Technical Report

Quantized Output-feedback Control of Nonlinear Systems: A Cyclic-small-gain Approach

Tengfei Liu, Zhong-Ping Jiang, and David J. Hill

Abstract

This paper presents a fundamentally novel approach to quantized output-feedback control of nonlinear systems. The advances made toward this challenging, yet important, problem are a seamless integration of dynamic quantization and the recently developed cyclic-small-gain theorem. The constructed output-feedback quantized controller guarantees the input-to-state stability (ISS) property for the closed-loop quantized system. More interestingly, an ISS-Lyapunov function is derived and is used in the implementation of output dynamic quantization, even with a 1-bit quantizer.

Index Terms

Quantized feedback control, output-feedback, nonlinear systems, input-to-state stability (ISS), small-gain.

I. INTRODUCTION

Over the past few years, quantized feedback control has gained intensive attention due to the convergence of controls and communications. Dynamic quantization was developed to deal with problems caused by the finite word length of the quantizers. The basic idea of dynamic quantization is to appropriately scale the quantization levels such that the input of the quantizer is always covered by the range of the quantizer, and at the same time, the state of the control system converges to the origin. Recent results of dynamically quantized control of networked systems and nonlinear systems can be found in [8], [10].

The input-to-state stability (ISS) methodology (see [16]) and ISS small-gain theorem [4], [3] has been introduced to quantized control to characterize the robustness of quantized control systems with respect to quantization errors in [10]. Quantized output feedback control has also been studied in [9], [11] for linear and nonlinear systems. Specifically, the author of [11] presented an ISS-based framework for quantized output-feedback control.

This paper attempts to develop a fundamentally new design tool for quantized output-feedback control of nonlinear systems, which is general enough to recover previous studies in the recent literature. The novelty of the proposed approach lies in the employments of: (i) set-valued maps to cover the uncertainty caused by the quantization error;

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(ii) modified gain assignment technique [4] as a tool for control design; (iii) our recent result of cyclic-small-gain theorem [6], [12] to guarantee stability and to construct an ISS-Lyapunov function. Dynamic quantization is designed based on the ISS-Lyapunov function. Our result shows that the output of the quantized control system can be steered to within an arbitrarily small neighborhood of the origin even with a 1-bit uniform quantizer.

To make the paper self-contained, recall that a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is positive definite if it is continuous, $\gamma(s) > 0$ for all $s > 0$ and $\gamma(0) = 0$. $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}$ function (denoted by $\gamma \in \mathcal{K}$) if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty$ function (denoted by $\gamma \in \mathcal{K}_\infty$) if it is a $\mathcal{K}$ function and also satisfies $\gamma(s) \to \infty$ as $s \to \infty$. Id represents the identity function. I_n represents the identity matrix of size n. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ mean the largest and the smallest eigenvalues of a real and symmetric matrix P, respectively.

II. PROBLEM FORMULATION

Consider the disturbed output-feedback nonlinear system with quantized output:

$$\dot{x}_i = x_{i+1} + f_i(y, d), \quad 1 \le i \le n-1 \tag{1}$$

$$\dot{x}_n = u + f_n(y, d) \tag{2}$$

$$y = x_1 \tag{3}$$

$$y^q = q(y, \mu) \tag{4}$$

where $[x_1, \dots, x_n]^T \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}^{n_d}$ represents external disturbance inputs, $y \in \mathbb{R}$ is the output, $q(y, \mu)$ is the output quantizer with variable $\mu > 0$, $y^q \in \mathbb{R}$ is the quantized output, $[x_2, \dots, x_n]^T$ is the unmeasured portion of the state, and f_i's ($1 \le i \le n$) are uncertain locally Lipschitz continuous functions.

Remark 1: It is worth noting that the generality and importance of output-feedback nonlinear systems (1)–(3) has been well outlined in [7] and numerous references therein. However, little research has been done from a viewpoint of quantized feedback control due to technical challenges.

