Probability of Multipath Signature Separability

Haley M. Jones
Department of Engineering
Faculty of Engineering and Information Technology
The Australian National University
ACT 0200, AUSTRALIA

Abstract—Elimination of co-channel interference in a multiple antenna receiver scenario has long been considered a beamforming problem. Such a solution becomes largely ineffective when the interference source is almost co-located with the desired signal source. Interference elimination cannot thus be achieved via beamforming alone. If the multipath signatures of the desired and interfering signals can be discriminated, they can be effectively utilized to separate the desired signal from the interfering signal at high probabilities for acceptably low MMSE levels. Such attractive results can be achieved for users as close as $\lambda/10$.

I. INTRODUCTION

Multipath is an inevitable part of any practical wireless system and must be taken into account in any realistic analysis. Traditionally, multipath has been viewed as an encumbrance to accurate signal recovery. Most often in the literature antenna arrays have been employed as beamformers in which sharp nulls are formed to eliminate, or at least substantially reduce, the signal contributions of multipath components, as well as spatially separated co-channel interferers. However, there are several circumstances in which this is made difficult, as discussed next.

Interfering signals may be spread over a range of angles rather than being at several discretely resolvable angles. This situation is tackled in [7] in which interfering signals spatially separated from the desired signal and with an angular spread too large to be eliminated using a sharp null are substantially removed using controlled broad nulls. Further, if there are more interfering signals than degrees of freedom in the array or if interfering signals arrive at the array with angles of arrival (AOAs) indistinguishable from that of the desired signal, beamforming affords little advantage by itself and other techniques, generally temporally based, must be used. In [8] only spatial diversity techniques are used, for which performance is degraded when too many in-beam co-channel interference (CCI) signals are present. Beamforming techniques are also the primary method used in [5] where experimental results show that when two users are placed very close together (within $1^\circ$ with respect to the receiver array), with very little angular spread of multipath, the performance is quite poor in terms of diversity gain. In [2] an approximation to a maximum likelihood sequence estimator is used in conjunction with an antenna array. While improvements over simple beamforming are indicated, degradation in performance is obvious when interfering signals are not spatially separated from the desired signal.

With increasing interest in the use of antenna arrays [3], [4] and the spatial diversity offered by them, multipath is beginning to be seen in a more positive light as a natural source of signal diversity to be exploited. That is, the presence of multipath in a co-channel multiuser environment can be an advantage, if dealt with appropriately. The layered space-time architecture for fixed wireless systems developed at Bell Labs embodies this principle [1]. For example, if, apart from their actual transmitted data, all other aspects of multiple users’ signals are indistinguishable (e.g., transmission carrier frequency and time slot), the users’ multipath signatures at the receiver will still, in general, contain uncorrelated components.

It is commonly assumed that, in a multipath-rich environment, if users are at least $\lambda/2$ apart, where $\lambda$ is the transmission wavelength, then their signals may be recovered adequately using appropriate signal processing [12]. In [6] we developed the concept of separability which can be applied to give a quantitative measure of the accuracy with which a desired signal can be detected in a given transmission environment, in the presence of interference and noise.

Our particular interest is in determining the limits of the probit of multipath as a source of signal diversity in terms of how close two signal sources can be. The multipath signatures of sources of the order of $\lambda$ apart are effectively identical at the receiver. The only difference is a phase shift between each corresponding pair of multipaths. In [6] we showed that $\lambda/2$ was conservative by showing that users can be as close as $\lambda/10$ with meaningful signal separation possible. At a carrier frequency of 1GHz this corresponds to 3cm.

These results were simulation based and depended upon specific scatterer geometries. It is important that we determine a more generally applicable, theoretical basis for these results. In this paper, we present such a theoretical formulation, based upon the spatial minimum mean square error (MMSE) detector. We consider a multipath signalling environment with a desired user and a close interfering user, with a 30dB SNR.

The MMSE detector performance is considered from a statistical viewpoint. Our results give the probability of reaching at least a given MMSE level, $\varepsilon$, for a given scattering environment and distance $d$ between the desired and interfering users. We show that under the given signal conditions, and when multipath signatures are used as the sole distinguishing feature between the users’ signals, the users can be as close as $\lambda/10$.

