Controllability and Observability of Nonlinear Systems

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Abstract

These tutorial notes discuss the basic ideas in the theory of controllability and observability for nonlinear control systems. The theory treated is primarily due to Hermann and Krener. The first section gives a short overview of the issues, followed by section 2 which reviews distributions, codistributions, and the Frobenius Theorem. Section 3 deals with controllability. Chow’s Theorem is presented, before beginning the Hermann–Krener theory. Finally, section 4 discusses the Hermann–Krener formulation of observability. A number of examples and illustrations are provided.

Key words: Controllability, observability, nonlinear systems.
Controllability and Observability of Nonlinear Systems

1. Introduction

Let $M$ be a smooth manifold of dimension $n$. Denote by $U$ an open neighborhood of $x^0 \in M$. We consider a control system $\Sigma$, described in local coordinates by

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t), u(t)) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t), \\ y(t) = h(x(t)), \\ x(0) = x^0, \end{cases}$$

where $t \mapsto u_i(t)$ is a control function with values in a convex set $\Omega \subset \mathbb{R}^r$, $t \mapsto x(t)$ is the state trajectory with $x(t) \in M$ and $t \mapsto y(t)$ is the output curve with $y(t) \in \mathbb{R}^p$.

Given system $\Sigma$ initialized at $x^0$, the map

$$\mathcal{S}_{x^0} : \{ t \mapsto u(t), \ t \in [0, T] \} \longrightarrow \{ t \mapsto y(t), t \in [0, T] \}$$

is called the input-output map.

Given a control $t \mapsto u(t)$, let $\gamma_u$ denote the corresponding flow:

$$x(t) = \gamma_u(t)x^0.$$ 

We consider the following two concepts, with an emphasis on local ideas.

a. **Controllability**

Suppose we are given a system $\Sigma$ and an initial state $x^0$. Let $x^1$ be another state. Is it possible to choose a control $t \mapsto u(t)$ to steer $\Sigma$ from $x^0$ to $x^1$?

![Diagram](image_url)

$x^0 \rightarrow \gamma_u(t^*)x^0 = x^1$ (This is often referred to as reachability, here $x^1$ is reachable from $x^0$.)

b. **Observability**

A control system $\Sigma$ is observable if $y(t)$ uniquely determines $x(t)$ for all $t \geq 0$. This means that the output $y(t)$ provides sufficient information to reconstruct the state $x(t)$ uniquely.

In summary, controllability deals with steering the system from one state to another, while observability concerns the ability to determine the state from the output.
If so, $x^1$ is accessible from $x^0$. What are the accessible states? Is $x^0$ accessible from $x^1$? Is $x^1$ accessible from $x^0$ locally?

$\Sigma$ is controllable if every state is accessible from every other state. What criteria (for example, algebraic) tell us when $\Sigma$ is controllable (or has some weaker property)?

b. Observability

This time we are given an output "record" $t \mapsto y(t)$, $t \in [0, T]$. We ask what information about the states can be obtained from such a record.

Two initial states $x^0, x^1$ are indistinguishable if no matter what control we use, the corresponding trajectories always produce the same output record.

What are the indistinguishable/distinguishable states? Can states be locally distinguished?

$\Sigma$ is observable if any state is distinguishable from any other state. What criteria is available here, perhaps for weaker concepts?

In a sense, observability is a "dual" notion to controllability.

What follows is based on Hermann and Krener [1], and Isidori [2].
2. Distributions and Codistributions

Notation:

\[ \mathcal{F}(M) = \text{ring of smooth functions } M \to \mathbb{R} \]
\[ \mathcal{X}(M) = \text{smooth vector fields on } M. \]

(a Lie algebra and a module over \( \mathcal{F}(M) \))

\[ \mathcal{X}^*(M) = \text{smooth 1-forms on } M \]

(a module over \( \mathcal{F}(M) \))

Definitions

1. A distribution \( \mathcal{D} \) is a submodule of \( \mathcal{X}(M) \).

2. \( D(x) \equiv \{X(x) : X \in \mathcal{D}\} \leq T_xM \) (subspace of)

3. \( \mathcal{D} = \bigcup_{x \in M} D(x) \) (a (singular) subbundle of \( TM \))

4. If \( \dim D(x) \) is constant, we say \( D \) is nonsingular.

5. A set of linearly independent vector fields \( \{X_1, \ldots, X_k\} \) in a neighborhood \( U \) of \( x \) is called a local basis (frame) if

\[ D(y) = \text{span}\{X_1(y), \ldots, X_k(y)\} \text{ for all } y \in U. \]

6. A point \( x \) is called a regular point of \( \mathcal{D} \) if \( \dim D(y) = \dim D(x) \) for all \( y \in U, U \) a neighborhood of \( x \). Otherwise, \( x \) is a singular point.

7. \( \Gamma(\mathcal{D}) = \{X \in \mathcal{X}(M) : X(x) \in D(x) \forall x \in M\} \) (a distribution)

8. \( \mathcal{D} \) is complete if \( \mathcal{D} = \Gamma(\mathcal{D}) \). We shall assume that all distributions are complete.

Lemma. A point \( x \) is a regular point of \( \mathcal{D} \) if and only if there exists a local basis in a neighborhood of \( x \).

Definitions

1. A codistribution \( \mathcal{E} \) is a submodule of \( \mathcal{X}^*(M) \). \( \mathcal{E} \) is sometimes called a Pfaffian system.

2. \( E(x) = \{\omega(x) : \omega \in \mathcal{E}\} \leq T^*_xM \)

3. \( E = \bigcup_{x \in M} E(x) \) (a (singular) subbundle of \( T^*M \))

4. Analogous definitions for nonsingularity, completeness, local frame, etc.
Duality

\[ D^\perp = \{ \omega \in \mathcal{X}^*(M) : \langle \omega, X \rangle = 0 \ \forall \ X \in D \} \]

\[ \mathcal{E}^\perp = \{ X \in \mathcal{X}(M) : \langle \omega, X \rangle = 0 \ \forall \ \omega \in \mathcal{E} \} \]

\( D^\perp \) (resp. \( \mathcal{E}^\perp \)) is called the annihilator of \( D \) (resp. \( \mathcal{E} \)) and is a codistribution (distribution).

Invariance Let \( X, Y \in \mathcal{X}(M), \quad \omega \in \mathcal{X}^*(M) \).

\[ ad_X Y = L_X Y = [X, Y] \quad \text{(Lie derivative)} \]

\( D \) is \( ad_X \) invariant (or, invariant under \( X \)) if \( Y \in D \) implies \( ad_X Y \in D \).

\[ ad_X \omega = L_X \omega \quad \text{(Lie derivative)} \]

\( \mathcal{E} \) is \( ad_X \) invariant (invariant under \( X \)) if \( \omega \in \mathcal{E} \) implies \( ad_X \omega \in \mathcal{E} \).

