Feedback Control of Quantum Systems

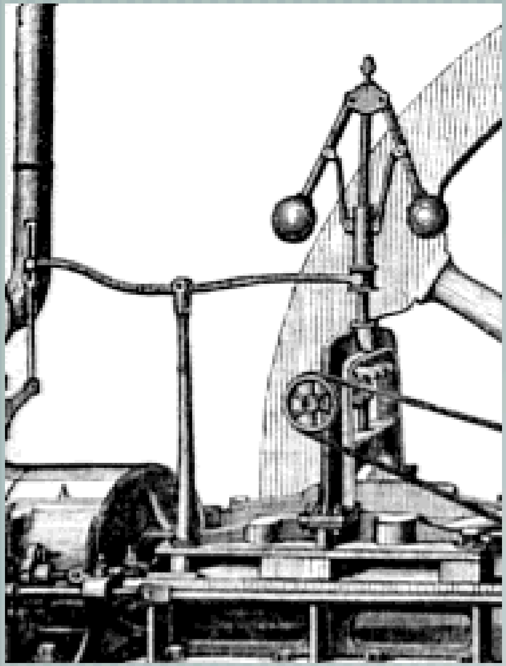
Matt James
ANU

Dedicated to Slava Belavkin
who pioneered the way

Outline

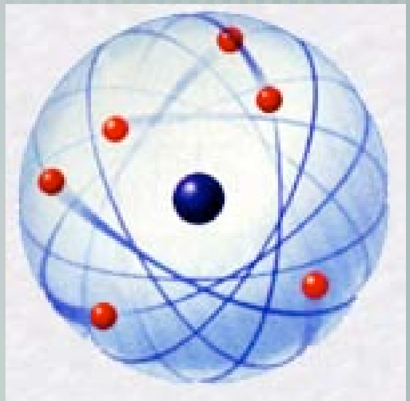
- What is “quantum control”?
- A Little History
- Optimal Control Using Quantum Langevin Equations
- H-Infinity Control for Linear Quantum Systems
- Discussion

What is "quantum control"?



Watt used a governor to control steam engines - very macroscopic.

Now we want to control
things at the nanoscale - e.g. atoms.



Quantum Control:

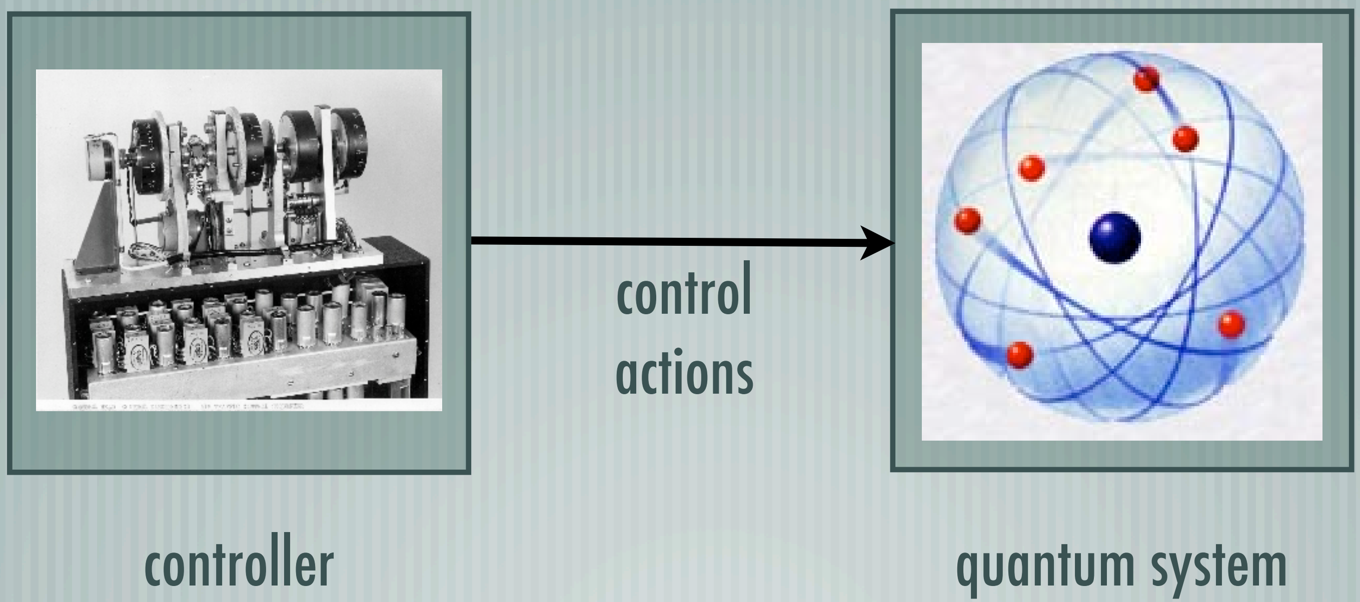
Control of physical systems whose behaviour is dominated by the laws of quantum mechanics.

2003: Dowling and Milburn,

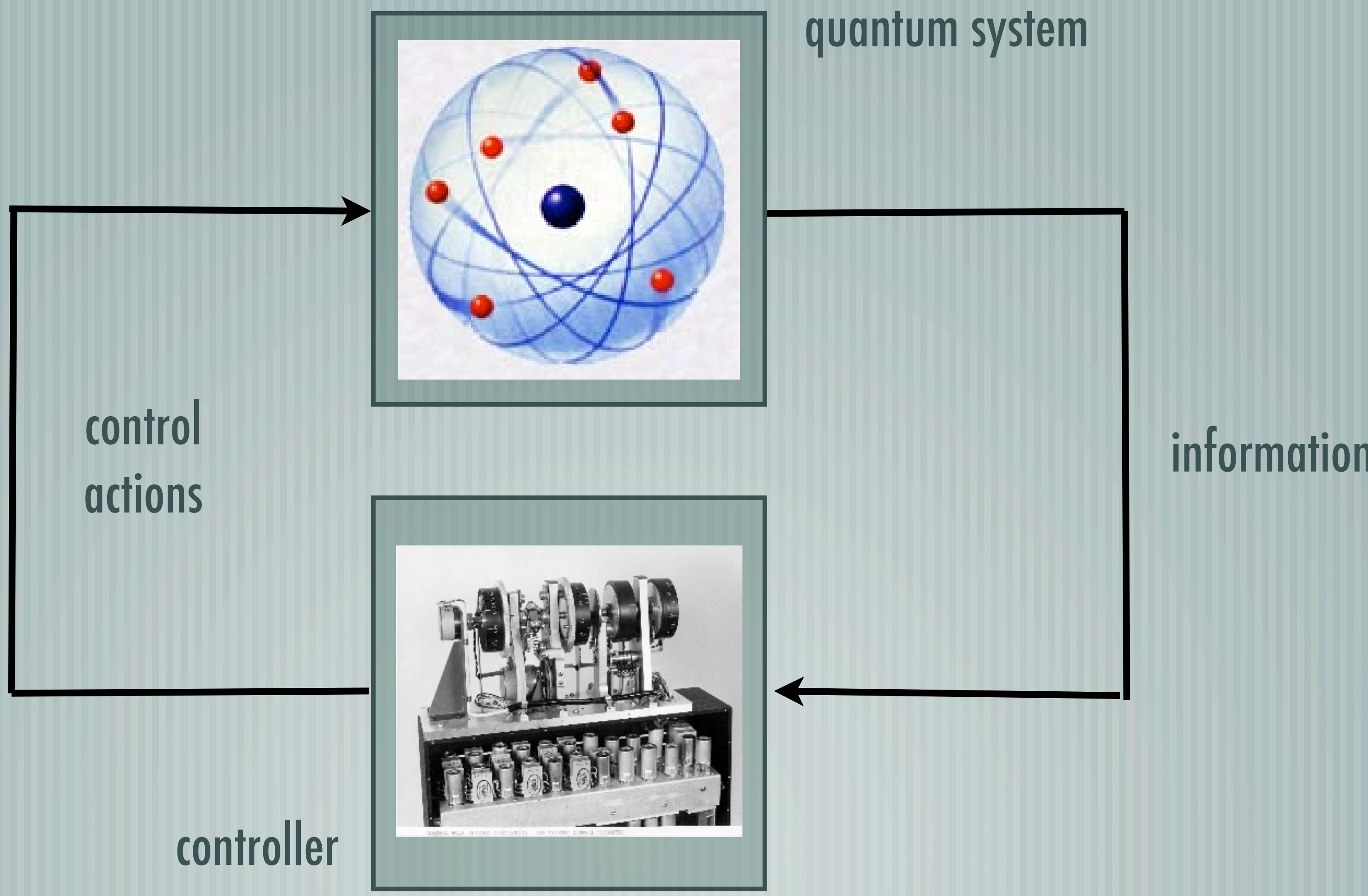
"The development of the general principles of quantum control theory is an essential task for a future quantum technology."

Types of Quantum Control:

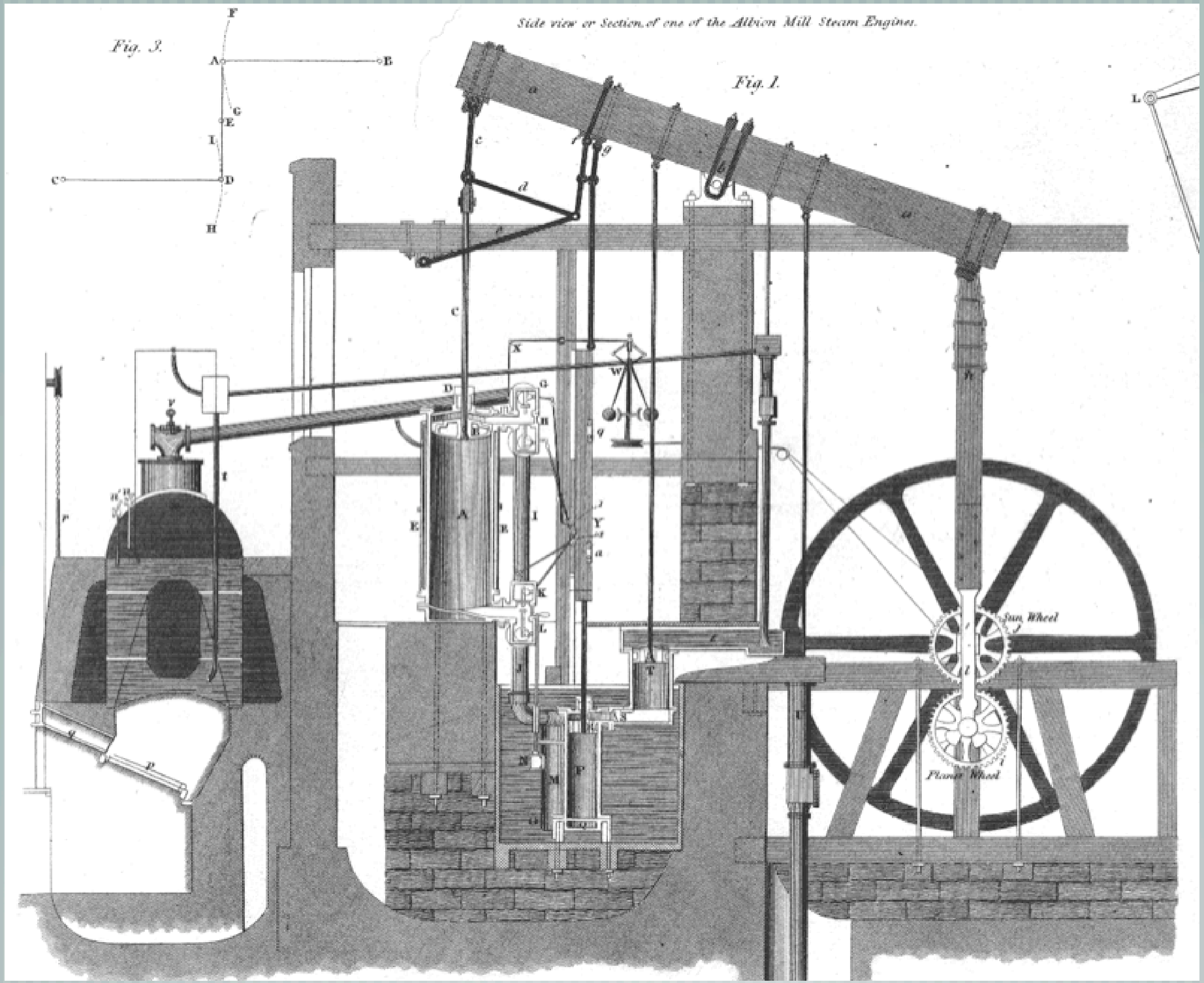
- <u>Open loop</u> - control actions are predetermined, no feedback is involved.