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while maintaining acceptable performance using an antenna array receiver, backing up our previous simulation results.

II. DISCUSSION OF SEPARABILITY

In [6] we introduced the concept of separability as a formal mechanism for measuring the performance of a multiple input, multiple output communications system. We presented formal definitions for achievement of ε-separability under a given metric. For example, if we choose our metric to be the MSE and our desired performance level to be an MSE of, at most, 0.01 then our value of ε is 0.01. Then we can say that our two signals are MSE 0.01-separable for a given channel, if the performance is MSE < 0.01.

The concept of separability is particularly suited to the determination of performance level limits achievable for given types of signal transmission environments. We used the concept of separability to determine to what extent the distance between two nearly co-located users limits the usefulness of multipath as a source of multiuser signal diversity as a function of space.

In [6], our results were largely simulation based using very specific scattering models meaning that, while we believed them to be, our results were not obviously generally applicable. In this work we generalise the results by formulating a theoretical probabilistic framework. The results we achieve in this work match very well those in our simulation based work providing a solid foundation for their general applicability.

III. MODEL FORMULATION

A. Multipath Signatures

We consider a system with one desired signal, one interfering signal and noise, in a given time instant. We assume that the desired and interfering users’ transmitted signals are identical in all respects, apart from the actual bit sequences, which are uncorrelated. Thus, we assume that the users are transmitting simultaneously with the same carrier frequencies and transmitted signal powers. The signal modulation for both users is assumed to be binary phase shift keying (BPSK) [9]. The only distinguishing feature, then, is the multipath signature of each user. A user’s multipath signature is defined as consisting of the set of 1) amplitudes, 2) phases and 3) AOAs of the multipath signals for that user. For simplicity, we assume that the multipath delay spread for each user is negligible with respect to the symbol length so that ISI may be ignored.

Let the desired user be user 1, with signal \( b_1 \) and the interfering user be user 2 with signal \( b_2 \). We form the user signal vector \( \mathbf{b} = [b_1, b_2]^T, b_k \in \{-1, 1\}, k \in \{1, 2\} \). Let each user’s signal have \( P \) multipath components. We assume that the users are close enough that their signatures can be considered identical in amplitude and AOA but different in phase, for each corresponding pair of multipath components.

1 If we were considering time delays, the time delay of each component could also form a part of the signature.

B. Receiver Array Processing

Multipath as a source of diversity is only effective if there are multiple sensors at the receiver. We therefore assume that the receiver is an antenna array. We assume that each sensor has an isotropic omnidirectional gain response of \( 1/\theta \). That is, the sensor gains are such that they have no effect on the gain of any incoming signal. However, there is a phase imposed on each incoming signal at each sensor which is related to the AOA of the signal and the distance from the sensor to the zero-phase point in the array, assumed here to be the centre of the array. We can form an array response matrix \( \mathbf{G} \) of these imposed phases which are effectively complex gains of unity amplitude.

Let \( \theta_p \) be the AOA at the array of the \( p \)th multipath component for each user. Let \( L \) be the number of array sensors, indexed by \( \ell \), such that \( \psi_{\ell p} \) is defined as the additional phase imposed by sensor \( \ell \) upon a signal with an AOA of \( \theta_p \), due to the sensor’s physical position in the array. Then \( \mathbf{G} \) is an \( L \times P \) matrix

\[
\mathbf{G} = \begin{bmatrix}
e^{j\psi_{11}} & \cdots & e^{j\psi_{1p}} & \cdots & e^{j\psi_{1P}} \\
\vdots & & \ddots & & \vdots \\
e^{j\psi_{L1}} & \cdots & e^{j\psi_{Lp}} & \cdots & e^{j\psi_{LP}}
\end{bmatrix}.
\]

Comment: The elements \( e^{j\psi_{lp}} \) of \( \mathbf{G} \) are dependent only on the AOA of each signal so the array response \( \mathbf{G} \) is identical for both users.

