An Overview of Risk-Sensitive Stochastic Optimal Control

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Workshop:
Stochastic control, communications and numerics perspective
• A brief history
• Approaches to controller design
• Relationships among approaches
• Some limit results (state feedback)
• Robust interpretation
• Partially observed case
• Quantum systems
• References
A brief history


[Linear-Exponential-Quadratic-Gaussian (LEQG)]

Linear system with Gaussian noise:

\[ dx = (Ax + Bu)dt + dw \]

Minimize average of exponential of quadratic cost:

\[
J^\mu = \mathbb{E}_{x,t} \left[ \exp \mu \left( \int_t^T \frac{1}{2} [x'Qx + \frac{1}{2}|u|^2] ds + \frac{1}{2} x'Rx \right) \right]
\]

\( \mu > 0 \) - risk-sensitive

\( \mu < 0 \) - risk-seeking
Optimal state feedback control is

\[ u^*(x, t) = -B' P^\mu_t x \]

where

\[
\begin{aligned}
    \dot{P}^\mu_t &= -P^\mu_t A - A' P^\mu_t + P^\mu_t (BB' - \mu I) P^\mu_t - Q, \\
    P^\mu_T &= R
\end{aligned}
\]

[control type Riccati equation]
Limit $\mu \to 0$: recover standard LQG control (risk-neutral).

Linear system with Gaussian noise:

$$\frac{dx}{dt} = (Ax + Bu)dt + dw$$

Minimize average of quadratic cost:

$$E_{x,t}\left[\int_t^T \frac{1}{2}[x'Qx + \frac{1}{2}|u|^2] \, ds + \frac{1}{2}x'Rx\right]$$

**Solution:** Optimal state feedback control is

$$u^*(x, t) = -B'P_t x$$

where

$$\begin{align*}
\dot{P}_t &= -P_t A - A'P_t + P_t B B'P_t - Q, \\
P_T &= R
\end{align*}$$
partially observed case

Linear system with Gaussian noise:

\[
\begin{align*}
\dot{x} &= (Ax + Bu)dt + dw \\
\dot{y} &= Cxdt + dv
\end{align*}
\]

Minimize average of exponential of quadratic cost

\[
J^\mu = \mathbb{E}_{x,t} \left[ \exp \mu \left( \int_t^T \frac{1}{2} [x'Qx + \frac{1}{2} |u|^2] ds + \frac{1}{2} x'R x \right) \right]
\]

over causal maps \( y(\cdot) \mapsto u(\cdot) \).

Optimal partially observed control is

\[ u_t^* = -B' P_t^\mu \hat{x}_t^\mu \]

where

\[ d\hat{x}_t^\mu = ([A + \mu Y^\mu Q]\hat{x}^\mu + Bu)dt + Y^\mu C'(dy - C\hat{x}_t^\mu dt) \]

and

\[ \dot{Y}^\mu = AY^\mu + Y^\mu A' + \mu Y^\mu QY^\mu + I - Y^\mu C'C Y^\mu \]

Significantly, this is not the Kalman filter.

When \( \mu \to 0 \), LEQG \( \to \) LQG, whose solution is expressed in terms of the Kalman filter.
Some subsequent developments:

• LEQG. Connection with dynamic games. (Jacobson, 1973)
• LEQG. Connection with robust control ($H^\infty$). (Glover-Doyle, 1988)
• Robustness properties of risk-sensitive criterion. Also stochastic IQC. (James, Fleming, Boel, Dupuis, Petersen, 1995, 1998)
• Risk-sensitive control for finance. (Nagai 2000)
• Risk-sensitive control for quantum systems. (James 2004)
The basic aim of *robust control* is:

Design a controller to achieve "satisfactory performance" in the presence of disturbances and uncertainty.