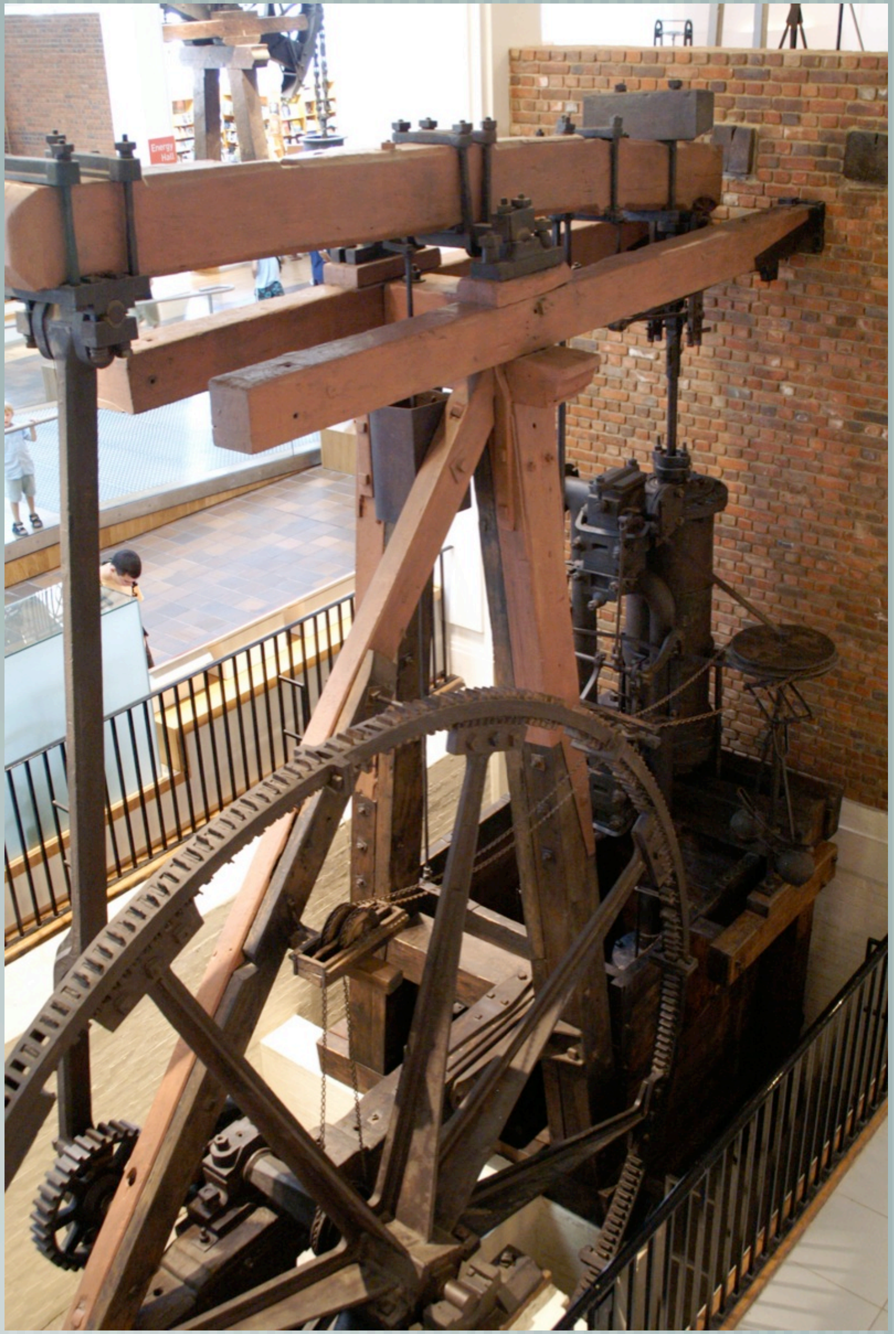


Closed loop - control actions depend on information gained as the system is operating.



Closed loop means feedback, just like in Watt's steam engines.



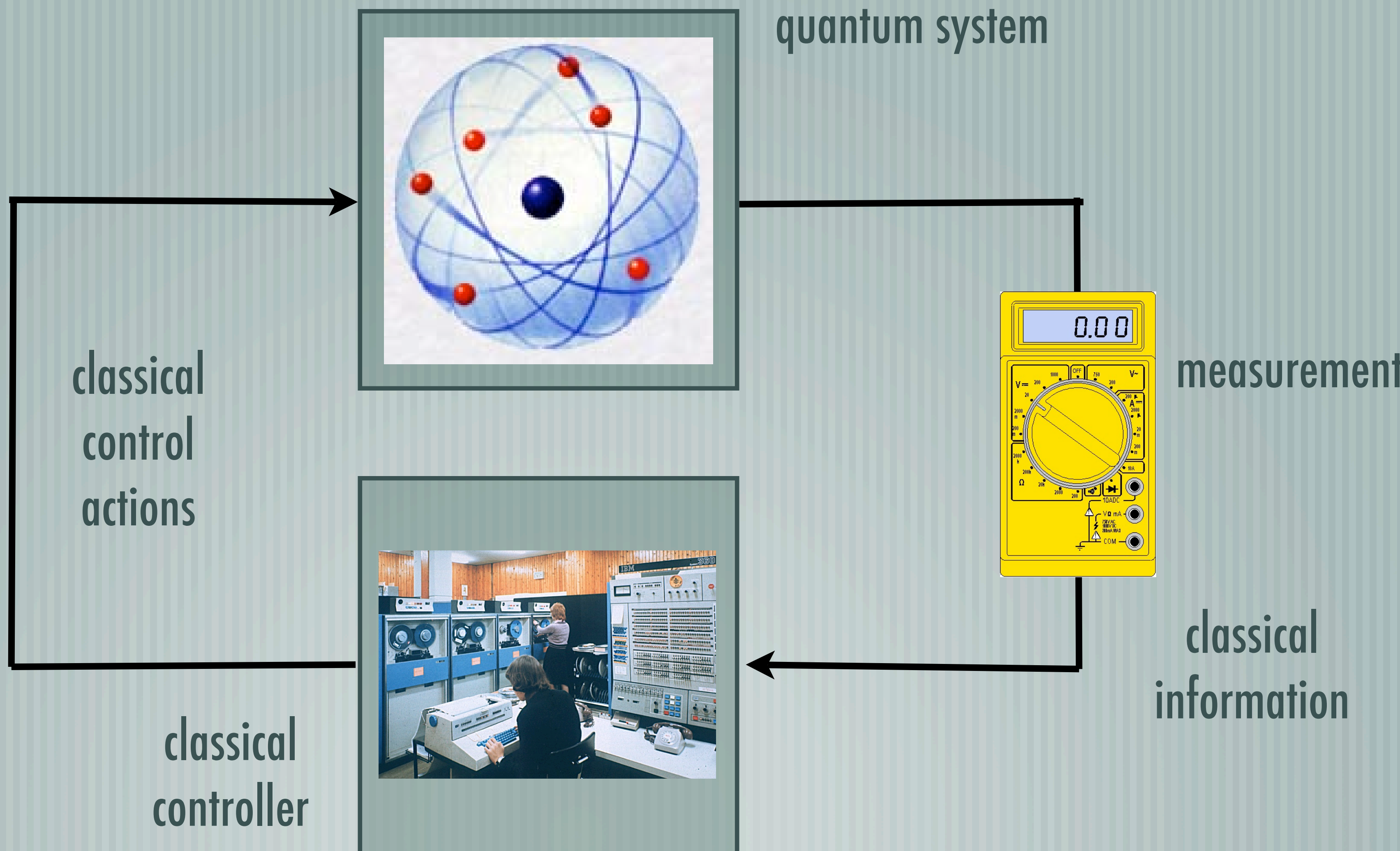


Boulton and Watt, 1788 (London Science Museum)

Types of Quantum Feedback:

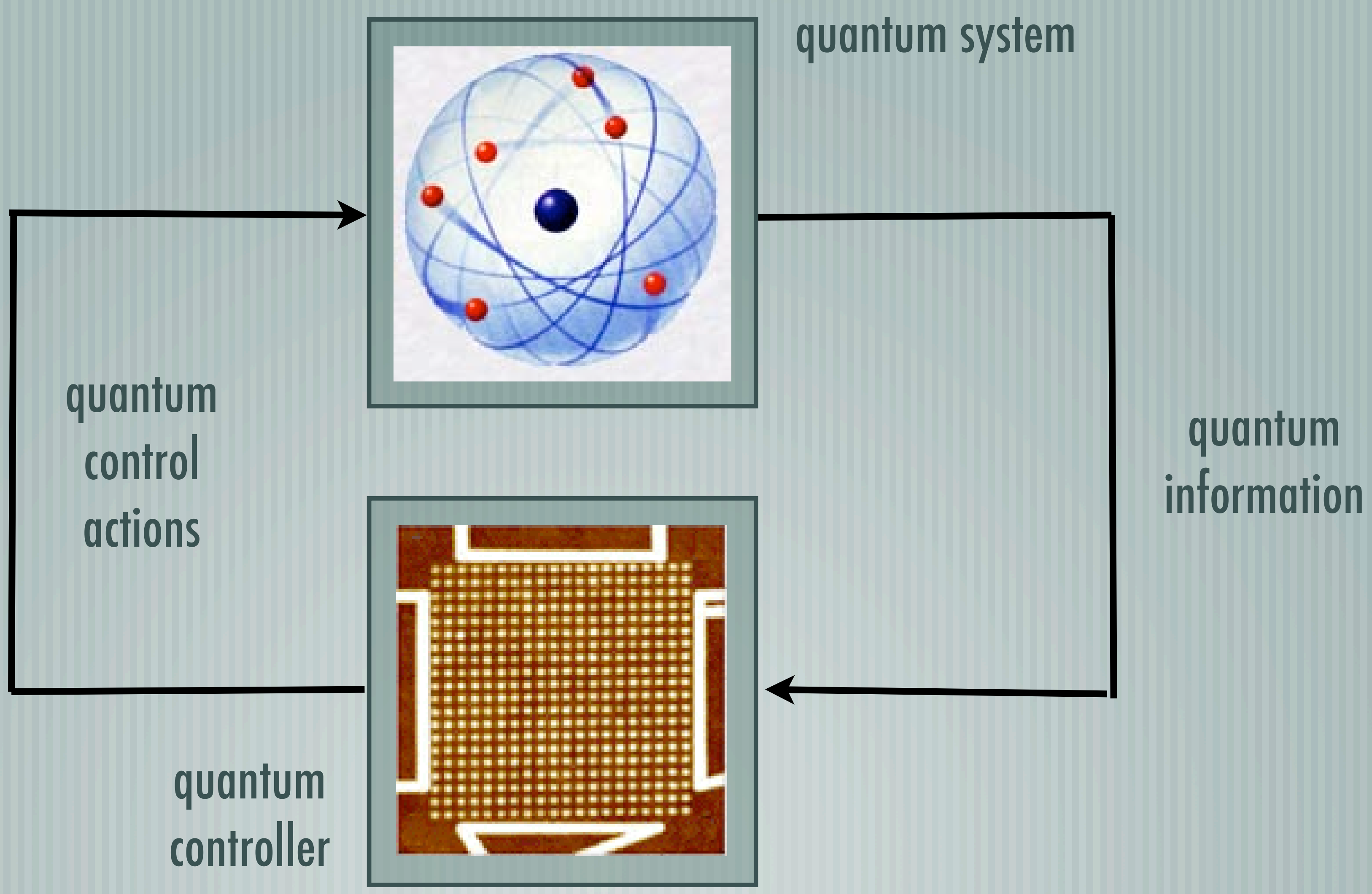
Using measurement

The classical measurement results are used by the controller (e.g. classical electronics) to provide a classical control signal.

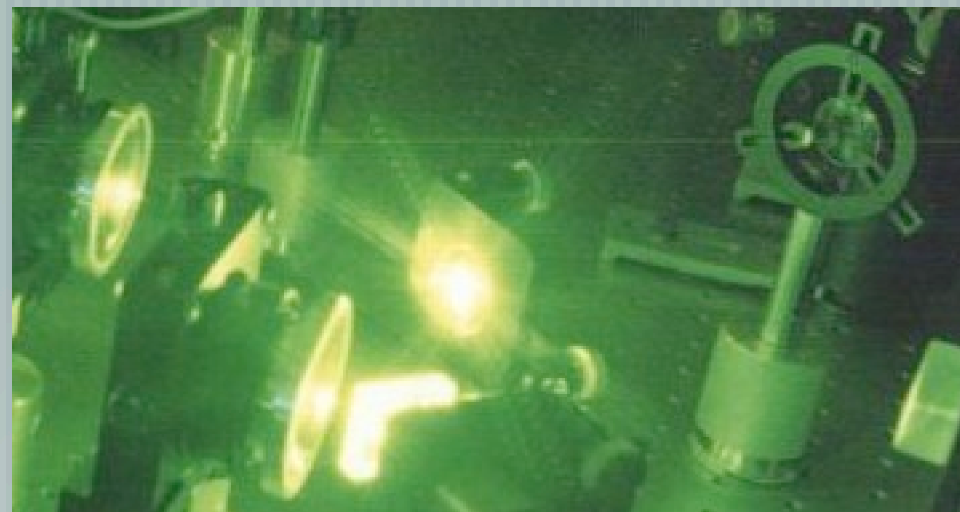


Not using measurement

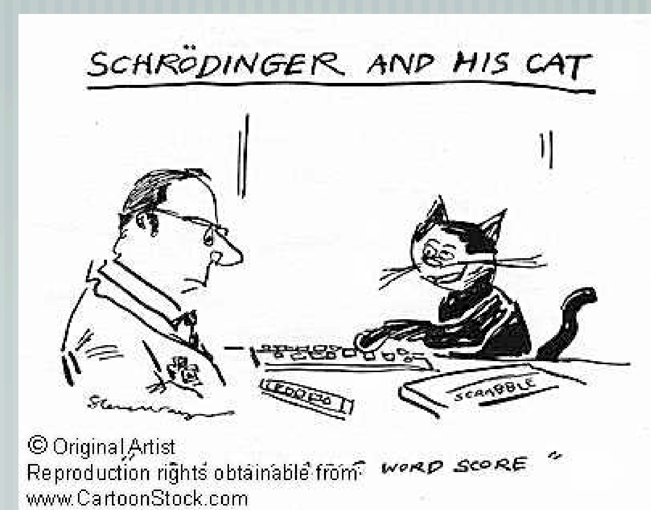
The controller is also a quantum system, and feedback involves a direct flow of quantum information.



The study of quantum feedback control has *practical*



and *fundamental* value.



A Little History

- To the best of my knowledge, Slava Belavkin was the first to publish results concerning quantum feedback control (late 1970's).
- However, there were independent pioneers in the physics community in the 1990s including Wiseman, Milburn, Doherty and Jacobs, who also made fundamental contributions.

Quantum Feedback Chronology

Late 1970's	Belavkin	Linear, Gaussian filtering and control
Early 1980's	Belavkin	Optimal Control using Quantum Operations
Late 1980's	Belavkin	Optimal Filtering and Control using Quantum Stochastic Differential Equations
Early 1990's	Wiseman, Milburn	Quantum optical measurement feedback
Mid-late 1990's	Doherty, Jacobs Mabuchi et al	LQG optimal control Experiments
2000's	Ahn, Belavkin, D'Helon, Doherty, Edwards, Gough, Bouten, van Handel, James, Kimura, Lloyd, Thomsen, Petersen, Schwab, Wiseman, Yanigisawa, and others	Optimal control Lyapunov control robust control applications experiments

From Slava Belavkin's webpage...

Quantum Filtering and Control (QFC)

as a dynamical theory of quantum feedback was initiated in my end of 70's papers and completed in the preprint [1]. This was my positive response to the general negative opinion that quantum systems have uncontrollable behavior in the process of measurement. As was showen in this and the following discrete [2] and continuous time [3] papers, the problem of quantum controllability is related to the problem of quantum observability which can be solved by means of quantum filtering. Thus the quantum dynamical filtering first was invented for the solution of the problems of quantum optimal control. The explicit solution [4] of quantum optimal linear filtering and control problem for quantum Gaussian Markov processes anticipated the general solution [5, 6] of quantum Markov filtering and control problems by quantum stochastic calculus technics. The derived in [5, 6] quantum nonlinear filtering equation for the posterior conditional expectations was represented also in the form of stochastic wave equations. The general solution of these filtering equations in the renormalized (in mean-square sense) linear form was constructed for bounded coefficients in [7] by quantum stochastic iterations. Quantum Filtering Theory (QFT) and the corresponding stochastic quantum equations have now wide applications for quantum continuos measurements, quantum dynamical reductions, quantum spontaneous localizations, quantum state diffusions, and quantum continuous trajectories. All these new quantum theories use particular types of stochastic Master equation which was initially derived from an extended quantum unitary evolution by quantum filtering method.

Slava Belavkin's quantum feedback control papers (pre-2000)

1 V. P. Belavkin: Optimal Measurement and Control in Quantum Dynamical Systems. *Preprint Instytut Fizyki* **411** 3--38. Copernicus University, Torun', 1979. quant-ph/0208108, PDF.

2 V. P. Belavkin: Optimization of Quantum Observation and Control. In *Proc of 9th {IFIP} Conf on Optimizat Techn.* Notes in Control and Inform Sci **1**, Springer-Verlag, Warszawa 1979.

3 V. P. Belavkin: Theory of the Control of Observable Quantum Systems. *Automatica and Remote Control* **44** (2) 178--188 (1983). quant-ph/0408003, PDF.

4 V. P. Belavkin: Non-Demolition Measurement and Control in Quantum Dynamical Systems. In *Information Complexity and Control in Quantum Systems* 311--329, Springer--Verlag, Wien 1987.

5 V. P. Belavkin: Non-Demolition Measurements, Nonlinear Filtering and Dynamic Programming of Quantum Stochastic Processes. *Lecture notes in Control and Inform Sciences* 121 245--265, Springer--Verlag, Berlin 1989.

6 V. P. Belavkin: Non-Demolition Stochastic Calculus in Fock Space and Nonlinear Filtering and Control in Quantum Systems. In *Stochastic Methods in Mathematics and Physics* 310--324, World Scientific, Singapore 1989.

7 V. P. Belavkin: Stochastic Equations of Quantum Filtering. In *Proc of 5th International Conference on Probability Theory and Mathematical Statistics* 10--24. Vilnius 1990.

8 V. P. Belavkin: Continuous Non-Demolition Observation, Quantum Filtering and Optimal Estimation. In *Quantum Aspects of Optical Communication* Lecture notes in Physics **45** 131 -145, Springer, Berlin 1991. quant-ph/0509205, PDF.

9 V. P. Belavkin: Dynamical Programming, Filtering and Control of Quantum Markov Processes. In *Stochastic Analysis in Mathematical Physics* Lecture notes in Physics 9--42, World Scientific, Singapore 1998.

10 V. P. Belavkin: Measurement, Filtering and Control in Quantum Open Dynamical Systems. *Rep. on Math. Phys.* **43** (3) 405-425 (1999).

For instance, let try to choose the optimal measurement of the controlled quantum oscillator (2.2) with transmission line (2.3), so that to minimize its energy $\Omega\langle x^*(t)x(t)\rangle$ at the final instant of time $t=\tau$ by means of the control strategy, the norm $\int_0^\tau |u(t)|^2 dt$ of which should not be too great. Assume, that the initial state x is Guassian with the mathematical expectation $\langle x\rangle = Z$ and

$$\langle (x-Z)(x-Z)\rangle = 0, \quad \langle (x-Z)^*(x-Z)\rangle = \hbar\Sigma ,$$

v(t) is quantum Gaussian white noise [19] which is described by the following correlations

$$\langle v(t)\ v(t')\rangle = 0, \quad \langle v(t)^*\ v(t')\rangle = \hbar\sigma\tilde{\delta}(t-t'),$$

what corresponds to the equilibrium state with the temperature T: $\sigma = \gamma(\exp(\hbar\Omega/kT) - 1)^{-1}$. This problem is characterized by the quality criterion

$$\Omega\langle x(\tau)^* x(\tau)\rangle + \int_0^\tau \langle\theta|u(t)|^2 + \omega(x(t)-u(t))^*(x(t)-u(t))\rangle dt, \quad (2.6)$$

where θ, $\omega \geq 0$ are parameters responsible for the measurement quality: when $\theta = \Omega = 0$ (2.6) corresponds to the problem of pure filtration, when $\omega = 0$, $\theta \neq 0$ it corresponds to the control problem.