Output quantizer $q(y, \mu)$ is defined as $q(y, \mu) = \mu q_0\left(\frac{y}{\mu}\right)$, where $q_0 : \mathbb{R} \to \Omega$ is a piece-wise constant function satisfying $|q_0(r) - r| \le 1$ if $|r| \le M$. We directly use the following property of $q(y, \mu)$:

$$|y| \le M\mu \Rightarrow |q(y, \mu) - y| \le \mu, \tag{5}$$

where $M > 0$, $M\mu$ is the quantization range and $\mu > 0$ is the maximum quantization error when $|y| \le M\mu$.

Remark 2: In [9], quantizers are formulated as $|y| \le M\mu \Rightarrow |q(y, \mu) - y| \le \Delta\mu$ with $\Delta > 0$. It is of interest to note that (5) is in accordance with the formulation in [9]. Indeed, if $\Delta \ne 1$, define $\mu' = \Delta\mu$, $M' = M/\Delta$ and $q'(y, \mu') = \frac{\mu'}{\Delta} q_0\left(\frac{\Delta y}{\mu'}\right)$. Then, $|y| \le M'\mu' \Rightarrow |q'(y, \mu') - y| \le \mu'$.

Given fixed M, the basic idea of dynamic quantization is to dynamically update μ (and thus $M\mu$) in the process of quantized control to improve the control performance. Variable μ is referred to as zooming variable. Increasing μ enlarges the range and is called zooming-out; decreasing μ reduces the range and is called zooming-in.

In this paper, it is assumed that μ is right-continuous with respect to time and is updated in discrete-time as:

$$\mu(t_{k+1}) = Q(\mu(t_k)), \quad k \in \mathbb{Z}_+ \tag{6}$$

where $Q : \mathbb{R}_+ \to \mathbb{R}_+$ represents the dynamic quantization logic, and $t_k \geq 0$ with $k \in \mathbb{Z}_+$ are updating times satisfying $t_{k+1} - t_k = d_t$ with $d_t > 0$.

Assumptions 1–4 are made throughout the paper.

Assumption 1: System (1)–(3) with $u = 0$ is forward complete and small-time norm-observable with y as the output.

Remark 3: Assumption 1 is needed only for zooming-out. Refer to [1] and [11] for small-time norm-observability and its application in dynamic quantization.

Assumption 2: For each $f_i(y, d)$ ($1 \leq i \leq n$) in (1)–(2), there exists a known $\psi_{f_i} \in \mathcal{K}_\infty$ such that

$$|f_i(y, d)| \leq \psi_{f_i}(|(y, d)|), \quad \forall y \in \mathbb{R}, \ \forall d \in \mathbb{R}^{n_d}. \tag{7}$$

Assumption 3: The external disturbance d in (1)–(2) is allowed to take values in $\bar{d}\mathcal{B}$ with constant $\bar{d} \geq 0$ and $\mathcal{B}$ the unit ball in $\mathbb{R}^{n_d}$.

Assumption 4: Quantizer $q(y, \mu)$ satisfies (5) with $M > 1$.

Remark 4: Assumption 4 is not restrictive. $M > 1$ means that quantization range $M\mu$ is strictly larger than the maximal quantization error μ when $|y| \leq M\mu$. This is satisfied even by the 1-bit uniform quantizer [8] shown in Fig. 1.

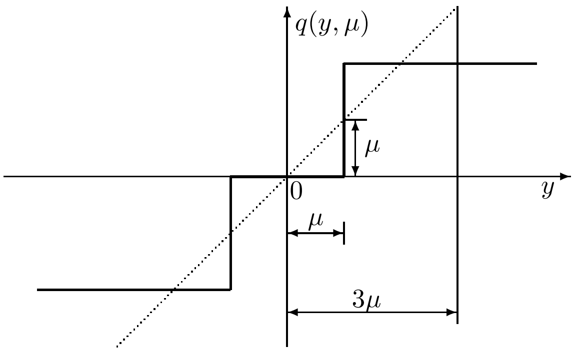

Fig. 1. The 1-bit uniform quantizer satisfying (5) with $M = 3$.

