ELECTROMAGNETISM SUMMARY

➤ Magnetostatics
➤ Ampere’s law and Faraday’s law
➤ Maxwell’s equations
➤ Waves
Magnetostatics: The static magnetic field

➤ Gauss’s law for the magnetic field:

\[ \oint_{A} \mathbf{B} \cdot d\mathbf{A} = 0 \]

➤ There is no static sink or source of the magnetic field. Also generally true.

➤ However **current** is the source of a static magnetic field,

\[ \oint_{\gamma} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \]

The line integral of \( \mathbf{B} \) around a closed circuit \( \gamma \) bounding a surface \( A \) is equal to the current flowing across \( A \). **Ampere’s law**
Important Correction to Ampere’s Law

A time varying Electric field is also a source of the time varying magnetic field,

\[ \oint_{\gamma} B \cdot dl = \int_{A} \mu_0 \left( j + \epsilon_0 \frac{\partial E}{\partial t} \right) . dA \]
Why this Correction to Ampere’s Law?

➤ Ampere’s law without E contradicts charge conservation

$$\oint_{\gamma} \mathbf{B} \cdot d\mathbf{l} = \int_{A} \mu_0 \left( \mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{A}$$

➤ Consider the new Ampere’s law on a close surface area, A.
Why this Correction to Ampere’s Law?

If we shrink the closed contour $\gamma$ on the left hand side to zero then we obtain

$$0 = \oint A \mu_0 \left( j + \varepsilon_0 \frac{\partial E}{\partial t} \right) \cdot dA$$

However the two terms on the right hand side are

$$\oint A E \cdot dA = \frac{q}{\varepsilon_0}$$

and

$$\oint A j \cdot dA = -\frac{\partial q}{\partial t}$$
Faraday’s Law

- The law of electromagnetic induction or Lenz’s law
- **A time varying magnetic field** is also the source of the electric field,

\[ \oint_{\gamma} \mathbf{E} \cdot d\mathbf{l} = - \int_{A} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \]
Maxwell’s Equations: Integral Form

➤ **Gauss’s law for the electric field.** Charge is the source of electric field:

\[
\oint_{A} \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0}
\]

➤ **Faraday’s law.** A changing magnetic flux causes an electromotive force:

\[
\oint_{\gamma} \mathbf{E} \cdot d\mathbf{l} = -\int_{A} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}
\]

➤ **Gauss’s law for the magnetic field.** Magnetic fields are source free:

\[
\oint_{A} \mathbf{B} \cdot d\mathbf{A} = 0
\]

➤ **Ampere’s law:**

\[
\oint_{\gamma} \mathbf{B} \cdot d\mathbf{l} = \int_{A} \mu_0 \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{A}
\]
Maxwell’s Equations: Differential Form

➤ **Gauss’s law for the electric field.**

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

➤ **Faraday’s law.**

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

➤ **Gauss’s law for the magnetic field.**

\[ \nabla \cdot \mathbf{B} = 0 \]

➤ **Ampere’s law:**

\[ \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \]
EM Waves

➤ Start with Faraday’s law and Ampere’s law in VACUO.

\[
\oint_{\gamma} \vec{E} \cdot d\vec{l} = - \int_{A} \frac{\partial \vec{B}}{\partial t} \cdot dA
\]

\[
\oint_{\gamma} \vec{B} \cdot d\vec{l} = \int_{A} \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot dA
\]

➤ Recall that the line integral along \( \gamma \) is on the perimeter of the surface \( A \).

➤ Thus a one dimensional \( \vec{E} \) and \( \vec{B} \) must have \( \vec{E} \cdot \vec{B} = 0 \).

➤ Moreover \( \vec{E} \times \vec{B} \) points in the direction of propagation.
Monochromatic Waves 1

- Apply F and A to the diagrams below and note that A points in the direction of the right hand screw rule with respect to the direction of γ.
Monochromatic Waves 2
Monochromatic Waves 3

➤ Integrating $\mathbf{E}$ around $\gamma$ in the left figure...

$$[E(z + dz) - E(z)] L = -\frac{\partial B}{\partial t} L \Delta z$$

$$\frac{\partial E}{\partial z} L \Delta z = -\frac{\partial B}{\partial t} L \Delta z$$

➤ Integrating $\mathbf{B}$ around $\gamma$ in the right figure...

$$[B(z + dz) - B(z)] L = -\varepsilon_0 \mu_0 \frac{\partial E}{\partial t} L \Delta z$$

$$\frac{\partial B}{\partial z} L \Delta z = -\varepsilon_0 \mu_0 \frac{\partial E}{\partial t} L \Delta z$$
Monochromatic Waves 4

Two equations for $E$ and $B$

\[
\frac{\partial E}{\partial z} = -\frac{\partial B}{\partial t}
\]
\[
\frac{\partial B}{\partial z} = -\varepsilon_0\mu_0\frac{\partial E}{\partial t}
\]

Simultaneous solution gives the wave equation for each field:

\[
\frac{\partial^2 E}{\partial z^2} = \varepsilon_0\mu_0\frac{\partial^2 E}{\partial t^2}
\]
\[
\frac{\partial^2 B}{\partial z^2} = \varepsilon_0\mu_0\frac{\partial^2 B}{\partial t^2}
\]
Solution of the Wave Equation 1

➤ Two equations for \( E \) and \( B \)

\[
E = E(X); \quad B = B(X)
\]

where \( X = z \pm v.t \)

\[
\frac{\partial^2 E}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 E}{\partial t^2}
\]

\[
\frac{\partial^2 B}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 B}{\partial t^2}
\]

➤ The solution of the wave equation is an arbitrary function of the displacement argument \( X = z \pm v.t \)
Solution of the Wave Equation 1

\[ \frac{\partial E}{\partial z} = -\frac{\partial B}{\partial t} \Rightarrow E'(X) = \mp vB'(X) \]

\[ \frac{\partial B}{\partial z} = -\varepsilon_0\mu_0 \frac{\partial E}{\partial t} \Rightarrow B'(X) = \mp v\varepsilon_0\mu_0 E'(X) \]

Thus \( E(X) = cB(X) \) and \( v = c = \sqrt{\frac{1}{\varepsilon_0\mu_0}}. \)

The solution of the wave equation is an arbitrary function of the displacement argument \( X = z \pm c.t \)
Solution of the Wave Equation 2

- Solution as a function of z for four different times...