Analysis of the MMSE Detector in the Spatial Domain

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Abstract—Linear detectors can be based upon signal spatial, rather than temporal, characteristics and used to weight the sensors in an antenna array receiver. Such a receiver allows us to take advantage of the signal diversity offered by the multipath components of a signal. This provides an additional technique for distinguishing the signals of co-channel users with different enough multipath signatures. Analysis of the behaviour of spatially based linear detectors in sparse multipath fields with respect to change in SNR, SIR and SINR, offers insights into the most desirable relative multipath characteristics for good performance. We use the MMSE detector as an example linear detector and compare its performance with that predicted for a general linear detector. We show why it is desirable to have multipath propagation directions as diverse as possible and show that for maximum possible phase diversity between two close users’ multipath signatures, at least 11 multipaths are required.

Index Terms—MMSE, wireless, multipath, linear detector, antenna array, spatial diversity

I. INTRODUCTION

Multipath channels are most often considered in a statistical sense. For example, a Rayleigh or Rice distribution is often assumed for the received signal magnitude [1], [2]. Such channels have, until recently, been considered negatively due to inherent signal fading. Even the recent high-capacity multiple transmitter/multiple receiver systems which rely on multipath rich environments in which two users were separated by a distance $d \lesssim \lambda$. As such, their multipath signatures could be considered identical at the receiver apart from a phase difference in each corresponding pair of multipaths. With an antenna array receiver, and known multipath signatures, it was shown that the users’ signals could be distinguished with an appropriately high performance measure without the benefit of any other multiuser access technique, when they were as close as $\lambda/10$. However, these results were largely simulation based using specific scatterer geometries.

To understand the nature of multipath channels in a more general sense, in this paper we analyse the effect of a linear detector on received multipath signals. We perceive the detector as manipulating the multipath field to achieve its detection goals. We consider the possible effects of a general linear detector on the signal to noise, interference, and noise and interference ratios (SNR, SIR, SINR) for two users transmitting in a multipath channel. We consider the performance of the minimum mean squared error (MMSE) detector as a particular example of a linear detector. We consider only one and two multipath fields as these afford particular insights while not being overly mathematically cumbersome. We find that in conditions of asymptotic SNR the MMSE detector acts to minimize the interfering user’s power as opposed to maximizing the desired user’s power. Our analyses also give us sound theoretical bases for some commonly acknowledged desirable multipath channel properties. In particular we show why it is better to have multipaths which are well spread in their directions of propagation. This leads us to a result which shows that for almost guaranteed maximum phase diversity between two users’ multipath signatures, at least 11 multipaths are required in the channel.

II. BACKGROUND

Figure 1 shows a two dimensional (2D) contour plot of the variation of average power in a typical multipath rich field. The field is generated by the superposition of several multipaths with random magnitudes, phases and propagation directions. In generating this field it was assumed that the region was small enough that the magnitudes could be considered constant.

We view Fig. 1 as a mapping. In the context of a communication system the region of space depicted is where the transmitter resides. The average power at each point is the average

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Typical contour plot of the average power of the superposition of many distinct plane waves propagating in random directions in a region of space.}
\end{figure}
power that would be received at a given fixed receiver if the transmitter was situated at that point. For close users the mapping applies equally to the received signals of the desired and interfering users. For example, if the desired user is at the origin and the interfering user is at \((0, \lambda/2)\), for the scatterer scenario in Fig. 1, the desired user’s average received signal power is higher than that for the interfering user.

In this paper we analyse how a linear detector behaves with respect to the scattering environment, that is, the power field distribution, such as that in Fig. 1. We first establish some basic properties of such fields.

III. REVIEW OF PLANE WAVES

We briefly review the concept of plane waves [5], their properties and how they can represent a multipath transmission environment. A plane wave is generally associated with a spherical or cylindrical sinusoidal signal source. If the receiver is a far enough distance from such a source, such that it is considered to be in the far-field of the source, the wavefront curvature at the receiver can be considered to be flat such that the wavefront is effectively planar. Hence, it is called a plane wave.