Definition 1 ([16]): For a continuous-time nonlinear system $\dot{\varsigma} = f(\varsigma, \varpi_1, \ldots, \varpi_m)$, with state $\varsigma \in \mathbb{R}^{n_\varsigma}$, external inputs $\varpi_i \in \mathbb{R}^{n_{\varpi_i}}$ ($i = 1, \ldots, m$) and locally Lipschitz vector field f, a function $V_\varsigma : \mathbb{R}^{n_\varsigma} \to \mathbb{R}_+$ is an ISS-Lyapunov function if it is positive definite, radially unbounded and differentiable almost everywhere, and there exist $\chi_\varsigma^{\varpi_1}, \ldots, \chi_\varsigma^{\varpi_m} \in \mathcal{K}_\infty$ and α_ς positive definite such that

$$\begin{aligned} & V_\varsigma(\varsigma) \geq \max_{i=1,\ldots,m} \{\chi_\varsigma^{\varpi_i}(|\varpi_i|)\} \\ \Rightarrow \ & \nabla V_\varsigma(\varsigma) f(\varsigma, \varpi_1, \ldots, \varpi_m) \leq -\alpha_\varsigma(V_\varsigma(\varsigma)) \end{aligned} \tag{8}$$

holds wherever V_ς is differentiable. Function $\chi_\varsigma^{\varpi_i}$ is referred to as the ISS gain from ϖ_i to $V_\varsigma(\varsigma)$.

The objective of this paper is to design a quantized output feedback controller with dynamic quantization to steer the output y to within an arbitrarily small neighborhood of the origin.

III. A NEW TOOL FOR QUANTIZED OUTPUT-FEEDBACK CONTROL DESIGN

By convenience, denote $w = y^q - y$ as the quantization error.

A. *Reduced-order Observer Design*

We construct the following reduced-order observer:

$$\dot{\xi}_i = \xi_{i+1} + L_{i+1}y^q - L_i(\xi_2 + L_2 y^q),\ 2 \le i \le n-1 \tag{9}$$

$$\dot{\xi}_n = u - L_n(\xi_2 + L_2 y^q) \tag{10}$$

where ξ_i is an estimate for the unmeasured state $x_i - L_i y$ for each $2 \le i \le n$. Define $e_0 = [x_2 - L_2 y - \xi_2, \ldots, x_n - L_n y - \xi_n]^T$ as the observation error. Then, from (1)–(4) and (9)–(10), we get the observation error system:

$$\dot{e}_0 = Ae_0 + \phi_0(y, d, w) \tag{11}$$

where

$$A = \begin{bmatrix} -L_2 & & & \\ \vdots & & I_{n-2} & \\ -L_{n-1} & & & \\ -L_n & 0 & \cdots & 0 \end{bmatrix}, \tag{12}$$

$$\begin{aligned} \phi_0(y, d, w) = & \begin{bmatrix} -L_2 & \\ \vdots & I_{n-1} \\ -L_n & \end{bmatrix} \begin{bmatrix} f_1(y, d) \\ \vdots \\ f_n(y, d) \end{bmatrix} \\ & + \begin{bmatrix} L_2^2 - L_3 \\ \vdots \\ L_{n-1}L_2 - L_n \\ L_n L_2 \end{bmatrix} w. \end{aligned} \tag{13}$$

Real constants L_i's in (12) are chosen so that A is Hurwitz, and thus there exists a matrix $P = P^T > 0$ satisfying $PA + A^T P = -2I_{n-1}$. For ϕ_0 defined in (13), using Assumption 2, we can find $\psi_{\phi_0}^y, \psi_{\phi_0}^d, \psi_{\phi_0}^w \in \mathcal{K}_\infty$ such that $|\phi_0(y, d, w)|^2 \le \psi_{\phi_0}^y(|y|) + \psi_{\phi_0}^d(|d|) + \psi_{\phi_0}^w(|w|)$ holds for all y, d, w.

