# An Introduction to Quantum Control

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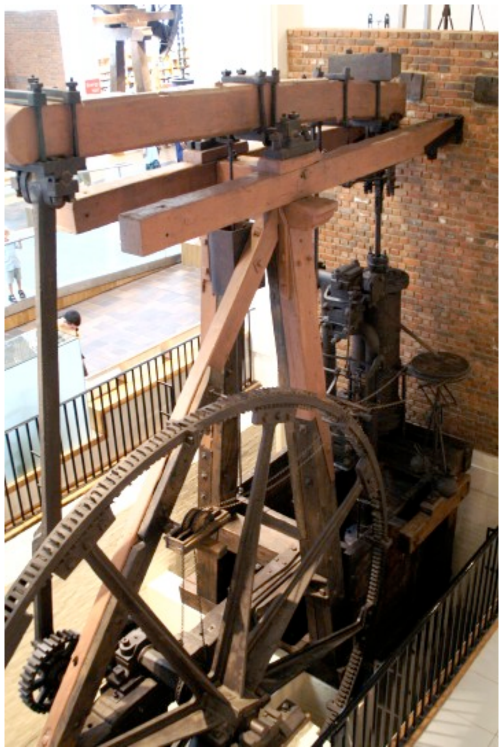
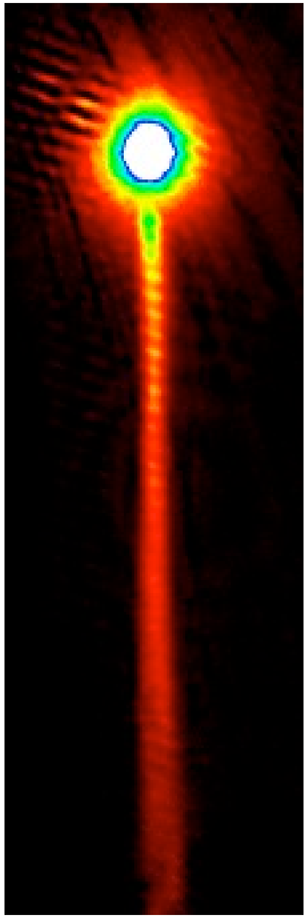
# Outline

- What is “Quantum control” and why?
- Quantum mechanics
- Some of our recent work

# What is "quantum control", and why?



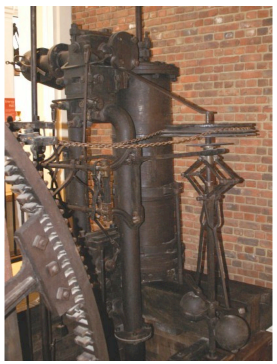
Watt used a governor to control steam engines
- very macroscopic.



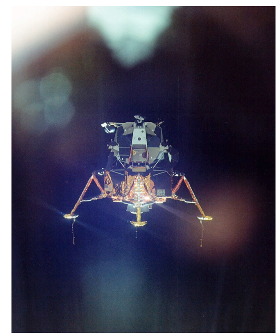
Now we want to control things at the quantum level
- e.g. atoms

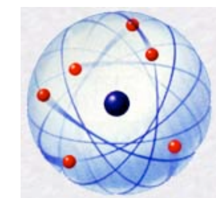
# Control Engineering Timeline

## classical control



## modern control





1788 — 1960 — 2000+

PID, Bode diagrams,
gain and phase margins,
Nyquist stability criteria,
root locus, etc

optimal control, LQG,
Kalman filtering, estimation,
multivariable control, adaptive control,
robust control, nonlinear control,
stochastic control, quantum control,
systems biology, networks, etc

## Quantum Control:

*Control of physical systems whose behaviour is dominated by the laws of quantum mechanics.*

2003: Dowling and Milburn,

*“The development of the general principles of quantum control theory is an essential task for a future quantum technology.”*

## Quantum Technology:

Quantum technology is the application of quantum science to develop new technologies. This was foreshadowed in a famous lecture:

1959:

Richard Feynman, "*Plenty of Room at the Bottom*"

> "What I want to talk about is the problem of manipulating and controlling things on a small scale."

# Key drivers for quantum technology:

- *Miniaturization* - quantum effects can dominate

  Microelectronics

  - feature sizes approaching 20nm within 10 years (Moore's Law)

  Nanotechnology

  - nano electromechanical devices have been made sizing 10's nm

- *Exploitation of quantum resources*

  Quantum Information

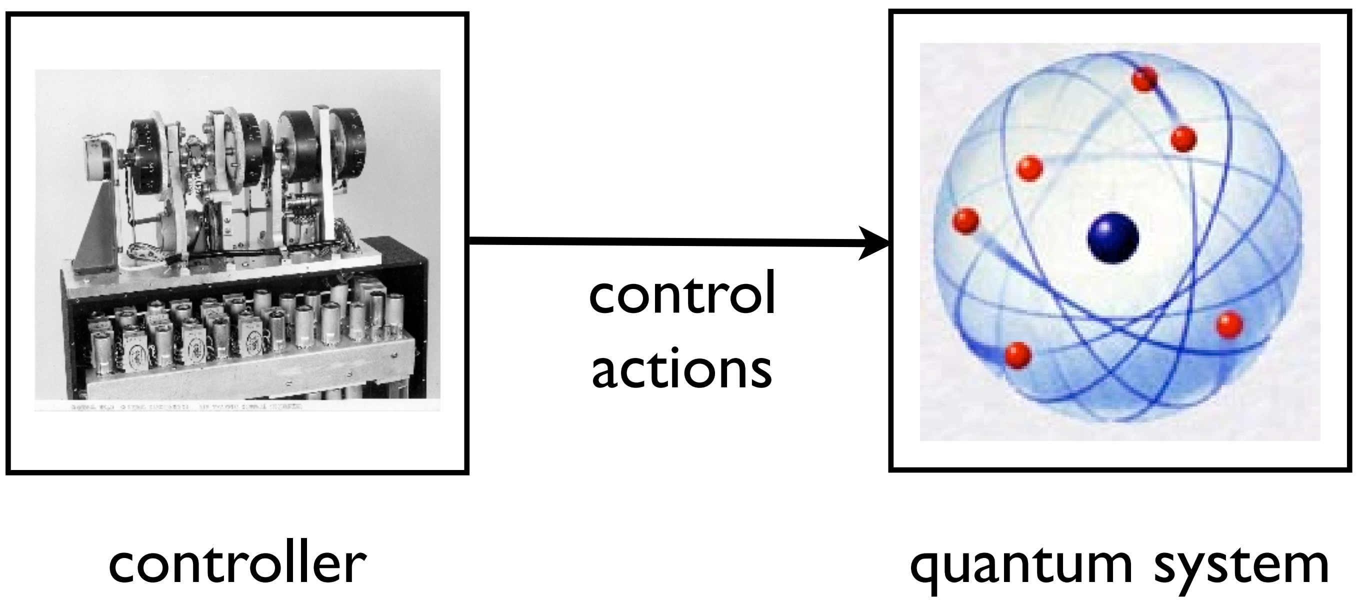
  - (ideally) perfectly secure communications

