Let \( x_1, x_2, \ldots, x_n \) be random variables.

\( \mathcal{E}(\mathbb{R}^n) \)

is a \( \mathcal{S} \)-algebra on \( \mathcal{B}(\mathbb{R}^n) \).

(Why?)

\[
\mathcal{E}(\mathbb{R}^2) = \{ \mathcal{E}(\mathbb{R}) \times \mathcal{E}(\mathbb{R}) \} \cap \mathcal{B}(\mathbb{R}^2)
\]

(Why?)

The collection

\[
\mathcal{F}_{\{0,1\}} = \{ \{0\}, \{1\} \}
\]

\[
\mathcal{F}_{\{0,1\}} = \{ \mathcal{F}_{\{0\}}, \mathcal{F}_{\{1\}} \}
\]

\[
\mathcal{F}_{\{0,1\}}(x) = \begin{cases} 
1 & x \leq 0 \\
0 & x > 1
\end{cases}
\]
Equivalently, \( X_1, \ldots, X_N \) independent iff
\[
P( X_1 \in A_1, \ldots, X_N \in A_N )
= P( X_1 \in A_1 ) \cdots P( X_N \in A_N )
\]
for any choice \( A_i \in \mathcal{F}_i \).

Note: If \( X \) and \( Y \) are independent then
\[
E[XY] = E[X]E[Y].
\]

---

Conditional Expectation

In many applications we need to compute quantities like
\[
E[ X \mid Y ],
\]
the conditional expectation of \( X \) given \( Y \), i.e., "given that \( Y = y \) has occurred, \( E[X \mid Y = y] \) is the most likely value of \( X \)."

Recall that \( \mathcal{Y} = \pi(Y) \), the \( \mathcal{F} \)-algebra generated by \( Y \), summarises the information about \( Y \). We will define
\[
E[ X \mid Y ] = E[ X \mid Y = y ]
\]
where the RHS is the
Conditional expectation of $X$ given the $\sigma$-algebra $\mathcal{Y}$.

**Def.** Let $X$ be an integrable r.v. defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{Y} \subset \mathcal{F}$ be any sub-$\sigma$-algebra. The conditional expectation of $X$ given $\mathcal{Y}$ is any $\mathcal{Y}$-measurable r.v. $\tilde{X}$ (i.e. $\tilde{X}^{-1}(E) \in \mathcal{Y}$ for $E \in \mathcal{B}(\mathbb{R})$) such that

$$E[X I_A] = E[\tilde{X} I_A] \quad \forall A \in \mathcal{Y}.$$ 

**Eq.** If $\tilde{X}$ is any other $\mathcal{Y}$-measurable r.v. satisfying the above, then $\tilde{X} = X$ a.s.

**Ex.** 

\[
(\mathcal{G}, \mathcal{Y}, \mathbb{P}) = (\mathcal{G}, \mathcal{G}', \mathbb{P}) \\
\mathcal{Y} = \sigma\left\{ [0, \frac{1}{3}), \left[\frac{1}{3}, \frac{2}{3}\right), \left[\frac{2}{3}, 1]\right\}.
\]

\[
\begin{array}{ccc}
0 & 1 & 0 \\
\frac{1}{3} & \frac{2}{3} & 1
\end{array}
\]
$$X(\omega) = \omega$$

Note

$$\mathcal{Y} = \{ (\omega, 0, 1), (0, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, 1), \frac{1}{3}, 1, \frac{1}{3}, 3 \}$$

The three (generating) sets (also \(\mathcal{Y}\))

$$[0, \frac{1}{3}) [\frac{1}{3}, \frac{2}{3}) [\frac{2}{3}, 1]$$

are called atoms of \(\mathcal{Y}\).

Let’s compute \(X = E[X|Y]\).

Since \(X\) is \(Y\)-measurable,

$$X = a \mathbb{I}_{[0, \frac{1}{3})} + b \mathbb{I}_{[\frac{1}{3}, \frac{2}{3})} + c \mathbb{I}_{[\frac{2}{3}, 1]}$$

for some constants \(a, b, c\). (Why?)

Use defining property \(\mathbb{E}\).

Let \(A = [0, \frac{1}{3}) \in \mathcal{Y}\), then

$$\mathbb{E}[X^2 \mathbb{I}_{[0, \frac{1}{3})}] = \mathbb{E}X^2 \mathbb{I}_{[0, \frac{1}{3})}$$
\[ \cos \omega = \int_{0}^{1} \omega \Gamma \left[ \frac{4}{3}, \omega \right] \, d\omega \]
\[ = \int_{0}^{\frac{1}{3}} \omega \, d\omega = \frac{1}{18} \]

\[ \sin \omega = \int_{0}^{1} \left( a \Gamma \left[ \frac{2}{3}, \frac{2}{3} \right] + b \Gamma \left[ \frac{1}{3}, \frac{2}{3} \right] + c \Gamma \left[ \frac{1}{3}, \frac{1}{3} \right] \right) \, d\omega \]
\[ = a \int_{0}^{\frac{1}{3}} \, d\omega = a \cdot \frac{1}{3} \]
\[ \therefore \frac{1}{3} = a \cdot \frac{1}{3} \quad a = \frac{1}{6} \]

Similarly,
\[ A = \left[ \frac{2}{3}, \frac{1}{3} \right] \Rightarrow b = \frac{1}{2} \]
\[ A = \left[ \frac{2}{3}, \frac{1}{2} \right] \Rightarrow c = \frac{5}{6} \]

\[ \hat{X}(\omega) = \frac{1}{6} \Gamma \left[ \frac{2}{3}, \frac{1}{3} \right] (\omega) \]
\[ + \frac{1}{2} \Gamma \left[ \frac{1}{3}, \frac{2}{3} \right] (\omega) \]
\[ + \frac{5}{6} \Gamma \left[ \frac{1}{3}, \frac{1}{2} \right] (\omega) \]

\[ \hat{X} = \mathbb{E}[X|Y] \]

Check that \( \mathbb{E}[X] \) satisfies for
\[ A = \left[ \frac{1}{3}, \frac{2}{3}, \frac{1}{2} \right] \left[ \frac{2}{3}, \frac{1}{3}, \frac{1}{2} \right] \]
Thus \( E[X|Y] \) is the \( Y \)-measurable r.v. obtained by "averaging\( X \) over the atoms of \( Y \)."

**Properties**

1. \( E[\alpha X_1 + \beta X_2 | Y] = \alpha E[X_1|Y] + \beta E[X_2|Y] \)

2. \( E[X|\emptyset] = \emptyset \)

3. \( X_1 \geq X_2 \Rightarrow E[X_1|Y] \geq E[X_2|Y] \)

4. If \( A \) is an atom of \( Y \), then
   \[
   E[X|Y](\omega) = \frac{1}{P(A)} E[X I_A]
   \]
   for all \( \omega \in A \)
   if \( P(A) \neq 0 \).

5. Let \( A, B \in \mathcal{F} \) be events. Then
   letting
   \[
   Y = \{ \emptyset, A, A^c \}
   \]
   \[
   X = I_B
   \]

   we see that
   \[
   P(B|A) = E[X|Y](\omega) \quad \text{if } \omega \in A
   \]
   \[
   = \frac{1}{P(A)} E[I_B I_A] = \frac{1}{P(A)} P(A \cap B)
   \]
6. If \( Y_1 < Y_2 < f \) then
\[
E[E[X | Y_2] | Y] = E[X | Y]
\]

7. If \( Y = \{ X, 2X \} \), then
\[
E[X | Y] = E[X].
\]

(\( + \) \( + \) are "smoothing" properties.)

8. If \( X \) is \( Y \)-measurable, then
\[
E[X | Y] = X
\]

9. Jensen's inequality: if \( f \) is convex,
\[
f( E[X | Y] ) \leq E[f(X) | Y]
\]

10. Minimum variance (least square property)

\( E[X | Y] \) is the unique \( Y \)-measurable rv.

minimizing the criterion
\[
E[|X - E[X | Y]|^2] \quad \text{over} \quad X \quad \text{\( Y \)-measurable}
\]

\[
E[|X - E[X | Y]|^2] = \min_X \quad \text{\( Y \)-measurable}
\]

The \( E[X | Y] \) is the "optimal" estimate.
Convergence of Random Variables

Let \( X \) be a r.v. and \( \{X_n\} \) a sequence of r.v.s. Suppose \( X, X_n \) are defined on \((\Omega, \mathcal{F}, \mathbb{P})\), a real valued.

What does it mean to say
\[
\lim_{n \to \infty} X_n = X
\]

Fix \( \omega \in \Omega \) and consider the sequence of real numbers \( X_n(\omega) \).
Then
\[
\lim_{n \to \infty} X_n(\omega) = X(\omega)
\]
means that for every \( \varepsilon > 0 \), \( n \geq N \) implies
\[
|X_n(\omega) - X(\omega)| < \varepsilon.
\]

(Almost sure convergence)

Remark: In many applications, this mode of convergence is too strong, and is not satisfied even though \( X_n \to X \) in some reasonable sense, e.g., \( X_n \) may converge \( L^p \)
"except on a set of probability zero".
Almost Sure Convergence

\[ \lim_{n \to \infty} X_n = X \text{ a.s.} \]

\[ \Rightarrow X_n \to X \text{ a.s.} \]

\[ P \left( \lim_{n \to \infty} X_n(\omega) = X(\omega) \right) = 1 \]

\[ \forall \ \exists \ A \in \mathcal{F} > 0, \quad P(A) = 0 \]

\[ \lim_{n \to \infty} X_n(\omega) = X(\omega) \quad \forall \omega \not\in A \]

Note: Sometimes, "a.s." is replaced by "with probability one" or "w.p.1" (with probability one).