Define $V_0(e_0) = e_0^T P e_0$. Define $\underline{\alpha}_0(s) = \lambda_{\min}(P)s^2$ and $\overline{\alpha}_0(s) = \lambda_{\max}(P)s^2$ for $s \in \mathbb{R}_+$. Then, $\underline{\alpha}_0(|e_0|) \leq V_0(e_0) \leq \overline{\alpha}_0(|e_0|)$ holds for all e_0. Direct computation yields:

$$\begin{aligned}\nabla V_0(e_0)\dot{e}_0 =& -2e_0^T e_0 + 2e_0^T P\phi_0(y,d,w)\\ \leq& -e_0^T e_0 + |P|^2|\phi_0(y,d,w)|^2\\ \leq& -\frac{1}{\lambda_{\max}(P)}V_0(e_0)\\ &+|P|^2(\psi_{\phi_0}^y(|y|)+\psi_{\phi_0}^d(|d|)+\psi_{\phi_0}^w(|w|)).\end{aligned}$$

Define $\chi_0^y = 4\lambda_{\max}(P)|P|^2\psi_{\phi_0}^y$, $\chi_0^d = 4\lambda_{\max}(P)|P|^2\psi_{\phi_0}^d$ and $\chi_0^\mu = 4\lambda_{\max}(P)|P|^2\psi_{\phi_0}^w$. Then, we have

$$\begin{aligned}& V_0(e_0) \geq \max\{\chi_0^y(|y|), \chi_0^d(|d|), \chi_0^\mu(|w|)\}\\ \Rightarrow\quad & \nabla V_0(e_0)\dot{e}_0 \leq -\alpha_0(V_0(e_0))\end{aligned} \tag{14}$$

where $\alpha_0(s) = \frac{1}{4\lambda_{\max}(P)}s$ for $s \in \mathbb{R}_+$.

B. *Modified Gain Assignment Lemma*

This subsection presents a modified version of the gain assignment technique in [4], [2], [15] for recursive quantized control design. Consider the following first-order system:

$$\dot{\eta} = \phi(\eta, \delta_1, \ldots, \delta_m) + v \tag{15}$$

where $\eta \in \mathbb{R}$ is the state, $v \in \mathbb{R}$ is the control input, $\delta_1, \ldots, \delta_m \in \mathbb{R}$ represent the disturbance inputs, the nonlinear function $\phi(\eta, \delta_1, \ldots, \delta_m)$ satisfies

$$|\phi(\eta, \delta_1, \ldots, \delta_m)| \leq \psi_\phi(|(\eta, \delta_1, \ldots, \delta_m)|) \tag{16}$$

with $\psi_\phi \in \mathcal{K}_\infty$. Define $V_\eta(\eta) = \alpha_V(|\eta|)$ with $\alpha_V(s) = \frac{1}{2}s^2$ for $s \in \mathbb{R}_+$. Recall that sgn represents the standard sign function.

Lemma 3.1: Consider system (15). For any constant $0 < c < 1$, $\epsilon > 0$, $\iota > 0$ and $\chi_\eta^1, \ldots, \chi_\eta^m \in \mathcal{K}_\infty$, one can find a smooth, odd, strictly decreasing and radially unbounded κ such that if

$$v \in \{\kappa(\eta + a\delta_{m+1} + \operatorname{sgn}(\eta)\delta_{m+2}) : |a| \leq 1\} \tag{17}$$

with $\delta_{m+1}, \delta_{m+2} \in \mathbb{R}_+$, then it holds that

$$\begin{aligned}& V_\eta(\eta) \geq \max_{k=1,\ldots,m}\left\{\chi_\eta^k(|\delta_k|), \alpha_V\left(\frac{\delta_{m+1}}{c}\right), \epsilon\right\}\\ \Rightarrow\quad & \nabla V_\eta(\eta)\dot{\eta} \leq -\iota V_\eta(\eta).\end{aligned} \tag{18}$$

The proof of Lemma 3.1 is in Subsection IV-A.