  Quantum Computing

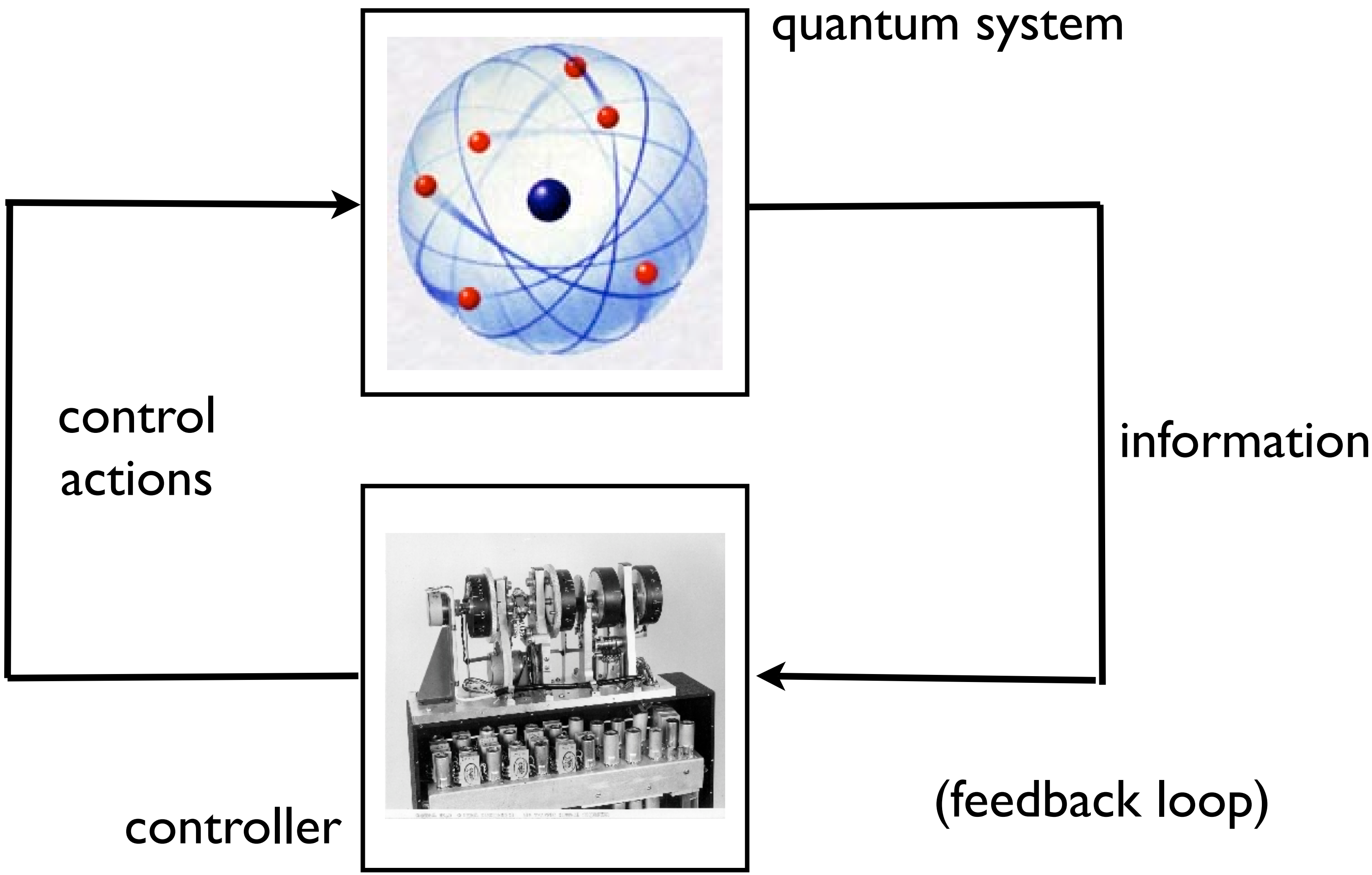
  - algorithms with exponential speed-ups

## Types of Quantum Control:

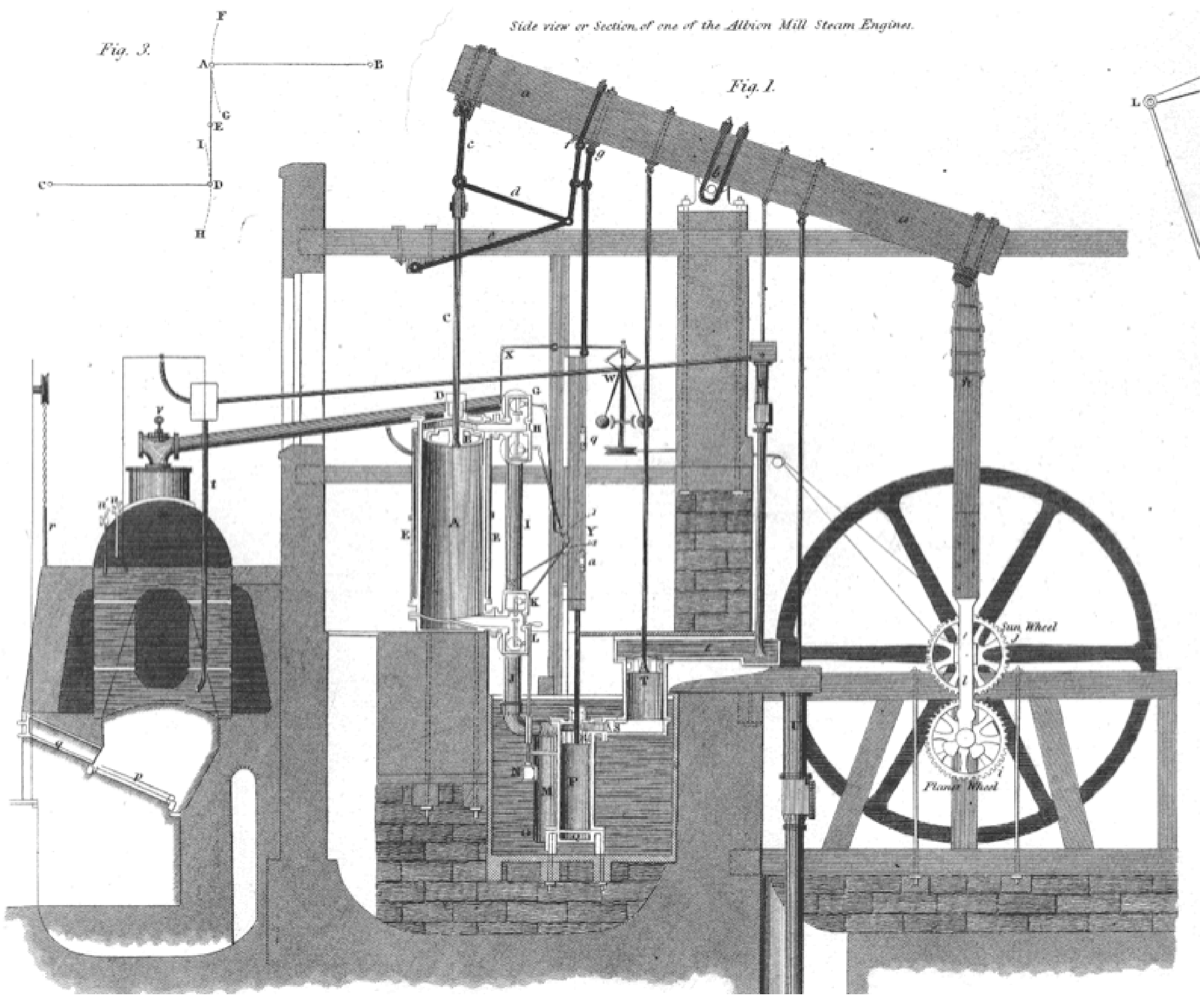
- Open loop - control actions are predetermined, no feedback is involved.



- <u>Closed loop</u> - control actions depend on information gained as the system is operating.