Convergence in Probability

\[ \lim_{n \to \infty} X_n = X \text{ in probability} \]

\[ \Rightarrow X_n \overset{p}{\to} X \]

Means...
$b > c,$

$$\lim_{n \to \infty} P \left( |X_n - X| > c \right) = 0.$$  

**Convergence in $p$-th mean (in $L^p$)**

$$\lim_{n \to \infty} X_n = X \quad \text{in $p$-th mean} \quad \Rightarrow \quad X_n \overset{L^p}{\to} X$$

**Means:**

$$\lim_{n \to \infty} E \left[ |X_n - X|^p \right] = 0$$

**Convergence in Distribution**

$$\lim_{n \to \infty} X_n = X \quad \text{in distribution} \quad \Rightarrow \quad X_n \overset{d}{\to} X$$

**Means:**

At every point $x$ where $F_X$ is continuous,

$$\lim_{n \to \infty} F_{X_n}(x^n) = F_X(x)$$

(Here, $F_{X_n}, F_X$ are the distribution functions $X_n, X$.)
Weak convergence

\[ \lim_{n \to \infty} X_n = X \quad \text{weakly (or in distribution)} \]

\[ X_n \xrightarrow{w} X \quad \Rightarrow \quad X_n \xrightarrow{d} X \]

\[ P(X_n) \to P_X \]

i.e., for all continuous bounded functions \( \phi \)

\[ \int \phi dP(X_n) \to \int \phi dP_X \]

\[ \in \mathbb{E} \left[ \phi(X_n) \right] \to \mathbb{E} \left[ \phi(X) \right] \]

Relationship:

\[ \psi \quad \text{d.s.} \]

\[ \Rightarrow \quad \text{in } p.m. \quad \iff \quad \text{in } L^p \]

\[ \downarrow \]

in distribution
2. Stochastic Processes

Heuristic: A stochastic process is a random function of time.

\[ X_n(w) \]

\[ 1 \leq n \leq \infty \]

\[ X_1(w), X_2(w), \ldots \]

\[ X_n(w) \]

\[ \in \mathbb{R} \]

Eq 7. Noise

Form A stochastic process \( \{X_n\}_{n=0}^{\infty} \) is a sequence of random variables \( X_n \) defined, say, on \((-\infty, \infty, \Phi)\).

For each \( w \in \mathbb{R} \),

\[ X_n(w) = \{ X_1(w), X_2(w), \ldots \} \]

determines a sample path.

(If \( \{X_n\}_{n=0}^{\infty} \) is real valued, then \( \{X_n\}_{n=0}^{\infty} \) can be viewed as a random variable with values in \( \mathbb{R}^\infty \).)
It is important to quantify the information history or flow corresponding to a stochastic process. This is achieved using filtrations.

**Def**: A filtration of a measurable space \((\Omega, \mathcal{F})\) is a family \(\{\mathcal{F}_n\}_{n=0}^\infty\) of sub-\(\sigma\)-algebras \(\mathcal{F}_n \subset \mathcal{F}\) such that \(\mathcal{F}_n \subset \mathcal{F}_{n+1}\) for \(n = 0, 1, \ldots\).

If \(\{X_t\}\) is a stochastic process, \(\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)\) defines a filtration \(\{\mathcal{F}_n\}\), \(\mathcal{F}_0 \subset \mathcal{F}\), called the history or filtration generated by \(\{X_t\}\).

**Def**: A stochastic process \(\{X_n\}\) is adapted to a filtration \(\{\mathcal{F}_n\}\) if \(X_n\) is \(\mathcal{F}_n\)-measurable for all \(n = 0, 1, 2, \ldots\).

**Rem**: \(\mathcal{F}_n = \sigma(\text{events that have occurred up to time } n)\).
2.3

1. A stopping time \( \tau \) is defined as a non-decreasing function of time:

\[ \tau : \Omega \rightarrow \mathbb{R}_{\geq 0} \]

such that

\[ \{ \omega : \tau(\omega) = n \} \in \mathcal{F}_n \]

for all \( n \in \mathbb{N} \).

2. \( X_n = \begin{cases} 1 & \text{if } \tau = n \\ -1 & \text{if } \tau > n \end{cases} \)

\[ Y_n = \sum_{k=0}^{\tau} X_k \]

\( \{ X_n \} \) is a stochastic process adapted to

\[ \mathcal{F}_n = \sigma( X_0, \ldots, X_n ) = \sigma( X_0, \ldots, Y_n ) \]

Let

\[ N = \min \{ n : Y_n \geq 5 \} \]

= first time that \( Y_n \geq 5 \).

Then \( N \) is a stopping time relative to the filtration \( \{ \mathcal{F}_n \} \).
Define \( X_n \) and stepping from \( N \),

\[ X_n, \quad T_n \]

\[ X_n \mid \omega = X(\omega)_{N(\omega)} \]

\[ \mathcal{F}_n = \{ A \in \mathcal{F} \mid A \text{ is } \sigma \text{-algebra} \text{ on } \mathbb{X}_n \} \]

\( X_N \) is the value of \( \{X_n\} \) "jumped" at \( N \).

The process \( \{X_{N\mid n}\}_{n=0}^{\infty} \) is the "stepped process".

\[ X_{N\mid n} = \begin{cases} X_n & n \leq N \\ X_N & n > N \end{cases} \]
Martingales

A stochastic process \( \{X_n\} \) defined as \((\mathcal{F}, F, (\mathcal{F}_n))\) is called

a supermartingale / martingale / submartingale

if

(i) \( \{X_n\} \) is adapted to \( F\)

(ii) \( \mathbb{E}[|X_n|] < +\infty \)

(iii) \( \mathbb{E}[X_{n+1} | \mathcal{F}_n] \leq X_n \)

\[ \mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n \]

\[ \mathbb{E}[X_{n+1} | \mathcal{F}_n] \geq X_n \]

\(\forall n\)

(A martingale corresponds intuitively to a fair game of chance.)

Note for \( m > n \),

\[ \mathbb{E}[X_m | \mathcal{F}_n] = \mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n \]

if \( \{X_n\} \) is a martingale.

In general, \( \mathbb{E}[X_m | \mathcal{F}_n] \), \( m \geq n \),

is the predicted value of \( X_m \)

given the history \( \mathcal{F}_n \).
A supermartingale (submartingale) decreases (increases) in conditional mean.

**Theorem (Martingale Convergence)**

Suppose \( (X_n) \) is a supermartingale such that
\[
\sup_n \mathbb{E} |X_n| < +\infty.
\]
Then there exists a r.v. \( X_{\infty} \) such that
\[
\lim_{n \to \infty} X_n = X_{\infty} \quad \text{a.s.}
\]

Let \( \{\mathcal{F}_n\} \) be a filtration on \((-\infty, \infty)\) and define
\[
\mathcal{F}_{\infty} = \sigma(\mathcal{F}_n, n \in \mathbb{N})
\]
Let \( Y \in L_1((-\infty, \infty), \mathcal{F}_{\infty}, \mathbb{P}) \). Then
\[
X_n = \mathbb{E}[Y | \mathcal{F}_n]
\]
is a martingale. By MCT, and by the theorem,
\[
\lim_{n \to \infty} X_n = Y
\]
An important application of the MCT is in stability analysis of stochastic systems. Supermartingales play the role of Lyapunov functions.

There are also important inequalities associated with martingales. E.g.,

$$P\left( \sum_{n=1}^{\infty} |X_n| > x \right) \leq \frac{1}{x} \sum_{n=1}^{\infty} E|X_n|$$

(Doob's Maximal Thm.)

(Insert p. 272)

**Semi-martingales**

**Def** A stochastic process \( A = \{A_n\} \) is said to be predictable with respect to the filtration \( \{\mathcal{F}_n\} \) if \( A_n \) is \( \mathcal{F}_{n-} \)-measurable.

E.g. Suppose \( \mathcal{F}_n = \sigma(\mathcal{Z}_0, \mathcal{Z}_1, \cdots, \mathcal{Z}_n) \) for some stochastic process \( \{\mathcal{Z}_n\} \). If \( A \) is predictable w.r.t. \( \mathcal{F}_n \) then \( A_n \) is a function of \( \mathcal{Z}_0, \mathcal{Z}_1, \cdots, \mathcal{Z}_{n-1} \), i.e., the value of \( A_n \) is determined by the values of \( \mathcal{Z}_0, \mathcal{Z}_1, \cdots, \mathcal{Z}_{n-1} \).
\[ W_n = \begin{cases} 1 & w_1 \neq 2 \\ -1 & w_1 = 2 \end{cases} \quad \text{(i.d.d.)} \]

\[ X_n = \sum_{k=0}^{n} W_k \]

\[ = X_{n-1} + W_n \]

\[ X \text{ is a martingale w.r.t. } \mathcal{F}_n \]

\[ \mathcal{F}_n = \sigma(\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_n) \]

\[ E[X_{n+1} | \mathcal{F}_n] = E[X_n + W_{n+1} | \mathcal{F}_n] \]

\[ = X_n + E[W_{n+1} | \mathcal{F}_n] \quad \text{since } X_n \text{ is } \mathcal{F}_n \text{-meas.} \]

\[ = X_n + E[W_{n+1}] \quad \text{by independence} \]

\[ = X_n \]
Let $X$ be a stochastic process adapted to $\mathcal{F}_n$. If

$$X = A + M$$

where $A$ is $\mathcal{F}_n$-predictable and $M$ is an $\mathcal{F}_n$-martingale,

then $X$ is called a semimartingale.