Integrability

Definitions

1. A distribution \( D \) is involutive if for all \( X \in D \), \( D \) is \( ad_X \) invariant.

2. An integral submanifold \( N \) of \( D \) is a connected immersed submanifold \( N \subset M \) such that for all \( x \in N \), \( T_x N \leq D(x) \). A maximal integral submanifold is an integral submanifold not properly contained in any other integral submanifold.

3. A distribution \( D \) is integrable if its maximal integral submanifolds define a partition of \( M \). This is called a foliation, the maximal integral submanifolds being called leaves.

Theorem (Frobenius) Let \( D \) be a nonsinguar distribution. Then \( D \) is integrable if and only if \( D \) is involutive.

Following Boothby [6], we state and prove (sketch) a local version of the Frobenius theorem.

Theorem (A local version of Frobenius Theorem) Let \( p \in M \) be a regular point of \( D \). Then, on a neighborhood \( U \) of \( p \), \( D \) is (completely) integrable if and only if \( D \) is involutive.

Since \( p \) is regular point, if \( D \) is \( k \)-dimensional, there exists a local basis \( \{ X_1, \ldots, X_k \} \) in a neighborhood \( U \) of \( p \). To say that \( D \) is involutive means that

\[ [X_i, X_j] = \Sigma_{l=1}^k e_{ij}^l X_l, \quad 1 \leq i, j \leq k \]
for some \( c^i_{ij} \in \mathcal{F}(U) \).

\( \mathcal{D} \) is (completely) integrable if there exists local coordinates \( x_1, \ldots, x_n \) in \( U \) such that \( \{ \frac{\partial}{\partial x_i}, \ i = 1, \ldots, k \} \) form a local basis for \( \mathcal{D} \). In this case the \( k \)-dimensional submanifold

\[
\{ x_1, \ldots, x_n : x^{k+1} = c^1, \ldots, x^n = c^{n-k} \}
\]

is an integral submanifold of \( \mathcal{D} \) in \( U \). A submanifold expressed in this form is called a slice.

**Proof** If \( \mathcal{D} \) is (completely) integrable, then it is involutive, since

\[
[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0, \quad 1 \leq i, j \leq k.
\]

Next, suppose that \( \mathcal{D} \) is involutive. Proceed by induction. (The ideas are sketched).

\( k = 1 \). In this case \( \mathcal{D} \) is determined by a vector field on \( U \), call it \( X \). Let integral curves of \( X \) define the coordinate \( y_1 \), that is, choose coordinates \( y_1, \ldots, y_n \) such that \( X = \frac{\partial}{\partial y_1} \).

Hence \( \mathcal{D} \) is integrable. The integral submanifolds are the integral curves. So in this case the theorem is just existence of local flows.

Suppose that the theorem is true for distributions of dimensions \( 1, \ldots, k - 1 \).

Let \( X_1, \ldots, X_k \) be a local basis for \( \mathcal{D} \) and \( y_1, \ldots, y_n \) coordinates with \( X_1 = \frac{\partial}{\partial y_1} \). Change basis

\[
Y_1 = X_1
\]

\[
Y_i = X_i - (X_i y_1)X_1, \quad 2 \leq i \leq k.
\]

Then \( Y_2, \ldots, Y_k \) are involutive and set \( N_0 = \{ y_1 = 0 \} \). Change coordinates on \( N_0 : y_2, \ldots, y_n \mapsto x_2, \ldots, x_n \) so that

\[
\text{span} \{ Y_2, \ldots, Y_k \} = \text{span} \{ \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_k} \},
\]

(by induction hypothesis). Extend this to a change of coordinates on \( U \) by setting \( X_1 = y_1 \).

Then check that \( \{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \} \) forms a local basis for \( \mathcal{D} \) on \( U \). \( \square \)
Definitions

1. We say that a codistribution $\mathcal{E}$ is integrable if its annihilator $\mathcal{E}^\perp$ is integrable.

2. Let $h : M \to \mathbb{R}^p$ be smooth.

\[ R(dh) = \text{codistribution spanned by } dh_1, \ldots, dh_p. \]

Lemma $R(dh)$ is integrable.

Proof First, we check if $\mathcal{E}$ is $ad_X$ invariant, then $\mathcal{E}^\perp$ is $ad_X$ invariant.

Let $\omega \in \mathcal{E}$, $Y \in \mathcal{E}^\perp$. Then $\langle \omega, Y \rangle = 0$ and $ad_X \omega \in \mathcal{E}$. Now

\[
0 = L_X \langle \omega, Y \rangle = \langle \omega, Y \rangle + \langle \omega, L_X Y \rangle \\
= 0 + \langle \omega, L_X Y \rangle.
\]

Hence $L_X Y \in \mathcal{E}^\perp$. So we must show that

\[ ad_X R(dh) \subset R(dh) \text{ for all } X \in R(dh)^\perp. \]

But this follows from

\[ ad_X dh_i = \langle dh_i, X \rangle = 0. \]

Remarks

1. A converse is also true. A nonsingular distribution is integrable if its annihilator is locally spanned by exact 1-forms. (See Isidori, p.21.)

2. If $\mathcal{E}$ is integrable, then it defines a foliation via $\mathcal{E}^\perp$. If $\omega \in \mathcal{E}$ its restriction to a leaf is the zero 1-form.

3. If $X \in R(dh)^\perp$, then $X$ is tangent to leaves of the foliation. In particular, $\langle dh_i, X \rangle = 0$, so that $h_i$ is constant on leaves.