Stochastic control has similar aims.
Dynamics

\[
\begin{aligned}
\dot{x}_s &= f(x_s) + g(x_s)d_s, \quad 0 < s < T, \\
z_s &= h(x_s)
\end{aligned}
\]

\[f(0) = 0, \quad h(0) = 0, \quad x_0 = 0\]

Disturbance—Output Map

\[T_{d, z} : d(\cdot) \mapsto z(\cdot)\]

\[H_\infty \text{ Norm}\]

\[\| T_{d, z} \|_\infty \triangleq \sup_{d \in L_2[0, T]} \frac{\| z \|_2}{\| d \|_2}\]

Note:

\[\| T_{d, z} \|_\infty \leq \gamma \iff \int_0^T |z(t)|^2 dt \leq \int_0^T \gamma^2 |d(t)|^2 dt \quad \forall d \in L_2[0, T]\]
D-H$_2$  

_**deterministic optimal control**_

**disturbance:** none

**controller:** gives optimal performance without special consideration of disturbances.

Optimal cost

\[
W(x,t) = \inf_u \left[ \int_t^T L(x_s, u_s) \, ds + \Phi(x_T) \right]
\]

Dynamics

\[
\begin{cases}
\dot{x}_s = b(x_s, u_s), & t < s < T, \\
x_t = x
\end{cases}
\]
S-$H_2$ (RN)

**stochastic optimal control**

**disturbance:** noise

**controller:** gives optimal AVERAGE performance

Optimal cost

$$W^\varepsilon(x, t) = \inf_u \mathbb{E}_{x,t} \left[ \int_t^T L(x_s^\varepsilon, u_s) \, ds + \Phi(x_T) \right]$$

**Dynamics**

\[
\begin{align*}
\left\{
\begin{array}{l}
 dx^\varepsilon_s &= b(x^\varepsilon_s, u_s) \, ds + \sqrt{\varepsilon} \, dB_s, \quad t < s < T, \\
x^\varepsilon_t &= x
\end{array}
\right.
\]

($\varepsilon > 0$ - noise intensity)
stochastic risk-sensitive optimal control

**disturbance:** noise

**controller:** gives optimal AVERAGE performance using exponential cost

(heavily penalizes large values)

**Optimal cost**

\[
S^{\mu,\varepsilon}(x, t) = \inf_u \mathbb{E}_{x,t} \left[ \exp \frac{\mu}{\varepsilon} \left( \int_t^T L(x_{s}^{\varepsilon}, u_s) \, ds + \Phi(x_T^{\varepsilon}) \right) \right]
\]

**Dynamics**

\[
\begin{cases}
    dx_s^{\varepsilon} = b(x_s^{\varepsilon}, u_s) \, ds + \sqrt{\varepsilon} \, dB_s, & t < s < T, \\
    x_t^{\varepsilon} = x
\end{cases}
\]

(\(\mu > 0 - \text{risk sensitivity}\))
$H_\infty$ robust control

**disturbance:** $L_2$ signal

**controller:** achieves specified $H_\infty$ norm constraint

(worst-case design)

Find $u(\cdot)$ such that

$$\| T_{d, z} \|_\infty \leq \gamma$$

**Dynamics**

\[
\begin{aligned}
\dot{x}_s &= b(x_s, u_s) + d_s \quad t < s < T \\
x_t &= x \\
z_s &= h(x_s)
\end{aligned}
\]

($\gamma = 1/\sqrt{\mu} - H_\infty$ constraint)
D-DG

deterministic differential game

disturbance: \( L_2 \) signal

gives optimal performance subject to opposing efforts of disturbance (worst-case design)

Optimal cost

\[
W^\mu(x, t) = \sup_{d} \inf_{u} \left[ \int_{t}^{T} L(x_s, u_s) - \frac{1}{2\mu} |d_s|^2 \, ds + \Phi(x_T) \right]
\]

Dynamics

\[
\begin{align*}
\dot{x}_s &= b(x_s, u_s) + d_s, \quad t < s < T \\
x_t &= x
\end{align*}
\]
S-DG

**disturbances:** noise

$L_2$ signal

**controller:** gives optimal AVERAGE performance

subject to opposing efforts of disturbance (worst-case design)

Optimal cost

\[
W^{\mu,\varepsilon}(x, t) = \sup_d \inf_u \mathbb{E}_{x,t} \left[ \int_t^T L(x_s^{\varepsilon}, u_s) - \frac{1}{2\mu} |d_s|^2 \, ds + \Phi(x_T^{\varepsilon}) \right]
\]