Semimartingales are important in engineering:

$$X = \text{"signal" or "useful" + "noise"}$$

$A$ is sometimes called the "compensator" of $X$ (relative to $\mathcal{F}_n$).

Then (Doob decomposition)

If $X$ is an integrable adapted process, then $X$ is a semimartingale.

**Proof.** Define

$$A_n = \sum_{k=1}^{n} \mathbb{E} [X_k - X_{k-1} \mid \mathcal{F}_{k-1}]$$

$$M_n = X_n - A_n$$

By definition, $A$ is $\mathcal{F}_n$-predictable.

Next,

$$\mathbb{E} [M_n \mid \mathcal{F}_{n-1}] =$$
\[ E[X_n | \mathcal{F}_n] = \sum_{L=1}^{n} E[L(X_L - X_{L-1}) | \mathcal{F}_{L-1}] / \sigma^2 \]

\[ = E[X_n | \mathcal{F}_n] - \left( \frac{E[L E[X_n - X_{n-1}] | \mathcal{F}_{n-1}]}{\sigma^2} \right) \]

\[- \frac{E[L]}{\sigma^2} E[L(X_L - X_{L-1}) | \mathcal{F}_{L-1}] \]

\[ = X_{n-1} - A_{n-1} \]

\[ = M_{n-1} \]

\[ M \Rightarrow \{ \mathcal{F}_n \} \rightarrow \text{ mixing, } \]

**Rmk:** Write \( \Delta X_n = X_n - X_{n-1} \)

Then \( \Delta X_n = \Delta A_n + \Delta M_n \)

\( \Delta A_n = E[\Delta X_n | \mathcal{F}_{n-1}] \)

\( E[\Delta M_n | \mathcal{F}_{n-1}] = 0 \)

\( \in \text{ innovations, represent new information.} \)

\[ X_n = \sum_{k=0}^{n} W_k \quad (W_k \text{ iid } \mathcal{N}(0, 1)) \]

\[ \Delta X_n = W_n = \Delta M_n, \quad \Delta A_n = 0. \]
Corollary If $X$ is a submartingale adapted to $\{\mathcal{F}_n\}$, then there exists an increasing predictable process $A$ and a martingale $M$ s.t.

$$X = A + M$$

Proof. By Doob's theorem, $X = A + M$ where $A$ is predictable, $M$ a g. We must show $\Delta A_n \geq 0$.

Since $X$ is a submartingale,

$$\Delta A_n = E[X_n - X_{n-1} | \mathcal{F}_{n-1}] \geq 0$$

\[ Q.E.D. \]

The **(optional) quadratic variation** process $[X, X]$ of a stochastic process $X$ is defined by

$$[X, X]_n = X_n^2 + \sum_{k=1}^{n} (\Delta X_k)^2$$

The **predictable quadratic variation** is

$$<X, X>_n = X_n^2 + \sum_{k=1}^{n} E[(\Delta X_k)^2 | \mathcal{F}_{k-1}]$$

Note $[X, X]$, $<X, X>$ are increasing

$[X, X]$ and $<X, X>$ are adapted

$<X, X>$ is predictable.
If \( M \) is a martingale, then
\[
\langle M, M \rangle_t \text{ is the 'empirical' of } M^2.
\]

Cross terms are defined also; e.g.,
\[
\{X, Y\} = \frac{1}{2} \left( \{X + Y, X + Y\} - \{X, X\} - \{Y, Y\} \right)
\]
or
\[
\{X, Y\}_n = \sum_{k=1}^{n} \{X_k, Y_k\} + X_0 Y_0.
\]

E.g., let \( \mathcal{F}_n \) be a filtration and \( \{W_n\} \) a sequence of iid \( N(0, \sigma^2) \) r.v.'s s.t. \( W_n \) is independent of \( \mathcal{F}_{n-1} \). Then set
\[
X_n = \sum_{k=1}^{n} W_k.
\]
The
\[
\langle X, X \rangle_n = n \sigma^2,
\]
the variance of \( X \).
**Markov Processes**

Let $X$ be a stochastic process defined on $(\mathbb{S}, \mathcal{F}, \mathbb{P})$ and taking values in a measurable space $(S, \mathcal{S})$ and adapted to $\mathcal{F}_n$. $X$ is called a **Markov process** if

$$
P(X_m \in A \mid \mathcal{F}_n) = P(X_m \in A \mid X_n)
$$

for $A \in \mathcal{S}$, $m \geq n$.

The conditional probabilities of the future behavior of $X$ given the whole past depend only on the current value.

$\mathbb{S} = \mathbb{R}$, $\mathcal{S} = \mathcal{B}(\mathbb{R})$

$W$ is a process

$b : \mathbb{R}^2 \to \mathbb{R}$

$X_{n+1} = b(X_n, W_n)$

$X$ is a **Markov process** with

$\mathcal{F}_n = \sigma(W_0, W_1, \ldots, W_{n-1})$

$$E[A(X_{n+1}) \mid \mathcal{F}_n] = E[E[A(b(X_n, W_n)) \mid \mathcal{F}_n]]$$

$$= \int_{-\infty}^{\infty} \psi(b(X_n, w)) f_{W_n}(dw)$$

$$= E[E[\psi(X_{n+1}) \mid X_n]]$$

Setting $\psi = I_A$ we see that $X$ is Markov,
The (one-step) transition probabilities of a Markov process \( X \) are given by

\[
P_{X_{n+1} | X_n}(A | x) = \mathbb{P}(X_{n+1} \in A \mid X_n = x)
\]

(transition kernel)

\[
P_{X_{n+1} | X_n}(A | x) = \int_{-\infty}^{\infty} \mathcal{I}_A(b(x, w)) \, P_X \, (dw)
\]

Proposition (Chapman-Kolmogorov)

\[
P_{X_{n+1}}(A) = \int_S P_{X_{n+1} | X_n}(A | x) \, P_X \, (dx)
\]

\[
\mathbb{E}[\psi(X_{n+1})] = \int_S \psi(x') \, P_{X_{n+1}}(dx') = \int_S \psi(x') \int_S P_{X_{n+1} | X_n}(dx' \mid x) \, P_X(dx)
\]

\[
\text{Proof:} \quad P_{X_{n+1}}(A) = \mathbb{E}[\mathbb{I}_A(X_{n+1})]
\]

\[
= \mathbb{E}[\mathbb{E}[\mathbb{I}_A(X_{n+1}) \mid X_n]] = \mathbb{E}[\mathbb{E}[P_{X_{n+1} | X_n}(A | x) \mid X_n]]
\]

\[
= \int_S P_{X_{n+1} | X_n}(A | x) \, P_X(dx)
\]

\[
= \int_S P_{X_{n+1} | X_n}(A | x) \, P_X(dx)
\]
eq (Markov chain) \( S = \{s_1, \ldots, s_N\} \)

Let \( P \) be an \( N \times N \) matrix such that

\[
\sum_{j=1}^{N} P_{i,j} = 1, \quad a_{ij} \geq 0.
\]

Define

\[
P_{X_{n+1}, X_n}(s_j, s_i) = P_{ij}
\]

\[
p_n(\cdot) = P_{X_n}(\cdot | s_i) = P(X_n = \cdot | s_i)
\]

Then

\[
p_{n+1}(s_j) = \sum_{i=1}^{N} P_{ij} \cdot p_n(i)
\]

\[
p_{n+1} = P^* p_n
\]

\[
p_n = (P^*)^n p_0
\]

where \( p_0(s_i) = P(X_0 = s_i) = \pi(s_i) \) is the initial distribution.

Thus, the prob dist \( p_n \) of \( X_n \) satisfies the dynamics

\[
\begin{cases}
    p_{n+1} = P^* p_n \\
    p_0 = \pi
\end{cases}
\]

Futhermore, we regard \( p_n \) as a column vector. If, instead, we regard \( p_n \) as a row vector, then

\[
p_{n+1} = p_n P.
\]
In general, suppose \( f_{x_{n+1}}/x_n \) is independent of time \( n \). Write
\[
p(A|x) = f_{x_{n+1}}(A|x)
\]
\[
\rho(A) = f_{x_n}(A)
\]
\[
\rho_n(A) = P_{x_n}(A)
\]

Then
\[
\begin{align*}
\rho_{n+1} &= \rho_n^* \\
\rho_0 &= \rho
\end{align*}
\]

where
\[
(\rho_n^* \rho)(A) = \int S p(A|x) \rho(dx)
\]

**Observation Processes**

Let \( X \) be a Markov process with values in \( S \). An observation process is an \( S \)-valued process defined by the observation probabilities \( p_{y|x} \):
\[
P(\gamma_n \in B) = \int S p_{y|x}(B|x) \rho_n(dx)
\]
\[\forall B \in \mathcal{B}(R)\]

**Note** For each \( x \in S \), \( p_{y|x}(\cdot | x) \) is a probability measure on \((\emptyset, \mathcal{B}(\emptyset))\).
Let \( \{V_n\} \) be independent of \( X \).

Define

\[
Y_n = h(X_n, V_n)
\]

Then

\[
P_{Y/X}(y/x_n) = P(Y_n \in B | x_n)
\]

\[
= P(h(x_n, V_n) \in B | x_n)
\]

\[
= \int_B I_B(h(x_n, v_n)) p_{V_n}(dv)
\]

---

**Linear Representation of a Markov Chain**

Replace \( S \) by \( S' = \{e_1, \ldots, e_N\} \),

where \( e_i \) is the \( i \)th unit vector in \( \mathbb{R}^n \).

Then

\( X_n = e_i \) for some \( i = 1, \ldots, N \).

**Lemma.** Let \( X_n \) be a Markov chain

with probability matrix \( P \).

Then

\[
\Delta M_{n+1} = X_{n+1} - P X_n
\]

is a martingale increment.

