BOUNDS ON CAPACITY IMPROVEMENTS USING SPATIAL FILTERING

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ABSTRACT

The relationship between the information capacity and the corresponding beam pattern of a multi-sensor array is investigated. From this, bounds on the information capacity improvements possible utilizing spatial filtering in a mulituser environment are established.

1. INTRODUCTION

The demands on the mobile network are continually increasing. Wide-spread demand for mobile access to internet data such as video and pictures is not far off. The increase in information capacity required for such use is around three orders of magnitude. Because of this, much effort is currently going into research into new techniques of improving the mobile network capacity. One area in which this effort is being focussed is that of spatial filtering using antenna arrays. [5] investigates the performance improvement, in terms of receiver gain, of 12 ad 24 beam antenna arrays with fixed beam positions, over a standard 3-sector dual diversity configuration, via field tests. Chan [1] looks at improving spectrum efficiency in a sectorized cellular system by decreasing cell size.

Studies of capacity improvements obtainable via spatial filtering techniques include, [7] and [4] where capacity improvement is related to decreased frequency reuse factors and [3] where capacity improvements are cited in terms of increases in numbers of users. The results of these studies are based on practical systems. We, by contrast, take an analytical approach and establish a basic relationship between information capacity and the array beam pattern. From this we are able to establish some bounds on the possible improvements to information capacity using beamforming.

2. BEAMFORMING

2.1. Antenna Arrays

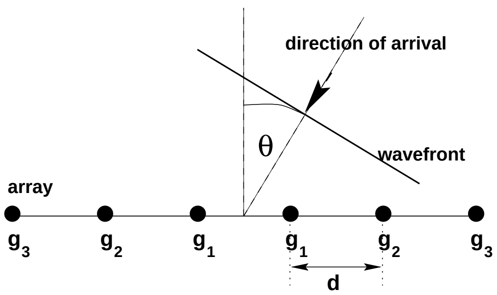

Figure 1: Schematic Diagram of a Symmetric Antenna Array

Consider a plane electromagnetic wave of wavelength λ impinging on a linear, uniformly spaced array of sensors from an angle θ, with respect to the broadside of the array, as shown in Fig. 1. The wave consists of a sinusoidal carrier modulated by either an analogue or a digital information signal.

A symmetric array with an even number, $2N$, of sensors, each a distance, d, from its neighbours, is considered. Let n, $\{n = -N \ldots N, n \neq 0\}$, index the sensors. The distance of sensor n from the centre of the array is given by, $\frac{(2|n|-1)d}{2}$.

Each sensor has a complex weight and a carrier phase associated with it. The wavefront arrives at different sensors in the array at different times. The phase of the carrier at different sensors at any one time is therefore different. Let the carrier phase at the centre of the array be 0. The relative carrier phase of sensor n is given by,

$$\phi_n = \left[\frac{(2|n|-1)d}{2}\right] \sin\theta \frac{2\pi}{\lambda}. \tag{1}$$

Throughout this paper a separation of $d = \frac{\lambda}{2}$ is assumed. Let,

$$u = \frac{\pi}{2}\sin\theta. \tag{2}$$

Then (1) can be simplified to,

$$\phi_n = (2|n|-1)u \tag{3}$$

2.2. Choosing the Sensor Weights

The power gain afforded a signal wavefront arriving at the array is dependent upon the direction of arrival of the wavefront. The wavefront in Fig. 1 is arriving from an angle of θ. Fig 2 shows a typical array power gain function over the range of angles from $-90°$ to $90°$. The narrow band of angles afforded the maximum gains form the *main beam*. The power gain function is also referred to as the *beam pattern*. Hence the term *beamforming*. The position of the main beam can be varied by varying the amplitudes and phases of the complex sensor weights, w_n. In general, the gain function is given by the following equation.

$$|G_{2N}(u)|^2 = \left|\sum_{n=1}^{N} w_n e^{j(2n-1)u} + \sum_{n=1}^{N} w_n e^{-j(2n-1)u}\right|^2 \tag{4}$$

The general gain function can also be written in terms of $\cos u$ [2] and further manipulated to be represented by a polynomial of order $(2N-1)$ in $x = \cos u$. This will be useful when the Chebyshev weights are considered later. Slightly different notation has been used here to that used in [2].