Remark 5: The existence of $\delta_{m+2} \geq 0$ in (17) does not influence the ISS property of the η-system. Intuitively, the term $\operatorname{sgn}(\eta)\delta_{m+2}$ with $\delta_{m+2} \geq 0$ consolidates the negative feedback strength of η. See the proof of Lemma 3.1 for details.

C. Recursive Control Design

Define $e_1 = y$. Consider the $(e_0, e_1, \xi_2, \dots, \xi_n)$-system:

$$\dot{e}_0 = Ae_0 + \phi_0(e_1, d, w) \tag{19}$$

$$\dot{e}_1 = \xi_2 + \phi_1(e_0, e_1, d) \tag{20}$$

$$\dot{\xi}_i = \xi_{i+1} + \phi_i(e_1, \xi_2, w), \quad 2 \le i \le n-1 \tag{21}$$

$$\dot{\xi}_n = u + \phi_n(e_1, \xi_2, w) \tag{22}$$

where

$$\phi_1(e_0, e_1, d) = L_2 y + (x_2 - L_2 y - \xi_2) + f_1(y, d)$$

$$\phi_i(e_1, \xi_2, w) = L_{i+1} y^q - L_i(\xi_2 + L_2 y^q), \quad 2 \le i \le n-1$$

$$\phi_n(e_1, \xi_2, w) = -L_n(\xi_2 + L_2 y^q).$$

We get (20) from the x_1-subsystem (1) using the fact that $(x_2 - L_2 e_1 - \xi_2)$ is the first element of vector e_0. We get (21) and (22) from (9) and (10) by using $y^q = y + w = e_1 + w$.

We will construct a new $(e_0, e_1, \dots, e_n)$-system consisting of ISS subsystems obtained through a recursive design of the $(e_0, e_1, \xi_2, \dots, \xi_n)$-system. The ISS-Lyapunov function V_0 for the e_0-subsystem is defined in Subsection III-A. For $1 \le i \le n$, each e_i-subsystem will be designed with an ISS-Lyapunov function candidate

$$V_i(e_i) = \alpha_V(|e_i|) \tag{23}$$

where $\alpha_V(s) = \frac{1}{2}s^2$ for $s \in \mathbb{R}_+$. In the following discussions, we simply use V_i instead of $V_i(e_i)$ for $0 \le i \le n$. Denote $\bar{e}_i = [e_0^T, e_1, \dots, e_i]^T$ and $\bar{\xi}_i = [\xi_2, \dots, \xi_i]^T$.

In this subsection, we suppose that μ is constant and consider only the case of $|e_1| = |y| \le M\mu$. From (5), this means $|w| = |y^q - y| \le \mu$.

1) The e_0-subsystem: Define $\gamma_0^1 = \chi_0^y \circ \alpha_V^{-1}$ and $\chi_0^\mu = \chi_0^w$. Then, from (14), we have

$$\begin{aligned} & V_0 \ge \max\{\gamma_0^1(V_1), \chi_0^d(|d|), \chi_0^\mu(\mu)\} \\ \Rightarrow \quad & \nabla V_0 \dot{e}_0 \le -\alpha_0(V_0). \end{aligned} \tag{24}$$

2) The e_1-subsystem: The e_1-subsystem can be rewritten as

$$\begin{aligned} \dot{e}_1 &= \xi_2 - e_2 + (\phi_1(e_0, e_1, d) + e_2) \\ &:= \xi_2 - e_2 + \phi_1^*(\bar{e}_2, d) \end{aligned} \tag{25}$$

with the new state variable e_2 to be defined below. From Assumption 2 and the definition of ϕ_1, we can find a $\psi_{\phi_1^*} \in \mathcal{K}_\infty$ such that $|\phi_1^*(\bar{e}_2, d)| \le \psi_{\phi_1^*}(|(\bar{e}_2, d)|)$.