Let \( a_k = [a_{1k} e^{j\phi_{1k}}, \ldots, a_{pk} e^{j\phi_{pk}}]^T \), where \( \{\} \)\(^T\) denotes transpose, be the vector of complex gains for the \( k \)th user’s multipaths. Then the desired user’s multipath gain vector can be written as

\[
a_1 = [a_1 e^{j\phi_1}, \ldots, a_P e^{j\phi_P}]^T
\]

where \( a_P \) and \( \phi_P \) are the amplitude and phase, respectively, of the desired user’s \( p \)th multipath. The interfering user’s multipath gain vector can similarly be written as

\[
a_2 = [a_1 e^{j(\phi_1 + \Delta\phi_1)}, \ldots, a_P e^{j(\phi_P + \Delta\phi_P)}]^T
\]

where \( \Delta\phi_p \) is the phase difference between the desired user’s \( p \)th multipath and the interfering user’s \( p \)th multipath.

Comment: The only differences between these two gain vectors are the phase differences \( \Delta\phi_p \) between each corresponding pair of multipaths.

We assume that the multipath signature for each user is time-invariant over the time interval of interest. The \( k \)th user’s multipath signature is represented by \( \mathbf{c}_k = [c_{1k}, \ldots, c_{Lk}]^T \) given by

\[
\mathbf{c}_k = \mathbf{G} \mathbf{a}_k.
\]

Comment: \( \mathbf{a}_k \) incorporates the amplitudes and phases of the \( k \)th user’s multipath signature and \( \mathbf{G} \) incorporates the AOAs.

The array output vector is \( \mathbf{y} = [y_1, \ldots, y_L]^T \) with one output for each sensor. Let \( \mathbf{C} = [c_1, c_2]^T \) be the multipath signature matrix. And let \( \mathbf{n} = [n_1, \ldots, n_L]^T \) be the vector
of additive noise components from each sensor where the components are statistically independent, zero mean, white, Gaussian noise (AWGN) each with variance $\sigma_n^2$. Then the array output is

$$y = Ga_1b_1 + Ga_2b_2 + n = Cb + n.$$  \hspace{1cm} (5)

Having established the model for the array sensor outputs $y$, we next develop equations for our chosen detector, the spatially based MMSE detector.

IV. SPATIAL CHARACTERISTIC BASED MMSE DETECTOR

In [6] we compared the separability performances of three common linear detectors, the matched filter, decorrelating and minimum mean square error (MMSE) detectors. We base our work in this paper on the minimum mean square error (MMSE) detector as it, predictably, gave the best results in our previous simulation based work. The MMSE detector takes into account both noise and interference and tries to compensate for both, maximizing the SINR [11].

A memoryless linear detector generates an estimate $\hat{b}_1$ of the desired signal $b_1$ from a raw estimate $v_1$ which is a linear combination of a chosen set of received signals. The set of received signals chosen to generate the estimate in our model are the $L$ array sensor outputs $y$. Let $h_1$ be the estimate coefficient vector for the desired user for a given linear detector. That is, the elements of $h_1$ are the coefficients in the linear combination of received signals used to generate the raw estimate $v_1$. Then $y$ is the input to the linear detector, and the general equation for the linear detector output, or raw estimate, $v_1$ is given by

$$v_1 = h_1^Hy.$$  \hspace{1cm} (6)

Comment : The raw estimate, $v_1$, produced by the MMSE detector is such that $E\{ (\hat{b}_1 - v_1)^2 \}$ is minimized where $E\{ \cdot \}$ is the expectation operator.

It can be shown that the estimate coefficient vector giving the MMSE for the desired user is given by

$$h_1 = (c_1c_1^H + c_2c_2^H + \sigma_n^2I_L)^{-1}c_1.$$ \hspace{1cm} (7)

and the MMSE, $\sigma_e^2$, itself is given by

$$\sigma_e^2 = 1 - h_1^H c_1.$$ \hspace{1cm} (8)

Comment : To obtain (7) we used the assumption that the user bit sequences are uncorrelated.

We next establish a probabilistic framework for determining expected MMSE $\varepsilon$-separability performance outcomes.