Consider a plane wave, indexed by \(p\), propagating through a region in space, with wavelength, \(\lambda\). Let \(\eta_p = [\cos \theta_p, \sin \theta_p]\) be the propagation direction unit vector and \(\psi_p\) be the phase at the origin. Let \(r_0 = [r_0, \theta_0]\) be the position vector of a random point in space in polar coordinates. The difference between the phases of the propagating wave at the origin and at \(r_0\) is \(\psi_{p0} = -\langle 2\pi / \lambda \rangle (r_0, \eta_p)\) where \(\langle \cdot \rangle\) denotes inner product. If \(a_p > 0\) is the plane wave magnitude then the complex amplitude of the plane wave at \(r_0\) is

\[
A_{p0}(t) = a_p e^{-j(\frac{2\pi}{\lambda}(r_0, \eta_p) + \psi_{p0})} = a_p e^{-j\phi_{p0}} \tag{1}
\]

where \(\phi_{p0} = \psi_{p0} + \psi_{p0}\) is the absolute phase at \(r_0\). We have ignored time dependence as it has no bearing on the results.

In this paper we assume a 2D model in which all plane waves are travelling in the \(x-y\)-plane. This models the situation where the scatterers are local to the transmitter and both are in the far-field of the receiver, as is commonly the case in wireless communications.

A. General Average Power

Consider the superposition of \(P\) plane waves of a common single frequency in a region of space. If the region is small enough that the amplitude of each can be considered to remain constant then the average power at a given point, \(r_0\), in this region can be shown to be

\[
P_{av} = \frac{1}{2} \sum_{p=1}^{P} a_p^2 + \sum_{p=1}^{P-1} \sum_{q=p+1}^{P} a_p a_q \cos(\phi_{p0} - \phi_{q0}). \tag{2}
\]

B. Average Power for \(P = 1\)

We include the \(P = 1\) case to show that a spatial diversity receiver offers little benefit in this case, emphasizing the utility of a multipath environment. Taking \(P = 1\) in (2), the second term vanishes to give a single plane wave, \(a_1 e^{-j\phi_1}\), an average power of \(P_{av} = a_1^2/2\) at every point in the region. The average power is independent of position as we would expect.

C. Zero Average Power Conditions for \(P = 2\)

We consider the \(P = 2\) case for the insights it offers into the importance of the relative values of the complex amplitudes and propagation directions of the constituent multipaths which can be extended to more complex fields. Taking \(P = 2\) in (2) the average power at a point \(r_0\), is given by

\[
P_{av} = \frac{1}{2} (a_1^2 + a_2^2) + a_1 a_2 \cos(\phi_{10} - \phi_{20}). \tag{3}
\]

It can be shown that points of zero power are achieved only when the following conditions hold

\[
a_1 = a_2, \tag{4}
\]

\[
\phi_{10} = \psi_{20} + (2m + 1)\pi. \tag{5}
\]

We use these conditions later to determine how a linear detector must weight the individual paths to zero the interfering user’s output power.

D. Minimizing \(P_{av}\) Peak Separation, \(P = 2\)

The average power in a field generated by two distinct plane waves has a sinusoidal variation. Earlier, we compared the behaviour of a linear detector, based upon spatial characteristics, to manipulation of the average field power. In fact, with desired and interfering users with almost identical multipath signatures in a two multipath field, we show later that the MMSE detector manipulates the field so that the average power at the lowest point of the sinusoidal troughs is zero, using the conditions from the previous section, and that a zero power point coincides with the position of the interfering user. We can think of this as the linear detector effectively “shifting” the field until a trough coincides with the interfering user’s position.

This perfect shifting only occurs for asymptotic SNRs. For more practical SNR levels, noise inhibits the amount the detector can shift the field, as we show later. Intuitively, then, the smaller the shortest horizontal distance between adjacent peaks or troughs, the easier it is for a detector to shift a trough to coincide with the position of the interfering user. If the peak power levels are also kept constant, the power slope is steeper so a smaller shift brings about a greater drop in power, aiding in attaining a higher SINR in low SNR conditions. Thus, it is of interest to determine the conditions which deliver the shortest distance between adjacent peaks or troughs.