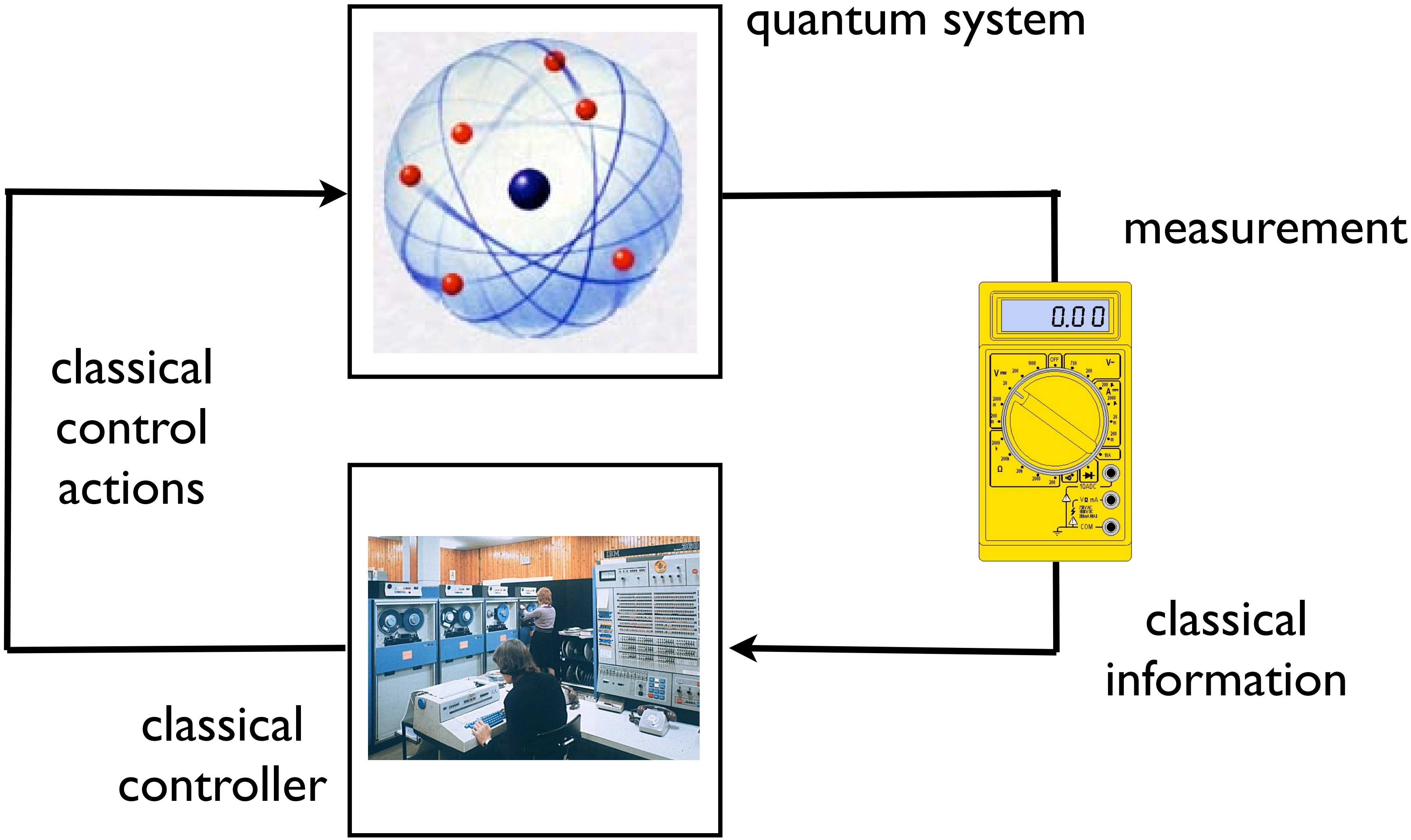
Closed loop means feedback, just as in Watt's steam engines.



# Types of Quantum Feedback:

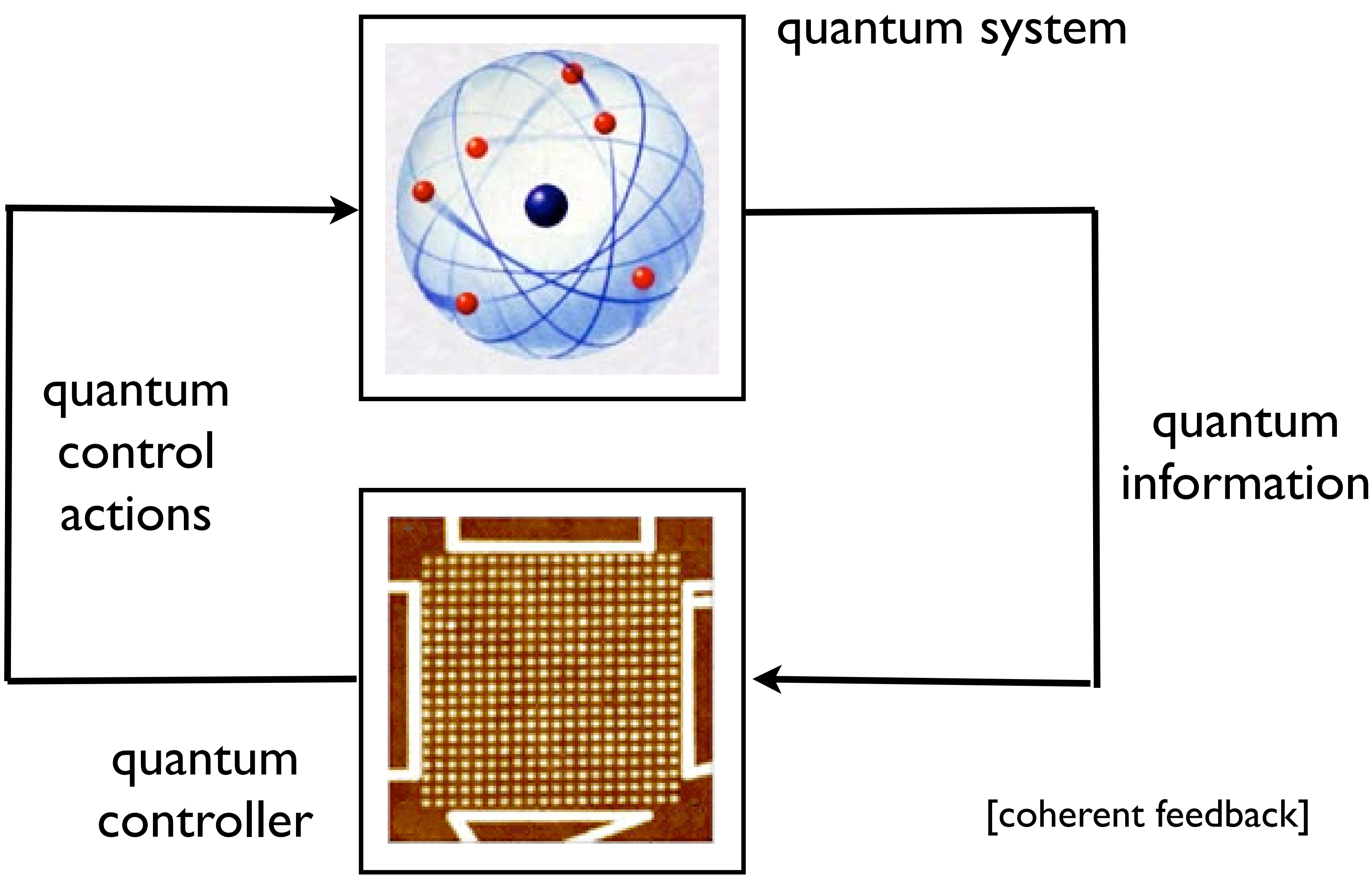
- Using measurement

The classical measurement results are used by the controller (e.g. classical electronics) to provide a classical control signal.