V. PROBABILISTIC FRAMEWORK FOR SEPARABILITY

In our previous work [6] we showed via Monte Carlo simulations that MMSE $\varepsilon$-separability for $\varepsilon = 0.01$ is achievable when the two users are as close as $\lambda/10$. That is, the MMSE can still be below 0.01 when the users are as close as $\lambda/10$ for many simulated multipath environments. These results were based upon simple scattering models which were not necessarily generally applicable.

In this work we seek to generalise these results by developing a probabilistic measure of MMSE $\varepsilon$-separability performance which can be applied to any scatterer distribution. And, in keeping with our previous work, our particular interest is in the relationship between the MMSE $\varepsilon$-separability and the distance, $d$, between the users.

So, to formulate the probabilistic model we begin by establishing what are the parameters in our system which are pertinent to evaluating the MMSE.

Comments :

1) We observe from (7) and (8) that the MMSE is dependent upon the users’ multipath signatures and the noise variance.

2) Therefore, for close users in a given scattering environment, characterised by $c_i$, and given SNR (or noise variance) the MMSE is only dependent upon the phase differences, $\Delta \phi_p$, between each of the users’ corresponding multipaths.

3) All other values in the MMSE equation (8) are known.

Fig. 1. Relative positions of the desired and interfering users depicted by distance $d$ and angle $\theta_i$. The angle $\alpha_p$ associated with the $p^{th}$ scatterer, determines the phase difference $\Delta \phi_p$ between the $p^{th}$ multipaths of the two users.

Now, the $\Delta \phi_p$ are directly dependent upon the position, $(d, \theta_i)$, of the interfering user relative to the desired user where $\theta_i$ is the relative angle of the position of the interfering user with respect to the position of the desired user. Keeping in mind our interest in the relationship between the MMSE and $d$, we note that this relative position can be expressed in terms of $d$ and spatial angle $\theta_i$, as shown in Fig. 1. We make the following observations.

1) For a given $d$, the MMSE is solely dependent upon $\theta_i$.

2) As the interfering user is equally likely to be at any angle $\theta_i$ from the desired user, we can model $\theta_i$ as a random variable with uniform distribution between $-\pi$ and $\pi$.

So, for a given scattering environment and SNR the MMSE, as a function of $d$, is dependent upon the spatial angle, $\theta_i$, of the position of the interfering user. And, as $\theta_i$ is a random variable, the MMSE is also a random variable with a particular (unknown) probability density function (pdf).

Now, we say that MMSE $\varepsilon$-separability is achieved for a given scenario if the MMSE $\leq \varepsilon$. As we are now dealing with the MMSE as a random variable, we must express the achievability of MMSE $\varepsilon$-separability as a probability. That is, MMSE $\varepsilon$-separability is achieved for a given desired user multipath signature, with close interfering user, and noise variance, $\sigma_n^2$, with conditional probability, $P\{ \text{MMSE} \leq \varepsilon|c_1, \sigma_n^2 \}$. We
can expand and rearrange this in terms of \( d \) or, more generally, the phase term \( 2\pi d/\lambda \), by reexpressing the argument of the probability expression

\[
P\{\text{MMSE} \leq \varepsilon | c_1, \sigma_n^2 \} = P\{ f \left( \frac{2\pi d}{\lambda} \right) \geq X | c_1, \sigma_n^2 \} \quad (9)
\]

where \( f(\cdot) \) is a function, defined below, \( X \) is a constant and \( \geq \) implies that we do not yet know the relative nature of the relationship between \( f(\cdot) \) and \( X \). To determine \( f(\cdot) \) and \( X \) we expand the LHS of (9) using (7) and (8), to obtain

\[
P\{\text{MMSE} \leq \varepsilon | c_1, \sigma_n^2 \} = P\{ 1 - h_1^H c_1 \leq \varepsilon | c_1, \sigma_n^2 \}
\]

\[
= P\{ 1 - c_1^H (c_1 c_1^H + 2c_2^H + \sigma_n^2 I_L) - 1 \leq \varepsilon | c_1, \sigma_n^2 \}
\]

\[
= P\{ g(c_2) \leq \varepsilon | c_1, \sigma_n^2 \}
\]

where \( g(\cdot) \) is a function, defined in (13) in terms of \( c_2 \) because all other terms in (12) are constant. Now, if we substitute the RHS of (13) into the LHS of (9) then we have

\[
P\{ g(c_2) \leq \varepsilon | c_1, \sigma_n^2 \} = P\{ f \left( \frac{2\pi d}{\lambda} \right) \geq X | c_1, \sigma_n^2 \}. \quad (14)
\]