**Corollary.** \( X \) has the semimartingale or

innovation representation

\[
X_{n+1} = P X_n + \Delta M_{n+1}
\]

or

\[
\Delta X_{n+1} = (P - I) X_n + \Delta M_{n+1}
\]
3. **State Space Models**

**Technical Note:**

A **Borel space** is a measurable space \( \mathcal{S} = (\mathcal{S}, \mathcal{B}) \) obtained by taking a topological space \( (\mathcal{S}, \tau) \) and defining \( \mathcal{B} \) to be the Borel \( \sigma \)-algebra generated by the open sets \( \tau \). Hence \( \mathcal{B} = \mathcal{B}(\tau) \).

\[ \mathcal{B}(\tau) = \{ S \} \cup \{ \bigcup S \} \cup \{ \bigcap S \} \quad \forall S \subseteq \tau \]

We will define stochastic systems in terms of controlled transition probabilities and output probabilities.

**Def:** A **stochastic system** \((\mathcal{S}, U, O, P, Q, \rho)\) is specified by

- **Borel space** \(\mathcal{S}, U, O\) (state, control, or input, observation or output)

- **Transition probability** \(P\)

\[ P(\cdot | x, u) \text{ is a probability measure on } \mathcal{S} \]

\[ \forall (x, u) \in \mathcal{S} \times U \]
- Output probability $Q$

$$Q(\cdot | x)$$ is a probability measure on $O$ \forall x \in S.

- Initial distribution $f$ of $X_0$.

The evolution of the system is as follows. If $X_n = x \in S$ is the state at time $n$, and if $U_n = u \in U$ is the control input applied at time $n$, then the system moves to a new state $X_{n+1}$ according to the probability measure $Q(\cdot | x, u)$ and produces an output $Y_n$ according to $Q(\cdot | x)$.

**Example:** Nonlinear system

- $S = \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$, $O = \mathbb{R}^p$
- $\{W_n\}$, $\{V_n\}$ i.i.d., independent, values in $\mathbb{R}^p$, $\mathbb{R}^s$
- $X_0 \sim f$ independent of $W, V$

$$b : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \quad h : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^p$$

$$\begin{cases} X_{n+1} = b(X_n, U_n, W_n) \\ Y_n = h(X_n, V_n) \end{cases}$$
\[ P(A | x, u) = \int_{\mathbb{R}^n} I_A (b(x, u, w)) \ p_w (dw) \]

\[ \forall \ A \in \mathcal{B} = \mathcal{B}(\mathbb{R}^n) \]

\[ \forall \ (x, w) \in \mathbb{R}^n \times \mathcal{C}. \]

\[ Q(B | x) = \int_{\mathbb{R}^n} I_B (h(x, u)) \ p_v (dw) \]

\[ \forall \ B \in \mathcal{G} = \mathcal{B}(\mathbb{R}^n) \]

\[ x \in \mathcal{S} \]

**Example**

**Linear Systems**

Same as nonlinear except \( b, h \) are linear and \( W, V \) and \( p \) are Gaussian:

\[ W_n \sim N(0, Q) \quad \quad V_n \sim N(0, R) \]

\[ p \sim N(\bar{x}_0, \Sigma_0) \]

\[ X_{n+1} = AX_n + BU_n + GW_n \]

\[ Y_n = CX_n + HV_n \]

**Example**

**Controlled Markov Chains** (Finite state)

\[ S = \{ s_1, s_2, \ldots, s_N \} \quad \quad U = \{ u_1, u_2, \ldots, u_M \} \]

\[ O = \{ o_1, o_2, \ldots, o_p \} \]

\( P(\cdot | x, u) \) is defined by a controlled transition matrix

\[ P(u) \]
\[ \sum_{j=1}^{N} p_{ij}(u) = 1 \quad \forall \ u \in U \quad \forall \ s_i \in S \quad p_{ij}(u) \geq 0 \]

\[ P(s_j \mid x, u) = p_{ij}(u) \]

\[ Q( \cdot \mid x) \text{ is defined by an output probability matrix } Q: \]

\[ Q(o_j \mid s_i) = Q_{ij} \]

\[ \sum_{j=1}^{N} Q_{ij} = 1 \quad \forall \ s_i \in S \quad Q_{ij} \geq 0 \]

Feedback Control Laws or Policies

A deterministic sequence \( u = \{u_0, u_1, u_2, \ldots \} \) of control values is called an open loop control.

A feedback or closed loop control depends on the observations.

More precisely, let \( y = \{y_0, y_1, y_2, \ldots \} \) be a sequence of functions \( g_n : D^{n+1} \rightarrow U \)

The value \( u_n \) of the control at time \( n \) is given by
\[ u_n = g_n(y_n) \]

where \[ y_n = \{ y_0, y_1, \ldots, y_n \} \].

Such a sequence \( g \) is called a feedback control policy or law (closed loop).

A \underline{closed loop} control is a special case of a \underline{closed loop} policy.

A policy \( g \) determines a probability measure \( \mu_g \) in the \underline{canonical} sample or path space \( (\Omega, \mathcal{F}, \mathbb{F}) \),

\[ \Omega = (S \times U \times Y)^\infty \]

\[ \mathcal{F} = \mathcal{B}(\Omega) \]

Note: \( u \in \Omega \) is of the form

\[ \omega = \{ x_0, u_0, y_0, x_1, u_1, y_1, \ldots \} \]

Let \( \mathcal{F}_n = \sigma( x_0, y_0, x_1, y_1, \ldots, x_n, y_n ) \)

\[ Y_n = \sigma( y_0, y_1, \ldots, y_n ) \]

Then \( U_n = g_n(Y_n) \) is adapted to \( Y_n \) and \( X_n, Y_n \) is adapted to \( \mathcal{F}_n \).

Note: \( Y_n = \sigma( y_0, y_1, \ldots, y_n, u_0, u_1, \ldots, u_n ) \)

since \( g \) is well.
Partial and Full Information

In general, the only information available to the controller is contained in the output or observation process $Y$. $X$ is in general unavailable. This is the case of partial information.

If $Q = S$ and $Q(\cdot | x) = S_x$ (state measure), then $Y \equiv X$, and one has full state information.

Central problems are much easier to solve when one has full state information. In the general case of partial information, one must do some filtering as well.
4. **Filtering**

**Introduction**

Consider the (non-dynamic) filtering problem

\[
\begin{align*}
X &= \mu + W \\
Y &= CX + V
\end{align*}
\]

where \( W \sim N(0, Q) \), \( V \sim N(0, R) \) are independent.

Let's compute

\[
\hat{X} = E[X | Y]
\]

a. Using "innovations"

We will see that \( \hat{X} \) has the representation

\[
\hat{X} = \mu + C \hat{Y}
\]

where \( \hat{Y} = Y - C \mu \) is the "innovation".

Note

\[
\bar{X} = E[X] = \mu
\]

\[
\bar{Y} = E[Y] = C \mu
\]
Covariances:

\[ \Sigma_x = \text{cov}(X) = \mathbb{E} [(X - \mu)(X - \mu)^\top] \]
\[ = \mathbf{Q} \]

\[ \Sigma_y = \text{cov}(Y) = \mathbb{E} [(Y - \mathbf{c}_p)(Y - \mathbf{c}_p)^\top] \]
\[ = \mathbf{C} \mathbf{Q} \mathbf{C}^\top + \mathbf{R} \]

\[ \Sigma_{xy} = \text{cov}(X, Y) = \mathbb{E} [(X - \mu)(Y - \mathbf{c}_p)^\top] \]
\[ = \mathbf{Q} \mathbf{C}^\top \]

Claim

\[ \hat{X} = \mu + \Sigma_{xy} \Sigma_y^{-1} (Y - \mathbf{c}_p) \]

Proof

Write

\[ \hat{Y} = \mu + \Sigma_{xy} \Sigma_y^{-1} (Y - \mathbf{c}_p) \]

\[ \hat{Y} = X - \hat{X} \]

Then

\[ \mathbb{E} \hat{Y} = \mathbf{0} \]

\[ \mathbb{E} \hat{Y} (Y - \mathbf{c}_p) = \mathbf{0} \]

since Gaussian \( \hat{Y} \) and \( Y \) are independent.

\[ \hat{X} = \mathbb{E} \left[ X - \hat{Y} + \hat{Y} \mid Y \right] \]

\[ = \mathbb{E} \left[ \hat{Y} (Y) + \hat{Y} (Y) \right] \]

\[ = \mathbf{0} + \mathbf{R} \]
This gives the "Kalman Filter" for this problem:

\[
\hat{X} = \mu + QC(CQC' + R)^{-1}(Y - C\mu)
\]

b. Using "reference probability"

Under a probability measure \( P^* \):

\[
\begin{align*}
\lambda &= \mu + W \\
Y &= X + V
\end{align*}
\]

Joint density of \( X, Y \):

\[
p_{X,Y}(x,y) = p_X(x) p_{Y|X}(y|x)
\]

\[
p_X(x) = (2\pi|\Sigma|)^{-1/2} \exp \left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)
\]

\[
p_{Y|X}(y|x) = (2\pi|\Theta|)^{-1/2} \exp \left(-\frac{1}{2}(y-Cx)^T\Theta^{-1}(y-Cx)\right)
\]

Using a new "reference" probability measure \( P^* \):

\[
\begin{align*}
X &= \mu + W \\
Y &= V^*
\end{align*}
\]

where \( V^* \sim N(0_R) \), \( W \sim N(0, \Theta) \)