Dynamics

\[
\begin{cases}
    dx_s^{\varepsilon} &= (b(x_s^{\varepsilon}, u_s) + d_s) \, ds + \sqrt{\varepsilon} \, dB_s, \quad t < s < T \\
x_t^{\varepsilon} &= x
\end{cases}
\]
RS, S-DG, D-DG, and S-H$_2$ are "perturbations" of D-H$_2$. 
\[ \text{RS} \equiv \text{S-DG} \]
\[ \equiv \text{D-}H_2 \]
\[ + \text{ (noise variance term)} \]
\[ + \text{ (disturbance energy term)} \]

Equivalence via
- logarithmic transformation,
- dynamic programming PDEs, and
- representation theorem

Expansion valid for
- small noise intensity and
- small risk-sensitivity
\[ S - H_2 \equiv D - H_2 + \text{(noise variance term)} \]

\[ H_\infty \equiv D - H_2 + \text{(disturbance energy term)} \]

\[ RS \equiv H_2 + H_\infty \]
Some limit results (state feedback)

Assumptions

- \( b(x, u) = f(x) + g(x)u \), where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( g : \mathbb{R}^m \rightarrow \mathbb{R}^n \) and their first order derivatives \( Df \), \( Dg \) are bounded and uniformly Lipschitz continuous.

- Control values: \( U \subset \mathbb{R}^m \) is compact and convex.

- Disturbance values: \( D = \mathbb{R}^n \).

- \( \Phi : \mathbb{R}^n \rightarrow \mathbb{R} \) is non-negative; \( \Phi \) and \( D\Phi \) are bounded and uniformly Lipschitz continuous.
• \( L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \) is \( C^2 \), non-negative; \( L(\cdot, u) \) and \( D_x L(\cdot, u) \) are bounded and uniformly Lipschitz continuous, uniformly in \( u \in U \); and \( D_u^2 L(x, \cdot) > 0 \), uniformly in \( x \in \mathbb{R}^n \).

• There exists locally Lipschitz \( U^*(x, p) \) achieving maximum in

\[
\min_{u \in U} [p \cdot b(x, u) + L(x, u)]
\]

• Use Elliott–Kalton formulation for two–player zero–sum differential games (strategies, upper and lower values, etc)

• For later use, assume also
  
  \( - \quad L(x, u) = \frac{1}{2}|u|^2 + V(x) \), and the functions \( f, g, V \) and \( \Phi \) are of class \( C^\infty \) and have compact support.
  
  \( - \quad U^*(x, p) \) is of class \( C^2 \).  

Viscosity Solutions

Viscosity solutions introduced by Crandall-Lions 1983.

The definition for a fully nonlinear parabolic PDE

\[
\begin{align*}
\frac{\partial v}{\partial t} + F(x, Dv, D^2v) &= 0 \text{ in } \mathbb{R}^n \times (0, T) \\
v(x, T) &= \Phi(x) \text{ in } \mathbb{R}^n.
\end{align*}
\]

is as follows.
An upper semicontinuous (u.s.c.) function $\bar{v}$ (resp. l.s.c. function $v$) is called a \textit{viscosity subsolution} (resp. \textit{supersolution}) if

$$\bar{v} \leq \Phi \quad (\text{resp. } v \geq \Phi) \text{ on } \mathbb{R}^n \times \{T\}$$

and

$$\frac{\partial}{\partial t} \phi(x, t) + F(x, D\phi(x, t), D^2\phi(x, t)) \geq 0 \quad (\text{resp. } \leq 0)$$

for every smooth function $\phi$ and any local maximum (resp. minimum) $(x, t) \in \mathbb{R}^n \times [0, T)$ of $\bar{v} - \phi$ (resp. $v - \phi$).