Positions of maximum (peak) or minimum (trough) power can be determined by taking the derivative of (3) with respect to \(r_0\) and \(\theta_0\). It can be shown [6] that the distance between two extrema is bounded by

\[
d \geq 0.25\lambda \tag{6}
\]

Equality is obtained in (6) when the propagation directions of the two multipaths are related by

\[
|\theta_1 - \theta_2| = (2n + 1)\pi \quad n \in \mathbb{Z}. \tag{7}
\]

Adjacent extrema implies that one must be a maximum and the other a minimum. Thus, the shortest distance between two peaks, or troughs, in the average power of the sum of a pair
of plane waves in 2D space is \(\lambda/2\) when the plane waves are travelling in opposite directions. The closer their propagation directions are, the larger the distance between average power extrema becomes. This is illustrated in Fig. 2 for random \(\theta_1\) with \(|\theta_1 - \theta_2| = 5^\circ, 20^\circ, 60^\circ\) and \(180^\circ\). The propagation directions are indicated in the top right corner of each graph. For \(|\theta_1 - \theta_2| = 180^\circ\), the minimum distance between adjacent troughs or peaks is, indeed, \(\lambda/2\).

The implication for linear detection is that the further apart are the angles of arrival (AOAs) of two multipaths at the receiver, the closer together are the extrema and the easier it is to shift the multipath field so that the interfering user’s position coincides with a minimum. This implies that for \(P > 2\) the paths should have uniformly distributed propagation directions.

### IV. Notation Preliminaries

Before using the results of the previous section to analyse linear detector behaviour, we establish some notation.

#### A. Representation of Users’ Multipath Signals

If \(P\) is the number of significant scatterers in the signal environment, then the received signal attributable to any user is the superposition of \(P\) multipaths. The desired user and an interferer a distance, \(d \sim \lambda\), from it have almost identical multipath signatures at the receiver. The magnitudes and AOAs of each corresponding pair of multipaths are identical, with only the phases differing by a factor \(\Delta \phi_p\). We may write the complex amplitudes of the desired and interfering users’ \(p^{th}\) multipath signals, at the receiver, as

\[
(a_1)_p = a_p e^{-j\phi_p}, \quad p \in \{1, \cdots, P\}
\]

\[
(a_2)_p = a_p e^{-j(\phi_p + \Delta \phi_p)},
\]

where \(a_p\) is the magnitude and \(\phi_p\) is the phase of the desired user’s \(p^{th}\) multipath signal. This thus establishes notation for the users’ signals to be used in later sections.

#### B. Receiver Signal Processing

The antenna array output and raw estimate (i.e. prior to quantization to a bit value) are given by

\[
y = Ga_1b_1 + Ga_2b_2 + n = c_1b_1 + c_2b_2 + n \quad (10)
\]

\[
v = h_1^Hy,
\]

where \(G\) is the \(L \times P\) array response matrix, \(L\) is the number of array sensors, \(b_1\) and \(b_2\) are the information bits of the desired and interfering users’ signals, respectively, for the given time interval, \(n\) is an AWGN vector and \(h_1\) is the \(L \times 1\) estimate coefficient vector of the incumbent linear detector. Having established notation, we can now proceed to analysis of linear detector behaviour in a multipath environment.

#### V. Detector Behaviour, \(P = 1\)

While not strictly a multipath scenario, we consider the behaviour of a generic linear detector in a one path field with respect to its effect on the SIR and SNR.

##### A. Input Power Ratios

From (8), (9) and (2) with \(P = 1\), and with noise variance, \(\sigma_n^2\), the average input power of each of the users’ signals and the input SNR are given by

\[
P_{\text{in}} = a_1^2/2 \quad \text{SNR}_{\text{in}} = a_1^2/2\sigma_n^2 \quad (12)
\]

The input SIR is unity. We use these values to analyse the behaviour of a linear detector in a one path field.