It will be shown below (see §3) that the optimal measurement minimising criterion (2.6) is statistically equivalent to the measurement of the stochastical process $Z(t) = \hat{x}(t) + \dot{x}(t)$ described by the Kalman-Bucy filter:

$$d\hat{x}(t)/dt + \alpha\hat{x}(t) = \gamma u(t) + \varkappa(t)(y(t) - \gamma(\hat{x}(t) - u(t))). \quad (2.7)$$

Here $\hat{x}(0) = Z$, $\varkappa(t) = (\gamma\Sigma(t) - \sigma)/(\mu + \sigma)$, $\Sigma(t)$ is the solution of the equation

$$d\Sigma(t)/dt = (\sigma - \gamma\Sigma(t))(\mu + \gamma\Sigma(t))/(\mu + \sigma), \quad \Sigma(0) = \Sigma,$$
$$d\dot{x}(t)/dt + \alpha\dot{x}(t) = \varkappa(t)(\dot{v}(t) - \gamma\dot{x}(t)), \quad \dot{x}(0) = 0, \quad (2.8)$$

V.P. Belavkin, 1987

[1985 conference paper based on 1979 paper] determining the controllable Markov dynamics of the quantum system (1.9), (2.10) in the absence of observations are not described by the propagators $T_t^{\tau}(u_t^{\tau})$, with the excpetion of the degenerate case in which the a posteriori states coincide mod μ_t^{τ} with a priori states, i.e., actually do not depend on the results of the observations v_t^{τ}.

3. OPTIMAL QUANTUM CONTROL

We now discuss the optimal control of a quantum dynamical system with observation $\{\Pi_t^{\tau}\}$, the performance of which as a function of the initial time $t \in \mathbb{R}$ is determined by the mathematical expectation $\langle \rho_t, Q_t(u_t, dv_t)\rangle$ of a certain physical quantity $Q_t(u_t, dv_t) \in \mathfrak{A}$ depending on the input state $u_t = \{u(t+\tau)\}_{\tau \geq 0}$ continuously* and on the output event $dv_t = d\{v(t+\tau)\}_{\tau > 0}$ according to the equation in t

$$Q_t(u_t, dv_t) = \Pi_t^{\tau}(u_t^{\tau}, dv_t^{\tau}) Q_{t+\tau}(u_{t+\tau}, dv_{t+\tau}) + S_t^{\tau}(u_t^{\tau}, dv_t^{\tau}). \tag{3.1}$$

Here $S_t^{\tau}(u_t^{\tau}, dv_t^{\tau}) \in \mathfrak{A}$ denotes Hermitian operators having the integral form†

$$S_t^{\tau}(u_t^{\tau}, dv_t^{\tau}) = \int_0^{\tau} \Pi_t^{\tau'}(u_t^{\tau'}, dv_t^{\tau'}) S(u(t+\tau'), t+\tau') d\tau', \tag{3.2}$$

where $S(u, t) = S(u, t)^*$ denotes Hermitian operator functions completely determining (3.1) for a certain boundary condition $Q_T(u_T, dv_T) = Q$ at the final time $T > t$, corresponding to the specification of a terminal risk $\langle \rho_T, Q\rangle$ ($Q = Q^*$ is a certain Hermitian operator).

Definition 3. A measurable mapping $v_t \mapsto u_t(v_t) \in U_t$ is called a nonadvanced control strategy if its components $u(t+\tau, \cdot)$: $v_t \mapsto u(t+\tau)$ are determined by functions independent of $v_{t+\tau}$, and it is called a retarded control strategy if all $u(t+\tau, \cdot)$ are determined by functions $v_t^{\tau'} \mapsto u(t+\tau, v_t^{\tau'})$ for some measurable $\tau' = \tau'(t+\tau) < \tau$. A nonadvanced strategy $u_t(\cdot)$ is called admissible if the integral

$$Q_t[u(\cdot)] = \int Q_t(u_t(v_t), dv_t)$$

exists in strong operator topology, and it is called optimal for an initial state $\rho_t = \rho$ if it realizes the extremum

$$q(\rho, t) = \inf_{u_t(\cdot) \in U_t(\cdot)} \langle \rho, Q_t[u_t(\cdot)]\rangle, \tag{3.3}$$

where $U_t(\cdot)$ is a certain set of admissible strategies $u_t(\cdot)$ [ε-optimal if $\langle \rho, Q_t[u_t(\cdot)]\rangle$ exceeds (3.3) at most by ε].

We note that in accordance with (3.1) a strategy $u_t(\cdot)$ is admissible with respect to $Q_t(\cdot, \cdot)$ if and only if its segments $u_{t+\tau}(\cdot)$ for fixed v_t^{τ} are admissible strategies with respect to $Q_{t+\tau}(\cdot, \cdot)$ and for the segments $u_t^{\tau}(\cdot)$ there exist measures

$$\Pi_t^{u,\tau}(dv_t^{\tau}) = \int_{dv_t^{\tau}} \Pi_t^{\tau}(u_t^{\tau}(v_t^{\tau}), dv_t^{\tau}), \tag{3.4}$$

specifying operator-valued integrals

$$S_t^{\tau}[u_t^{\tau}(\cdot)] = \int_0^{\tau} \int_{V_t^{\tau}} \Pi_t^{u,\tau'}(dv_t^{\tau'}) S(t+\tau', u(t+\tau', v_t^{\tau'})). \tag{3.5}$$

The latter holds for any delayed strategy that is admissible for a given boundary condition $Q_T(\cdot, \cdot) = Q$.

THEOREM 3. Let the sets $U_t^{\tau}(\cdot)$ of segments of admissible strategies satisfy the condition

$$U_t^{\tau}(\cdot) \times U_{t+\tau}^{\tau'}(\cdot) \subseteq U_t^{\tau+\tau'}(\cdot) \quad \forall t \in \mathbb{R}, \tau, \tau' > 0. \tag{3.6}$$

Then the minimum risk (3.3) as a function of the state ρ and the time t satisfies the functional equation

$$q(\rho, t) = \inf_{u_t^{\tau}(\cdot) \in U_t^{\tau}(\cdot)} [\langle \rho, S_t^{\tau}[u_t^{\tau}(\cdot)]\rangle + \int \pi_{\rho t}^{u\tau}(dv_t^{\tau}) q(\hat{\rho}, t+\tau)], \tag{3.7}$$

where $\pi_{\rho t}^{u\tau}(\cdot) = \langle \rho, \Pi_t^{u\tau}(\cdot)1\rangle$, $\hat{\rho} = \rho\Phi_{\rho t}^{\tau}(u_t^{\tau}(v_t^{\tau}), v_t^{\tau})$ denotes probability measures (1.7) and a posteriori states (1.8) corresponding to an admissible strategy $u = u_t^{\tau}(\cdot)$ and an initial state $\rho = \rho_t$.

*In strong operator topology.

†The conditions for the existence of the integral (3.2), its continuous dependence on u_t^{τ}, and its σ-additivity with respect to dv_t^{τ}, requiring of the operator function $(u, t) \to S(t, u) \in \mathfrak{A}$ continuity in $u \in U$ and measurability with respect to $t \in \mathbb{R}$ under strong operator topology, are presumed to be fulfilled.

V.P. Belavkin, 1983

$u(t)$ is defined by strategies $u_t = d_t(w^t, v_t) = \{\alpha_t(s, w^t, v_t^-, v(s)) \mid s \geq t\}$, where $w^t = (u^t, v^t)$, $v^t = v_0^t$, $v_r^s = \{v(t) \mid r \leq t < s\}$ are the results of nondemolition measurements $v(t)$ on the interval $[r, s[$, $v_t = v_t^\infty$, described by a commutative process $Y(t)$, satisfying the equations either $dY(t) = \hat{a}(t)dN_t + \hat{c}(t)dt$, or $dY(t) = 2\mathrm{Re}\,\hat{b}(t)dB_t + \hat{c}(t)dt$ with invertable $\hat{a}(t, u^t)$, $\hat{b}(t, u^t)$, $\hat{c}(t, u^t) \in \mathcal{B}^t = \{Y(s) \mid s \leq t\}''$ defined by the corresponding real-valued functions $a(w^t, v(t))$, $b(w^t, v(t))$, $c(w^t, v(t))$.

Let us consider the optimal control problem with the operator-valued risk $u \in \mathcal{U} \longmapsto R_t(u) \in \mathcal{A}_t(u) = \{\iota_s(\mathcal{A}, u^s) \mid s \geq t\}''$ satisfaing the equation

$$R_{t_0}(u) = \int_{t_2}^{t_1} S_t(u^t, u(t))dt + R_{t_1}(u), \qquad (4.9)$$

where $S_t(u^t, u(t)) = \iota_t(S(u(t)), u^t)$ for a continuous $\mathcal{A}$-valued self-adjoint function $S(u(t)) = S(u(t))^*$. The optimal control strategy d_t^o of the extremal problem

$$\langle \phi \otimes \omega, R_t(u^t, d_t(w^t, v_t)) \rangle = \inf, \qquad (4.10)$$

where ϕ is an initial normal state on $\mathcal{A}$, and ω is the vacuum state on $\mathcal{N} = \mathcal{B}(\mathcal{F})$. This solution can be found by the dynamic programming method as a solution of the following Bellman continuous inverse-time equation.

THEOREM 10. Let $R(t, w^t, d_t)$ be the averaged $\mathcal{A}$-valued risk uniquely defined for the strategy d_t by

$$\langle \phi \otimes \omega, \iota_t(u^t, R(t, w^t, d_t)) \rangle = \langle \phi \otimes \omega, R_t(u^t, d_t(w^t, v_t)) \rangle$$

due to the Markov condition for $X_t(u^t)$ with respect to $\omega = \omega^t \otimes \omega_t$, and

$$\hat{r}_t(w^t, d_t) = \varepsilon^t[R_t(u^t, d_t(w^t, v_t))] = \langle \hat{\varphi}_t(w^t), R(t, w^t, d_t) \rangle$$

be the posterior risk, corresponding to the strategy d_t, where ε^t is the conditional expectation on $\mathcal{A}_t \vee \mathcal{B}^t$ with respect to the commutativ algebra $\mathcal{B}^t$ and

$$\hat{\phi}_t(w^t, v(t)) = \varepsilon^t \cdot \iota_t(u^t)(v^t)$$

be controlled a posterior state on $\mathcal{A}$ for a $w^t = (u^t, v^t)$. Then

$$\inf_t \langle \hat{\phi}_t(w^t, v(t)), R(t, w^t, d_t) \rangle = s(t, \hat{\varphi}_t(w^t, v(t))),$$

where the functional $s(t, \phi)$ satisfies the following Bellman equation:

$$-\partial_t s(\phi) = \inf_{u(t)} \langle \phi, S(u(t)) + \Lambda(u(t), \delta)s(\phi) \rangle + \langle \phi, Z^+Z \rangle \Delta s(\phi) \qquad (4.11)$$

in case of counting observation and

$$-\partial_t s(\phi) = \inf_{u(t)} \langle \phi, S(u(t)) + (u(t), \delta)s(\phi) \rangle + \tfrac{1}{2}\langle \phi, \Theta(\delta) \rangle^2 s(\phi) \qquad (4.12)$$

in case of diffusion observation.