A continuous function is called a \textit{viscosity solution} if it is both a subsolution and a supersolution.
Comparison Theorem

If \( \bar{v} \) is a subsolution and if \( v \) is a supersolution, then

\[
\bar{v} \leq v \text{ in } \mathbb{R}^n \times [0, T].
\]

Thus any continuous viscosity solution is unique.
Dynamic Programming PDEs

RS

\[
\frac{\partial S^{\mu,\varepsilon}}{\partial t} + \frac{\varepsilon}{2} \Delta S^{\mu,\varepsilon} + \min_{u \in U} [DS^{\mu,\varepsilon} \cdot b(x, u) + \frac{\mu}{\varepsilon} L(x, u) S^{\mu,\varepsilon}] = 0 \text{ in } \mathbb{R}^n \times (0, T)
\]

\[S^{\mu,\varepsilon}(x, T) = \exp \frac{\mu}{\varepsilon} \Phi(x) \text{ in } \mathbb{R}^n\]

Logarithmic transformation: (Fleming, 1970’s)

\[
W^{\mu,\varepsilon}(x, t) = \frac{\varepsilon}{\mu} \log S^{\mu,\varepsilon}(x, t)
\]
\[ \frac{\partial W^{\mu,\varepsilon}}{\partial t} + \frac{\varepsilon}{2} \Delta W^{\mu,\varepsilon} + \min_{u \in U} \max_{d \in D} [DW^{\mu,\varepsilon} \cdot (b(x, u) + d) + L(x, u) - \frac{1}{2\mu} |d|^2] = 0 \text{ in } \mathbb{R}^n \times (0, T) \]

\[ W^{\mu,\varepsilon}(x, T) = \Phi(x) \text{ in } \mathbb{R}^n. \]

Note:

\[ \min_{u \in U} \max_{d \in D} \left[ p \cdot (b(x, u) + d) + L(x, u) - \frac{1}{2\mu} |d|^2 \right] = \min_{u \in U} [p \cdot b(x, u) + L(x, u)] + \frac{1}{2} \mu |p|^2 \]
Representation theorem

Viscosity formulation of stochastic differential games developed by


**Theorem** (James, 1992; Fleming-McEneany 1992) *The function $W^{\mu,\varepsilon}$ defined by the logarithmic transformation is the optimal cost function of the stochastic differential game defined above.*

\[ i.e. \ RS \equiv \ S-DG \]

**Proof:**  
- Methods of Fleming (1971) imply
  \[ |DW^{\mu,\varepsilon}(x,t)| \leq C^* \text{ for all } (x,t) \in \mathbb{R}^n \times [0,T]. \]

- In view of this bound, the set of disturbance values $D$ can be taken as bounded.

- The result now follows from Fleming–Souganidis (1989). \(\square\)
Limit PDEs

Obtained as $\varepsilon \downarrow 0$ and/or $\mu \downarrow 0$.

\textbf{D-DG}

\begin{align*}
\frac{\partial W^\mu}{\partial t} + \min_{u \in U} \max_{d \in D} [DW^\mu \cdot (b(x, u) + d)] \\ + L(x, u) - \frac{1}{2\mu} |d|^2 \end{align*}
\begin{align*}
= 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T) \\
W^\mu(x, T) = \Phi(x) \quad \text{in} \quad \mathbb{R}^n.
\end{align*}
\[ S-H_2 \quad (\varepsilon > 0, \; \mu \downarrow 0) \]

\[ \frac{\partial W^\varepsilon}{\partial t} + \frac{\varepsilon}{2} \Delta W^\varepsilon + \min_{u \in U} [D W^\varepsilon \cdot b(x, u) + L(x, u)] = 0 \text{ in } \mathbb{R}^n \times (0, T) \]

\[ W^\varepsilon(x, T) = \Phi(x) \text{ in } \mathbb{R}^n. \]

\[ D-H_2 \quad (\varepsilon \downarrow 0, \; \mu \downarrow 0) \]

\[ \frac{\partial W}{\partial t} + \min_{u \in U} [D W \cdot b(x, u) + L(x, u)] = 0 \text{ in } \mathbb{R}^n \times (0, T) \]

\[ W(x, T) = \Phi(x) \text{ in } \mathbb{R}^n. \]
Limit theorems depend on general viscosity limit techniques developed by