##### B. Output Power Ratios

To determine the output powers we need to know the raw estimate. Using (8)-(11) the raw estimate is

\[
v = h_1^Hy = h_1^H G a_1 e^{-j\phi_1}b_1 + h_1^H G a_2 e^{-j(\phi_1 + \Delta \phi_1)}b_2 + h_1^H n \quad (13)
\]

Now, at the receiver, the detector’s estimate coefficient vector, \(h_1\), weights the array response, \(G\). We refer to this as the weighted array response, \(h_1^H G\). In the one path field case, for each of the users’ signals it is \(h_1^H G = w_1 e^{j\xi_1}\). Now, we rewrite (13) as follows

\[
v = u_1 e^{j\gamma_1}b_1 + u_1 e^{j(\gamma_1 - \Delta \phi_1)}b_2 + h_1^H n \quad (14)
\]

where \(u_1 = w_1 a_1\) is the magnitude of both the desired and interfering users’ output signals and \(\gamma_1 = \xi_1 - \phi_1\) is the phase of the desired user’s output signal.

To determine the effect of a linear detector on the SIR and SNR, we must compare input and output powers. From (14), the output powers of both users’ signals are identical and equal to

\[
P_{\text{out}} = u_1^2/2 \quad (15)
\]

so that the output SIR, like the input SIR, is equal to 1. Thus, the SIR is unchanged by a linear detector when close users are in a one path field, as we would expect.

From (13) the output SNR is

\[
\text{SNR}_{\text{out}} = \frac{a_1^2 ||h_1^H G||^2}{2\sigma_n^2 ||h_1||^2} \neq \text{SNR}_{\text{in}}. \quad (16)
\]

Thus, a linear detector has a non-trivial effect on the SNR when there is only one multipath.
C. MMSE Detector Behaviour, \( P = 1 \)

We now show the effect of the MMSE linear detector with a two sensor array on the SNR in a one path field, based upon the results in the previous section. We choose the two sensor case for the insights it gives us.

A two sensor array with a MMSE detector has

\[
G = \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} = \begin{bmatrix} e^{j\phi_1} \\ e^{-j\phi_1} \end{bmatrix}, \quad h_1^H = \frac{c_1}{4a_1^2 + \sigma_n^2}.
\]

That is, without loss of generality, we assume that the array sensors impart a unity gain to the incoming signal and equal, but opposite, phases \( \psi_1 \) and \(-\psi_1\).

Now, from (13) and (16)

\[
\text{SNR}_{\text{out}} = \frac{\|h_1^H G a_1\|^2}{2\sigma_n^2\|h_1\|^2} = 2\text{SNR}_{\text{in}}.
\]

Thus, with 2 sensors in a one path field the MMSE detector doubles the SNR. Simulations, show that with \( L \) sensors, the SNR improves \( L \)-fold. So, while with only one path the SIR cannot be changed by an antenna array with a linear detector, the SNR benefits from the increased receiver signal power over a single sensor.

VI. DETECTOR BEHAVIOUR, \( P = 2 \)

We examine the behaviour of a generic linear detector in a two multipath field as this offers insights into the desirable attributes of a general multipath field.

A. Zeroing the Interfering User’s Output Power

We now derive expressions for the users’ output powers. From (8) and (9), the desired and interfering users’ transmitted signal vectors for \( P = 2 \) are given by

\[
a_1 = \begin{bmatrix} a_1 e^{-j\phi_1} \\ a_2 e^{-j\phi_2} \end{bmatrix}, \quad a_2 = \begin{bmatrix} a_1 e^{-j(\phi_1 + \Delta\phi_1)} \\ a_2 e^{-j(\phi_2 + \Delta\phi_2)} \end{bmatrix}.
\]