Local Representation (Isidori, pp.25)

Lemma Let $\mathcal{D}$ be a nonsingular involutive distribution of dimension $k$ and assume $\mathcal{D}$ is invariant under a vector field $X$. Then at each $p \in M$, there exist coordinates $(U, \xi)$ in
which the vector field \( X \) can be represented as

\[
X(\xi) = \begin{bmatrix}
X_1(\xi_1, \ldots, \xi_k, \xi_{k+1}, \ldots, \xi_n) \\
\vdots \\
X_k(\xi_1, \ldots, \xi_k, \xi_{k+1}, \ldots, \xi_n) \\
X_{k+1}(\xi_{k+1}, \ldots, \xi_n) \\
\vdots \\
X_n(\xi_{k+1}, \ldots, \xi_n)
\end{bmatrix}
\]

Proof By Frobenius’ theorem, there exist coordinates \((U, \xi)\) about \( p \in M \) such that

\[
\mathcal{D}(q) = \text{span}\{ \frac{\partial}{\partial \xi_1}(q), \ldots, \frac{\partial}{\partial \xi_k}(q) \}, \quad q \in U.
\]

Now \( ad_X \mathcal{D} \subset \mathcal{D} \) implies

\[
ad_X \frac{\partial}{\partial \xi_j} \in \text{span} \{ \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_k} \}.
\]

But

\[
ad_X \frac{\partial}{\partial \xi_j} = -\sum_{i=1}^{n} \left( \frac{\partial X_i}{\partial \xi_j} \right) \frac{\partial}{\partial \xi_i}.
\]

Hence must have

\[
\frac{\partial X_i}{\partial \xi_j} = 0 \text{ for } i = k + 1, \ldots, n; \quad j = 1, \ldots, k.
\]

Remark If \( \mathcal{E} \) is a nonsingular involutive codistribution of dimension \((n - k)\), there exist local coordinates such that

\[
\mathcal{E} = \text{span} \{ d\xi_{k+1}, \ldots, d\xi_n \}.
\]

If also \( \mathcal{E} \) is invariant under \( X \), then we can choose \( \xi_1, \ldots, \xi_k \) such that \( X \) has a representation of the above form.

These representations are useful in visualising controllability and observability.

Notation

\[
\langle ad_X | \mathcal{D} \rangle = \text{smallest } ad_X \text{ invariant distribution containing } \mathcal{D}.
\]

\[
\langle ad_X | \mathcal{E} \rangle = \text{smallest } ad_X \text{ invariant codistribution containing } \mathcal{E}.
\]
Refering to our control system $\Sigma$, we say that $D$ (or $E$) is $ad_f$ invariant if $D$ (or $E$) is $ad_f(\cdot,u)$ invariant for all $u \in \Omega^m$.

$$\mathcal{R}(f) = \text{distribution spanned by} \{ f(\cdot,u) : u \in \Omega^m \}$$

$$\langle ad_f | \mathcal{R}(f) \rangle$$ is called the controllability distribution.

$$\langle ad_f | \mathcal{R}(dh) \rangle$$ is called the observability codistribution.

**Examples**

1. $M = IR^2$
   
   $\Delta_1 = \text{sp} \left\{ \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right\}$
   
   $\Delta_2 = \text{sp} \left\{ (1 + x_1) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right\}$

   These are smooth distributions.

   ![Diagram](image)

   However,

   $$\Delta_1 \cap \Delta_2(x) = \begin{cases} 
   \{0\} & \text{if} \ x_1 \neq 0 \\
   \Delta_1(x) = \Delta_2(x) & \text{if} \ x_1 = 0 
   \end{cases}$$

   is not a smooth distribution.

2. $M = IR$, $\Delta = \text{sp} \left\{ x \frac{\partial}{\partial x} \right\}$.

   $$\text{dim} \Delta(x) = \begin{cases} 
   0 & \text{if} \ x = 0 \\
   1 & \text{if} \ x \neq 0 
   \end{cases}$$

   So $x = 0$ is a singular point of $\Delta$.

3. $M = IR$, $\Delta = \text{sp} \left\{ x \frac{\partial}{\partial x} \right\}$. Then

   $$\Delta^\perp(x) = \begin{cases} 
   \{0\} & \text{if} \ x \neq 0 \\
   T_x^*M & \text{if} \ x = 0 
   \end{cases}$$

   Then $\Delta^\perp$ is not a smooth codistribution.
4. $M = \mathbb{R}^2$, $\Delta = \{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\}$. Then

$$\Delta^\perp = \text{sp}\{dx_1 - dx_2\}$$

is a smooth codistribution.

3. Controllability

3.1. Chow’s Theorem

Consider a system $\Sigma$, defined in a neighborhood $U$, without drift that is $f(x) \equiv 0$. Define

$$\mathcal{A}(x^0, U) = \{\gamma_u(s)x^0 : s \in \mathbb{R}, \ r \mapsto u(r)$$

piecewise constant, $u(r) \in \Omega^m$, $\gamma_u(r)x^0 \in U$

for all $0 \leq |r| \leq s.$}

This is the set of states in $U$ accessible from $x^0 \in U$. Define $\mathcal{F}$ to be the Lie algebra spanned by the vector fields $g_1, \ldots, g_m$ on $U$. Note that $\mathcal{F}$ is a distribution.

Assume that $\mathcal{F}$ is nonsingular. Let $N$ be the corresponding (maximal in $U$) integral submanifold of $\mathcal{F}$ passing through $x^0$.

**Theorem** Suppose $\dim \mathcal{F}(x) = k \leq n$ on $U$. Then $\mathcal{A}(x^0, U) \subset N$ contains a relatively open subset of $N$.

If $\dim \mathcal{F}(x) = n$ on $U$, then $N = U$ and we have:

**Corollary** (Classical Chow’s theorem) If $\dim \mathcal{F}(x) = n$ on $U$, then $\mathcal{A}(x^0, U)$ contains an open subset of $U$. 

[Diagram of N, U, A(x^0, U) with a vector field and a submanifold.]
The Lie algebra $\mathcal{F}$ gives the possible directions in which the system can evolve. Let $u^1$ and $u^2$ be two controls, and $f^i = \sum_{i=1}^{m} g_i u_i^i$ be the corresponding vector fields. Now $\Sigma$ can evolve in the directions $f^1, f^2$. The theorem says it can evolve in the direction $[f^1, f^2] \in \mathcal{F}$ also (Brockett [5]).

All such trajectories lie in the integral submanifold $N$.

**Proof** (Based on Krener [3], Isidori [2], p. 43) Choose $u^1 \in \Omega^m$ and set $f^1(x) = g(x)u^1$. We can assume $f^1(x_0) \neq 0$, otherwise choose another $u^1$. So $f^1(x) \neq 0$ for $x$ near $x^0$, and there exists $\delta_1 > 0$ such that $\phi_1 : V_1 = (-\delta_1, \delta_1) \to U$, where $\phi_1(s_1) = \gamma_1(s_1)x^0$, is an injective immersion. Thus $N_1 = \phi_1(V_1)$ is a 1-dimensional integral submanifold of $\mathcal{F}$ in $U$. Suppose we have constructed $N_{j-1} = \phi_{j-1}(V_{j-1})$ and $j \leq k$. Note dim $N_{j-1} = j - 1$.