**Theorem** (James, 1992) We have

\[
\lim_{\varepsilon \downarrow 0} W^{\mu;\varepsilon}(x, t) = W^{\mu}(x, t)
\]

uniformly on compact subsets, where \( W^{\mu} \in C(\mathbb{R}^n \times [0, T]) \) is the unique bounded continuous viscosity solution of the corresponding PDE and is the optimal cost function of the deterministic differential game defined above.

i.e. \( \text{S-DG} \rightarrow \text{D-DG} \) as \( \varepsilon \downarrow 0 \).
\textbf{Proof:} \quad \bullet \text{ Assumptions imply } \\
0 \leq W^{\mu, \varepsilon}(x, t) \leq C \text{ for all } (x, t) \in \mathbb{R}^n \times [0, T], \ \mu, \varepsilon > 0 \\

\bullet \text{ The function } \\
\bar{v}(x, t) = \limsup_{\varepsilon \downarrow 0, \ y \to x, \ s \to t} W^{\mu, \varepsilon}(y, s). \\
\text{is u.s.c. and a viscosity subsolution.} \\

\bullet \text{ The function } \\
\underline{v}(x, t) = \liminf_{\varepsilon \downarrow 0, \ y \to x, \ s \to t} W^{\mu, \varepsilon}(y, s) \\
\text{is l.s.c and a viscosity supersolution.} \\

\bullet \text{ By construction} \\
\underline{v} \leq \bar{v} \\

\bullet \text{ By the } \textit{comparison theorem} \\
\bar{v} \leq \underline{v}
• Therefore

\[
\lim_{\varepsilon \downarrow 0} W^{\mu,\varepsilon}(x, t) = \bar{v} = v \triangleq v
\]

• From *Evans–Souganidis* (1984) the value \( W^\mu \) of the D-DG is the unique viscosity solution, and hence

\[
W^\mu = v.
\]

---

Expansion of optimal cost for S-DG

\[
W^{a\varepsilon, b\varepsilon}(x, t) = W(x, t) + \varepsilon (W_g(x, t), W_n(x, t)) \cdot \begin{pmatrix} a \\ b \end{pmatrix} + \ldots
\]

[methods of Fleming, 1970's]
Robust interpretation

(Boel, Dupuis, James, Petersen 1998, 2000)

We wish to interpret the finiteness of the risk-sensitive criterion for a nominal system $G_0$ in terms of a performance bound for a perturbed system $G$.

\[ w \rightarrow G_0 \rightarrow Z \]

*nominal system*
Nominal system dynamics:

\[ G_0 : \begin{cases} 
  dX_t &= \bar{b}(X_t)dt + \varepsilon \frac{1}{2} \sigma(X_t)dw_t \\
  Z_t &= h(X_t) 
\end{cases} \]

Risk-sensitive criterion:

\[ S(x, T) \doteq E_x \left[ \exp \frac{1}{2\gamma^2 \varepsilon} \int_0^T |Z_t|^2 dt \right] \]

\[ \gamma^2 = \frac{1}{\mu} \]
Key tool is the representation [1997]

\[
\gamma^2 \varepsilon \log S(x, T) = \sup_{v \in \mathcal{V}_T} E_x \left[ \frac{1}{2} \int_0^T (|\bar{Z}_t|^2 - \gamma^2 |v_t|^2) \, dt \right]
\]

where \( \bar{X}_t \) is given by perturbed system dynamics:

\[
G : \begin{cases} 
    d\bar{X}_t = \bar{b}(\bar{X}_t) dt + \sigma(\bar{X}_t)v_t dt + \frac{\varepsilon}{2} \sigma(\bar{X}_t) dw_t \\
    \bar{Z}_t = h(\bar{X}_t)
\end{cases}
\]
perturbed system

\[ \begin{align*}
E_x \left[ \frac{1}{2} \int_0^T |\bar{Z}_t|^2 dt \right] & \leq \gamma^2 E_x \left[ \frac{1}{2} \int_0^T |v_t|^2 dt \right] + \gamma^2 \varepsilon \log S(x, T) \\
\text{(analogous to } H^\infty \text{ inequality)}
\end{align*} \]
Partially observed (output feedback) case