Similarly to the one multipath case in (13), we can write the raw estimate for a two multipath field as

\[
v = (u_1 e^{j\gamma_1} + u_2 e^{j\gamma_2})b_1 + (u_1 e^{j(\gamma_1 - \Delta\phi_1)} + u_2 e^{j(\gamma_2 - \Delta\phi_2)})b_2 + h_1^H n
\]

where \( u_p e^{j\gamma_p} \) and \( u_p e^{j(\gamma_p - \Delta\phi_p)} \) are the complex output signal amplitudes of the desired and interfering users’ \( p \)-th multipaths, respectively. It can be seen from (20) that the complex output signal amplitude for each user is the sum of a pair of plane waves. Thus, from (3) the average output powers of the users’ signals are

\[
P_{1_{\text{out}}} = \frac{1}{2}(u_1^2 + u_2^2) + u_1 u_2 \cos(\gamma_1 - \gamma_2)
\]

\[
P_{2_{\text{out}}} = \frac{1}{2}(u_1^2 + u_2^2) + u_1 u_2 \cos[(\gamma_1 - \Delta\phi_1) - (\gamma_2 - \Delta\phi_2)]
\]

where \( P_{1_{\text{out}}} \) and \( P_{2_{\text{out}}} \) are the average output powers, respectively, of the desired and interfering users’ signals.

Now, comparing (22) with (3) and using the conditions in (4) and (5) we deduce that for \( P_{2_{\text{out}}} \) to be 0, the following conditions must hold.

\[
u_1 = u_2 \quad \Rightarrow \quad \frac{w_1}{w_2} = \frac{a_2}{a_1}
\]

\[
(\gamma_1 - \Delta\phi_1) - (\gamma_2 - \Delta\phi_2) = (2r + 1)\pi, \quad r \in \mathbb{Z}
\]

Thus, from (23) the ratio of the sensor weights must be in inverse proportion to the ratio of the multipath amplitudes. And, from (24), the phases imposed by the receiver must be such that the cosine term in (22) cancels the two squared amplitude terms. This occurs when the following relationships hold.

\[
\gamma_1 - \gamma_2 = \Delta\phi_1 - \Delta\phi_2 + (2r + 1)\pi, \quad r \in \mathbb{Z}
\]

\[
\xi_1 - \xi_2 = (\phi_1 - \phi_2) - (\Delta\phi_1 - \Delta\phi_2) + (2r + 1)\pi
\]

We use these relationships in later sections.

B. Maximizing Desired User’s Output \( P_{aw} \)

In addition to minimizing \( P_{2_{\text{out}}} \) it is of interest to know under what conditions a linear detector maximizes \( P_{1_{\text{out}}} \). From (21), the following condition must hold.

\[
\gamma_1 - \gamma_2 = 2r\pi, \quad r \in \mathbb{Z}
\]

From (25) and (27) to both zero \( P_{2_{\text{out}}} \) and maximize \( P_{1_{\text{out}}} \), we require

\[
\Delta\phi_1 - \Delta\phi_2 = (2s + 1)\pi, \quad s \in \mathbb{Z}.
\]

That is, if \( P_{2_{\text{out}}} \) is zeroed, then \( P_{1_{\text{out}}} \) is maximized when we have a maximum difference, in a circular sense, in the phase differences for the two pairs of multipaths. As \( \Delta\phi_p \) is dependent upon the distance \( d \) between the users, we can determine the minimum \( d \) at which (28) is achieved. It can be shown [6] that

\[
d \geq 0.25\lambda.
\]

So, the two users must be at least a 0.25\( \lambda \) apart before it is possible to both zero \( P_{2_{\text{out}}} \) and maximize \( P_{1_{\text{out}}} \). This dual goal is achieved when the difference between the phase differences of the two pairs of paths is equal to \( \pi \). This is analogous to the result in (6) where it was found that the minimum distance between a peak and a trough in the average power pattern of the superposition of two multipaths was \( \lambda/4 \) when the directions of propagation of the two paths were also \( \pi \) apart. Note that minimizing \( P_{2_{\text{out}}} \) is the priority of the MMSE detector and maximization of \( P_{1_{\text{out}}} \) occurs only by chance, as it is dependent upon the quantities \( \Delta\phi_1 \) and \( \Delta\phi_2 \) which are beyond the control of the detector.