Claim: Given $x \in N_{j-1}$, we can choose $u^j \in \Omega^m$ such that

$$f^j(x) = g(x)u^j \notin T_xN_{j-1}.$$ 

Suppose not. Then $g(x)u \in T_xN_{j-1} \forall u \in \Omega^m$. This implies $\mathcal{F}(x) \subset T_xN_{j-1}$ for all $x \in N_{j-1}$. Define

$$\overline{\mathcal{F}}(x) = \begin{cases} T_xN_{j-1} & \text{if } x \in N_{j-1} \\ \mathcal{F}(x) & \text{if } x \in U \setminus N_{j-1} \end{cases}$$

Then by construction $g_1(x), \ldots, g_m(x) \in \overline{\mathcal{F}}(x), x \in U$, and $\overline{\mathcal{F}} \subset \mathcal{F}$. Let $X_1, X_2$ be vector fields on $U$ with $X_i(x) \in \overline{\mathcal{F}}(x)$. Then $X_1, X_2$ are vector fields on $N_{j-1}$, so

$$[X_1, X_2](x) \in T_xN_{j-1} = \overline{\mathcal{F}}(x), \quad x \in N_{j-1}.$$
Also,
\[ [X_1, X_2](x) \in \mathcal{F}(x) \text{ for } x \in U \setminus N_{j-1}. \]

Hence \( \overline{\mathcal{F}} \) is involutive, and so \( \overline{\mathcal{F}} = \mathcal{F} \).

But \( k = \dim \mathcal{F}(x) \leq \dim \overline{\mathcal{F}}(x) = \dim T_x N_{j-1} = j - 1, \ x \in N_{j-1}. \)

This is a contradiction since \( j \leq k \), proving the claim.

By continuity, we can shrink \( V^{j-1}, N^{j-1} \) if necessary so that \( f^i(x) \notin T_x N_{j-1} \) for all \( x \in N_{j-1} \). Also, shrinking further if necessary, there exits \( \delta_j > 0 \) such that \( \gamma_j(s_j) x \in U \) for \( x \in N_{j-1}, \ s_j \in (-\delta_j, \delta_j) \). Set \( V_j = V_{j-1} \times (-\delta_j, \delta_j) \) and define
\[
\phi_j(s_1, \ldots, s_j) = \gamma_j(s_j) \phi_{j-1}(s_1, \ldots, s_{j-1}).
\]

By assumption, \( \phi_{j-1} \) has rank \( j - 1 \). It remains to verify that \( \phi_j \) has rank \( j \), from which it follows that \( N_j \) is a \( j \)-dimensional integral submanifold of \( \mathcal{F} \) in \( U \).

Now \( \phi_j = \gamma_j \circ \phi_{j-1} \). So for \( 1 \leq i \leq j - 1 \),
\[
\phi_j \left( \frac{\partial}{\partial s_i} \right)(s_1, \ldots, s_{j-1}, 0) = i d \phi_{j-1} \left( \frac{\partial}{\partial s_i} \right)(s_1, \ldots, s_{j-1})
\]
\[
\phi_j \left( \frac{\partial}{\partial s_j} \right)(s_1, \ldots, s_{j-1}, 0) = \gamma_j(0) id \left( \frac{\partial}{\partial s_j} \right) = f^i(\bar{x}),
\]
where \( \bar{x} = \phi_{j-1}(s_1, \ldots, s_{j-1}) \in N_{j-1}. \) Thus if \( \delta_j \) is sufficiently small,
\[
\{ \phi_j \left( \frac{\partial}{\partial s_i} \right)(s_1, \ldots, s_j) \}_{i=1}^j
\]
are \( j \) linearly independent vectors at \( \phi_j(s_1, \ldots, s_j) \in N_j \).

This process terminates at \( j = k \), and therefore \( N_k \) is the desired relatively open subset of \( N \).

The situation is more complicated when \( \Sigma \) has drift, that is, \( f(x) \neq 0 \). In particular, distinction must be made between forward and reverse time. Without drift, reversing time amounts to replacing \( u \) by \( -u \).
Define

\[ \mathcal{A}(x^0, U) = \{ \gamma_u(s)x^0 : s \geq 0, \ r \mapsto u(r) \ \text{piecewise constant}, \ u(r) \in \Omega^m, \ \gamma_u(r)x^0 \in U \text{ for all } 0 \leq r \leq s \} \]

This is the set of states in \( U \) accessible from \( x^0 \in U \) by going \textit{forward} in time only. Let \( \mathcal{F} \) be the Lie algebra generated by \( f, g_1, \ldots, g_m \) on \( U \).

Again, assume that \( \mathcal{F} \) is nonsingular on \( U \) and let \( N \) be the corresponding (maximal in \( U \)) integral submanifold of \( \mathcal{F} \), passing through \( x^0 \).

The following generalization of Chow's theorem is due to Krener, [3].

**Theorem (Krener)** Suppose \( \dim \mathcal{F}(x) = k \) on \( U \). Then \( \mathcal{A}(x^0, u) \subset N \) contains a relatively open subset of \( N \).

**Proof** Refer to [3]. \( \Box \)

This time asymmetry is reflected in the following. Define

\[ \mathcal{C}(x^0, U) = \{ \gamma_u(s)x^0 : s \leq 0, \ r \mapsto u(r) \ \text{piecewise constant} \ u(r) \in \Omega^m, \ \gamma_u(r)x^0 \in U \text{ for all } s \leq r \leq 0 \} . \]

If \( x^1 \in \mathcal{C}(x^0, U) \), then it is possible to steer the system from \( x^1 \) to \( x^0 \) by going forward in time, that is going backwards from \( x^0 \) to \( x^1 \). This is sometimes stated "\( x^1 \) is \textit{controllable} to \( x^0 \)," distinguishing between accessibility (reachability) and (this notion of) controllability.

In general \( \mathcal{A}(x^0, U) \neq \mathcal{C}(x^0, U) \), however they coincide when \( \Sigma \) has no drift, or when \( f \in \text{span} \{ g_1, \ldots, g_m \} \).

### 3.2. Hermann-Krener Formulation

We summarize the ideas and results of Hermann and Krener [1], modified a little by the more recent ideas in Krener [4].

**Controllability and Local Controllability**

A state \( x^1 \) is \textit{U-accessible} from \( x^0 \) if there exits a bounded measurable control \( t \mapsto u(t) \), defined on some interval \([0, T]\), such that the corresponding trajectory \( t \mapsto x(t) \), \( x(t) \in \)
$U$, for all $t \in [0,T]$, $x(0) = x^0$, $x(T) = x^1$. We define the accessible sets by

$$
\mathcal{A}(x^0, U) = \{ x^1 \in U : x^1 \text{ is } U\text{-accessible from } x^0 \},
$$

$$
\mathcal{A}(x^0) = \mathcal{A}(x^0, M).
$$

If $x^1 \in \mathcal{A}(x^0, U)$, in general it is not true that $x^0 \in \mathcal{A}(x^1, U)$. So accessibility is a reflexive, transitive but not symmetric relation.