Now, we use the matrix inversion lemma \((A + UV)^{-1} = A^{-1} - A^{-1} U(I + V^H A^{-1} U)^{-1} V^H A^{-1} \) in (7), to obtain

\[
h_1^H c_1 = \frac{\sigma_n^2 \|c_1\|^2 + \|c_2\|^2}{\sigma_n^2 + \sigma_c^2 \|c_1\|^2 + \|c_2\|^2} c_1 - \frac{\sigma_c^2 \|c_2\|^2}{\sigma_n^2 + \sigma_c^2 \|c_1\|^2 + \|c_2\|^2} c_2^2.
\]

Expanding (15), in terms of \( 2\pi d/\lambda \), in the LHS of (14) allows us to derive expressions for \( f \left( \frac{2\pi d}{\lambda} \right) \) and \( X \) in the RHS of (14).

We do this next for the simplest case of two sensors in the antenna array, \( L = 2 \).

**VI. THE TWO SENSOR CASE (\( L = 2 \))**

The simplest case of two sensors (one sensor does not usefully constitute an array) limits the lengths of both \( c_1 \) and \( c_2 \) to two, \( c_1 = [c_{11}, c_{21}]^T, c_2 = [c_{12}, c_{22}]^T \). For notational simplicity we represent \( 2\pi d/\lambda \) by \( d' \). To determine an expression for \( f \left( \frac{2\pi d}{\lambda} \right) = f(d') \), in (14), we must expand \( c_1 \) and \( c_2 \) in terms of the multipath signature components, amplitudes, phases and AOAs, as represented by \( a_1, a_2 \) and \( G \). We note that the \( \ell \)th component of \( c_2 \) can be written as

\[
c_2 = \sum_{p=1}^{P} e^{j\psi_{pq}} a_p e^{j(\Delta_{pq} + \phi_{pq})} = \sum_{p=1}^{P} e^{j\psi_{pq}} a_p e^{j(\phi_{pq} + d' \sin \alpha_p)}.
\]

For \( L = 2 \), (14) becomes

\[
P\{ g(c_2) \leq \varepsilon | c_1, \sigma_n^2 \} = P\{ f(d') \geq X | c_1, \sigma_n^2 \}
\]

where \( f(d') \) and the lower bound, \( X \), are given by

\[
f(d') = \sum_{p=1}^{P} A_{pp} \cos \gamma_{pp} + \sum_{p=1}^{P-1} \sum_{q=p+1}^{P} B_{pq} \cos[d'(\sin \alpha_p - \sin \alpha_q) - \delta_{pq}]
\]

\[
X = \sigma_n^4 \left( \frac{1 - \varepsilon}{\varepsilon} - \frac{\sigma_c^2 |c_{11}|^2 + |c_{21}|^2}{\varepsilon} \right)
\]

Next, we consider the separability performance of a particular desired user multipath signature, using our probabilistic formulation for determining the likelihood of achieving a particular MMSE \( \varepsilon \)-separability level.
Our results indicated that the probability of an overall MMSE $\epsilon$ less than $0.0005$. Our scattering environment consisted of 20 multipaths where $L_{su}$ both users' multipath signatures and the noise variance. For smaller values of $\epsilon$ the distance threshold at which separability remains effectively constant is closer to $\lambda/2$. Thus, our probabilistic formulation allows us to determine the probability of achieving a given separability level for a given multipath environment for a desired user with one interfering user.

VIII. CONCLUSIONS

We have presented a probabilistic formulation for determining the likelihood of achieving a given MMSE $\epsilon$-separability level for a given multipath environment for a desired user in the presence of an almost co-located interfering user. While an exact expression for the desired pdf could not be found, further work is focussed on finding an upper bound. Our theoretically based results have clearly backed up previous simulation based results showing that users can be as close as $0.1 \lambda$ while still achieving acceptable levels of signal separability.

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