Here $\partial_t = \partial/\partial t$, $\delta = \delta/\delta\phi$, $\Delta s(\phi) = s(Z\phi Z^*/\langle \phi, Z^*Z \rangle) - s(\phi)$,

$$\Lambda(u(t), \delta)s(\phi) = \delta s(\phi) - 2\mathrm{Re}K(u(t))^* \delta s(\phi)$$

$$\Gamma(u(t), \delta)s(\phi) = Z^*\delta s(\phi)Z - 2\mathrm{Re}K(u(t))^* \delta s(\phi)$$

$$\Theta(\delta)s(\phi) = 2\mathrm{Re}(Z - \langle \phi, Z \rangle)^* \delta s(\phi)$$

and $K(u(t)), K(u(t))^*$ are defined by $\pm iK(u(t)) + Z^*Z/2$ and $\phi_t(w_t)$ is a posterior state on $\mathcal{A}$ for controlled and observed data $w^t = (u^t, v^t)$ satisfying the corresponding nonlinear filtering equations: either (4.3) or (4.7) written respectively in the form:

$$d\hat{\phi}_t + \hat{\phi}_t \Lambda(u(t))dt = \hat{\phi}_t \Delta\, dN_t \;; \quad d\hat{\phi}_t + \hat{\phi}_t \Gamma(u(t))dt = \hat{\phi}_t \Theta\, dQ_t$$

where $\hat{\phi}_0 = \phi$.

The linear programming for Gaussian ϕ, canonical Z and quadratic $S(u(t))$ was considered in [3]. The general formulation of quantum dynamic programming for the partially observable controlled quantum objects in operational approach was given in [16].

References

1. V.P.Belavkin, Optimal quantum filtration of Markovian signals. Problems of Control and Information Theory, v.7(5),p.345-360 (1978).
2. V.P.Belavkin, Quantum filtering of Markovian signals with quantum white noises. Radiotechnika i Elektronika, v.25 (7), p.1445-1453 (1980) (in Russian).
3. V.P.Belavkin, Nondemolition measurement and control in quantum dynamic systems. In: Information complexity and control in quantum physics, ed. A.Blaquiere, S.Diner, G.Lochak, Springer-Verlag, Wien - New York, p.311-336 (1987).

V.P. Belavkin, 1989

[1988 conference paper]

Optimal Control Using Quantum Stochastic Differential Equations

- Quantum Langevin equations (QLE) provide a general framework for open quantum systems and contain considerable physical information.
- QLEs expressed in terms of quantum stochastic differential equations (QSDE) are very well suited for control engineering.
- Quantum operation models arise naturally after suitable conditioning
- The system Hamiltonian may depend on a classical control parameter.

Model:

Belavkin, 1988; Bouten-van Handel, 2005; Edwards-Belavkin, 2005; Gough-Belavkin-Smolyanov, 2005; James, 2005

Quantum probability space $(\mathscr{B} \otimes \mathscr{W}, \rho \otimes \phi)$, where $\mathscr{B} = \mathscr{B}(\mathsf{h})$ is the algebra of system operators with state ρ, and $\mathscr{W} = \mathscr{B}(\mathsf{F})$ is the algebra of operators on Fock space with vacuum state ϕ.

Unitary dynamics

$$dU_t = \Big\{ L dA_t^* - L^* dA_t - \frac{1}{2} L^* L dt - iH(u(t)) dt \Big\} U_t, \qquad U_0 = I$$

System operators $X \in \mathscr{B}$ evolve according to

$$X_t = j_t(X) = U_t^* X U_t$$

$$dj_t(X) = j_t(\mathcal{L}^{u(t)}(X))\, dt + j_t([L^*, X])\, dA_t + j_t([X, L])\, dA_t^*$$

where

Lindblad

$$\mathcal{L}^u(X) = i[H(u), X] + L^* X L - \frac{1}{2}(L^* L X + X L^* L)$$

Measurements $Y_t = U_t^\dagger (A_t + A_t^\dagger) U_t$,

$$dY_t = j_t(L + L^\dagger)\, dt + dA_t + dA_t^\dagger.$$

$\mathscr{Y}_t = \mathrm{vN}\{Y_s,\ s \leq t\}$ is a commutative family of von Neumann algebras, so Y is equivalent to a classical process, which can be measured via homodyne detection.

$\mathscr{C}_t = \mathrm{vN}\{Z_s = A_s + A_s^*,\ s \leq t\}$ is also commutative, and

$$\mathscr{Y}_t = U_t^* \mathscr{C}_t U_t$$

Belavkin non-demolition condition:

$$[X, Z_s] = [X_t, Y_s] = 0, \quad 0 \leq s \leq t$$

so that

(commutants)

$$X \in \mathscr{C}_t', \quad j_t(X) \in \mathscr{Y}_t'$$

Thus conditional expectations

$$\mathbb{P}[X|\mathscr{C}_t], \quad \mathbb{P}[X_t|\mathscr{Y}_t]$$

are well-defined.

(hence filtering is possible)

With respect to the state $\mathbb{P}$, the process $Z_t = A_t + A_t^*$ is equivalent to a standard Wiener process. Furthermore, with respect to the state $\mathbb{P}_T^0$ defined by

$$\mathbb{P}_T^0[X] = \mathbb{P}[U_T X U_T^*], \; X \in \mathscr{B} \otimes \mathscr{W},$$

the measurement process Y_t is equivalent to a standard Wiener process (cf. *reference measure* in classical filtering).

A *controller* is a causal function from measurement data to control signals:

$$u(t) = \mathbf{K}(t, y_{[0,t]})$$

The coefficients of the stochastic differential equations are now adapted to the measurement filtration $\mathscr{Y}_t$. The filtering theory continues to apply.

Belavkin, 1988; Bouten-van Handel, 2005

The Optimal Control Problem:

Cost operators [non-negative, self-adjoint]

$$C_1(u), \quad C_2$$

used to specify performance integral

$$\int_0^T C_1(t)dt + C_2(T),$$

where $C_1(t) = j_t(u, C_1(u(t)))$, $C_2(t) = j_t(u, C_2)$.

We wish to minimize the expected cost

$$J(\mathbf{K}) = \mathbb{P}[\int_0^T C_1(t)dt + C_2(T)]$$

over all controllers $\mathbf{K}$.

Dynamic Programming:

First, we represent the expected cost in terms of filtered quantities, as follows.

Define $\tilde{U}_t$ by

Holevo, 1990; Belavkin, 1992; Bouten-van Handel, 2005; James, 2005

$$d\tilde{U}_t = \left\{ L(dA_t^* + dA_t) - \frac{1}{2} L^* L dt - iH(u(t))dt \right\} \tilde{U}_t, \qquad \tilde{U}_0 = I$$

Then $\tilde{U}_t^* X \tilde{U}_t \in \mathscr{C}_t'$ $(X \in \mathscr{B})$ and the unnormalized filter state

$$\sigma_t(X) = U_t^* \mathbb{P}[\tilde{U}_t^* X \tilde{U}_t | \mathscr{C}_t] U_t$$

is well-defined. It evolves according to

Belavkin, 1992

$$d\sigma_t(X) = \sigma_t\big(\mathcal{L}^{u(t)}(X)\big)dt + \sigma_t(L^* X + XL)dY_t$$

This is the form of the *Belavkin quantum filter* we use. It is analogous to the Duncan-Mortensen-Zakai equation of classical filtering.

σ_t is an *information state* (control theory terminology).

data $\mathbf{v}^t$ for the nondemolition observation (7) with a given initial wave function $\psi \in \mathscr{H}$ has the density

$$\varrho_t^\omega = \int |F_t(\omega)\psi|^2 \, d\mu(\omega \mid v_t) = (\psi \mid \hat{P}(v_t)\psi) \equiv \hat{\varrho}(v_t),$$

where $\hat{P}(v_t) = \hat{\Phi}_t\{I\}(v) = P_t^\omega$. The non-Gaussian measure $d\nu = \varrho d\mu$ defines the factorial generating functionals $\varrho_g^t = \langle \hat{e}_g^t(y) \rangle$ for the process $\mathbf{Y}^t$ as $(\psi \mid \Phi_g^t\{I\}\psi)$ and the mean values $\langle X(t) \rangle$ of the operators $X(t)$ at the initial states $\psi \in \mathscr{H}$ as $(\psi \mid \Phi_t^{(0)}\{X\}\psi)$ by the averaging

$$(\psi \mid \Phi_g^t\{X\}\psi) = \int e^{v_t(g) - g_t^2/2} \, \hat{\pi}_t\{X\}(v_t) \, d\nu(v_t) = \phi_g^t\{X\}$$

of the product $\hat{e}_g^t(v_t)\hat{\pi}_t\{X\}(v_t)$, where $\hat{\pi}_t\{X\}(v_t) = (\psi \mid \hat{\Phi}_t\{X\}(v_t)\psi)/\hat{\varrho}(v_t)$, over all the observed in the past trajectories $\mathbf{v}^t$. □

Let us derive the corresponding linear stochastic equation for the non-normalized posterior map (18) $X \mapsto \hat{\Phi}_t\{X\}$ defining the a posteriori transformation $\phi_0 \mapsto \phi_0 \circ \hat{\Pi}$ for any initial ϕ_0 by $\hat{\Pi}_t\{X\} = \hat{\Phi}_t\{X\}/\hat{\varrho}_t$, $\hat{\varrho}_t = \phi_0\{\hat{P}_t\}$. In the case of a complete nondemolition observation it can be obtained in the Schrödinger picture from (13) in the same way as (11) from (5) by using Ito's formula for $\hat{F}_t^* X \hat{F}_t = \hat{\Phi}_t\{X\}$:

$$d(\hat{F}_t^* X \hat{F}_t) + \hat{F}_t\Big(XK_t + K_t^* X - \int_\Lambda L_{t,\mathbf{x}}^* X L_{t,\mathbf{x}} \, d\lambda\Big)\hat{F}_t \, dt = \int_\Lambda \hat{F}_t^* (XL_{t,\mathbf{x}} + L_{t,\mathbf{x}}^* X)\hat{F}_t \, dY_t(d\mathbf{x}).$$

In the general case the stochastic differential equation for (18) gives the following

Theorem 2. *The conditional expectation* (18), *defining in* (17) *the absolutely continuous operational measure* $\hat{\Phi}_t\{X\}(v_t)\,d\mu(\omega)$ *with respect to the Wiener process* $v_t(\omega)$, *represented in Fock space by* $\mathbf{Y}_t = \{Y_t(E) \mid E \in \mathscr{B}\}$, *satisfies the linear stochastic equation*

$$d\hat{\Phi}_t\{X\} + \hat{\Phi}_t\Big\{XK_t + K_t^* X - \int_\Lambda L_{t,\mathbf{x}}^* X L_{t,\mathbf{x}} \, d\lambda\Big\} dt = \int_\Lambda \hat{\Phi}_t\{X\bar{L}_{t,\mathbf{x}} + \bar{L}_{t,\mathbf{x}}^* X\} \, dY_t(d\mathbf{x}) \qquad (19)$$

corresponding to Eq. (15) *for the Wick symbol* $\Phi_g^t\{X\} = \langle f \mid \hat{\Phi}_t \mid f \rangle$, $g = 2\Re \bar{f}_t$. *Here* $\bar{L}_{t,\mathbf{x}}$ *are* $\mathscr{B}$*-measurable operator-valued functions of* $\mathbf{x} \in \Lambda$, $\bar{L}_{t,\mathbf{x}} = 0$, *if* $\mathbf{x} \notin E$ *for any* $E \in \mathscr{B}$, *defined for almost all* t *as a conditional averaging of* $L_{t,\mathbf{x}}$ *with respect to* $\mathscr{B} \subsetneq \mathscr{A}$ *by*

$$\int_\Lambda \bar{L}_{t,\mathbf{x}} \, g(\mathbf{x}) \, d\lambda = \int_\Lambda L_{t,\mathbf{x}} \, g(\mathbf{x}) \, d\lambda$$

for any $\mathscr{B}$*-measurable square-integrable* $g: \Lambda \mapsto \mathbf{R}$ *and* $\bar{f}(t,\mathbf{x})$ *is defined similary by the averaging of* $f(t,\mathbf{x})$. *In particular,* $\bar{L}_{t,\mathbf{x}} = \frac{1}{\lambda(E_i)} \int_{E_i} L_{t,\mathbf{x}} \, d\lambda$ *for all* $\mathbf{x} \in E_i$, *if* $\mathscr{B} = \{E_i \in \mathscr{A} \mid i \in I\}$ *is a* σ*-partition* $M = \sum_{i \in I} E_i$ *of* $M \subsetneq \Lambda$ *and* $\lambda(E_i) \neq 0$.