\( V^* \) and \( W \) \( \text{i.i.d.} \).
Note that the conditional error covariance is given by

$$\Sigma_x = \Sigma_{xy} \Sigma_y^{-1}$$

$$= \Sigma - \Sigma_{xy} \Sigma_y^{-1} \Sigma_{xy}$$

**Claim**: \(\tilde{x} = x - \mu\) is the projection of \(\tilde{x} = X - \mu\) onto the subspace

\[Y = \{ x \tilde{y} : x \in \mathbb{R} \} \]

relative to the inner product \(E[\tilde{x} \tilde{y}] = \sigma(x, y)\),

i.e. \(X - (X - \mu) \perp Y\)

e.g. \((\tilde{x} - (X - \mu), \tilde{y}) = 0\)

since \(\tilde{x} = x \tilde{y}\), we get

\[(\tilde{x}, \tilde{y}) = x(\tilde{y}, \tilde{y})\]

\[x = (\tilde{x}, \tilde{y})(\tilde{y}, \tilde{y})^{-1} = \Sigma_{xy} \Sigma_y^{-1} \tilde{y} \]
The joint density \( f_{X,Y}(x,y) \) is given by:

\[
p_{x,y}^+(x,y) = p_x(x) p_y(y)
\]

where

\[
p_y(y) = (2\pi \sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} y^T \Sigma^{-1} y\right)
\]

Write

\[
\Lambda(x,y) = \exp\left(-\frac{1}{2} x^T \Sigma^{-1} x - y^T \Sigma^{-1} y\right)
\]

\[
\left( \text{aside: } \frac{d\Lambda}{d \Sigma} = \Lambda \right)
\]

Bayes's rule:

\[
E[X \mid Y] = \frac{E[X \mid Y] \Lambda(x,y)}{E[\Lambda \mid Y]}
\]

- Conditional density

\[
p_{X \mid Y}(x \mid y) = \frac{\Lambda(x,y) p(x)}{\int_{\mathbb{R}^n} \Lambda(x,y) p(x) \, dx}
\]

\[
= \frac{p_{X \mid Y}(x \mid y)}{\int_{\mathbb{R}^n} p_{X \mid Y}(x \mid y) \, dx}
\]

\( p_{X \mid Y} \) = "normalized conditional density"
\[ \mathbb{E}(x|x) = \int_{\mathbb{R}^n} x \cdot f_{x|x}(x|y) \, dx \]

\[ = \int_{\mathbb{R}^n} x \cdot \Phi_{x|x}(x|y) \, dx \]

\[ = \mu + \Phi_{c}((C\Phi_{c} + R)^{-1}(y - \mu)) \]

**Remark:** The main techniques for nonlinear filtering are the innovations approach and the reference probability method.
The Kalman Filter

The proof of the following technical lemma follows from the discussion in the introduction (KV, p 97).

Lemma. Let \( X, Y \) be jointly Gaussian random variables, with

\[
X \sim N(\bar{X}, \Sigma_x), \quad Y \sim N(\bar{Y}, \Sigma_y)
\]

\[
\Sigma_{xy} = \text{cov}(X,Y)
\]

Write

\[
\hat{X} = \mathbb{E}(X | Y), \quad \tilde{X} = X - \hat{X}
\]

Then

\[
\hat{X} = \bar{X} + \Sigma_{xy} \Sigma_y^{-1} (Y - \bar{Y})
\]

and

\[
\tilde{X} \text{ is indp. of } Y \text{ and of } \hat{X}.
\]

Also,

\[
\tilde{X} \sim N(0, \Sigma_{\tilde{X}}),
\]

where

\[
\Sigma_{\tilde{X}} = \Sigma_x - \Sigma_{xy} \Sigma_y^{-1} \Sigma_{xy}'
\]

Coroll. Let \( X, Y, Z \) be jointly Gaussian with \( Y \) and \( Z \) indp. Write

\[
\hat{X} = \mathbb{E}(X | Y, Z), \quad \tilde{X} = X - \hat{X}
\]

Then

\[
\hat{X} = \bar{X} + \Sigma_{xy} \Sigma_y^{-1} (Y - \bar{Y}) + \Sigma_{xz} \Sigma_z^{-1} (Z - \bar{Z})
\]

\[
\Sigma_{\tilde{X}} = \Sigma_x - \Sigma_{xy} \Sigma_y^{-1} \Sigma_{xy}' - \Sigma_{xz} \Sigma_z^{-1} \Sigma_{xz}'
\]
Stochastic system:

\[
\begin{align*}
X_{k+1} &= AX_k + GW_k \\
Y_k &= CX_k + HV_k
\end{align*}
\]

\[
X_0 \sim N(\overline{X}_0, \Sigma_c), \quad W_k \sim N(0, Q_k)
\]

\[V_k \sim N(0, R) \quad \text{and mutually independent (Q_k, R_j, } \\
\Sigma_c, \Sigma_e)\]

Filtering problem: to compute conditional expectations of the form

\[
\hat{X}_k = E[\phi(X_k) \mid Y_{0:k}]
\]

Since all the random variables are Gaussian, the conditional distribution

\[p_{X_k \mid Y_{0:k}}(X_k \mid Y_{0:k}) = p(X_k \in A \mid Y_{0:k})\]

is Gaussian with density

\[
p_{X_k \mid Y_{0:k}}(X_k \mid Y_{0:k}) = p_{k,k}(X_k \mid Y_{0:k})
\]

\[
= \left(\frac{1}{2\pi|\Sigma_{k,k}|}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2} (X_k - \hat{X}_k) \Sigma_{k,k}^{-1} (X_k - \hat{X}_k)\right)
\]

where

\[
\hat{X}_{k,k} = E[X_k \mid Y_{0:k}]
\]

\[
\Sigma_{k,k} = E \left[ (X_k - \hat{X}_{k,k})(X_k - \hat{X}_{k,k})' \mid Y_{0:k} \right]
\]
Note that the conditional distribution is completely determined by the finite dimensional quantiles:

\[ X_{k|k} \in \mathbb{R}^n \]
\[ \Sigma_{k|k} \in \mathbb{R}^{n \times n} \]

The filtering problem will be solved if we can find simple recursions for these two quantities.

Thus,

\[ \hat{X}_k = \int_{\mathbb{R}^n} \phi(x) \ p_{k|k}(x | y_{0:k}) \, dx \]

**Theorem (Kalman Filter)**

The conditional density \( p_{k|k} \sim \mathcal{N}(X_{k|k}, \Sigma_{k|k}) \) is obtained from:

\[ X_{k+1|k} = AX_{k|k} + L_{k+1} (Y_{k+1} - CAX_{k|k}) \]
\[ X_{0|0} = L_0 Y_0 \]
\[ \Sigma_{k+1|k} = (I - L_{k+1}C) \Sigma_{k+1|k} \]
\[ \Sigma_{k+1|k} = A \Sigma_{k|k} A^T + GG^T \]
\[ \Sigma_{0|0} = (I - L_0C) \Sigma_0 \]
\[ L_k = \Sigma_{k|k-1} C'(C\Sigma_{k|k-1}C' + \nu RH\nu')^{-1} \]
\[ L_0 = \Sigma_0 C'(C\Sigma_0 C' + \nu RH\nu')^{-1} \]

\text{Notation:} \ p_{k+1|k}(x|y_{0:k}) \text{ denotes the conditional density of } x_{k+1} \text{ given } y_{0:k}.

\[ x_{k+1|k} = E[x_{k+1} | y_{0:k}] \]
\[ \Sigma_{k+1|k} = E[(x_{k+1} - x_{k+1|k})(x_{k+1} - x_{k+1|k})'] | y_{0:k} \]

\[ p_{k+1|k} \sim N(x_{k+1|k}, \Sigma_{k+1|k}) \]

\text{Proof: Step 1}

\[ x_{k+1|k} = E\left[ \hat{x} x_k + Cw_k | y_{0:k} \right] \]
\[ = A E[x_k | y_{0:k}] + Cw_k \]
\[ x_{k+1|k} = A \hat{x}_k \]

(1)

\[ \tilde{x}_{k+1|k} = x_{k+1} - x_{k+1|k} \]
\[ = A (x_k - x_{k|k}) + Cw_k \]
\[ \tilde{x}_{k|k} \]
\[ \Sigma_{k+1[k]} = A \Sigma_{k[k]} A' + G Q G' \] 

\[ \text{since } \tilde{X}_{k|k}, \tilde{W}_k, \tilde{v}_k \]

**Step 2**

\[ Y_{k|k-1} = E [ Y_k | Y_{0:k-1}] \]
\[ = CE [ X_k | Y_{0:k-1}] + HE [ Y_{k|k-1} ] \]
\[ = C \tilde{X}_{k|k-1} \]

\[ \tilde{Y}_{k|k-1} = Y_k - Y_{k|k-1} \]
\[ = C \tilde{X}_{k|k-1} + H v_k \]

\[ E [ X_k \tilde{Y}_{k|k-1} ] = \Sigma_{k|k-1} C' \]  

\[ \text{since } X_k - X_{k|k-1} + \tilde{X}_{k|k-1} \]
\[ \text{and since } X_k, v_k \text{ independent.} \]
Step 3

\[
X_{k|k} = E[X_k | y_{0:k-1}, \tilde{y}_{k|k-1}]
\]

\[
= E[X_k | y_{0:k-1}]
\]

\[
+ E[X_k \tilde{y}_{k|k-1}] (\Sigma_{k|k-1}^{-1}) \tilde{y}_{k|k-1}
\]

Using Cor. 6.

\[
X_{k|k} = X_{k|k-1} + \Sigma_{k|k-1} C' \left( C \Sigma_{k|k-1} C' + HRH' \right)^{-1} \tilde{y}_{k|k-1}
\]