2.3. Uniform Sensor Weights

The simplest choice of sensor weights is uniform. That is, identical amplitudes and phases. Let all of the sensor weights

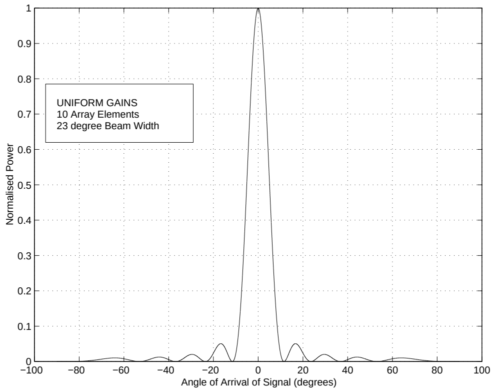

Figure 2: Normalised Beam Pattern for Uniform Weights

equal 1. The uniform beam pattern function is given by,

$$|U_{2N}(u)|^2 = \left|\sum_{n=1}^{N} e^{j(2n-1)u} + \sum_{n=1}^{N} e^{-j(2n-1)u}\right|^2 \quad (5)$$

$$= \left|\frac{\sin(2Nu)}{\sin(u)}\right|^2. \quad (6)$$

3. INFORMATION CAPACITY FOR ONE USER

The information capacity for one user's signal is considered. The power gain can be characterised as a function of the angle of arrival, θ, of a signal, as $P(\theta)$. (For the uniform case, $P(\theta) = |U_{2N}(u)|^2$.) Let the angle, $\theta_{\max}$, of the apex of the sensor array's main beam be set to be coincident with the direction of arrival of the desired user's signal[1]. Let the power gain afforded signals arriving from this direction be $P_{\theta_{\max}}$.

Consider the case of one interfering user with received signal power identical to that of the desired user. Let the interfering user's signal be arriving from a direction, θ. The capacity for the desired user is given by, [8],

$$C(\theta) = B\log_2\left(1 + \frac{P_{\theta_{\max}}}{P(\theta)+\sigma^2}\right) \text{ bits/s} \quad (7)$$

Fig. 3 shows the variation of capacity with the direction of arrival, θ, of the interfering user's signal. To obtain this graph, because it is relative values rather than absolute values that are of interest here, the bandwidth, B, was set to 1 and the power gain function was normalised such that the maximum power gain, $P_{\theta_{\max}}$, was also 1. The signal-to-noise power ratio (SNR $= \frac{P_{\theta_{\max}}}{\sigma^2}$) was set to 100 and the number of array sensors used was 10. These values were used consistently in all simulations to obtain the graphical results throughout this paper.

3.1. Analysis of Capacity with One Interfering User

3.1.1. Maximum Capacity with One Interfering User

Two special cases are worth considering. The first is when the angle of arrival of the interfering user's signal is coincident with a *null* (zero power gain) in the beam pattern. That is, the effective power of the interfering user's signal is zero. Let the capacity in this case be $C_{\theta_{\text{null}}}$. It is given by,

[1]This is without loss of generality because the angle of the main beam can be moved as desired by introducing appropriate phase delays in the sensors. However, the main beam is narrowest at an angle of 0 and this will give the best possible resolution between multiple users

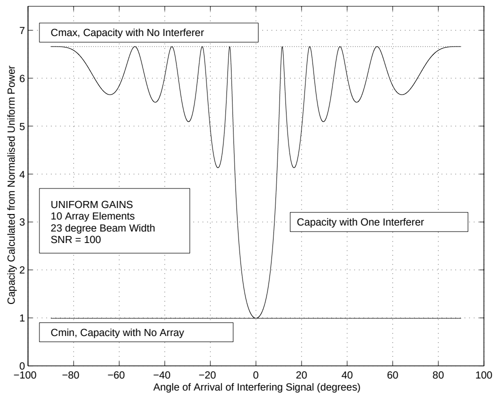

Figure 3: Capacity with One Interferer for Uniform Weights

$$\begin{aligned} C_{\theta_{\text{null}}} &= B\log_2\left(1 + \tfrac{P_{\theta_{\max}}}{\sigma^2}\right) \\ &= B\log_2\left(1 + \text{SNR}\right) \text{ bits/s.} \end{aligned} \quad (8)$$

This is equivalent to the case where there is no interfering user. Fig. 3 shows the level of the capacity when there is no interfering user. It coincides with the capacity function in several places and is, in fact, the maximum value achieved by the capacity function, in the presence of one interferer.