We say $\Sigma$ is controllable at $x^0$ if $\mathcal{A}(x^0) = M$, and controllable if $\mathcal{A}(x^0) = M$ for all $x^0 \in M$.

However, it may be necessary to go a long way or for a long time to reach points near $x^0$:

A stronger notion of controllability would require that the trajectory stay near $x^0$:

Thus, we say that $\Sigma$ is locally controllable at $x^0$ if for all neighborhoods $U$ of $x^0$, $\mathcal{A}(x^0, U)$ is also a neighborhood of $x^0$. $\Sigma$ is locally controllable if $\Sigma$ is locally controllable at every $x^0 \in M$.

The above definitions consider the ability of $\Sigma$ to steer from one state to another.

**Accessibility Property**

We noted above that $U$-accessibility is not an equivalence relation. According to Hermann-Krener, it is possible to define an equivalence relation on $U$ containing all $U$-accessible pairs. They call this weak $U$-accessibility.
Write
\[ \mathcal{A}(x^0, U) = \{ x^1 \in U : \ x^1, x^0 \ \text{weakly \ U-accessible}\}, \]
\[ \mathcal{A}(x^0) = \mathcal{A}(x^0, \mathcal{M}). \]
Here, \( x^1 \in \mathcal{A}(x^0, U) \) if and only if \( x^0 \in \mathcal{A}(x^1, U) \).
Analogously, one can define concepts of (local) weak controllability.

Another aspect of controllability is the ability of controls to influence all modes. Thus:
We say that \( \Sigma \) has the accessibility property if \( A(x^0) \) has nonempty interior for all \( x^0 \in \mathcal{M} \).
\( \Sigma \) has the local accessibility property if for every \( x^0 \in \mathcal{M} \), and every neighborhood \( U \) of \( x^0 \),
\( A(x^0, U) \) has nonempty interior.

**Theorem**  If \( \Sigma \) is locally weakly controllable, then \( \Sigma \) has the local accessibility property.

**Proof**  Suppose \( \Sigma \) is locally weakly controllable. The argument is similar to that used in
the proof of Chow’s theorem, only the Claim is true for a different reason.

If \( f(x, u) \in T_x N_{x-1} \) for all \( u \in \Omega^m \), then \( \gamma_u(t)x \in N_{x-1} \) for all \( t \), for all \( u \in \Omega^m \). This
contradicts local weak controllability.

**Controllability Rank Condition**

We say that \( \Sigma \) satisfies the controllability rank condition at \( x^0 \) if in a neighborhood
of \( x^0 \), \( \dim(af|R(f))(x) = n \). If this holds for all \( x^0 \in \mathcal{M} \), we say that \( \Sigma \) satifies the
controllability rank condition.

**Theorem**  If \( \Sigma \) satisfies the controllability rank condition at \( x^0 \in \mathcal{M} \), then \( \Sigma \) has the local
accessibility property at \( x^0 \).

**Proof**  By assumption, \( x^0 \) is a regular point for \( \langle ad_f|R(f) \rangle \). By Chow’s theorem, \( A(x^0, U) \)
contains an open subset of \( U \).

There is almost a converse:

**Theorem**  If \( \Sigma \) has the local accessibility property, then the controllability rank condition
is satisfied generically.

**Proof**  Suppose there exits \( U \subset \mathcal{M} \) such that \( \dim(af|R(f))(x) = k < n \) for \( x \in U \). Let
\( x^0 \in U \) and \( U' \) be the corresponding maximal integral submanifold passing through \( x^0 \).
Then \( A(x^0, U) \subset U' \), contradicting the local accessibility property.
Remarks
1. The rank condition is an algebraic test for a form of controllability.
2. We have the following implications:

\[
\begin{array}{c}
\text{local controllability} \quad \implies \quad \text{controllability} \\
\downarrow \\
\text{local accessibility property} \quad \implies \quad \text{accessibility property}.
\end{array}
\]

3.3. Local Representations (Isidori, p. 29, 40)

Write
\[
P = \langle ad_f|sp\{g_1, \ldots, g_m\}\rangle, \\
R = \langle ad_f|R(f)\rangle.
\]

If \(x\) is a regular point of \(P + sp\{f\}\), then
\[
(P + sp\{f\})(x) = R(x).
\]

Let \(r = \dim R\).

Theorem Let \(P, P + sp\{f\}, R\) be nonsingular, and suppose \(P \subset R, P \neq R\). Then, at each \(p \in M\), there exist coordinates \((U, \xi)\) in which \(\Sigma\) is represented by:

\[
\begin{align*}
\dot{\xi}_1 &= f_1(\xi_1, \ldots, \xi_n) + \sum_{i=1}^{m} g_{i}(\xi_1, \ldots, x_n) u_i \\
& \quad \vdots \\
\dot{\xi}_{r-1} &= f_{r-1}(\xi_1, \ldots, \xi_n) + \sum_{i=1}^{m} g_{i,r-1}(\xi_1, \ldots, \xi_n) u_i \\
\dot{\xi}_r &= f_r(\xi_r, \ldots, \xi_n) \\
\dot{\xi}_{r+1} &= 0 \\
& \quad \vdots \\
\dot{\xi}_n &= 0
\end{align*}
\]

Proof According to the representation theorem in §2, there exits local coordinates such that
\[
P = \{ \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_{r-1}} \}, \\
R = \{ \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_{r-1}}, \frac{\partial}{\partial \xi_r} \}.
\]
The first \((r - 1)\) coordinates represent the (maximal in \(U\)) integral submanifolds of \(P\), while the first \(r\) coordinates represent those of \(R\). Since \(f, g_1, \ldots, g_m \in R\), the components for \(r + 1, \ldots, n\) are zero. □

This gives us a geometrical picture of the behavior of \(\Sigma\). All trajectories of \(\Sigma\) are contained in slices of the form

\[
N_{x^0} = \{\xi_{r+1} = c_1, \ldots, \xi_n = c_{n-r}\}, \quad (r \text{ - dimensional})
\]

depending on the initial condition. The controls can affect the \((r-1)\) directions \(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_{r-1}}\) only. The drift \(f\) causes \(\Sigma\) to move from one \((r-1)\) dimensional slice \(N' = \{\xi_r = c_0, \ldots, \xi_n = c_{n-r}\}\) to another. In this sense \(\xi_r\) is analogous to time.

If \(f_r = 0\), all trajectories are contained in an \((r - 1)\) dimensional slice \(N'\), and there is no drift effect. This corresponds to \(f(x) = 0\) or \(f \in R\).