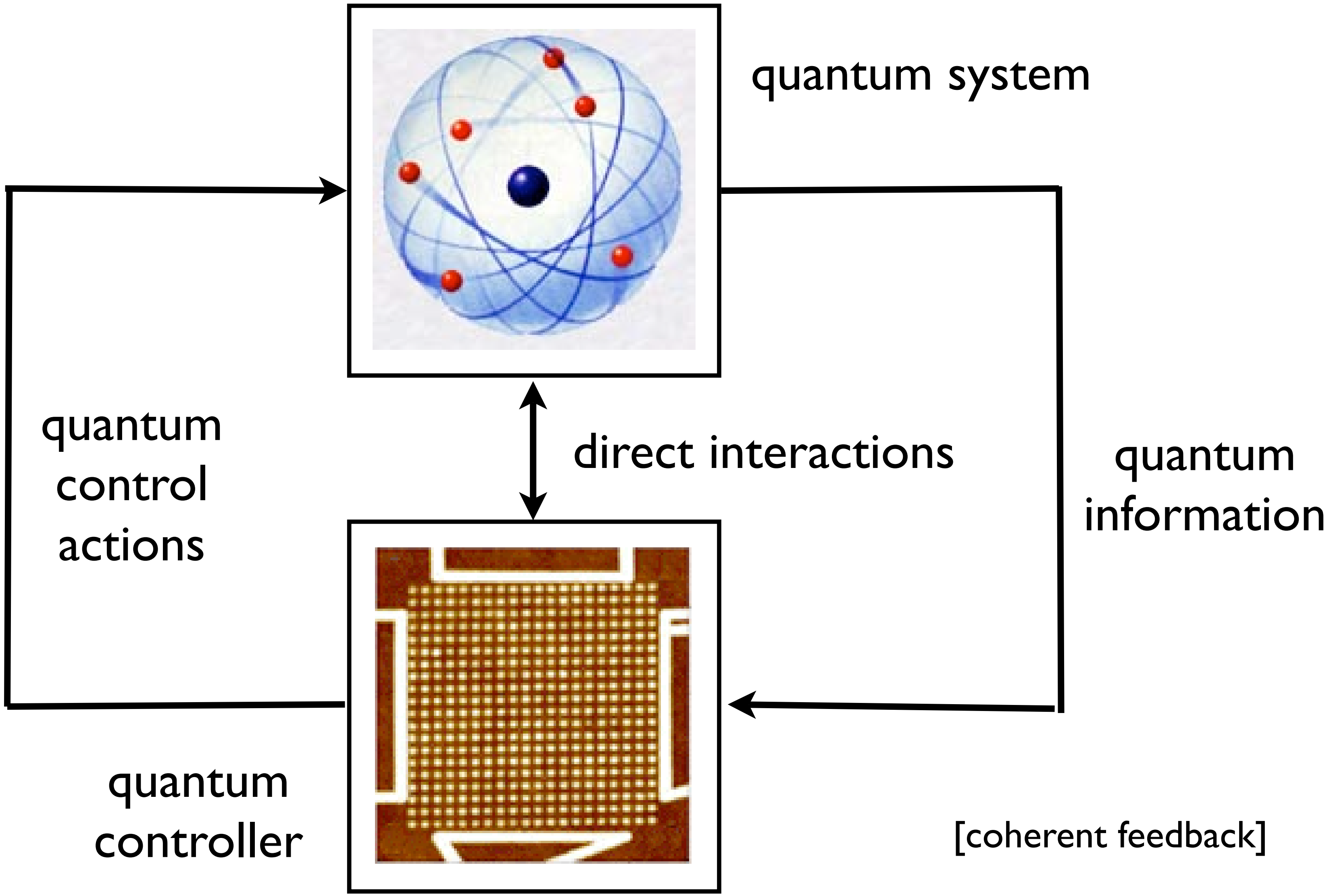
- # Not using measurement

The controller is also a quantum system, and feedback involves a direct flow of quantum information.



- Direct interactions

The controller is also a quantum system, and feedback may also include direct interactions.

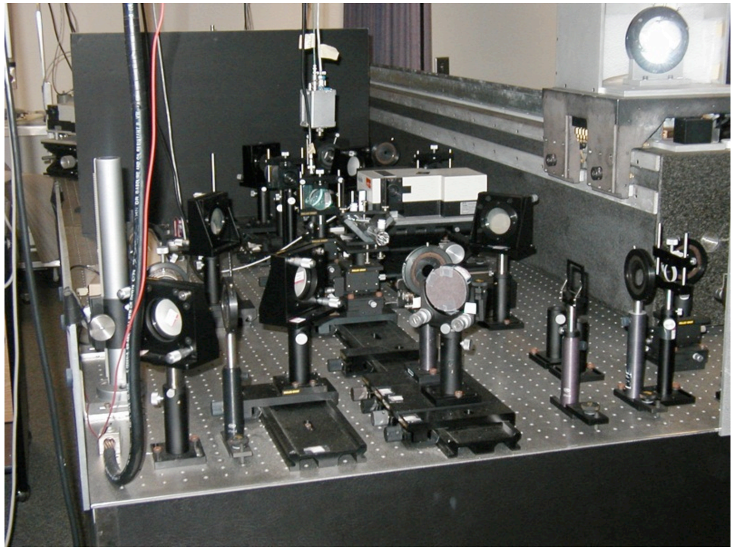



[coherent feedback]

# Why quantum control?

- Ordinary (or classical, or non-quantum) control is based on the laws of classical physics.
- Classical physics is not capable of correctly describing physical behavior at the nano scale and below.
- Quantum technology takes into account and/or exploits quantum behavior.
- Many technical issues (e.g. decoherence).
- Design of complex systems require some form of control.
- There is emerging a need a control theory that can cope with quantum models and potentially exploit quantum resources.
- Quantum technology, therefore, is presenting challenges to control theory.

The study of quantum feedback control has *practical*



and *fundamental* value.

# Quantum Mechanics

non-commuting observables

$$[Q, P] = QP - PQ = i\hbar\, I$$

expectation

$$\langle Q \rangle = \int q |\psi(q,t)|^2 dq$$

Heisenburg uncertainty

$$\Delta Q \Delta P \geq \frac{1}{2} |\langle i[Q,P] \rangle| = \frac{\hbar}{2}$$

Schrodinger equation

$$i\hbar \frac{\partial \psi(q,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(q,t)}{\partial q^2} + V(q)\psi(q,t)$$

quantum probability

$$(\mathscr{N}, \mathbb{P})$$

## Some mathematical preliminaries.

Hilbert space with inner product $\langle \cdot, \cdot \rangle$

We take $\mathbf{H} = \mathbf{C}^n$, $n$-dimensional complex vectors, $\langle \psi, \phi \rangle = \sum_{k=1}^n \psi_k^* \phi_k$.

Vectors are written (Dirac's *kets*)

$$\phi = |\phi\rangle \in \mathbf{H}$$

Dual vectors are called *bras*, $\psi = \langle \psi | \in \mathbf{H}^* \equiv \mathbf{H}$, so that

$$\langle \psi, \phi \rangle = \langle \psi || \phi \rangle = \langle \psi | \phi \rangle$$

Let $\mathscr{B}(\mathbf{H})$ be the Banach space of *bounded operators* $A : \mathbf{H} \to \mathbf{H}$.

For any $A \in \mathscr{B}(\mathbf{H})$ its *adjoint* $A^* \in \mathscr{B}(\mathbf{H})$ is an operator defined by

$$\langle A^* \psi, \phi \rangle = \langle \psi, A\phi \rangle \quad \text{for all } \psi, \phi \in \mathbf{H}.$$

Also define

$$\langle A, B \rangle = \mathrm{Tr}[A^* B], \quad A, B \in \mathscr{B}(\mathbf{H})$$

An operator $A \in \mathscr{B}(\mathbf{H})$ is called *normal* if $AA^* = A^*A$. Two important types of normal operators are *self-adjoint* ($A = A^*$), and *unitary* ($A^* = A^{-1}$).