(4)

\[
\Sigma_{k|k} = \Sigma_{k|k-1} - \Sigma_{k|k-1} C' \left( C \Sigma_{k|k-1} C' + HRH' \right)^{-1} C \Sigma_{k|k-1}
\]

(5)

Step 4

Combine (4) - (5) to get the Kalman filter
Filtering, Prediction and Smoothing

Consider

\[ X_{k|c} = E[X_k | Y_{0:c}] \]

Filtering \( k = c \)

Prediction \( k > c \)

Smoothing \( k < c \)

1-step ahead prediction: \( X_{k+1|k} \)

Then \( P_{k+1|k} (x | X_{0:k}) \) is \( N(X_{k+1|k}, \Sigma_{k+1|k}) \)

where

\[ X_{k+1|k} = A X_k | k-1 + A L_k (Y_k - C X_{k|k-1}) \]

\[ X_{0|1-1} = E X_0 = \bar{X}_0 \]

\( L_k, \Sigma_{k+1|k} \) given earlier.
The Steady-State Kalman Filter

**Theorem**

Let \( \Gamma = \sqrt{GG'G'} \) and define:

i. \((A, \Gamma)\) reachable
ii. \((A, C)\) observable
iii. \(HRH' > 0\)

Then:

a. \[ \lim_{k \to \infty} \Xi_{k+1|k} = \Xi \]

\(\Xi \in \mathcal{C}\) is the unique solution to the steady-state or algebraic Riccati equation

\[ \Xi = A\left( \Xi - \Xi C' (C \Xi C' + HRH')^{-1} C \Xi \right) A + GQG' \]

for any initial condition \( \Xi_{0|-1} \in \mathcal{C} \).

b. The time-invariant predictor

\[ \tilde{X}_{k+1|k} = A \tilde{X}_{k|k-1} + A \Xi (Y_k - C \tilde{X}_{k|k-1}) \]

with \( \Xi = \Xi C' (C \Xi C' + HRH')^{-1} \) and arbitrary initial condition \( \tilde{X}_{0|-1} \) gives:

\[ \lim_{k \to \infty} C\Xi (X_k - \tilde{X}_{k|k-1}) = \Xi \]
Proof (Main Steps)

Step 1  \( \Sigma_{0l-1} = \Sigma_{0l-1} \Rightarrow \Sigma_{k+l+k} \leq \Sigma_{k+l+k} \cup k \)  

(Inductively proves.) \hspace{3cm} \text{(Lemma 2.35, KV)}

Step 2  \((A, C)\) observable  \(\Rightarrow\)  \(\Sigma_{l+k} \cup k \leq \Sigma_{l+k} \cup k \)

\(\frac{1}{2}\) Recall that observability is equivalent to the positive definiteness of the Grammian  

\[ M = \Sigma_{l+k} (A' A + C C') \]

This is used to verify that  

\[ \Sigma_{l+n+1} = \Sigma \]

where \( \Sigma \) is only \( k \). \hspace{3cm} \text{(see p.19, KV)}

Step 3  \((A, C)\) observable and \( \Sigma_{0l-1} = \) 0  

\(\Rightarrow\)  \(\Sigma_{k+l+k} \leq \Sigma \).

\hspace{3cm} \text{(page 100, KV)}
Step 4 Assume $\Sigma_{0^{-1}} = \delta I$, $\delta > 0$, $\Rightarrow$

$\Sigma_{t+1} \rightarrow \Sigma$.

Then is $\Sigma_{t+1} \rightarrow \Sigma$ is $\Sigma_{0^{-1}}$.

ii. There is only one solution $\geq 0$ to the ARE

\[ \text{Pf:} \] Follows from where and

\[ 0 \leq \Sigma_{0^{-1}} \leq \delta I \]

Step 5 $(A, \Gamma)$ reachable $\Rightarrow$

\[ A - \tilde{R} \tilde{C} \text{ stable} \]

where $\tilde{R} = A \tilde{L}$, $\tilde{L} = \Sigma C' (CE \tilde{C}' + H \Lambda H')^{-1}$

$\Sigma = \lim_{t \to \infty} \Sigma_{t+1}$ given $\Sigma_{0^{-1}} = 0$.

(Note: $\Sigma$ exist if $(A, C)$ observable.)

(\text{Pf 6:} $\Sigma_{0^{-1}} = \delta I$, $\delta > 0$, $\Rightarrow$

$\Sigma_{t+1} \rightarrow \Sigma$.

\[ \text{Pf:} \]

$\Sigma_{t+1} - \Sigma = (A - \tilde{R} \tilde{C}) (\Sigma_{t+1} - \Sigma) (A - \tilde{R} \tilde{C})'$

stable

(\text{Pf:} $\Sigma_{t+1} \rightarrow \Sigma$)
Note: The steady-state KF is not the optimal filter!

The Kalman Filter for Controlled Linear Systems

Let $\gamma = \{g_1, g_2, \ldots\}$ be a feedback policy, possibly nonlinear. This determines:

$$x_{k+1}^g = A x_k^g + B u_k^g + C w_k^g$$

$$y_k^g = C x_k^g + H v_k^g$$

$$u_k^g = g_k(Y_{0:k}^g)$$

The processes $x^g, y^g$ need not be Gaussian. However, the conditional densities $p(x_{k+1}^g | x_k^g)$ are Gaussian.
The conditional means $X_{k|k}$ and $X_{k+1|k}$ are given by

$$X_{k+1|k+1} = \begin{bmatrix} A \end{bmatrix} X_{k|k} + \begin{bmatrix} B \end{bmatrix} U_{k} + L_{k+1} \left( Y_{k+1} - C \left( \begin{bmatrix} A \end{bmatrix} X_{k+1|k} + \begin{bmatrix} B \end{bmatrix} U_{k} \right) \right)$$

$$X_{k+1|k} = \begin{bmatrix} A \end{bmatrix} X_{k|k} + \begin{bmatrix} B \end{bmatrix} U_{k} + L_{k} \left( Y_{k} - C X_{k|k} \right)$$

$$\Sigma_{k+1|k+1} = \Sigma_{k|k+1}$$

and $L_{k}$ are given as before.

Write

$$X_{k} = \begin{bmatrix} \bar{X} \end{bmatrix} + X_{k}, \ \ Y_{k} = \begin{bmatrix} \bar{Y} \end{bmatrix} + Y_{k}$$

where

$$\begin{bmatrix} \bar{X} \end{bmatrix}_{k+1} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \bar{X} \end{bmatrix}_{k} + \begin{bmatrix} B \end{bmatrix} U_{k}, \ \ \begin{bmatrix} \bar{X} \end{bmatrix}_{k} = \begin{bmatrix} \bar{X} \end{bmatrix}_{0}$$

$$\begin{bmatrix} \bar{Y} \end{bmatrix} = C \begin{bmatrix} \bar{X} \end{bmatrix}_{k}$$

$$X_{k+1|k} = \begin{bmatrix} A \end{bmatrix} X_{k|k} + C U_{k}, \ \ X_{0} \sim N(0, \Sigma_{0})$$

$$Y_{k} = C X_{k} + G U_{k}.$$

Then the conditional density of $X_{k}$ given $(Y_{0:k}, U_{0:k-1})$ is the conditional density of $X_{k}$ given $Y_{0:k}$ shifted.
by $R_h$. (See KV pg. 3 for details.)
The HMM Filter (Markov Chain)

Let \( X, Y \) be a HMM defined by a transition matrix \( P \) and an output probability matrix \( Q \):

\[
\begin{align*}
P(X_{k+1} = s_j \mid X_k = s_i) &= p_{ij} \\
Q(Y_k = o_j \mid X_k = s_i) &= q_{ij}
\end{align*}
\]

This determines a measure \( \mu \) on a measurable space \((S^2, \Sigma)\) (canonical path space, space).

Filtering: We wish to compute the conditional probabilities

\[
P(X_k = s_j \mid Y_{0:k}).
\]

If we use the linear representation, we want the conditional vector

\[
\mathbf{P}(k) \triangleq \mathbf{E}L X_k \mid Y_{0:k}
\]

(a column vector)

(Recall that in the linear repn,

\[
X_k = \mathbf{P} \ast X_{k-1} + \Delta M_k
\]

where \( \Delta M_k \) is a martingale increment w.r.t. \( \mathcal{F}_k = \sigma(X_{0:k}, Y_{0:k}) \).)
Since $Y_k$ depends on $\Delta M_k$, it is not clear how to proceed.

We will use a reference probability $p^+$ under which $Y_k$ is independent of $\Delta M_k$.

Under $p^+$, the joint density of $X_{0:k}, Y_{0:k}$ is

$$p_{X_{0:k}, Y_{0:k}}(x_{0:k}, y_{0:k}) = p_{X_{0:k}}(x_{0:k}) p_{Y_{0:k}|X_{0:k}}(y_{0:k}|x_{0:k})$$

where

$$p_{X_{0:k}}(x_{0:k}) = \prod_{t=0}^{k-1} p_{X_t, X_{t+1}}(x_t, x_{t+1})$$

and initial

$$p_{Y_{0:k} | X_{0:k}}(y_{0:k} | x_{0:k}) = \prod_{t=0}^{k} Q(x_t, y_t)$$

The new measure $p^+$ corresponds to the joint density

$$p^+_{X_{0:k}, Y_{0:k}}(x_{0:k}, y_{0:k}) = p_{X_{0:k}}(x_{0:k}) p^+_{Y_{0:k} | X_{0:k}}(y_{0:k} | x_{0:k})$$

where

$$p^+_{Y_{0:k} | X_{0:k}}(y_{0:k} | x_{0:k}) = \frac{1}{p^+_{X_{0:k}}}$$

where $p = \#O$ is the number of output values.
Then

\[ \Lambda_k = \frac{dP}{dt} \bigg|_{y_k} \]

\[ = P^{t+1} \prod_{k=0}^{k} Q(x_k, y_k) \]

**Note:**
 Write \( \Psi_k \) for the diagonal matrix with entries \( p(x_i, y_k), i = 1, \ldots, n \).

Write

\[ \sigma_k = E^+ [ X_k \Lambda_k | Y_{0:k} ] \]

(a \( n \)-vector since \( X_k \) is an \( n \)-vector with the linear repn.)