3.4. Controllable Subsystems

As the above geometrical description suggests, it may be possible to restrict \(\Sigma\) to a submanifold on which it is controllable. We mention one result in this direction.

**Theorem** Suppose \(\dim(\text{ad}_f|\mathcal{R}(f))(x) = k \leq n\) for all \(x \in M\). Fix \(x^0 \in M\). Then there exists a system \(\Sigma'\) defined on the maximal integral submanifold \(N\) of \(\langle \text{ad}_f|\mathcal{R}(f) \rangle\) passing
through $x^0$ which has the local accessibility property. Further, $\Sigma'$ satisfies the controllability rank condition.

3.5. Examples

1. Linear systems. \[ \dot{x} = Ax + Bu, \quad M = IR^n. \]

\[ f(x) = Ax \quad g^i(x) = B_i \quad (B_i = \text{ith column of } B) \]

\[ [Ax, B_i] = -AB_i, \quad [Ax, [Ax, B_i]] = -A^2B_i, \text{ etc.} \]

\[ \mathcal{R}(f)(x) = \text{span } \{Ax, B_1, \ldots, B_m\}. \]

\[ \langle ad_f|\mathcal{R}(f)\rangle = \text{span } \{B_1, \ldots, B_m, AB_1, \ldots, AB_m, \ldots, A^{n-1}B_1, \ldots, A^{n-1}B_m\} \]

\[ \dim\langle ad_f|\mathcal{R}(f)\rangle = \text{rank } \{B, AB, \ldots, A^{n-1}B\} \]

Thus the controllability rank condition is equivalent to the requirement that rank \{B, AB, \ldots, A^{n-1}B\} = n. In this case, this is equivalent to global controllability.

2. Bilinear systems \[ \dot{x} = Ax + \sum_{i=1}^{m} (N_i x) u_i \]

\[ f(x) = Ax, \quad g^i(x) = N_i x \]

\[ [f, g^i] = -[A, N_i] \quad \text{(matrix commutator).} \]

\[ \mathcal{R}(f)(x) = \text{span } \{Ax, N_1 x, \ldots, N_m x\} \]

\[ \langle ad_f|\mathcal{R}(f)\rangle(x) = \text{span } \{\mathcal{R}(f)(x), [A, N^i](x), [A, [A, N^i]](x), [N^i, N^s](x), \text{ etc...}\} \]

Note that $\mathcal{R}(f)(0) = \langle ad_f|\mathcal{R}(f)\rangle(0) = 0$, so bilinear systems are not controllable at 0.

This example is discussed in detail in Brockett [8], where $N^i$ are skew symmetric matrices.

Then the trajectories evolve on the sphere $M: |x(t)| = |x(0)|$. Controllability is studied in terms of the matrix equations

\[ \dot{X}(t) = (A + \sum_{i=1}^{m} u_i(t)N_i)X(t) \quad , \quad X(0) = I. \]

It turns out that $X(t)$ is contained in a subgroup of $SO(n)$. If this subgroup acts transitively on $S^{n-1}$, then the bilinear systems is controllable on $M$. 
Now the Lie algebra of $SO(n)$ is $O(n)$, the real skew symmetric matrices. If

$$\{A, N_1, \ldots, N_m\}_{LA} = O(n),$$

then the above mentioned subgroup is $SO(n)$, which acts transitively on $S^{n-1}$, and so the bilinear system is controllable on $M$.

3. $M = \mathbb{R}^2$ 

$$\dot{x}_1 = u \quad u \in \mathbb{R}$$

$$\dot{x}_2 = x_1^2$$

$x^0 = (x_1(0) = 0, \; x_2(0) = 0)$.

$$f(x) = (0, x_1^2)^T, \quad g(x) = (1, 0)^T.$$

$\mathcal{R}(f)(x) = \begin{cases} \mathbb{R}, & x_1 = 0 \\ \mathbb{R}^2, & x_1 \neq 0 \end{cases}$

$$\langle ad_f \mathcal{R}(f) \rangle(x) = \mathbb{R}^2 \text{ for all } x.$$

Thus the controllability rank condition is satisfied everywhere. However,

$$\mathcal{A}(x^0) = \{(x_1, x_2) : x_2 > 0\} \cup \{x^0\}$$

![Diagram](image)

Clearly $\mathcal{A}(x^0)$ has non-empty interior.

This example is due to Crouch and Byrnes [9]. They remark that this system is invariant under the $\mathbb{Z}_2$ action on $\mathbb{R}^2$ defined by

$$(x_1, x_2) \mapsto (-x_1, x_2).$$
4. Observability

4.1. Hermann – Krener Formulation

Once again we review the ideas and results in [1], influenced by the more recent work in [4].

Observability and Local Observability

Two states \( x^0, x^1 \) are \( U \text{-distinguishable} \) if there exists a bounded measurable control \( t \mapsto u(t) \), defined on some interval \([0, T]\), such that the corresponding trajectories \( t \mapsto x^i(t) \) satisfy \( \dot{x}^i(0) = x^i, \quad x^i(t) \in U \) for all \( t \in [0, T] \), and \( h(x^1(t)) \neq h(x^2(t)) \) for some \( t \in [0, T] \). Define indistinguishability sets

\[
I(x^0, U) = \{ x^1 \in U : \text{ } x^1 \text{ is not } U \text{-distinguishable from } x^0 \},
\]

\[
I(x^0) = I(x^0, M).
\]

If \( x^1 \in I(x^0, U), \quad x^2 \in I(x^1, U) \), then in general \( x^2 \notin I(x^0, U) \). Thus indistinguishability defines a reflexive, symmetric but not transitive relation. However, when \( U = M \), we get an equivalence relation.

We say that \( \Sigma \) is observable at \( x^0 \) if \( I(x^0) = \{ x^0 \} \), and observable if \( I(x^0) = \{ x^0 \} \) for all \( x^0 \in M \).

Thus, for an observable system \( \Sigma \), all the input-output maps \( S_{x^0}, \quad x^0 \in M \), are distinct.

A stronger concept is the following. \( \Sigma \) is locally observable at \( x^0 \) if for all neighborhood \( U \) of \( x^0 \), \( I(x^0, U) = \{ x^0 \} \); and \( \Sigma \) is locally observable if this is true for all \( x^0 \in M \).

Notice that this requires that states be distinguishable by local experiments.

Distinguishability Property

It may suffice to distinguish locally between points, either by local or global experiments. We shall discuss an appropriate equivalence relation in section 4.4.

We say that \( \Sigma \) has the distinguishability property if every \( x^0 \in M \) has an open neighborhood \( U \) such that \( I(x^0) \cap U = \{ x^0 \} \).