The *spectral theorem* for a self-adjoint operator $A$ says that (finite dimensional case) it has a finite number of real eigenvalues and that $A$ can be written as

$$A = \sum_{a \in \mathrm{spec}(A)} P_a$$

where $P_a$ is the projection onto the eigenspace corresponding to the eigenvalue $a$ (diagonal representation).

## The Postulates of Quantum Mechanics.

The basic quantum mechanical model is specified in terms of the following:

**Observables**.

Physical quantities like position, momentum, spin, etc., are represented by self-adjoint operators on the Hilbert space $\mathbf{H}$ and are called *observables*. These are the noncommutative counterparts of random variables.

**States**.

A state is meant to provide a summary of the status of a physical system that enables the calculation of statistical quantities associated with observables. A generic state is specified by a *density matrix* $\rho$, which is a self-adjoint operator on $\mathbf{H}$ that is positive $\rho \geq 0$ and normalized $\mathrm{Tr}[\rho] = 1$. This is the noncommutative counterpart of a probability density.

The expectation of an observable $A$ is given by

$$\langle A \rangle = \langle \rho, A \rangle = \mathrm{Tr}[\rho A]$$

Pure states: $\rho = |\psi\rangle\langle\psi|, \quad \psi \in \mathbf{H}$ so that

$$\langle A \rangle = \mathrm{Tr}\left[|\psi\rangle\langle\psi| A\right] = \langle \psi, A\psi \rangle = \langle \psi | A | \psi \rangle$$

## Measurement.

A *measurement* is a physical procedure or experiment that produces numerical results related to observables. In any given measurement, the allowable results take values in the spectrum $\mathrm{spec}(A)$ of a chosen observable $A$.

Given the state $\rho$, the value $a \in \mathrm{spec}(A)$ is observed with probability $\mathrm{Tr}[\rho P_a]$.

## Conditioning.

Suppose that a measurement of $A$ gives rise to the observation $a \in \mathrm{spec}(A)$. Then we must condition the state in order to predict the outcomes of subsequent measurements, by updating the density matrix $\rho$ using

$$\rho \mapsto \rho'[a] = \frac{P_a \rho P_a}{\mathrm{Tr}[\rho P_a]}.$$

This is known as the "projection postulate".

## Evolution.

A *closed* (i.e. isolated) quantum system evolves in a *unitary* fashion: a physical quantity that is described at time $t = 0$ by an observable $A$ is described at time $t > 0$ by [Heisenberg picture]

$$A(t) = U(t)^* A U(t),$$

where $U(t)$ is a unitary operator for each time $t$. The unitary is generated by the *Schrödinger equation*

$$i\hbar \frac{d}{dt} U(t) = H(t) U(t),$$

where the (time dependent) Hamiltonian $H(t)$ is a self-adjoint operator for each $t$.

States evolve according to [Schrodinger picture]

$$\rho(t) = U(t) \rho U^*(t)$$

The two pictures are equivalent (dual):

$$\langle \rho(t), A \rangle = \langle \rho, A(t) \rangle$$

$\langle A, B \rangle = \mathrm{Tr}[A^* B], \quad A, B \in \mathscr{B}(\mathbf{H})$

**Example.**

In a famous experiment, Stern and Gerlach fired some silver atoms through an inhomogeneous magnetic field. They discovered two beams emerging from the magnetic field, confirming the discrete nature of the magnetic moment of atoms. Atoms in the upper beam are said to have "spin up", while those in the lower beam have "spin down". Hence they had measured a physical quantity called *spin*.

Let $\mathbf{H} = \mathbf{C}^2$, and consider the observable

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

representing spin in the $z$-direction.

Measurements of this quantity take values in

$$\mathrm{spec}(\sigma_z) = \{-1, 1\}$$

which correspond to spin down and spin up, respectively. We can write

$$\sigma_z = P_{z,1} - P_{z,-1}$$

[spectral representation]

where

$$P_{z,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_{z,-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

Consider a pure state, given by the vector

$$\psi = \begin{pmatrix} c_1 \\ c_{-1} \end{pmatrix}$$

with $|c_1|^2 + |c_{-1}|^2 = 1$

If we observe $\sigma_z$, we obtain

- the outcome 1 (spin up) with probability $\langle \psi, P_{z,1}\psi \rangle = |c_1|^2$, or
- the outcome $-1$ with probability $\langle \psi, P_{z,-1}\psi \rangle = |c_{-1}|^2$.

## Compatible and incompatible observables.

One of the key differences between classical and quantum mechanics concerns the ability or otherwise to simultaneously measure several physical quantities. In general it is not possible to exactly measure two or more physical quantities with perfect precision if the corresponding observables do not commute, and hence they are *incompatible*.

A consequence of this is lack of commutativity is the famous *Heisenberg uncertainty principle*.

## Quantum probability.

*Definition* (finite dimensional case)

A pair

$$(\mathscr{N}, \mathbb{P}),$$

where $\mathscr{N}$ is a $*$-algebra of operators on a finite-dimensional Hilbert space and $\mathbb{P}$ is a state on $\mathscr{N}$, is called a (finite-dimensional) quantum probability space.

A $*$-algebra $\mathscr{N}$ is a vector space with multiplication and involution (e.g. $\mathscr{B}(\mathbf{H})$).

A linear map $\mathbb{P} : \mathscr{N} \to \mathbf{C}$ that is positive ($\mathbb{P}(A) \geq 0$ if $A \geq 0$) and normalized ($\mathbb{P}(I) = 1$) is called a state on $\mathscr{N}$.

Compare with classical probability:

$$(\Omega, \mathcal{F}, \mathbf{P})$$

**NB:** *$\sigma$-algebras (classical) and $*$-algebras (quantum) describe information*

*Spectral theorem* (finite dimensional case)

Let $\mathscr{C}$ be a commutative $*$-algebra of operators on a Hilbert space, and let $\mathbb{P}$ be a state on $\mathscr{C}$. Then there is a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a map $\iota$ from $\mathscr{C}$ onto the set of measurable functions on $\Omega$ that is a $*$-isomorphism, such that

$$\mathbb{P}[X] = E_{\mathbf{P}}[\iota(X)]$$

Concretely, if $X \in \mathscr{C}$, then [simultaneous diagonalization]

$$X = \sum_{\omega \in \Omega} \iota(X)(\omega) P^{\mathscr{C}}(\omega)$$

and

$$\begin{aligned} \mathbb{P}[X] &= \sum_{\omega \in \Omega} \iota(X)(\omega) \mathbb{P}[P^{\mathscr{C}}(\omega)] \\ &= \sum_{\omega \in \Omega} \iota(X)(\omega) \mathbf{P}(\omega) \\ &= E_{\mathbf{P}}[\iota(X)] \end{aligned}$$

**Example** (spin)

Set $\mathbf{H} = \mathbf{C}^2$ and choose $\mathscr{N} = \mathscr{B}(\mathbf{H}) = M_2$, the $*$-algebra of $2 \times 2$ complex matrices. The pure state is defined by $\mathbb{P}(A) = \langle \psi, A\psi \rangle = \psi^* A \psi$ (recall that $\psi = (c_1 \; c_{-1})^T$ with $|c_1|^2 + |c_{-1}|^2 = 1$).

The observable $\sigma_z$, used to represent spin measurement in the $z$ direction, generates a commutative $*$-subalgebra

$$\mathscr{C}_z \subset \mathscr{N}.$$

Now $\mathscr{C}_z$ is simply the linear span of the events (projections) $P_{z,1}$ and $P_{z,-1}$.