3.1.2. Minimum Capacity with One Interfering User

The second special case is when the angle of arrival of the interfering user coincides with that of the desired user. That is, the power gain of the interfering user's signal is also $P_{\theta_{\max}}$. The capacity in this case is given by,

$$\begin{aligned} C_{\theta_{\max}} &= B\log_2\left(1 + \tfrac{P_{\theta_{\max}}}{P_{\theta_{\max}}+\sigma^2}\right) \\ &= B\log_2\left(1 + \tfrac{1}{1+\frac{1}{\text{SNR}}}\right) \text{ bits/s.} \end{aligned} \quad (9)$$

This is equivalent to the case where there is only one sensor (no array). Fig. 3 shows the level of the capacity when there is only one sensor. It can be seen that it coincides with the capacity function in only one place and is, in fact, the minimum value achieved by the capacity function, in the presence of one interferer.

3.1.3. Directions of Capacity Maxima and Minima

Now, it is of interest to know exactly where these maximum and minimum capacity points occur. To determine this, simply take the derivative of the capacity with respect to θ and set it to 0. The derivative of the capacity is given by,

$$\frac{dC}{d\theta} = \frac{dC}{dP}\frac{dP}{d\theta}. \quad (10)$$

Now, from (7) it can be seen that the capacity is a monotonically decreasing function of $P(\theta)$. Thus, $\frac{dC}{dP}$ is always negative and never 0. Thus, from (10), the capacity can only have maxima and minima at the same angles as the power gain function, or beam pattern, has maxima and minima.

Because $\frac{dC}{dP}$ is always negative, (10) also indicates that the maxima in the capacity function coincide with the minima in the beam pattern and that the minima in the capacity function coincide with the maxima in the beam pattern.

3.1.4. Approximate Linear Relationship Between Capacity and dB Power

The power function is expressed in decibels (dB) via the following equation.

$$P^{\mathrm{dB}} = 10\log_{10}[P(\theta)] \quad \mathrm{dB} \tag{11}$$

Now, let the maximum power gain, $P_{\theta_{\max}}$ equal 1. (7) can be rewritten in terms of log base 10 as follows,

$$C(\theta) = \frac{B}{\log_{10} 2}\log_{10}\left(1 + \frac{1}{P(\theta)+\sigma^2}\right) \;\; \mathrm{bits/s.} \tag{12}$$

Let the following assumptions hold,

$$\begin{array}{lllllll} P(\theta) & \gg & \sigma^2 & \Rightarrow & \frac{1}{P(\theta)+\sigma^2} & \rightarrow & \frac{1}{P(\theta)} \\ P(\theta) & \ll & 1 & \Rightarrow & 1+\frac{1}{P(\theta)} & \rightarrow & \frac{1}{P(\theta)}. \end{array} \tag{13}$$

Then $C(\theta)$ can be approximated as follows,

$$\begin{array}{lll} C(\theta) & \rightarrow & \frac{B}{\log_{10} 2}\log_{10}\left[\frac{1}{P(\theta)}\right] \\ & \rightarrow & \frac{-B}{\log_{10} 2}\log_{10}[P(\theta)] \\ & \rightarrow & \frac{-B}{10\log_{10} 2}P^{\mathrm{dB}} \qquad \text{as} \quad P(\theta)\rightarrow 0 \\ & & \qquad\qquad\qquad\qquad \sigma^2 \rightarrow 0. \end{array} \tag{14}$$

Fig. 4 shows plots of the capacity function and the approximation, given by (14). It can be seen that the approximation fits very well at the first sidelobe and for most of the largest lobe of the capacity function. It fits less well at other points where the assumptions in (13) are not quite so valid, but well enough to allow some insights into the capacity function.

The capacity function has a one-to-one relationship with the beam pattern. Its salient features can be directly related to those in the beam pattern.