\( \Sigma \) has the local distinguishability property if every \( x^0 \in M \) has an open neighborhood \( V \) such that for all open neighborhoods \( U \) of \( x^0, \quad U \subset V \), one has \( I(x^0, U) \cap V = \{ x^0 \} \).
Note These concepts were called weakly observable and locally weakly observable in [1].

Observability Rank Condition

We say that $\Sigma$ satisfies the observability rank condition at $x^0$ if in a neighborhood of $x^0$, $\dim\langle ad_f, R(dh)\rangle(x) = n$. If this holds for all $x^0 \in M$, we say that $\Sigma$ satisfies the observability rank condition.

**Theorem** If $\Sigma$ satisfies the observability rank condition at $x^0 \in M$, then $\Sigma$ has the local distinguishability property at $x^0$.

**Proof** First, let $U$ be any neighborhood of $x^0$. Suppose that $I(x^0, U) \neq \emptyset$ and let $x^1 \in I(x^0, U)$. Then we claim that $\phi(x^0) = \phi(x^1)$ for every $\phi \in \mathcal{G}$, where $\mathcal{G} = \langle ad_f \{h_1, \ldots, h_p\} \rangle$.

(Note $d\mathcal{G} = \langle ad_f R(dh) \rangle$.)

To see this: let $u_1, \ldots, u_k \in \Omega^m$ and $s_1, \ldots, s_k \geq 0$ be sufficiently small. Since $x^1 \in I(x^0, U)$,

$$h_i(\gamma_{u_k}(s_k) \circ \ldots \circ \gamma_{u_1}(s_1)x^0) = h_i(\gamma_{u_k}(s_k) \circ \ldots \circ \gamma_{u_1}(s_1)x^1).$$

Differentiate with respect to $s_k, \ldots, s_1$ and evaluate at 0 gives

$$ad_{f_1} \circ \ldots \circ ad_{f_k}(h_i)x^0 = ad_{f_1} \circ \ldots \circ ad_{f_k}(h_i)x^1.$$

But $\mathcal{G}$ is spanned by such functions. Hence the claim.

Since $\dim d\mathcal{G} = n$ around $x^0$, there exits $\phi_1, \ldots, \phi_n \in \mathcal{G}$ such that $d\phi_1, \ldots, d\phi_n$ are linearly independent. Define

$$\Phi : x \mapsto (\phi_1(x), \ldots, \phi_n(x))^T.$$ 

Now $D\Phi(x^0)$ is nonsingular, so by the inverse function theorem, $\Phi$ is locally injective, say in a neighborhood $V$. Then if $U \subset V$ is a neighborhood of $x^0$, the claim implies $I(x^0, U) = \{x^0\}$. □

Once again, we have a partial converse:

**Theorem** If $\Sigma$ has the local distinguishability property, then the observability rank condition is satisfied generically.

**Proof** (Sketch). Suppose there exits $U \subset M$ such that $\dim\langle ad_f R(dh)\rangle(x) = k < n$ for $x \in U$. Let $x^0 \in U$, and consider $\Sigma$ restricted to $U$, that is, $\Sigma |_U$. Now $\langle ad_f R(dh) \rangle^\perp$ is a $n - k > 0$ dimensional integrable distribution. If $x^1$ is in the same leaf as $x^0$, then
$S_{x_0} |_U = S_{x_1} |_U$, so $\Sigma |_U$ does not have the local distinguishability property. Hence neither does $\Sigma$. \hfill \Box

Remarks

1. The rank condition is an algebraic test for a (weak) form of observability.

2. We have:

\[
\begin{array}{c}
\text{local observability} \\ \Downarrow \\ \text{local distinguishability property}
\end{array} \quad \implies \quad \begin{array}{c}
\text{observability} \\ \Downarrow \\ \text{distinguishability property}
\end{array}
\]

4.2. Local Representations (Isidori, p. 29, 50)

Write $Q = \langle ad_f | R(dh) \rangle$, \quad $s = \dim Q$, \quad $d = n - s$.

Theorem Let $Q$ be nonsingular. Then, at each $p \in M$, there exist coordinates $(U, \xi)$ in which $\Sigma$ is represented by

\[
\begin{align*}
\dot{\xi}_1 &= f_1(\xi_1, \ldots, \xi_n) + \sum_{i=1}^{m} g_{i,1}(\xi_1, \ldots, x_n)u_i \\
&\vdots \\
\dot{\xi}_s &= f_s(\xi_1, \ldots, \xi_n) + \sum_{i=1}^{m} g_{i,s}(\xi_1, \ldots, \xi_n)u_i \\
\dot{\xi}_{s+1} &= f_{s+1}(\xi_{s+1}, \ldots, \xi_n) + \sum_{i=1}^{m} g_{i,s+1}(\xi_{s+1}, \ldots, \xi_n)u_i \\
&\vdots \\
\dot{\xi}_n &= f_n(\xi_{s+1}, \ldots, \xi_n) + \sum_{i=1}^{m} g_{i,n}(\xi_{s+1}, \ldots, \xi_n)u_i \\
y_i &= h_i(\xi_{s+1}, \ldots, \xi_n), \quad i = 1, \ldots, p.
\end{align*}
\]

Proof This follows from the representation theorem in §2. \hfill \Box

Notice that the outputs depend only on $\xi_{s+1}, \ldots, \xi_n$. Leaves of $Q^\perp$ are $(n - s)$ dimensional, given by slices of the form

\[N = \{\xi_{s+1} = c_1, \ldots, \xi_n = c_{n-s}\}.
\]

If $x^0, x^1 \in N$, then the last $(n - s)$ coordinates of the trajectories $t \mapsto x^0(t)$, $t \mapsto x^1(t)$, agree at time $t$, for all $t$, that is $\xi_i^0(t) = \xi_i^1(t)$, $i = s+1, \ldots, n$. Hence they produce the same output, and are indistinguishable. $\Sigma$ moves slice to slice.
4.3. Duality

The above discussion parallels somewhat the discussion on controllability, and some
duality is evident. A duality is well known for linear systems, and a corresponding notion
for nonlinear systems is expressed in terms of the duality between vector fields and 1-forms.
This idea was developed by Krener and Hermann [7].

In linear system theory, a pair \((A, B)\) is controllable if

\[
\mathcal{C} = \text{span} \{B, AB, \ldots, A^{n-1}B\} = \mathbb{R}^n.
\]

This corresponds to the controllability rank condition, and \(\mathcal{C}\) can be identified with the
controllability distribution. A pair \((C, A)\) is observable if

\[
\mathcal{O} = \text{span} \{C, CA, \ldots, CA^{n-1}\} = \mathbb{R}^{n*}.
\]

This is the observability rank condition, and \(\mathcal{O}\) is the observability codistribution.
This is equivalent to requiring

\[
\mathcal{O}^\perp = \bigcap_{i=0}^{n-1} \ker(CA^i) = \{0\},
\]

which says that the annihilator of the observability codistribution is zero.