Spectral theorem: gives the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where

$$\Omega = \{1, 2\},$$

$$\mathcal{F} = \{\varnothing, \{1\}, \{2\}, \Omega\},$$

$\mathbf{P}(\{1\}) = |c_1|^2$, etc.,

$$\iota(P_{z,1}) = \chi_{\{1\}}, \quad \iota(P_{z,-1}) = \chi_{\{2\}},$$

and

$$\iota(\sigma_z) : (1, 2) \mapsto (1, -1).$$

The observable

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathscr{N}$$

(corresponding to spin in the $x$-direction) also generates a commutative $*$-subalgebra $\mathscr{C}_x = \mathrm{span}\{P_{x,1}, P_{x,-1}\}$ to which we can apply the spectral theorem.

However, as $\sigma_x$ and $\sigma_z$ do not commute, they cannot be jointly represented on a classical probability space through the spectral theorem. In other words, $\sigma_x$ and $\sigma_z$ are incompatible and their joint statistics are undefined; hence they cannot both be observed in the same realization.

## Conditional expectation

*Definition* (finite dimensional case)

Let $(\mathscr{N}, \mathbb{P})$ be a finite-dimensional quantum probability space and let $\mathscr{C} \subset \mathscr{N}$ be a commutative $*$-subalgebra. Then

$$\mathbb{P}(\cdot|\mathscr{C}) : \mathscr{C}' \to \mathscr{C}$$

is called (a version of) the conditional expectation from $\mathscr{C}'$ onto $\mathscr{C}$ if $\mathbb{P}(\mathbb{P}(B|\mathscr{C})A) = \mathbb{P}(BA)$ for all $A \in \mathscr{C}$, $B \in \mathscr{C}'$.

The set $\mathscr{C}' = \{B \in \mathscr{N} : AB = BA \; \forall A \in \mathscr{C}\}$, is called the *commutant* of $\mathscr{C}$ (in $\mathscr{N}$).

**NB:** In general, conditional expectations from one arbitrary non-commutative algebra to another do not necessarily exist.

But the conditional expectation from the commutant of a commutative algebra onto the commutative algebra does exist. [least squares]

## Example

Consider $\mathbf{H} = \mathbf{C}^3$, $\mathscr{N} = M_3$ and $\mathbb{P}(X) = \langle \psi, X\psi \rangle$ with $\psi = (1\ 1\ 1)^T/\sqrt{3}$. Define $C, B \in \mathscr{N}$ by

$$C = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 5 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let $\mathscr{C}$ be the $*$-algebra generated by $C$. Then [$\mathscr{C}'$ not commutative]

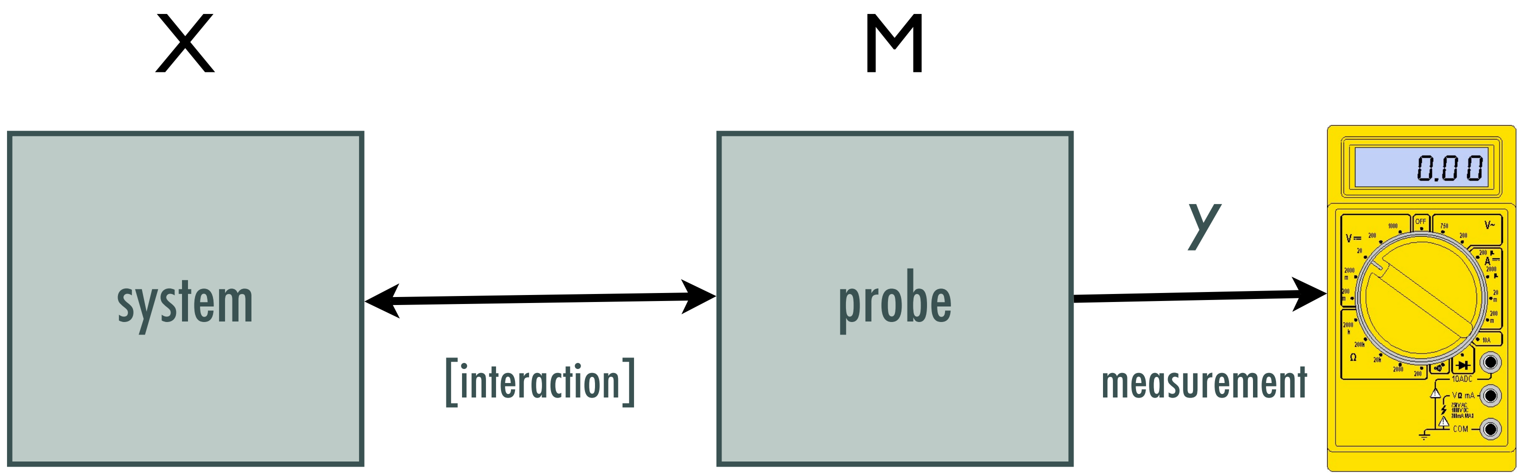
$$\mathscr{C}' = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & x \end{pmatrix} \ :\ a, b, c, d, x \in \mathbf{C} \right\}.$$

and

$$\mathbb{P}(B|\mathscr{C}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathscr{C}.$$

[orthogonal projection]

# Probe model for measurement




System observables $X$ commute with probe observables:

$$[X \otimes I, I \otimes M] = 0$$

The same is true after an interaction:

$$[U^*(X \otimes I)U, U^*(I \otimes M)U] = 0$$

In this way information about the system is transfered to the probe.

Probe measurement generates a commutative algebra

$$\mathscr{Y} = \mathrm{alg}\{U^*(I \otimes M)U\}$$

System observables belong to commutant:

$$U^*(X \otimes I)U \in \mathscr{Y}'$$

Therefore the conditional expectations

$$\mathbb{P}[U^*(X \otimes I)U|\mathscr{Y}]$$

are well defined.

This allows statistical estimation for system observables given measurement data $y$.

The von Neumann "projection postulate" is a special case.

In continuous time, this leads to *quantum filtering*.

# Filtering


**Classical:** Wiener, Kalman, Kushner, Stratonovich, Duncan, Mortensen, Zakai

$$\boxed{\begin{aligned} d\hat{\xi} &= A\hat{\xi}dt + PC^T(dy - C\hat{\xi}dt) \\ \dot{P} &= AP + PA^T - PC^TCP + BB^T \end{aligned}}$$

Kalman filter

**Quantum:**
- general quantum filtering - Belavkin
- generalised measurement theory - Davis,...
- stochastic master equations - Carmichael, ...

$$\boxed{d\rho_t = -\frac{i}{\hbar}[H,\rho_t]dt + \mathcal{D}[L]\rho_t dt + \mathcal{H}[L]\rho_t(dy_t - \mathrm{tr}[(L+L^\dagger)\rho_t]dt)}$$

quantum filter
(stochastic master equation)

# Quantum control

- The quantum probability framework is well-suited to quantum control engineering

- There is a theory of quantum stochastic differential equations

$$dU_t = \Big\{ L dA_t^* - L^* dA_t - \frac{1}{2} L^* L dt - iH(u(t))dt \Big\} U_t, \qquad U_0 = I$$

- The quantum filtering theory facilitates the design of classical controllers for quantum systems

- Not much is known about the design of quantum controllers for quantum systems

# Some of our recent and current work in quantum control

- Optimal quantum feedback control
  - risk-sensitive

[James, 2004, 2005]

$$J^{\mu}(\mathbf{K}) = \mathbb{P}[R^*(T)e^{\mu C_2(T)}R(T)]$$

$$\frac{dR(t)}{dt} = \frac{\mu}{2}C_1(t)R(t), \quad R(0) = I$$

$$\mathbf{K}^{\mu,\star} \ : \ \begin{cases} d\sigma_t^{\mu}(X) & = \sigma_t^{\mu}\big(\mathcal{L}^{\mu,u(t)}(X)\big)dt + \sigma_t^{\mu}(L^*X + XL)dY_t \\ u(t) & = \mathbf{u}^{\mu,\star}(\sigma_t^{\mu}, t) \end{cases}$$