Define a set-valued map S_1 as

$$S_1(e_1, \mu) = \{\kappa_1(e_1 + a\mu) : |a| \le 1\} \tag{26}$$

with κ_1 smooth, odd, strictly decreasing and radially unbounded, to be determined later. State variable e_2 is defined as

$$e_2 = \begin{cases} \xi_2 - \max S_1(e_1, \mu), & \text{if } \xi_2 \geq \max S_1(e_1, \mu); \\ \xi_2 - \min S_1(e_1, \mu), & \text{if } \xi_2 < \min S_1(e_1, \mu); \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Then, we have $\xi_2 - e_2 \in S_1(e_1, \mu)$.

For any $\gamma_1^0, \gamma_1^2 \in \mathcal{K}_\infty$, choose $\chi_1^0 = \gamma_1^0 \circ \underline{\alpha}_0$ and $\chi_1^2 = \gamma_1^2 \circ \alpha_V$. Then, $\gamma_1^0(V_0) = \chi_1^0 \circ \underline{\alpha}_0^{-1}(V_0) \geq \chi_1^0(|e_0|)$ and $\gamma_1^2(V_2) = \chi_1^2 \circ \alpha_V^{-1}(V_2) = \chi_1^2(|e_2|)$. With Lemma 3.1, for any specified $0 < c_1 < 1$, $\epsilon_1 > 0$, $\iota_1 > 0$, $\gamma_1^0, \gamma_1^2, \chi_1^d \in \mathcal{K}_\infty$, we can find a smooth, odd, strictly decreasing and radially unbounded κ_1 such that the e_1-subsystem with $\xi_2 - e_2 \in S_1(e_1, \mu)$ is ISS with V_1 satisfying

$$\begin{aligned} & V_1 \geq \max\left\{\gamma_1^0(V_0), \gamma_1^2(V_2), \chi_1^d(|d|), \chi_1^\mu(\mu), \epsilon_1\right\} \\ \Rightarrow \quad & \nabla V_1 \dot{e}_1 \leq -\iota_1 V_1 \end{aligned} \tag{28}$$

where $\chi_1^\mu(s) = \alpha_V(s/c_1)$ for $s \in \mathbb{R}_+$.

Remark 6: Since κ_1 is strictly decreasing, we can explicitly represent

$$\max S_1(e_1, \mu) = \kappa_1(e_1 - \mu), \tag{29}$$

$$\min S_1(e_1, \mu) = \kappa_1(e_1 + \mu). \tag{30}$$

A set-valued map S_1 with smooth, odd, strictly decreasing and radially unbounded κ_1 and the definition of e_2 are shown in Fig. 2.

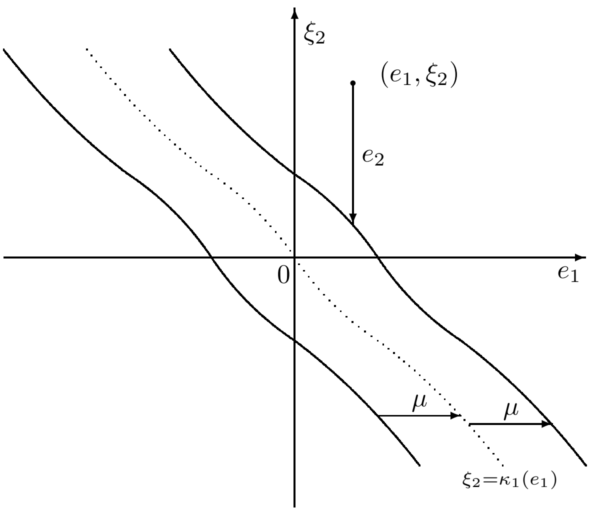

Fig. 2. Bounds of set-valued map S_1 and the definition of e_2.

Remark 7: In standard backstepping [7], ξ_2 is usually considered as the virtual control of the e_1-subsystem. To take the quantization error w into account, the virtual control law should be a function of w. However, the discontinuity of w leads to discontinuity of the virtual control law, which is hard to handle by directly taking the derivative. The set-valued map S_1 defined in (26) overcomes this problem. As shown in Fig. 2, we can just steer ξ_2 to the region with smooth bounds $\max S_1$ and $\min S_1$, instead of driving ξ_2 to some specific discontinuous virtual control law.