4.4. Observable Quotient Systems

Even if a system \(\Sigma\) on \(M\) is not observable, it may be possible to define a "quotient
system" \(\Sigma'\) on \(M/I\) which is observable, for an appropriate equivalence relation \(I\). The
equivalence classes ought to be the leaves of the above mentioned foliation.
In this section we shall simply state two results.

\(x^1Ix^0\) if and only if \(x^1 \in I(x^0)\), or equivalently, \(x^0 \in I(x^1)\). Then \(I\) is an equivalence
relation on \(M\). In fact, \(I\) is closed (continuity of ODEs on initial conditions). Then \(M/I\)
is Hausdorff. In general, \(I\) need not be regular. (Recall that \(I\) is regular if \(M/I\) admits a
\(C^\infty\) structure for which \(\pi : M \to M/I\) is a submersion.)

**Theorem (Sussman)** Let \(\Sigma\) be symmetric (that is, for all \(u \in \Omega^m\) there exits \(v \in \Omega^m\) such
that \(f(x,u) = -f(x,v)\) for all \(x \in M\)). If \((\Sigma, x^0)\) has the local accessibility condition,
then $I$ is closed and regular. Also, there exits a system $\Sigma'$ defined on $M' = M/I$ such that $(\Sigma', I(x^0))$ is observable, has the local accessibility property, and realizes the same input-output map.

We say that $x^0, x^1$ are strongly indistinguishable, written $x^0 SI x^1$, if there exits a continuous $\alpha : [0, 1] \to M$ such that $\alpha(0) = x^0$, $\alpha(1) = x^1$, and $x^0 I \alpha(s)$, for all $s \in [0, 1]$. Then $SI$ is an equivalence relation and $x^0 SI x^1$ implies $x^0 I x^1$. If $\Sigma$ has the local distinguishability property at $x^0$, then $SI(x^0) = \{x^1 : x^1 SI x^0\} = \{x^0\}$.

**Theorem** Suppose $\dim(ad_f|\mathcal{R}(dh))(x) = k \leq n$, for all $x \in M$. Then:

(i) $SI$ is a regular equivalence relation;

(ii) there exits a system $\Sigma'$ on $M' = M/\Sigma$ which has the local distinguishability property;

(iii) $(\Sigma, x^0)$ and $(\Sigma', SI(x^0))$ release the same input-output map, for all $x^0 \in M$;

(iv) if $\Sigma$ is (locally) controllable, then so is $\Sigma'$;

(v) if $\Sigma$ has the (local) accessibility property, then so does $\Sigma'$;

(vi) if $\Sigma$ satisfies the controllability rank condition, then so does $\Sigma'$, and moreover, $M'$ is Hausdorff.

### 4.5. Examples

1. 
\[
\begin{align*}
\dot{x} &= f(x), \quad x \in \mathbb{R}^n \\
y &= h(x), \quad y \in \mathbb{R}.
\end{align*}
\]

\[
f(x) = \sum_{i=1}^{n} f^i(x) \frac{\partial}{\partial x_i} \quad dh(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dx_i \\
L_f h(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (x) \quad L_f dh = dL_f h
\]

\[
\mathcal{G} = \text{span}\{h, L_fh, L_f^2 h, \text{ etc}\} = \mathcal{R}(h)
\]

\[
\langle ad_f|\mathcal{R}(dh)\rangle = d\mathcal{G} = \text{span}\{dh, L_f dh, L_f^2 dh, \text{ etc}\}
\]

2. Linear Systems
\[
\begin{align*}
\dot{x} &= Ax \\
y &= Cx
\end{align*}
\]

\[
f(x) = Ax \quad f^i(x) = \sum_{j=1}^{n} a_{ij} x_j \quad h(x) = Cx = \sum_{j=1}^{n} c_j x_j
\]
\[ L_{ij}h(x) = \sum_{i=1}^{n} (\sum_{j=1}^{n} a_{ij} x_j) C_i = \sum_{i=1}^{n} (A^T C^T) x^i \]
\[ L_{ikh}h(x) = \sum_{i=1}^{n} ((A^T)^k C^T) x_i \]
\[ L_{ij}dh = dL_fh = \sum_{i=1}^{n} ((A^T)^k C_i^T) dx_i \]
\[ \langle ad_f | \mathcal{R}(dh) \rangle = \text{span} \{ C, CA, \ldots, CA^{n-1} \} \]

Thus the observability rank condition is equivalent to the requirement that rank \( \{ C, CA, \ldots, CA^{n-1} \} = n \). Here, this is equivalent to global observability.

The Lie differentiation is just differentiating the output \((n - 1)\) times:

\[
\begin{align*}
x(t) &= e^{At} x_0, & y(t) &= C e^{At} x_0. \\
y(0) &= C x_0 \\
\dot{y}(0) &= CAx_0 \\
&\vdots \\
y^{(n-1)}(0) &= CA^{n-1} x_0 
\end{align*}
\]

This system of \(n\) equations can be solved uniquely for \(x_0\) in terms of \((y(0), \ldots, y^{(n-1)}(0))\) if rank \( \{ C, CA, \ldots, CA^{n-1} \} = n \).

3. \( M = \mathbb{R} \), \( \dot{x} = u \)

\[
y = \sin x \]

\[
f(x) = 0, \quad g(x) = 1, \quad h(x) = \sin x \]

\[ \mathcal{R}(f) = \text{sp} \{ \frac{\partial}{\partial x} \}, \quad \langle ad_f | \mathcal{R}(f) \rangle = \text{span} \{ \frac{\partial}{\partial x} \} \simeq \mathbb{R} \]

\[ dh = (\cos x) dx \quad L_f dh = (\sin x) dx \]

\[ \langle ad_f | \mathcal{R}(dh) \rangle(x) = \text{span} \{ (\cos x) dx, (\sin x) dx \} \simeq \mathbb{R} \]

Therefore both the controllability and observability rank conditions are satisfied. The system is in fact controllable, but not observable.

Let \( x_0 = 0 \). Then \( I(x^0) = \{ 2k\pi, k \in \mathbb{Z} \} \simeq \mathbb{Z} \).

However, on \( M' = S^1 = \mathbb{R}/\mathbb{Z} \), the system is observable. (It is again controllable, and we have "minimal realization".)
References