(James-Baras-Elliott, 1994)

Dynamics

\[
\begin{aligned}
\dot{x}_t &= b(x_t, u_t) \, dt + \sqrt{\varepsilon} \, dw_t \\
\dot{\tilde{y}}_t &= h(x_t) \, dt + \sqrt{\varepsilon} \, dv_t
\end{aligned}
\]

Objective

\[
J(u) = \mathbb{E} \left[\exp \frac{\mu}{\varepsilon} \left( \int_0^T L(x_t, u_t) \, dt + \Phi(x_T) \right) \right]
\]

\[\gamma^2 = 1/\mu\]
solution of RS

Information state \( \sigma_{t}^{\mu, \varepsilon}(x) \)

\[
\langle \sigma_{t}^{\mu, \varepsilon}, \eta \rangle = E^{0} \left[ \eta(x_{t}^{\varepsilon})Z_{t}^{\varepsilon} \exp \left( \frac{\mu}{\varepsilon} \int_{0}^{t} L(x_{s}^{\varepsilon}, u_{s}) \, ds \right) \mid Y_{t} \right]
\]

for all \( \eta \)

Dynamics

\[
\left\{ \begin{array}{l}
d\sigma_{t}^{\mu, \varepsilon} = (A^{u_{t}} + \frac{\mu}{\varepsilon} L^{u_{t}}) \sigma_{t}^{\mu, \varepsilon} \, dt + \frac{1}{\varepsilon} h\sigma_{t}^{\mu, \varepsilon} \, d\tilde{y}_{t}^{\varepsilon}, \\
\sigma_{0}^{\mu, \varepsilon} = \rho \quad \text{in} \quad L_{1}(\mathbb{R}^{n}),
\end{array} \right.
\]

Representation

\[
J(u) = E^{0} \left[ \langle \sigma_{T}^{\mu, \varepsilon}, \exp \frac{\mu}{\varepsilon} \Phi \rangle \right]
\]

Linear case: Bensoussan-van Schuppen
Dynamic programming

Value function

\[ S^{\mu, \varepsilon}(\sigma, t) = \inf_{u \in \tilde{U}_{t, T}} E^0_{\sigma, t} \left[ \langle \sigma_T^{\mu, \varepsilon}, e_{\varepsilon}^{\mu} \Phi \rangle \right] . \]

Dyn prog equation 

\begin{align*}
\frac{\partial}{\partial t} S^{\mu, \varepsilon} + \frac{1}{2\varepsilon} D^2 S^{\mu, \varepsilon}(h\sigma, h\sigma) \\
+ \inf_{u \in U} \left\{ DS^{\mu, \varepsilon} \cdot \left( A^u \sigma + \frac{\mu}{\varepsilon} L^u \sigma \right) \right\} &= 0 \\
S^{\mu, \varepsilon}(\sigma, T) &= \langle \sigma, e_{\varepsilon}^{\mu} \Phi \rangle
\end{align*}

Optimal controller

\[ u_t^*(\sigma) = \text{achieves min in d.p.e.} \]

information state feedback
Duality

\[ \langle \sigma, \nu \rangle \triangleq \int_{\mathbb{R}^n} \sigma(x)\nu(x)\,dx \]

max-plus

\[ (p, q) \triangleq \sup_{x \in \mathbb{R}^n} \{ p(x) + q(x) \} \]

large deviations

\[ \lim_{\varepsilon \to 0} \frac{\varepsilon}{\mu} \log\langle e^{\frac{\mu}{\varepsilon}p}, e^{\frac{\mu}{\varepsilon}q} \rangle = (p, q) \]
Information state

\[ \lim_{\varepsilon \to 0} \varepsilon \log \sigma_{t,\varepsilon}^\mu(x) = p_t^\mu(x), \]

Value function

\[ \lim_{\varepsilon \to 0} \varepsilon \log S_{\varepsilon,p,t}^\mu = W^\mu(p,t) \]

Some consequences

- Can solve output feedback deterministic dynamic games (D-DG) using analogous, though deterministic, information state methods (cf. max-plus).