3) The e_i-subsystem ($2 \le i \le n$): When $3 \le i \le n$, for each $2 \le k \le i-1$, a set-valued map S_k is defined as

$$S_k(e_1, \bar{\xi}_k, \mu) = \{\kappa_k(\xi_k - p_k) : p_k \in S_{k-1}(e_1, \bar{\xi}_{k-1}, \mu)\} \tag{31}$$

where κ_k is smooth, odd, strictly decreasing and radially unbounded; the new state variable e_{k+1} is defined as

$$e_{k+1} = \begin{cases} \xi_{k+1} - \max S_k(e_1, \bar{\xi}_k, \mu), & \\ \qquad \text{if } \xi_{k+1} \ge \max S_k(e_1, \bar{\xi}_k, \mu); & \\ \xi_{k+1} - \min S_k(e_1, \bar{\xi}_k, \mu), & \\ \qquad \text{if } \xi_{k+1} < \min S_k(e_1, \bar{\xi}_k, \mu); & \\ 0, \qquad \text{otherwise.} \end{cases} \tag{32}$$

Remark 8: It is worth noting that, since κ_k is strictly decreasing, it holds:

$$\max S_k(e_1, \bar{\xi}_k, \mu) = \kappa_k(\xi_k - \max S_{k-1}(e_1, \bar{\xi}_{k-1}, \mu)), \tag{33}$$

$$\min S_k(e_1, \bar{\xi}_k, \mu) = \kappa_k(\xi_k - \min S_{k-1}(e_1, \bar{\xi}_{k-1}, \mu)). \tag{34}$$

Lemma 3.2: Consider the $(e_0, e_1, \xi_2, \ldots, \xi_n)$-system in (19)–(22) with $|e_1| \le M\mu$. With S_k and e_{k+1} defined in (26), (27), (31) and (32) satisfied for each $2 \le k \le i-1$, for any variable e_{i+1}, when $e_i \ne 0$, the e_i-subsystem can be represented as

$$\dot{e}_i = \xi_{i+1} - e_{i+1} + \phi_i^*(\bar{e}_{i+1}, d, \mu, w, \bar{\xi}_i) \tag{35}$$

where

$$|\phi_i^*(\bar{e}_{i+1}, d, \mu, w, \bar{\xi}_i)| \le \psi_{\phi_i^*}(|(\bar{e}_{i+1}, d, \mu)|) \tag{36}$$

with $\psi_{\phi_i^*} \in \mathcal{K}_\infty$. Specifically, $\xi_{n+1} = u$.

The proof of Lemma 3.2 is in Subsection IV-B.

Define a set-valued map S_i as

$$S_i(e_1, \bar{\xi}_i, \mu) = \{\kappa_i(\xi_i - p_i) : p_i \in S_{i-1}(e_1, \bar{\xi}_{i-1}, \mu)\} \tag{37}$$

with κ_i smooth, odd, strictly decreasing and radially unbounded, to be defined later. Define e_{i+1} as

$$e_{i+1} = \begin{cases} \xi_{i+1} - \max S_i(e_1, \bar{\xi}_i, \mu), & \\ \qquad \text{if } \xi_{i+1} \ge \max S_i(e_1, \bar{\xi}_i, \mu); & \\ \xi_{i+1} - \min S_i(e_1, \bar{\xi}_i, \mu), & \\ \qquad \text{if } \xi_{i+1} < \min S_i(e_1, \bar{\xi}_i, \mu); & \\ 0, \qquad \text{otherwise.} \end{cases} \tag{38}$$

Then, we have $\xi_{i+1} - e_{i+1} \in S_i(e_1, \bar{\xi}_i, \mu)$.