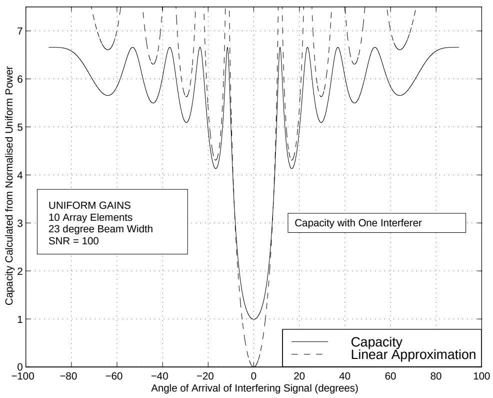

Figure 4: Approximation, Linear in Power (dB), of Capacity in Presence of One Interferer for Uniform Sensor Weights

3.1.5. The Significance of the Large Negative Pointing Lobe in the Capacity Function

It was noted earlier that the extrema of the capacity and power functions are coincident, but opposite. It is relatively straightforward to conclude that the large negative pointing lobe in the capacity function corresponds to the main lobe in the beam pattern. Thus, the large negative pointing lobe in the capacity function is identical in width to the main beam of the beam pattern.

The large negative pointing lobe indicates the drastic drop in capacity which occurs when the direction of arrival of the interfering user's signal approaches the direction of arrival of the desired user's signal within the main beam of the beam pattern.

3.1.6. The Significance of the Smaller Negative Pointing Lobes in the Capacity Function

The smaller negative pointing lobes in the capacity function have the same widths as the sidelobes in the beam pattern. (14) shows that the magnitude of the capacity function is inversely proportional to the magnitude of the beam pattern.

The smaller negative pointing lobes of the capacity function indicate the effect on capacity as the direction of arrival of the interfering user's signal moves through the sidelobes of the beam pattern. To have a minimum drop in capacity as the interfering user moves through the sidelobes, the maximum sidelobe level should be as low as possible.

3.2. Analysis of Capacity with Multiple Interfering Users

3.2.1. Maximum Capacity with Multiple Interfering Users

Let the number of interfering users be K. Let the directions of arrival of the signals of all of the interfering users coincide with nulls in the beam pattern. The capacity in this case is given by,

$$C_{\theta_{\mathrm{null}}} = B\log_2\left(1 + \frac{P_{\theta_{\max}}}{\sigma^2}\right) \;\; \mathrm{bits/s.} \tag{15}$$

(15) is simply a repeat of (8). This is the maximum possible capacity. It is independent of the number of interfering users.

3.2.2. Minimum Capacity with Multiple Interfering Users

Let the directions of arrival of the signals of all of the interfering users coincide with the main beam of the beam pattern. That is, $P(\theta_k) = P_{\theta_{\max}}\{k = 1,\ldots,K\}$. The capacity in this case is given by,

$$C_K(\theta_{\max}) = B\log_2\left(1 + \frac{P_{\theta_{\max}}}{KP_{\theta_{\max}}+\sigma^2}\right) \;\; \mathrm{bits/s.} \tag{16}$$

Similarly to the one interferer case, this is the minimum possible capacity for K interferers.

3.2.3. Significance of the Main Beam to Maximum Sidelobe Ratio

Let the angles of arrival of the signals of all of the interfering users coincide with one of the maximum level sidelobes. Let the power gain at these points be $P_{\theta_{\mathrm{SL}}}$. And let $r = \frac{P_{\theta_{\max}}}{P_{\theta_{\mathrm{SL}}}}$ be the ratio of the main beam to the maximum sidelobe level. The capacity in this case is given by,

$$\begin{array}{lll} C_{K\theta_{\mathrm{SL}}} & = & B\log_2\left(1+\frac{P_{\theta_{\max}}}{KP_{\theta_{\mathrm{SL}}}+\sigma^2}\right) \\ & = & B\log_2\left(1+\frac{1}{K\frac{P_{\theta_{\mathrm{SL}}}}{P_{\theta_{\max}}}+\frac{\sigma^2}{P_{\theta_{\max}}}}\right) \\ & = & B\log_2\left(1+\frac{1}{\frac{K}{r}+\frac{1}{\mathrm{SNR}}}\right) \;\; \mathrm{bits/s} \end{array} \tag{17}$$

Equating (17) with (9), yields,

$$\begin{array}{rcl} B\log_2\left(1+\frac{1}{1+\mathrm{SNR}}\right) & = & B\log_2\left(1+\frac{1}{\frac{K}{r}+\frac{1}{\mathrm{SNR}}}\right) \\ \Rightarrow K & = & r \end{array} \tag{18}$$

Specifically, (18) indicates that the number of interferers, with signals being received from the direction of one of the maximum sidelobes, required to reduce the capacity to the minimum level achievable in the presence of only one interferer is equal to r. More generally, the level of degradation of the capacity in the presence of multiple interferers is directly related to the ratios of the main beam to the sidelobes.