Proof. By the classical Ito's formula

$$d(\hat{e}_g^t \hat{\Phi}_t\{X\}) = d\hat{e}_g^t \hat{\Phi}_t\{X\} + \hat{e}_g^t d\hat{\Phi}_t\{X\} + d\hat{e}_g^t d\hat{\Phi}_t\{X\}$$
$$= \int_\Lambda g(t,\mathbf{x}) \, \hat{e}_g^t \hat{\Phi}_t\{X + X\bar{L}_{t,\mathbf{x}} + \bar{L}_{t,\mathbf{x}}^* X\} \, dY_t(d\mathbf{x}) - \hat{e}_g^t \hat{\Phi}_t\{XK_t + K_t^* X\} \, dt$$
$$+ \hat{e}_g^t \hat{\Phi}_t\Big\{\int_\Lambda (L_{t,\mathbf{x}}^* X L_{t,\mathbf{x}} + (X\bar{L}_{t,\mathbf{x}} + \bar{L}_{t,\mathbf{x}}^* X) \, g(t,\mathbf{x})) \, d\lambda\Big\} dt$$

Belavkin, 1992

We derive the following representation in terms of σ_t:

$$\begin{aligned}
J(\mathbf{K}) &= \mathbb{P}[\int_0^T \tilde{U}_t^* C_1(u(t))\tilde{U}_t dt + \tilde{U}_T^* C_2 \tilde{U}_T] \\
&= \mathbb{P}[\int_0^T \mathbb{P}[\tilde{U}_t^* C_1(u(t))\tilde{U}_t | \mathscr{C}_t] dt + \mathbb{P}[\tilde{U}_T^* C_2 \tilde{U}_T | \mathscr{C}_T]] \\
&= \mathbb{P}[\int_0^T U_t U_t^* \mathbb{P}[\tilde{U}_t^* C_1(u(t))\tilde{U}_t | \mathscr{C}_t] U_t U_t^* dt + U_T U_T^* \mathbb{P}[\tilde{U}_T^* C_2 \tilde{U}_T | \mathscr{C}_T] U_T U_T^*] \\
&= \mathbb{P}_T^0[\int_0^T \sigma_t(C_1(u(t)))dt + \sigma_T(C_2)]
\end{aligned}$$

This last expression is equivalent to classical expectation with respect to Wiener measure:

$$J(\mathbf{K}) = \mathbf{E}_T^0[\int_0^T \sigma_t(C_1(u(t)))dt + \sigma_T(C_2)]$$

The fact that the information state σ_t is computable from dynamics driven by the measured data means that the methods of dynamic programming are applicable.

Define the *value function*

$$S(\sigma,t) = \inf_{\mathbf{K}} \mathbf{E}^0_{\sigma,t}[\int_t^T \sigma_s(C_1(u(s)))ds + \sigma_T(C_2)]$$

which quantifies the optimal cost to go from a current state σ at time t.

The *dynamic programming principle* states that

$$S(\sigma,t) = \inf_{\mathbf{K}} \mathbf{E}^0_{\sigma,t}[\int_t^s \sigma_r(C_1(u(r)))dr + S(\sigma_s,s)]$$

At least formally, $S(\sigma,t)$ satisfies the *Hamilton-Jacobi-Bellman* equation

$$\begin{aligned} \frac{\partial}{\partial t}S(\sigma,t) + \inf_{u\in\mathbf{U}}\{\mathscr{L}^u S(\sigma,t) + C_1(u)\} &= 0 \\ S(\sigma,T) &= \sigma(C_2) \end{aligned}$$

where $\mathscr{L}^u$ is the Markov generator of σ_t (for fixed value $u \in \mathbf{U}$).

Optimal Controller:

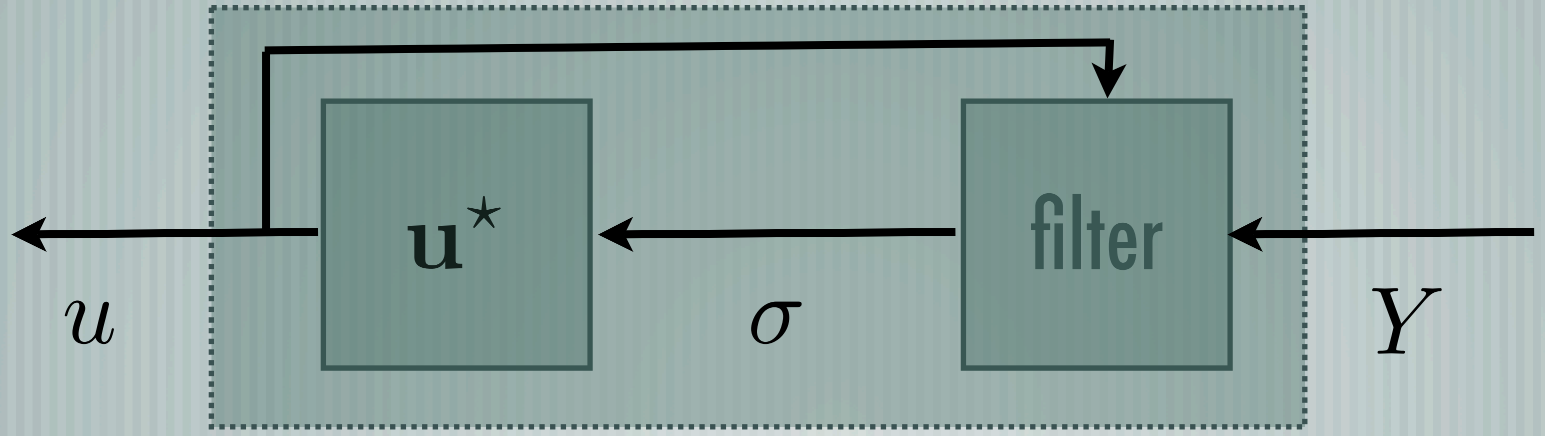
Suppose we have a solution $S(\sigma, t)$ of the HJB equation. Define

$$\mathbf{u}^\star(\sigma, t) = \arg\min_u \{\mathscr{L}^u S(\sigma, t) + C_1(u)\}$$

This defines the optimal feedback controller:

$$\mathbf{K}^\star \ : \ \begin{cases} d\sigma_t(X) & = \sigma_t\big(\mathcal{L}^{u(t)}(X)\big)dt + \sigma_t(L^* X + XL)dY_t \\ u(t) & = \mathbf{u}^\star(\sigma_t, t) \end{cases}$$

This controller has the *separation structure*, with filter dynamics the Belavkin quantum filter for the information state σ_t.



Another Type of Optimal Control Problem - Risk Sensitive

Classical: Jacobson, 1973; Whittle, 1980; Bensoussan-van Schuppen, 1985; James-Baras-Elliott, 1993

Classical criterion: [Note the exponential]

$$J^{\mu}(\mathbf{K}) = \mathbf{E}[\exp\{\mu(\int_0^t C_1(u(t),t)dt + C_2(T))\}]$$

Here, $\mu > 0$ is a risk parameter.

Quantum criterion: James, 2004, 2005

$$J^{\mu}(\mathbf{K}) = \mathbb{P}[R^*(T)e^{\mu C_2(T)}R(T)]$$

where $R(t)$ is the time-ordered exponential defined by

$$\frac{dR(t)}{dt} = \frac{\mu}{2}C_1(t)R(t), \quad R(0) = I$$

i.e.

$$R(t) = \overleftarrow{\exp}\left(\frac{\mu}{2}\int_0^t C_1(s)ds\right).$$

In general, it does not appear possible to solve this problem using the unnormalized conditional state σ_t. Accordingly, we introduce a *risk-sensitive information state*

$$\sigma_t^\mu(X) = U_t^* \mathbb{P}[\tilde{V}_t^* X \tilde{V}_t | \mathscr{C}_t] U_t, \; X \in \mathscr{B},$$

where $\tilde{V}_t \in \mathscr{C}_t'$ is given by

$$d\tilde{V}_t = \{L(dA_t^* + dA_t) - \frac{1}{2}L^*L - \frac{i}{\hbar}H + \frac{\mu}{2}C_1(u(t))\}\tilde{V}_t$$

We then have the representation

$$J^\mu(\mathbf{K}) = \mathbb{P}_T^0[\sigma_T^\mu(e^{\mu C_2})]$$

which facilitates dynamic programming.

The Hamilton-Jacobi-Bellman equation for this problem is

$$\begin{aligned} \frac{\partial}{\partial t} S^\mu(\sigma, t) + \inf_{u \in \mathbf{U}} \{\mathscr{L}^{\mu,u} S^\mu(\sigma, t)\} &= 0 \\ S^\mu(\sigma, T) &= \sigma(e^{\mu C_2}) \end{aligned}$$

where $\mathscr{L}^{\mu,u}$ is the Markov generator of σ_t^μ (for fixed value $u \in \mathbf{U}$).

Optimal Risk-Sensitive Controller:

Suppose we have a solution $S^\mu(\sigma, t)$ of the risk-sensitive HJB equation. Define

$$\mathbf{u}^{\mu,\star}(\sigma, t) = \arg\min_u \{\mathscr{L}^{\mu,u} S^\mu(\sigma, t)\}$$

This defines the optimal feedback controller:

$$\mathbf{K}^{\mu,\star} \ : \ \begin{cases} d\sigma_t^\mu(X) & = \sigma_t^\mu\big(\mathcal{L}^{\mu,u(t)}(X)\big)dt + \sigma_t^\mu(L^* X + XL)dY_t \\ u(t) & = \mathbf{u}^{\mu,\star}(\sigma_t^\mu, t) \end{cases}$$

where

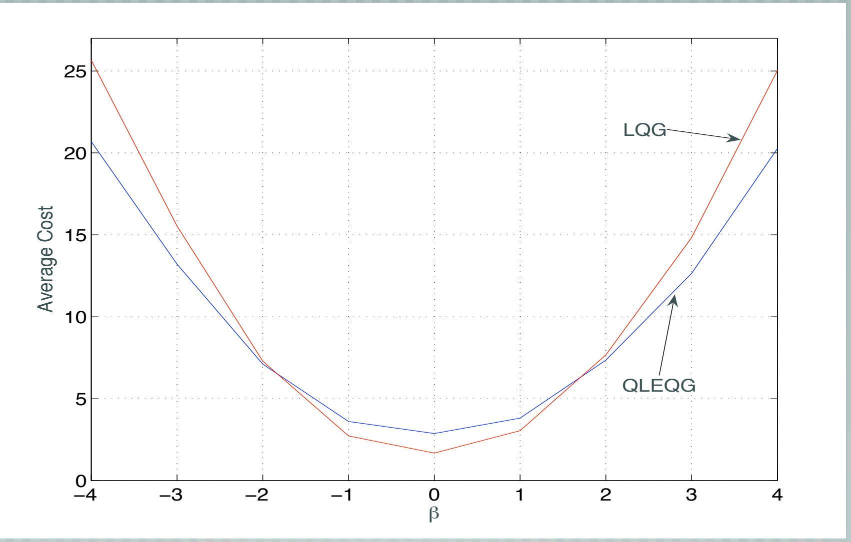
$$\mathcal{L}^{\mu,u}(X) = \mathcal{L}^u(X) + \frac{\mu}{2}(C_1(u)X + XC_1(u))$$

[Note the inclusion of the cost observable in the modified Lindblad.]