**Note:** Under \( P_t \), \( X \) is a Markov chain with transition matrix \( P \) and initial distn \( f \). \( Y \) is iid uniform, and \( X \) and \( Y \) are indp.

Bayes' rule:

\[ P(k|k) = \frac{\sigma_k}{<\sigma_k, 1>} \]

where \( <\sigma_k, 1> = E_{i \in I} \sigma_k(i) \)

\[ = E^+ [ \Lambda_k | Y_{0:k} ] \]

\( (<, >) \) = inner product in \( \mathbb{R}^n \)
**Theorem (H. W. F.: Filter)**

\[
\begin{align*}
\dot{p}_{kk} & = \frac{1}{c_k} \varphi_k p \ast p_{k-1k-1} \\
p_{D1o} & = \mathcal{J} \\
c_k & = \langle \varphi_k p \ast p_{k-1k-1}, 1 \rangle \\
c_0 & = 1
\end{align*}
\]

**Remark**

\[
\begin{align*}
\sigma_k & = \varphi_k p \ast \sigma_{k-1} \\
\sigma_0 & = \mathcal{J}
\end{align*}
\]

Linear recursion!

**Proof Step 1**

Recursion for \( \varphi_k \).

\[
\begin{align*}
\sigma_{k}(j) & = E^+ \left[ X_{k}(j \mid k) \mid Y_{0,k} \right] \\
& = E^+ \left[ \left( p \ast X_{k-1} + \Delta M \right) \Lambda_{k-1} \mid Y_{0,k} \right] \\
& \quad \text{by misclosure} \\
& = \varphi_k(j) \sum_{i \in \Omega} p_{ij} E^+ \left[ X_{k-1}(i) \Lambda_{k-1} \mid Y_{0,k-1} \right] \\
& = \varphi_k(j) \sum_{i \in \Omega} p_{ij} \sigma_{k-1}(i)
\end{align*}
\]
Step 2 by induction,

\[ <\sigma_k | 1> = \prod_{k=0}^{k} c_k \]

To see this,

\[ p_{k|k} = \frac{\sigma_k}{<\sigma_k | 1>} \]

\[ = \frac{\Phi_k p^* \sigma_{k-1}}{<\sigma_k | 1>} \]

\[ = \frac{\Phi_k p^* p_{k-1 | k-1} <\sigma_{k-1} | 1>}{<\sigma_k | 1>} \]

\[ c_k = <\sigma_k | 1> / <\sigma_{k-1} | 1> \]

\[ \prod_{k=1}^{k} c_k = \prod_{k=1}^{k} \frac{<\sigma_k | 1>}{<\sigma_{k-1} | 1> \cdots <\sigma_0 | 1>} \]

\[ = <\sigma_k | 1>. \]

\[/\]

Note: The (non-linear) recursion for \( p_{k|k} \) is just formula (6.2), p 83 of KV.
Filter for Controlled HMMs

Consider a controlled Markov chain with transition matrix \( P(u) \) and output probability matrix \( Q \).

Let \( q \) be a feedback policy:

\[
 u_k = q(Y_{0:k})
\]

We want to compute the conditional expectation:

\[
P( X_k = s_j \mid Y_{0:k}, U_{0:k-1})
\]

i.e.

\[
P^g_{k|k} = E \left[ X_k \mid Y_{0:k}, U_{0:k-1} \right]
\]

**Theorem (KV page 81)**: \( P^g_{k|k} \) does not depend on \( g \). It satisfies:

\[
\begin{cases}
P^g_{k|k} = \frac{1}{c_k} \sum_{k} P(u_k) P_{k-1|k-1} \\
P^g_{k|0} = g \\
\gamma_k = \langle \sum_{k} P^x \ P_{k-1|k-1} \rangle \\
c_0 = 1
\end{cases}
\]

as before.
Define a stochastic process $S = \{S_k\}$ is called an information state if

(i) $S_k$ is a function of $Y_k$, $U_k$, $U_{k-1}$

(ii) $S_{k+1}$ can be determined from $S_k$, $Y_{k+1}$, and $U_k$.

Example: $P_{klk} = P_{lkl}$ is an information state.

$S_{klk} = S_{lkl}$ is also an information state.

For linear systems, $X_{klk}$ is an information state.
5. Dynamic Programming

Case I: Complete State Information

The stochastic system is described by

\((S, U, P, \Phi)\)

A control policy \(g = \{g_0, g_1, \ldots\}\) determines control \(U\) and state \(X\) processes:

\(u_k = g_k(x_{k,k})\)

\(P(x_{k+1} \in A | x_{k}, u_{k}) = P(A | x_{k}, u_{k})\)

Let

\[G = \{g : g\text{ is a state feedback policy}\}\]

denotes the set of admissible controllers.

Cost function:

\[J(g) = E^g[\sum_{t=0}^{\infty} L(x_t, u_t) + \Phi(x_M)]\]

where

\[L : S \times U \to R\]

\[\Phi : S \to R\]

\[L \geq 0, \quad \Phi > 0,\]
Optimal control problem:

Find \( g^* \in G \) minimizing \( J \), i.e.

\[
J(g^*) = \min_{g \in G} J(g) \quad \forall \quad g \in G,
\]

We solve this problem using dynamic programming.

To this end, we define the value function

\[
V_k(x) = \inf_{g \in G_{k,m-1}} \mathbb{E}_{x|k} \left[ \sum_{k'=k}^{m-1} L(x_{k'}, u_{k'}) + \Phi(x_m) \right]
\]

for \( k = 0, \ldots, m-1 \)

\[
V_m(x) = \Phi(x)
\]

where \( G_{k,m} \) denotes policies \( g \in \{ g_k, g_{k+1}, \ldots, g_m \} \)

\( V_k(x) \) is the optimal "cost to go", given that we start at \( x \) at time \( k \):

i.e. \( x_k = x \).
The function $V$ satisfies the dynamic programming equation:

$$
\left\{ \begin{array}{l}
V_k(x) = \inf_{u \in U} \left\{ L(x,u) + \int_{S} V_{k+1}(z) P(k|x, u) \right\} \\
V_m(x) = g^*(x)
\end{array} \right.
$$

Further, let $\bar{u}_k^*(x)$ denote a control which achieves the minimum in (DP5), $k = 0, \ldots, m-1$. Then the policy $g^*$ defined by

$$
\bar{u}_k = g^*(X_{0:k}) = \bar{u}_k^*(X_k)
$$

is optimal, and

$$
J(g^*) = \int_{S} V_m(z) \rho(A | z).
$$

Remark: Policies $g$ for which $g_k$ is only a function of $X_k$ (and not $X_{0:k-1}$) are called Markov policies.

Then w.r.t. (DP5), when $g$ is a Markov policy, $X$ is a Markov process.

So the optimal policy $g^*$ above is Markov and the optimal stationary process $X$ is Markov:

$$
\rho^*_k\left( X_{0:k} \in A \mid \bar{u}_k \right) = \rho(A \mid X_k, g^*_k(x)).
$$
A first order algorithm for finding the optimal state feedback controller.

1. Set $k = M$. \( V_M(x_k) = \Phi(x_k) \)

2. Set $k = M-1$. Then solve (RPE) to get \( V_{m-1}(x) \) and \( u_{m-1}^*(x) \).

3. Set $k = M-2$. solve (RPE) to get \( V_{m-2}(x) \) and \( u_{m-2}^*(x) \).

4. Continue, until $k = 0$.

5. Set

\[
    g_k^*(x_0) = u_k^*(x_0) \\
    g_{k-1}^*(x_{k-1}) = u_{k-1}^*(x_{k-1}) \\
    \vdots \\
    g_1^*(x_0, x_1) = u_1^*(x_1) \\
    g_0^*(x_0, x_1, \ldots, x_{M-2}) = u_0^*(x_{M-2})
\]
Proof: Define \( W \) by

\[
\begin{aligned}
W_k(x) &= \inf_{u \in U} \left\{ L(x,u) + \int_{S} W_{k+1}(z) \rho(dz|x,u) \right\} \\
W_\infty(x) &= \Phi(x)
\end{aligned}
\]

Our goal is to prove \( W_k(x) = V_k(x) \), and \( g^* \) is optimal.

Define

\[
\bar{V}_k(x,g) = \mathbb{E}_{x,k} \left[ \sum_{t=k+1}^{\infty} L(X_t, U_t) + \Phi(X_\infty) \right]
\]

Claim:

\( W_k(x) = \inf_{g \in \mathcal{G}_{k,m-1}} \bar{V}_k(x,g) = \bar{V}_k(x,g^*) \)

Hence true for \( k = m \). Then for any \( g \in \mathcal{G}_{k,m-1} \)

\[
\begin{aligned}
\bar{V}_k(x,g) &= \mathbb{E}_{x,k} \left[ L(x,g_k) + \sum_{t=k+1}^{\infty} L(X_t, U_t) + \Phi(X_\infty) \right] \\
&= \mathbb{E}_{x,k} \left[ L(x,g_k) + \bar{V}_{k+1}(X_{k+1}, g_{k+1}) \right] \\
&\geq \mathbb{E}_{x,k} \left[ L(x,g_k) + W_{k+1}(X_{k+1}, g_{k+1}) \right] \\
&\geq W_k(x)
\end{aligned}
\]

by induction, with equality if \( g = g^* \). 