If it is important to guarantee that the minimum capacity level, when interferers are outside of the main beam, is above a given threshold, (18) is a useful design tool.

4. WEIGHTS FOR MINIMUM SIDELOBE LEVEL

4.1. Chebyshev Weights

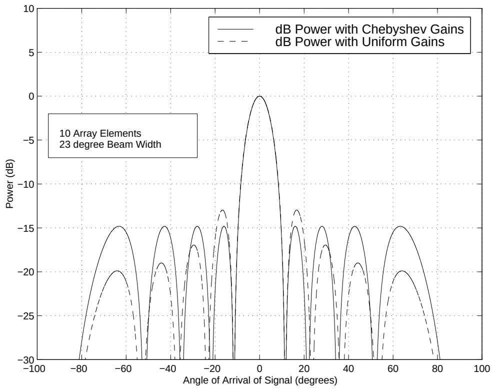

Figure 5: dB Beam Pattern Comparison

A sensor array with Chebyshev weights is known to give the lowest maximum sidelobe level for a given main beam width. The Chebyshev functions are a family of polynomials, in x, of arbitrary order. They are distinguished via the value of a design parameter, z_0. Recall that in section 2.2, x was defined as, $x = \cos u$. Let the position of the first null be θ_{null}. The Chebyshev function in terms of x, and the design parameter, z_0 are given by [6]

$$\begin{aligned} T_{2N-1}(z_0 x) &= \cos([2N-1]\cos^{-1} z_0 x) & |z_0 x| \leq 1 \\ &= \cosh([2N-1]\cosh^{-1} z_0 x) & |z_0 x| \geq 1 \\ z_0 &= \frac{1}{\cos(\frac{\pi}{2}\sin\theta_{\text{null}})}\cos\left(\frac{\pi}{2[2N-1]}\right). \end{aligned} \tag{19}$$

Dolph [2] shows how to equate the general gain function with the required Chebyshev function to obtain the Chebyshev sensor weights.

4.2. Comparisons of Gain and Capacity Functions with Chebyshev and Uniform Weights

Figs. 5 and 6 illustrate comparisons between the power gain and capacity functions, respectively, achieved using Chebyshev and uniform sensor weights. The capacity function illustrated in Fig. 6 assumes only one interfering user. With identical main beam widths, the Chebyshev weights afford a lower maximum sidelobe level than the uniform weights. This translates to a higher minimum capacity level in the presence of multiple interfering users when the angles of arrival of all interfering users' signals are in the sidelobe region.

5. CONCLUSIONS

A relationship has been established between the information capacity with respect to a desired user's signal and the sensor array beam pattern. If the angle of arrival of the desired user's signal coincides with the apex of the main beam in the sensor array beam pattern, the following can be said.

- The maximum possible capacity occurs when there are no interfering users. This is equivalent to the case when the angles of arrival of all interfering users coincide with nulls in the beam pattern, assuming perfect nulls.

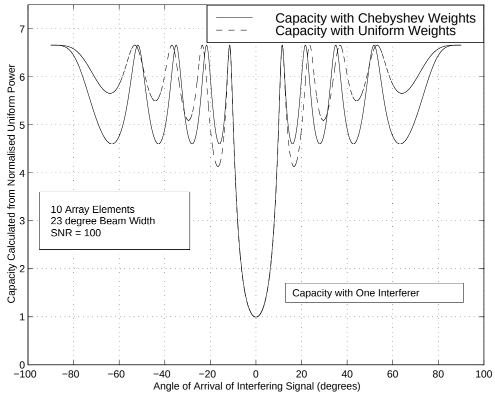

Figure 6: Capacity Function Comparison

- The minimum capacity occurs when there is no array, i.e. the sensor is omnidirectional. This is equivalent to the case when the angles of arrival of all interfering users coincide with the apex of the main beam in the beam pattern.
- The capacity function has a one-to-one relationship with the beam pattern and its salient features are directly related to those in the beam pattern.
- To have a minimum drop in capacity as the angles of arrival of any interfering users move through the sidelobes, the maximum sidelobe level must be as low as possible.

These observations form a bound for the improvements in information capacity possible via the use of spatial filtering.

6. REFERENCES

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