VII. MMSE DETECTOR BEHAVIOUR, \( P = 2 \)

We now consider the specific response of the MMSE linear detector to the signals of two close users in a two multipath environment using a two sensor array.
A. Zeroing Interfering User’s Output \( P_{\text{av}} \)

We can write the MMSE estimate coefficient vector as

\[
h_1 = (c_1 e_1^H + c_2 e_2^H + \sigma_n^2 I_L)^{-1} c_1. \tag{30}\]

The weighted array response vector for two close users in a two path field with asymptotic SNR can be written as

\[
h_1^H G = \begin{bmatrix} 2a_1a_2 e^{j\phi_1}(1 - \cos(\Delta\phi_1 - \Delta\phi_2))(1 - \cos 2(\theta_1 - \theta_2)) \\ 2a_1^2a_2 e^{j\phi_2}(1 - \cos(\Delta\phi_1 - \Delta\phi_2))(1 - \cos 2(\theta_1 - \theta_2)) \end{bmatrix}^T \tag{31}\]

where \( c \) is a constant, \( \theta_1 \) and \( \theta_2 \) are the phase shifts imposed on multipaths \( p = 1 \) and \( 2 \), respectively, by sensor \( 1 \) and \( w_1 \) and \( \zeta_p \) are defined in Section VI-A. The required relationships in (23) and (26) are indeed met for the MMSE detector in (31), so that \( P_{2\text{out}} \) is zeroed in conditions of asymptotic SNR.

As simplification of \( h_1^H G \) is already long-winded for 2 sensors, simulations were carried out for \( L > 2 \). The results for all simulations were also in accordance with the conditions in (23) and (26).

B. MMSE Behaviour with Varying SNR

We now examine the performance of the MMSE detector over the full range of SNR values. We again limit ourselves to the two sensor case in the name of computational simplicity.

It is well known that the temporally based MMSE maximizes the SINR [7]. We show that this is the case for the spatially based MMSE detector also. At low SNRs the noise terms dominate and the MMSE is effectively a matched filter (MF), maximizing the SNR [8].

\[
h_1 = 1/\sigma_n^2 [c_{11} c_{21}]^T. \tag{32}\]

At high SNRs the noise terms can be ignored and the MMSE is effectively a decorrelator, maximizing the SIR [7].

\[\text{Fig. 3. Changes in } (w_1/w_2)/(a_1/a_2), \text{ and distance, in degrees, of the cosine term argument of } P_{2\text{int}} \text{ in (22)} \mod \pi, \text{ with SNR, for } d = 0.25\lambda.\]

In Section VI-A we showed that to zero \( P_{2\text{out}} \) a linear detector must equalize the processed magnitudes of the two multipaths, \( \langle w_2/w_1 \rangle = (a_1/a_2) \). We have shown that the MMSE detector behaves in exactly this way under conditions of asymptotic SNR. Figure 3 shows the variation of this ratio with SNR for the MMSE detector. The horizontal line near the top of the graph indicates the expected value of this ratio at low SNRs when \( h_1 \) is effectively given by (32). As the SNR increases and the noise ceases to dominate the ratio approaches unity as expected.

It was also shown that to zero \( P_{2\text{out}} \) (25) must hold. Figure 3 also shows the deviation of the difference of the phase differences, \( \cos \pi \mod \pi \), from an odd integer multiple of \( \pi \) with SNR for the MMSE detector. As the SNR increases, the deviation from \( \pi \) approaches 0, as expected. Thus, under asymptotic SNR conditions, the MMSE detector performs as well as could be expected for a linear detector.

Figure 4 shows the response of the MMSE receiver to a two multipath field for two close users with increasing SNR values. The white bands indicate lines of maxima and minima with the minima flanked by darker strips than the maxima. For asymptotic SNR (100dB) \( P_{2\text{out}} \) is effectively zero while the SNR has decreased slightly to 95.78dB, similarly in the SNR \( = 30\)dB case, the SNR has decreased slightly to 25.79dB. For the SNR \( = 0\)dB case, \( P_{2\text{out}} \) is no longer zero but the output SNR is slightly larger than the input SNR, so the MMSE detector has tackled both the noise and the interference.

\[\text{Fig. 4. } P_{\text{av}}, \text{ for the desired (*) and interfering (o) users and output } P_{\text{av}}, \text{ for SNR: 100dB, } 30\text{dB and } 0\text{dB.}\]