[separation structure]

  - QLEQG

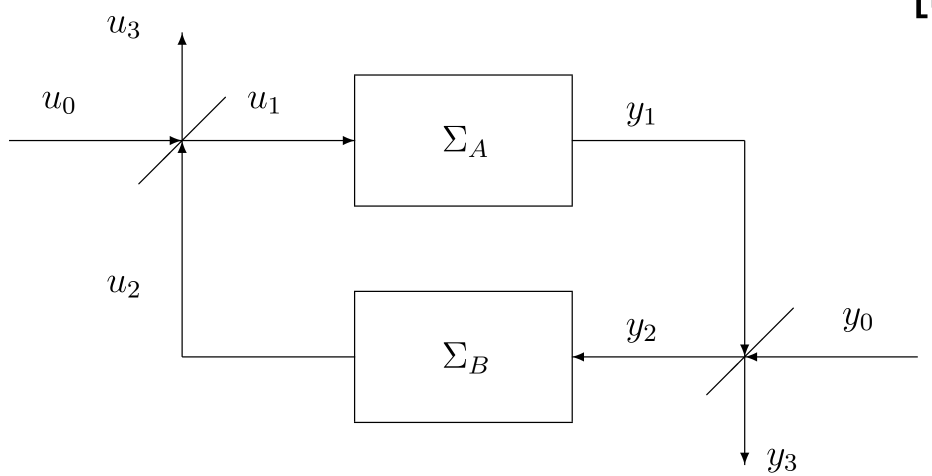
[D'Helon, Doherty, James, Wilson 2006]

[QLEQG $\neq$ CLEQG]

- Quantum networks

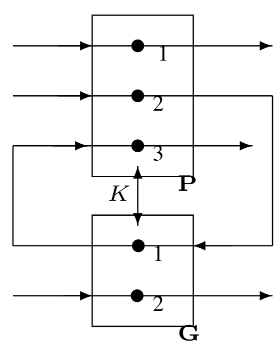
  - small gain for quantum networks

[D'Helon, James, 2006]




  - modelling of networks of quantum and classical components

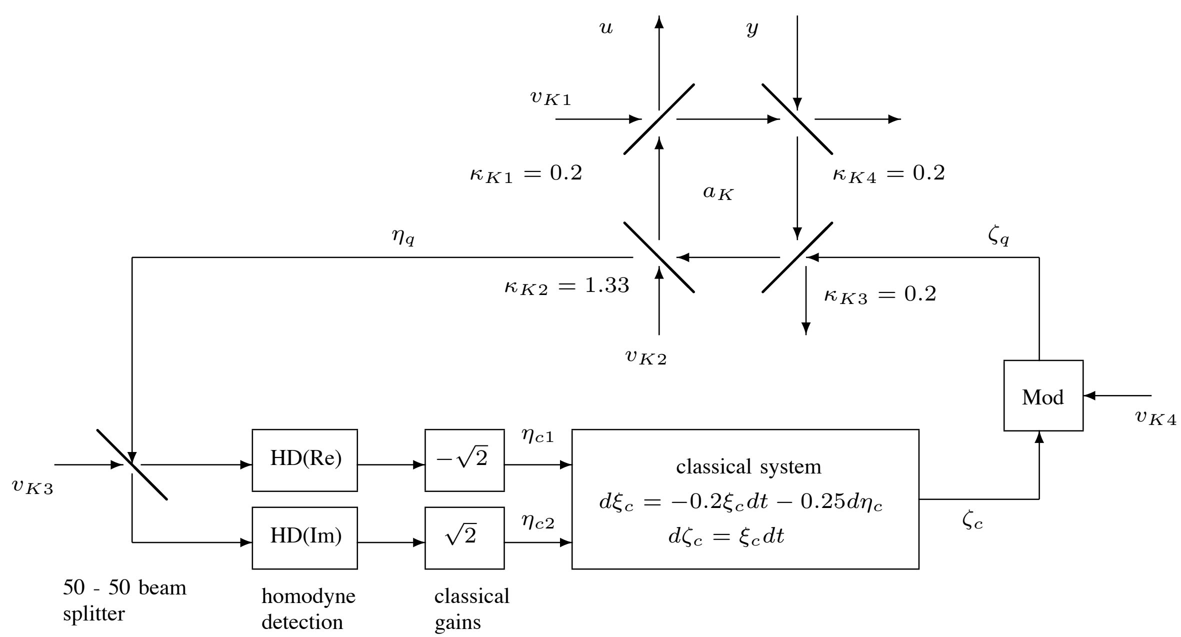
[Gough, James, 2007]



- Robust control

  - H-infinity synthesis with classical, quantum or mixed classical/quantum controllers

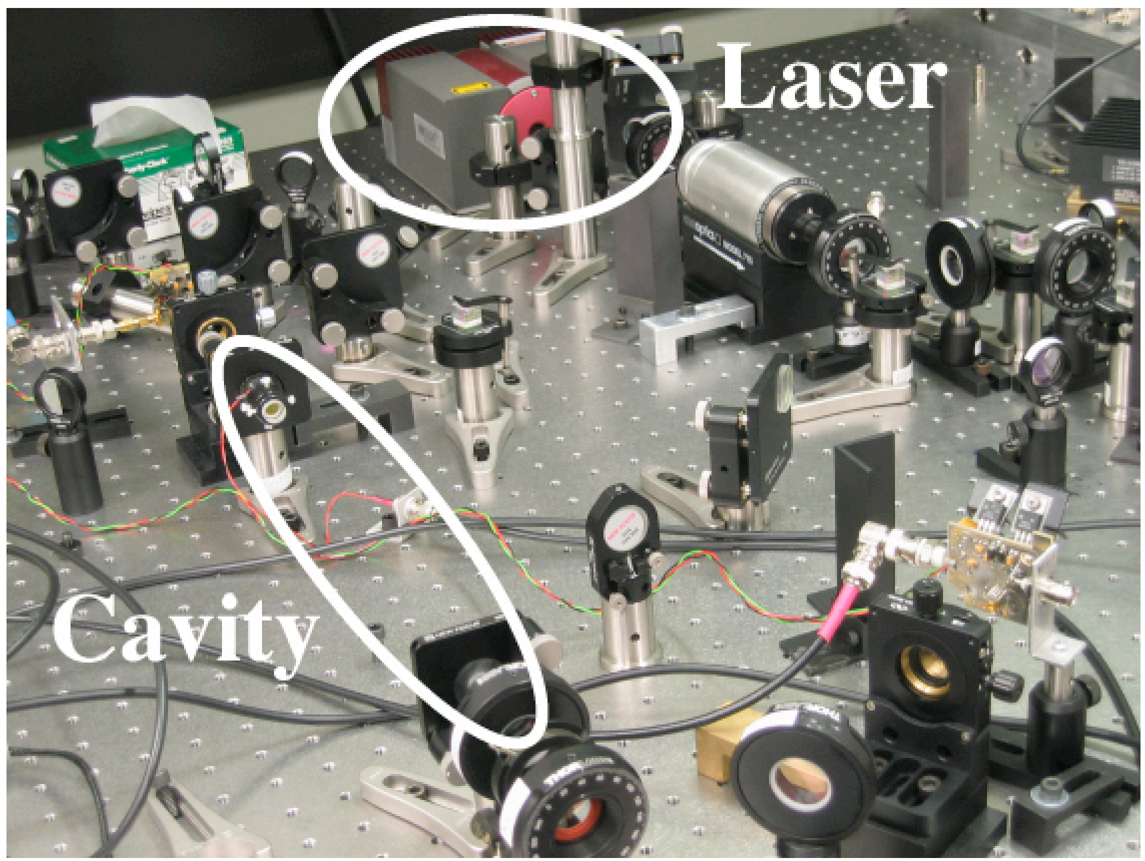
[James, Nurdin, Petersen 2006]