This controller also has the *separation structure*, with filter dynamics the modified Belavkin quantum filter for the risk-sensitive information state σ_t^μ.

Comments: The risk-sensitive problem is of interest for at least two reasons:

- **Robustness:** Risk-sensitive controllers have the practical benefit that they can cope with uncertainty better than standard.



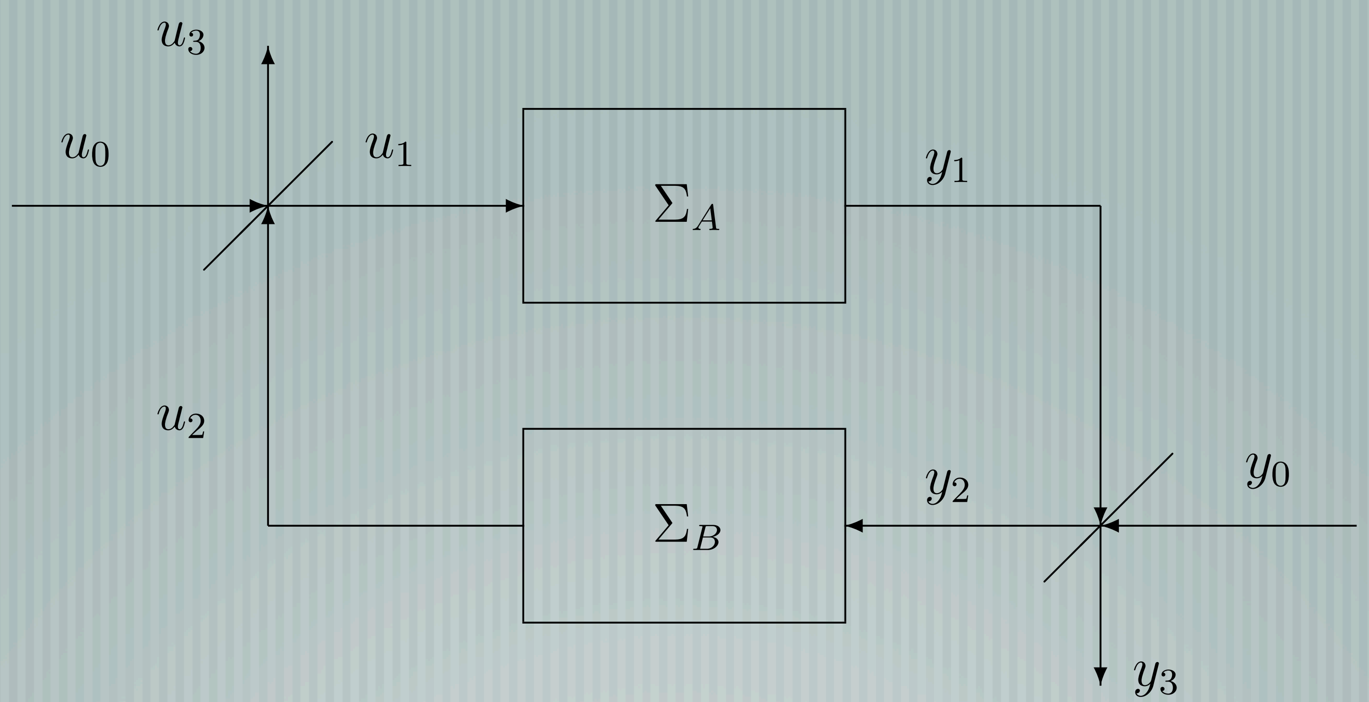
- **Fundamentals of quantum mechanics:** The risk-sensitive information state can be viewed as a subjective state that includes *knowledge* and *purpose*, extending the Copenhagen interpretation in the feedback context.

H-Infinity Control for Linear Quantum Systems

- "H-Infinity" refers to the Hardy space H^∞ which provides the setting for a frequency domain approach to robust control system design (initiated by Zames, late 1970's).
- Robustness refers to the ability of a control system to tolerate uncertainty, noise and disturbances, to some extent at least.
- Feedback is fundamental to this, and in fact is the *raison d'etra* for feedback.

Motivation: Feedback Stability

Even when individual components are stable, feedback interconnections need not be.



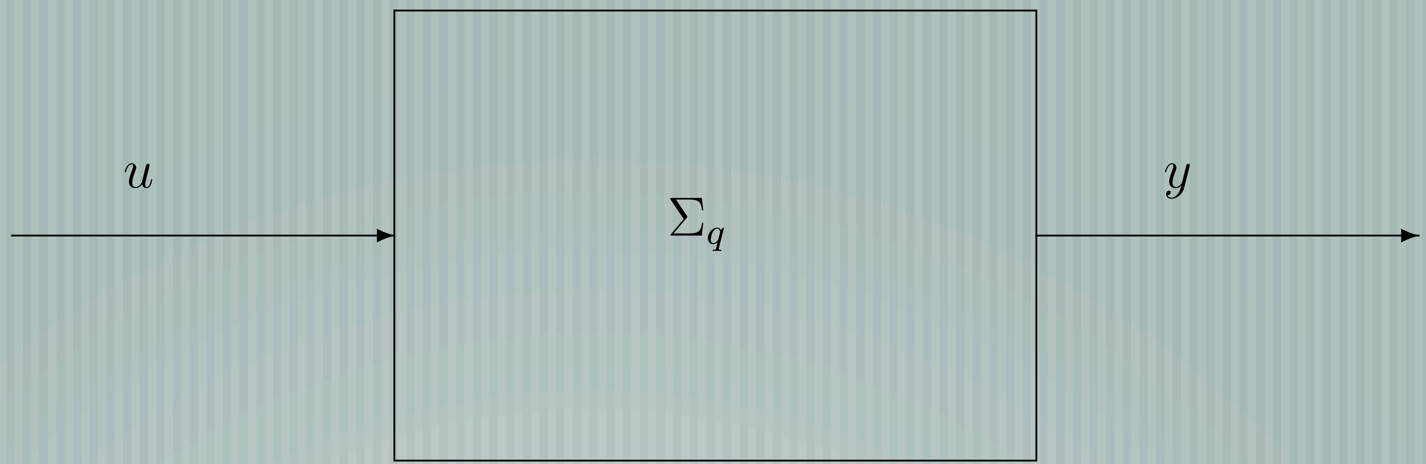
The small gain theorem asserts stability of the feedback loop if the loop gain is less than one:

$$g_A g_B < 1$$

Classical: Zames, Sandberg, 1960's

Quantum: D'Helon-James, 2006

Stability is quantified in a mean-square sense as follows.

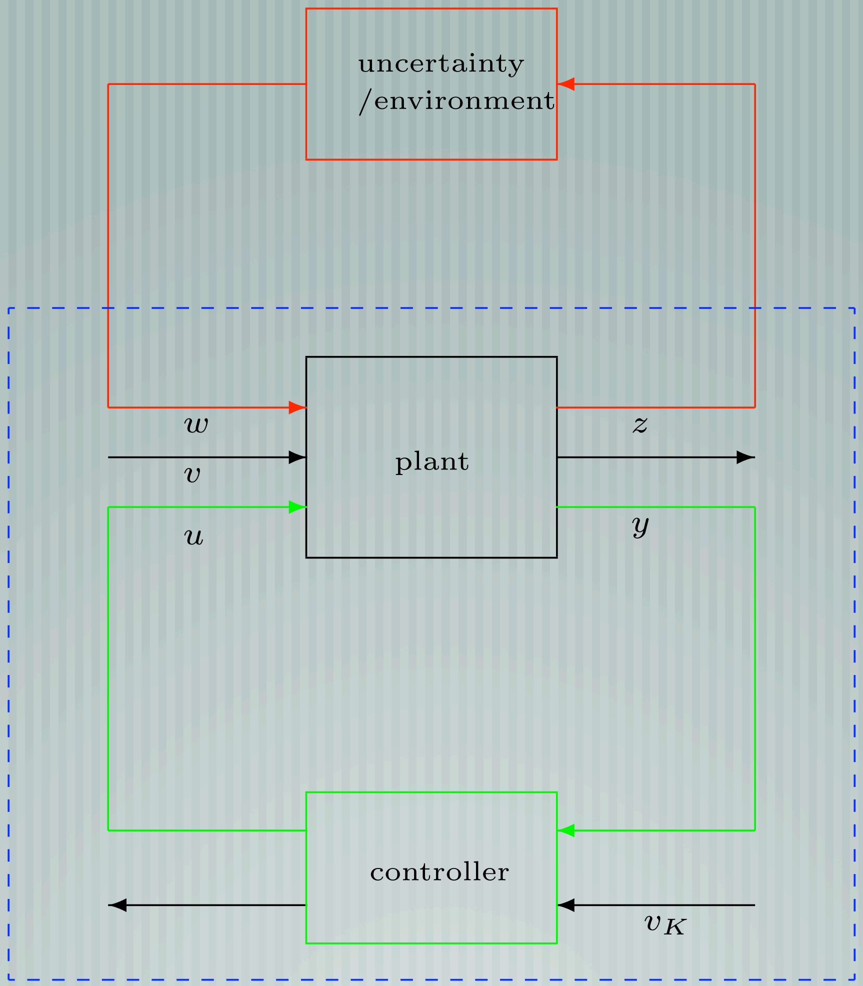


$$
\begin{aligned}
dU(t) &= \beta_u(t)dt + dB_u(t) \\
dY(t) &= \beta_y(t)dt + dB_y(t)
\end{aligned}
$$

$$
\boxed{\| \beta_y \|_t^2 \leq \mu + \lambda t + g^2 \| \beta_u \|_t^2}
$$

The H-Infinity Robust Control Problem:

Given a system (plant), find another system (controller) so that the gain from w to z is small. This is one way of reducing the effect of uncertainty or environmental influences.