\( \blacksquare \)
Eq. 1  Controlled Markov chain with
transition matrix \( P(u) \)

The dynamic programming equation is

\[
\begin{cases}
V_k(s_k) = \min_{u \in \mathcal{U}} \left\{ L(s_k, u) \\
+ \sum_{j=1}^{n} V_{k+1}(s_{k+1}) P_{ij}(u) \right\} \\
V_M(s_M) = \Phi(s_M)
\end{cases}
\]

If we regard \( V_k \) as a column vector
in \( \mathbb{R}^n \), etc., this is just

\[
\begin{cases}
V_k = \min_{u \in \mathcal{U}} \left\{ L^u + P(u) V_{k+1} \right\} \\
V_M = \Phi
\end{cases}
\]

This is a nonlinear **backward** recursion.

The optimal Markov policy \( \pi^* \) is

\[
\pi^*_k(x_k) = \tilde{u}^*_k(x_k)
\]

where \( \tilde{u}^*_k(s_k) \) achieves min in (OPE)

\( \tilde{u}^*_k \) can be viewed as a vector.
Eq. 2 \hspace{1cm} \text{Non-linear System:}

\[ X_{k+1} = b(X_k, U_k, W_k) \]

where \( W = \{ W_k \} \) i.i.d. \( \text{indep} \) of \( X_0 \sim \mathcal{P} \).

\[
\begin{cases}
V_k(x) = \min_{u \in U} \left\{ L(x, u) + \int_{\mathbb{R}^n} V_{k+1}(b(x, u, w)) P_w(du) \right\} \\
V_0(x) = \mathbb{E}(x) \\
\end{cases}
\]

\( V_k \) is a function \( \forall x \in \mathbb{R}^n \).

\( \tilde{u}_k^*(x) \) is the optimal function \( \forall x \in \mathbb{R}^n \).
\[ \begin{align*}
\text{Eg. 3 Linear systems} \quad & (LQG) \\
X_{k+1} &= AX_k + BU_k + GW_k \\
\text{Then} \quad & \\
V_k(x) &= \inf_{u \in \mathbb{R}^m} \sum_{i=1}^n L(x, u) + \int_{\mathbb{R}^n} V_{k+1}(A x + Bu + Gw) \left(2\pi I^{-1}\right)^{-\frac{1}{2}} \exp \left(-\frac{1}{2} w' Q^{-1} w \right) dw \right) \\
V_M(x) &= \tilde{\Phi}(x) \\
\text{Quadratic cost:} \quad & \\
L(x, u) &= \frac{1}{2} x' \Gamma x + \frac{1}{2} u' \Lambda u \\
\tilde{\Phi}(x) &= \frac{1}{2} x' \bar{P} x \\
\text{where} \quad & \Gamma \succ 0, \quad \bar{P} \succ 0, \quad \Delta \succ 0. \\
\text{To simplify, assume} \quad & \\
\Gamma = I, \quad \bar{P} = I, \quad Q = I, \\
X_{k+1} &= AX_k + BU_k + GW_k \\
\{W_k\} & \sim \mathcal{N}(0, I) \\
\end{align*} \]
Then

\[ V_k(x) = \frac{1}{2} x' P_k x + c_k \]

\[ \tilde{z}^*(x) = L_k x \]

where:

\[ \begin{aligned}
  P_k &= \Pi + A' \tilde{P}_{k+1} A - A' \tilde{P}_{k+1} B [ \Lambda + B' \tilde{P}_{k+1} B ]^{-1} B' \tilde{P}_{k+1} A \\
  P_m &= \bar{P} \\
  L_k &= - [ \Lambda + B' \tilde{P}_{k+1} B ]^{-1} B' \tilde{P}_{k+1} A \\
  \alpha_k &= \frac{1}{2} \sum_{i=k+1}^{M} \text{tr} P_i, \quad k = 2, \ldots, M-1, \quad \alpha_M = 0.
\end{aligned} \] 

\[ \text{Note 1. The Riccati equation (R) is a non-linear backward matrix recursion.} \]

2. The optimal Markov policy is

\[ u_k = g^*_k(x_{o,k}) = L_k x_k \]

which is linear.
Proof

Induction True for $k = N$, (trivial).

Assume true for $k + 1, \ldots, M$.

Plug $V_{k+1}(x) = \frac{1}{2} x' P_{k+1} x + x_{k+1}$ into (DPE) and evaluate the Gaussian integral (do this).

Then minimize over $u$.

This gives $u_k^*$. Plug back in to evaluate $V_k(x)$. This gives the Riccati equation (do this!)

With complete state information, the optimal controller is state feedback.
Case II Partial state Information

General stochastic system

\((S, U, O, F, G, p)\)

Control policies \(g \in G\) are active feedback

\[ u_k = g_k(Y_{0:k}) \]

Cost function:

\[ J_g = \mathbb{E}[\sum_{t=0}^{m-1} L(X_t, U_t) + \Phi(X_m)] \]

as before, except \(g\) is a function of \(Y\) only.

Partially observed optimal control:

Find an active feedback controller \(g^* \in G\) such that

\[ J(g^*) \leq J(g) \quad \forall \ g \in G. \]

Rather than treat the general case, we first do HMM's, and then linear systems.
Optimal Control of HMMs

Transition matrix $P(u)$
Output probability matrix $Q$.

To solve this problem, we could use any suitable information state.

Here, we will use

$$\pi_k = P_{1k}.$$ 

Recall that

$$(IS) \begin{cases} 
\pi_k = T(\pi_{k-1}, y_{k-1}, y_k) \\
\pi_0 = \phi
\end{cases}$$

where

$$T(\pi, u, y) = \mathcal{I}(y) P^*(u) \pi$$

We will show that optimal output feedback control of the HMM is equivalent to optimal control of (IS), a problem with full state information ($\pi_k$ is the new state, which is observed!).

To see how this is possible, write
\[ J(y) = E^y \left[ \sum_{i=0}^{m-1} L(X_i, U_i) + \Phi(X_m) \right] \]

\[ = E^y \left[ \sum_{i=0}^{m-1} E^\pi \left[ L(X_i, U_i) \mid Y_{0:i} \right] \right] \]

\[ + E^y \left[ \Phi(X_m) \mid Y_{0:m} \right] \]

\[ = E^y \left[ \sum_{i=0}^{m-1} \langle \pi, L^u \rangle + \langle \pi_m, \Phi \rangle \right] \]

where \( L^u = L(\cdot, u) \)

\[ \langle \pi, f \rangle = \sum_{i=0}^{m-1} \pi(i) f(i) \]

This \( J(y) \) has been expressed purely in terms of the information state \( \pi_k \).

**Thm** Define

\[ V_k(\pi) = \inf_{g \in G_{k,n-1}} E^y \left[ \sum_{i=k}^{m-1} \langle \pi, L^u \rangle + \langle \pi_m, \Phi \rangle \right] \]

\[ k = 0, \ldots, m-1 \]

\[ V_m(\pi) = \langle \pi, \Phi \rangle \]

Then \( V_k \) satisfies the dynamic programming equation

\[ \begin{cases} 
V_k(\pi) = \min_{u \in U} \left\{ \langle \pi, L^u \rangle + \sum_{y \in \Omega} V_{k+1}(T(\pi, u, y)) \rho(y \mid \pi, u) \right\} \\
V_m(\pi) = \langle \pi, \Phi \rangle 
\end{cases} \]
where
\[ \rho(y | \pi, u) = \sum_{i,j} \omega_{ij} y_f^j (u) \pi(i). \]

Further, if \( \overline{u}^*_k(\pi) \) achieves the minimum in \( \Pi_k \), then the policy
\[ u_k = g^*_k(Y_k, h) = \overline{u}^*_k(\pi_k) \]
is optimal.

\[ \text{Proof} \quad \text{See KF, page 85.} \]

Note: The optimal controller depends on the observation \( Y_k \) through the information state \( \pi_k \). Such controllers are called separated; i.e., separated into a filter plus a controller.
Optimal Control of Linear Systems (LQG)

\[
\begin{align*}
X_{k+1} &= AX_k + BU_k + GW_k \\
Y_k &= CX_k + HV_k
\end{align*}
\]

\[
L(x,u) = \frac{1}{2} x' \Pi x + \frac{1}{2} u' \Lambda u
\]

\[
\Phi(x) = \frac{1}{2} x' \Phi x
\]

where \( \Pi > 0 \), \( \Phi > 0 \), \( \Lambda > 0 \).

Assume for simplicity \( G = I \), \( H = I \), \( Q = I \), \( R = I \).

The conditional density

\[
\pi_k = p(h_k \mid x_k) \sim N(\mu_{k|k}, \Sigma_{k|k})
\]

is an information state. Since \( \Sigma_{k|k} \) is deterministic, \( x_{k|k} \) is itself an information state for the linear system. Thus we expect the optimal policy \( g^* \) to have the form

\[
\mu_k = g^*(x_{k|k}) = \tilde{u}^*(x_{k|k})
\]

for a suitable function \( \tilde{u}^*(x) \).

It turns out that the complete state information controller derived earlier,
The let $X_{k|k}$ be the conditional mean as determined by the Kalman filter. Let $L_k$ be the gain sequence determined by the state feedback linear regulator problem. Then

$$u_k = g^*(X_{0:k}) = L_k X_{k|k}$$

is the optimal policy for the partially observable LQG problem.

Note: This optimal controller is separated.

![Diagram of linear system with Kalman filter and controller](image)