- LQG [Nurdin, James, Petersen; current]

- Laser-cavity locking
  - LQG for experimental quantum optics

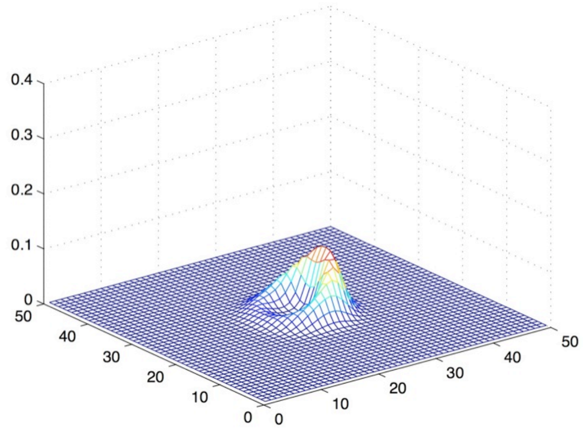
[Huntington, James, Petersen, Sayed Hassen ]



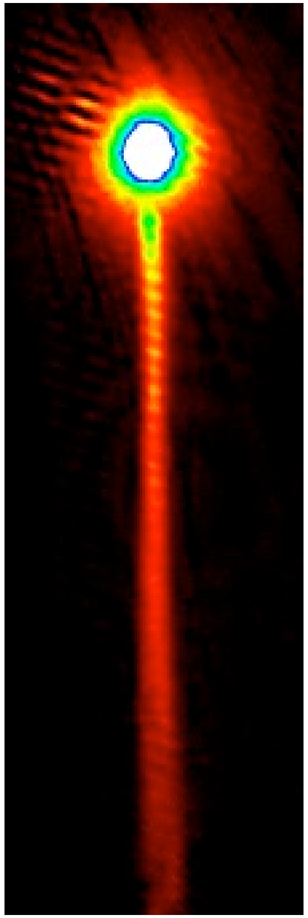
# • BECs and atom lasers

(builds on ANU ACQAO theory and experiment - Hope, Close et al)

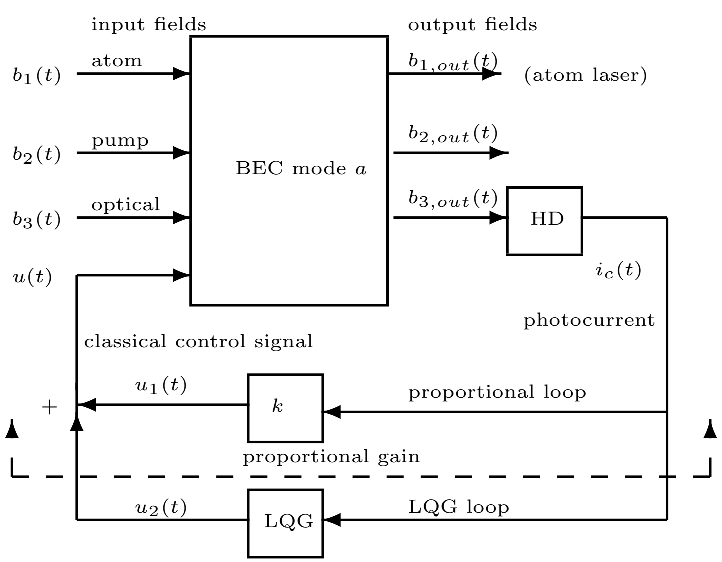
- stabilization via feedback




[Wilson, Carvalho, Hope, James 2007]



- multi-loop feedback control of atom lasers



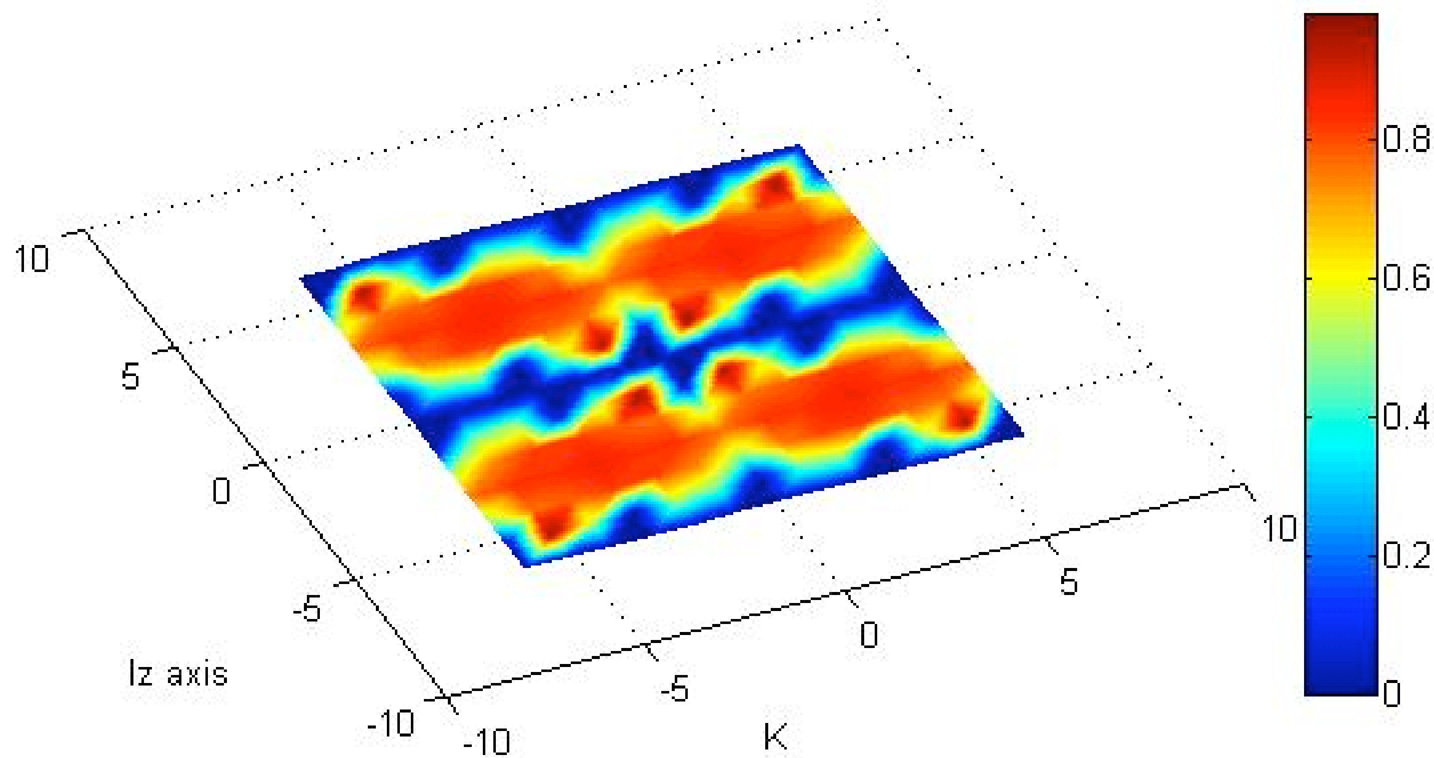

[Yanagisawa, James 2007]

- Optimal quantum open loop control

  - dynamic programming and HJB equations in the Lie group G=SU(2^n)

[Sridharan, James, new work in progress]

  * time optimal control for NMR (builds on Khaneja et al)



# Conclusion

- "Quantum control" is an embryonic, but growing field at the edge of physics, mathematics, and engineering
- Recent experimental and theoretical advances in quantum technology provide strong motivation for quantum control
- Quantum control likely to play a fundamental role in the development of new quantum technologies

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