- Output feedback nonlinear \( H^\infty \) control problem can be solved using the D-DG methods.
Risk-sensitive control of quantum systems

System: eg. atom, can be controlled, Hilbert space $H$

Bath: e.g. electromagnetic field, continuously monitored, Hilbert space $\Gamma$ (Fock space)
Dynamics on $\mathbf{H} \otimes \Gamma$: [Hudson-Parthasarathy quantum stochastic DE, 1980’s]

$$dU(t) = \left\{ -K(u(t))dt + LdB^\dagger(t) - L^\dagger dB(t) \right\} U(t)$$

with initial condition $U(0) = I$, where

$$K(u) = \frac{i}{\hbar} H(u) + \frac{1}{2} L^\dagger L$$

System operators evolve according to

$$X(t) = j_t(u, X) = U^\dagger(t)X \otimes IU(t)$$

i.e.

$$dX(t) + (X(t)K(t) + K^\dagger(t)X(t) - L^\dagger(t)X(t)L(t))dt = [X(t), L(t)]dB^\dagger(t) - [X(t), L^\dagger(t)]dB(t)$$

where

$$L(t) = j_t(u, L), \ K(t) = j_t(u, K(u(t)))$$
Measurements of real quadrature of field:

\[ Q_t = B_t + B_t^\dagger \]

\[ dY(t) = (L(t) + L^\dagger(t))dt + dQ(t) \]

output field \hspace{2cm} input field

Risk-sensitive cost:

\[ J^\mu = \langle \pi_0 \otimes vv^\dagger, R^\dagger(T)e^{\mu C_2(T)}R(T) \rangle \]

where

\[ \langle A, B \rangle = \text{tr}(A^\dagger B) \]

\[ \frac{dR(t)}{dt} = \frac{\mu}{2} C_1(t)R(t), \quad C_1(t) = j_t(u, C_1(u(t))), \quad C_2(t) = j_t(u, C_2) \]

\[ \pi_0 = \text{initial system state} \]

\[ vv^\dagger = \text{field vacuum state} \]
Stochastic representation:

\[ J^\mu = \mathbf{E}^0[\langle \sigma^\mu_T, e^{\mu C_2} \rangle] \]

where \( \mathbf{P}^0 \) is standard Wiener measure, and the information state (an unnormalized density operator) satisfies

\[ d\sigma^\mu_t + [K\sigma^\mu_t + \sigma^\mu_t K^\dagger - L\sigma_t L^\dagger]dt = \mu^\frac{1}{2}[C_1\sigma^\mu_t + \sigma^\mu_t C_1] dt + [L\sigma^\mu_t + \sigma^\mu_t L^\dagger]dy(t) \]

[modified stochastic master equation]

\[ \pi^\mu_t = \frac{\sigma^\mu_t}{\langle \sigma_t, 1 \rangle} \]

[unnormalized state, new to physics]

When \( \mu \to 0 \) recover the Belavkin quantum filtering equation (stochastic master equation) (1980’s) [unnormalized state \( \sigma_t \), normalized state \( \pi_t = \sigma_t/\langle \sigma_t, 1 \rangle \)].
The dynamic programming equation is, formally,

\[ \frac{\partial}{\partial t} S^\mu(\sigma, t) + \inf_{u \in U} \mathcal{L}^{\mu;u} S^\mu(\sigma, t) = 0, \quad 0 \leq t < T \]

\[ S^\mu(\sigma, T) = \langle \sigma, e^{\mu C_2} \rangle \]

Some related current work

- Discrete time (James 2004)
- Robustness interpretation, discrete time (James-Petersen 2004)
- Robustness, linear/quadratic case (Doherty, d’Helon, James, Petersen, Wilson)
References


York, 1996.


Conditions derived using concepts of loop gain, conicity, and positivity. part ii: Conditions


risk-sensitive control problems using measure-valued decompositions," Stochastics and


[22] M. Bardi and I. Capuzzo-Dolcetta, “Optimal Control and Viscosity Solutions of