Classical: Zames, late 1970's

Quantum: D'Helon-James, 2005; James-Nurdin-Petersen, 2006

Linear/Gaussian Model:

Plant:

$$\begin{aligned} dx(t) &= Ax(t)dt + B_0 dv(t) + B_1 dw(t) + B_2 du(t);\ \ x(0) = x; \\ dz(t) &= C_1 x(t)dt + D_{12} du(t); \\ dy(t) &= C_2 x(t)dt + D_{20} dv(t) + D_{21} dw(t) \end{aligned}$$

Controller:

$$\begin{aligned} d\xi(t) &= A_K \xi(t)dt + B_{K1} dv_K(t) + B_K dy(t) \\ du(t) &= C_K \xi(t)dt + B_{K0} dv_K(t) \end{aligned}$$

Signals: w, u, v, z, y, v_K are semimartingales, e.g.

$$dw(t) = \beta_w(t)dt + d\tilde{w}(t)$$

where $\tilde{w}(t)$ is the noise which is assumed white Gaussian with Ito table

$$d\tilde{w}(t)d\tilde{w}^T(t) = F_{\tilde{w}}dt$$

where $F_{\tilde{w}}$ is non-negative Hermitian.

[As in Belavkin's work]

Idea of Result:

Under some assumptions, then roughly speaking:

James-Nurdin-Petersen, 2006

(i) If the closed loop system regarded as an operator $w \mapsto z$ has gain less than g then there exists solutions X and Y to the algebraic Riccati equations

$$(A - B_2 E_1^{-1} D_{12}^T C_1)^T X + X(A - B_2 E_1^{-1} D_{12}^T C_1) + X(B_1 B_1^T - g^2 B_2 E_1^{-1} B_2')X \\ + g^{-2} C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 = 0;$$

$$(A - B_1 D_{21}^T E_2^{-1} C_2)Y + Y(A - B_1 D_{21}^T E_2^{-1} C_2) + Y(g^{-2} C_1^T C_1 - C_2^T E_2^{-1} C_2)Y \\ + B_1 (I - D_{21}^T E_2^{-1} D_{21}) B_1^T = 0.$$

satisfying stabilizability conditions and XY has spectral radius less than one.

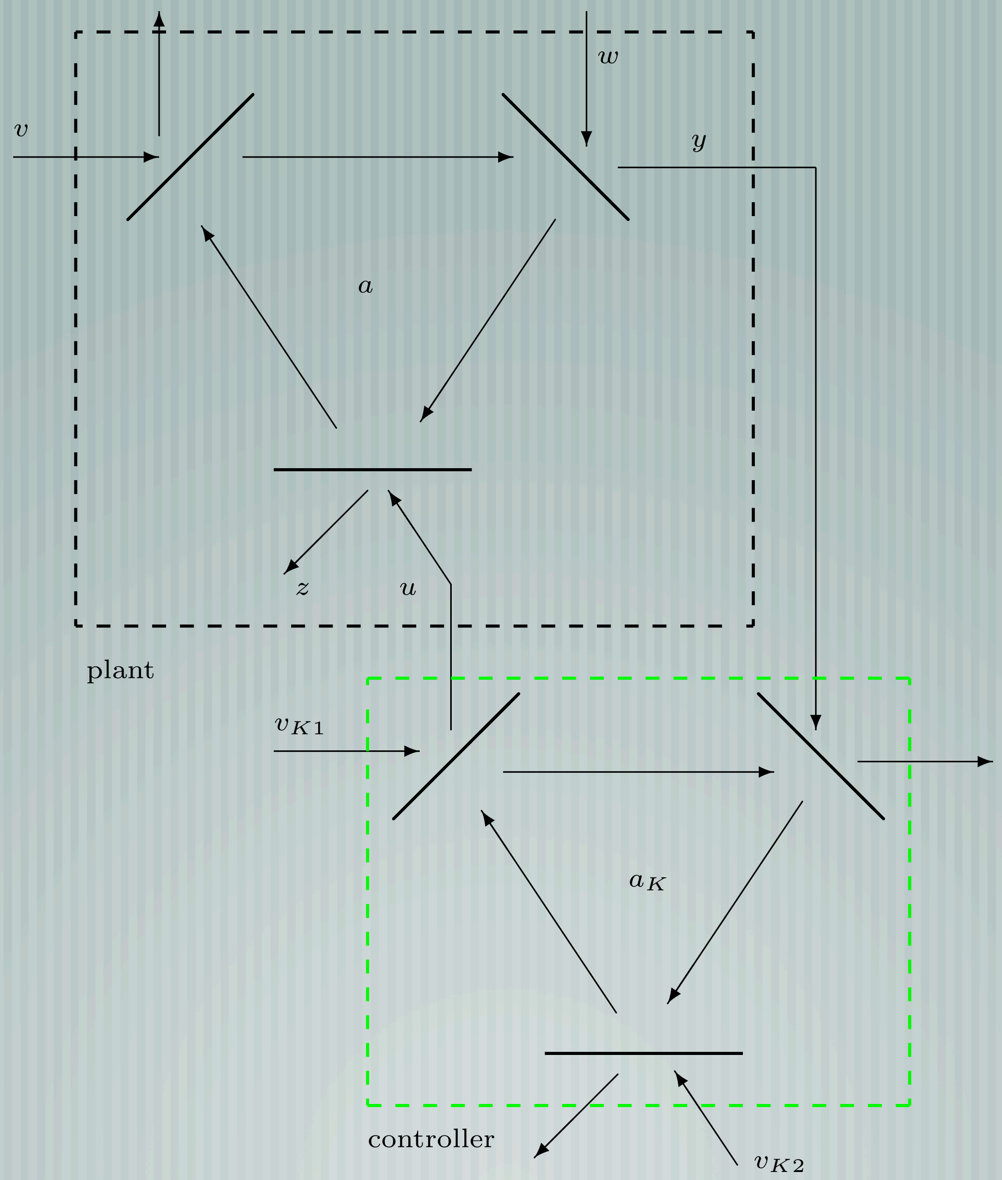
(ii) Conversely, if there exists solutions X, Y of these Riccati equations satisfying stabilizability conditions and XY has spectral radius less than one, then the controller defined by

$$\begin{aligned} A_K &= A + B_2 C_K - B_K C_2 + (B_1 - B_K D_{21}) B_1^T X; \\ B_K &= (I - YX)^{-1} (Y C_2^T + B_1 D_{21}^T) E_2^{-1}; \\ C_K &= -E_1^{-1} (g^2 B_2^T X + D_{12}^T C_1). \end{aligned}$$

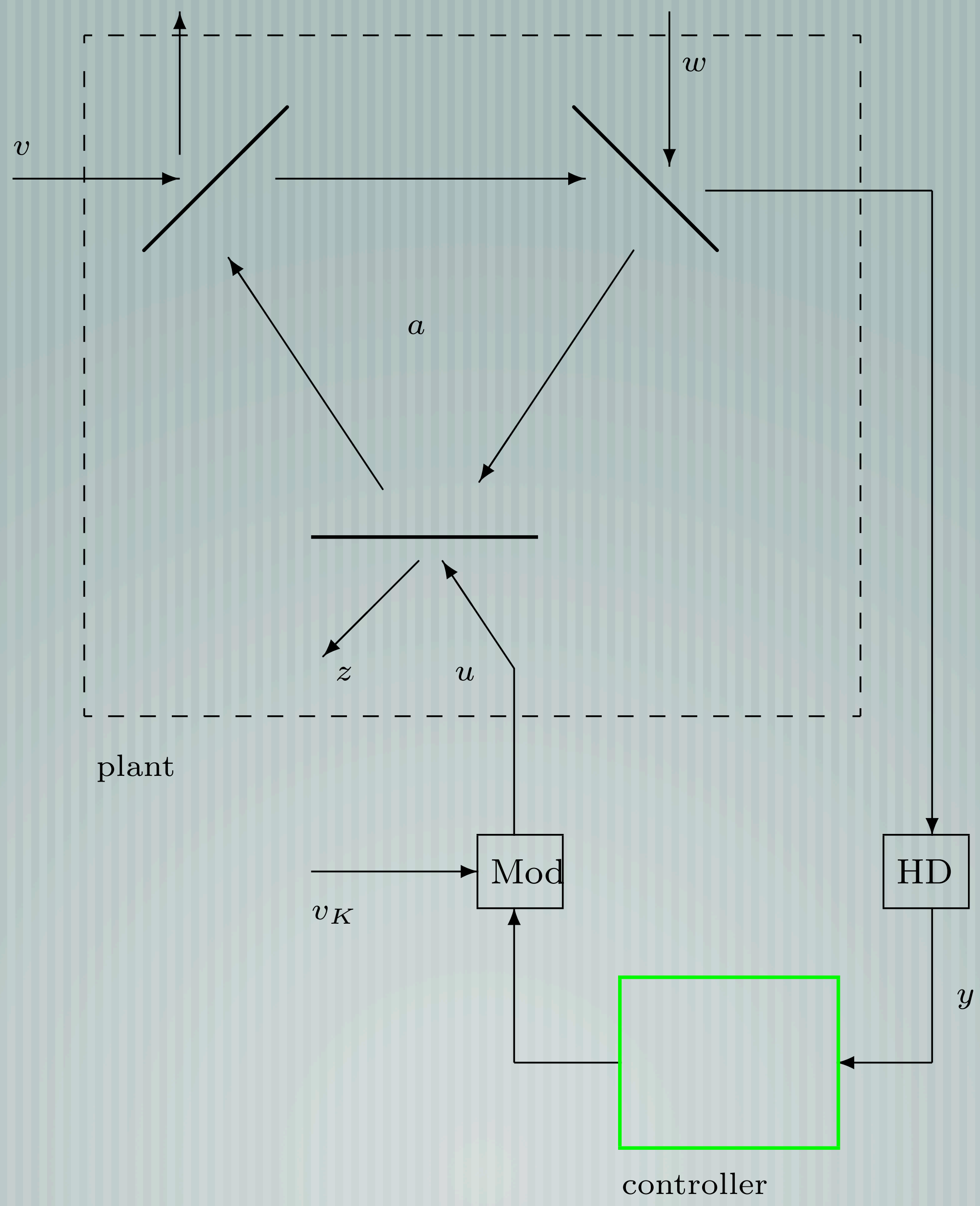
and an arbitrary choice of v_K, B_{K1}, B_{K0}, achieves a closed loop with gain less than g.

Note: Physical realizability may impose conditions on the controller noise terms v_K, B_{K1}, B_{K0}.

Example with Quantum Controller:



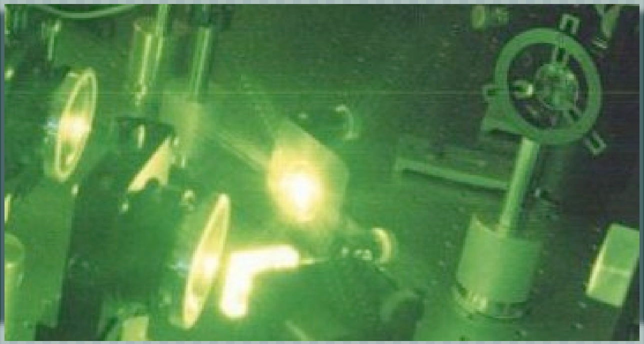
Example with Classical Controller:



Comments:

- These results provide the beginning of a robust control theory for quantum systems.
- Controllers themselves may be quantum or classical.
- It is important to note that the controller may need quantum noise inputs - this broadens the concept of controller, like randomization in classical optimal control.

Discussion



- We have sketched some recent work in quantum control that I have been involved with.
- Slava Belavkin's work was a crucial foundation.
- Quantum control has practical and foundational importance.
- Controllers may themselves be quantum, and may require additional quantum noise
- There is an important and exciting future for "feedback control of quantum systems."