VIII. Minimum \( P \) for Maximum Phase Diversity

In this section we seek the minimum number of paths for maximum phase diversity in the users’ multipath signature differences. For very close users, the only parameter that offers any diversity between their respective multipath signatures is the phase difference between each corresponding pair of multipaths. For maximum diversity, then, as diverse a range of phase differences as possible is required.

\[\text{Fig. 5. PDF and CDF of multipath propagation directions}\]

The phase difference, \( \Delta \phi_p \), between the interfering and desired users’ \( p^{th} \) paths, is related to the distance, \( d \), between
the users, and the angle, $\alpha_p$, as shown in Fig. 6, $\Delta \phi_p = (2\pi d/\lambda) \sin \alpha_p$. We can deduce that

$$-2\pi d/\lambda \leq \Delta \phi_p \leq 2\pi d/\lambda.$$  (33)

So, for the maximum range of phase differences for a given $d$, $\alpha_p$ must take on both the values $\pm \pi/2$. It can be seen from Fig. 6 that this can only occur when two scatterers are positioned $\pi$ radians from each other, with respect to the position of the desired user. That is, at least two multipaths must have propagation directions opposite to each other.

We would therefore like to ensure, to a chosen high probability, that, given the propagation direction of the first path, the propagation directions of the other $P-1$ multipaths are within a range no less than $\pi/2$ radians either side of it. We assume a uniform spread of multipath propagation directions between $-\pi$ and $\pi$. We define a field as offering sufficient phase diversity when the propagation directions of all paths are in an angular range of $\geq \pi$ radians is at least 99.9%.

![Fig. 6. Phase difference between the $p^{th}$ multipath of the desired and interfering users.](image)

The probability distribution function (PDF), $f(\theta)$, corresponding to the propagation direction of a random path is shown in Fig. 5. Let $F(\theta)$ be the cumulative distribution function (CDF) of the propagation direction such that

$$F(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\theta} dt = \frac{\theta + \pi}{2\pi}.$$  (34)

The probability that the second path is within $\pm \Delta \theta$ of the first path is given by

$$P(\theta_1 - \Delta \theta \leq \theta_2 \leq \theta_1 + \Delta \theta) = F(\Delta \theta) - F(-\Delta \theta) = \frac{\Delta \theta}{\pi}.$$  (35)

where the middle term in (35) holds because of the circular nature of $\theta$ and the linear nature of $F(\theta)$. Because the propagation directions of all paths are independently distributed the probability that $P-1$ paths are in the given angular range from the first path is given by

$$P(\theta_1 - \Delta \theta \leq \theta_2, \theta_3, \cdots, \theta_P \leq \theta_1 + \Delta \theta) = \left(\frac{\Delta \theta}{\pi}\right)^{P-1}. \quad (36)$$

We note that for $\Delta \theta = \pi/2$, the probability that all paths are within the range $\pm \pi/2$ from the first path is $2^{-(P-1)}$. That is, for each path that is added, the probability that all paths are within a total angular range of $\pi$ is halved.

Figure 7 shows plots of (36) for $\Delta \theta = \pi/4, \pi/2$ and $3\pi/4$. It can be seen that the minimum number of paths required for a probability of 99.9% (0.001) that all paths are within an angular range greater than $\Delta \theta = \pi/2$ is 11.

IX. CONCLUSIONS

In this paper we have analysed, in detail, the effects on the field power distribution of the relative values of multipath field parameters, amplitude, phase and propagation direction, in one and two multipath fields. We have applied the results of this analysis to the study of the behaviour of linear detectors in the spatial domain with particular emphasis on the MMSE detector. This analysis has given us a theoretical basis for the $P = 2$ case and insights into the important parameters in a general ($P > 2$) sparse multipath field. We observe that the path based approach, while insightful for fields with few multipaths, quickly becomes mathematically cumbersome as the number of multipaths is increased. A more global approach is required for fields with larger numbers of paths (multipath rich) and is addressed in [9].

REFERENCES