From the definition of e_i (i.e., e_{k+1} with $k = i-1$) in (32), in the case of $e_i \neq 0$, for all $p_i \in S_{i-1}(e_1, \bar{\xi}_{i-1}, \mu)$, it holds that $|\xi_i - p_i| \geq |e_i|$ and $\mathrm{sgn}(\xi_i - p_i) = \mathrm{sgn}(e_i)$, which means $\mathrm{sgn}(\xi_i - p_i - e_i) = \mathrm{sgn}(e_i)$, and thus $\xi_i - p_i = e_i + (\xi_i - p_i - e_i) = e_i + \mathrm{sgn}(e_i)|\xi_i - p_i - e_i|$. Note that $\xi_{i+1} - e_{i+1} \in S_i(e_1, \bar{\xi}_i, \mu)$. There always exists a $p_i \in S_{i-1}(e_1, \bar{\xi}_{i-1}, \mu)$ such that $\xi_{i+1} - e_{i+1} = \kappa_i(\xi_i - p_i) = \kappa_i(e_i + \mathrm{sgn}(e_i)|\xi_i - p_i - e_i|)$.

With Lemma 3.1, for any $\epsilon_i > 0$, $\iota_i > 0$, $\gamma_i^0, \ldots, \gamma_i^{i-1}, \gamma_i^{i+1}, \chi_i^d, \chi_i^\mu \in \mathcal{K}_\infty$, we can find a smooth, odd, strictly decreasing and radially unbounded κ_i such that the e_i-subsystem with $\xi_{i+1} - e_{i+1} \in S_i(e_1, \bar{\xi}_i, \mu)$ is ISS with V_i satisfying

$$\begin{aligned} & V_i \geq \max_{k=0,\ldots,i-1,i+1} \left\{ \gamma_i^k(V_k), \chi_i^d(|d|), \chi_i^\mu(\mu), \epsilon_i \right\} \\ \Rightarrow \quad & \nabla V_i \dot{e}_i \leq -\iota_i V_i. \end{aligned} \tag{39}$$

By default, $V_{n+1} := \alpha_V(|e_{n+1}|)$. The true control input $u = \xi_{n+1}$ occurs with the e_n-subsystem, and we set $e_{n+1} = 0$.

4) Realizable Quantized Controller: From (39) with $i = n$, our desired quantized controller u can be chosen in the following form:

$$p_2^* = \kappa_1(y^q) \tag{40}$$

$$p_i^* = \kappa_{i-1}(\xi_{i-1} - p_{i-1}^*), \quad 3 \leq i \leq n \tag{41}$$

$$u = \kappa_n(\xi_n - p_n^*). \tag{42}$$

In the case of $|y| \leq M\mu$, we have $|w| = |y^q - y| \leq \mu$ and thus $\kappa_1(y^q) = \kappa_1(y + w) = \kappa_1(e_1 + w) \in S_1(e_1, \mu)$. It is then directly checked that

$$\begin{aligned} p_2^* \in S_1(e_1, \mu) \Rightarrow \cdots \Rightarrow p_i^* \in S_{i-1}(e_1, \bar{\xi}_{i-1}, \mu) \\ \Rightarrow \cdots \Rightarrow u = \xi_{n+1} - e_{n+1} \in S_n(e_1, \bar{\xi}_n, \mu) \end{aligned}$$

where $e_{n+1} = 0$. Thus, if $|y| \leq M\mu$, then the quantized control law (40)–(42) guarantees (28) and (39).

D. Cyclic-small-gain based Synthesis

Denote $e = \bar{e}_n$. For $0 \leq i \leq n$, each e_i-subsystem has been made ISS (or more precisely, practically ISS). In this subsection, we fine tune the ISS-gains such that the e-system satisfies the cyclic-small-gain condition [6], [12]. The system graph of the e-system is shown in Fig. 3.

According to the recursive design, given the $\bar{e}_{i-1}$-subsystem, by designing the set-valued map S_i for the e_i-subsystem, we assign the ISS gains γ_i^k $(1 \leq k \leq i-1)$ such that

$$\gamma_k^{k+1} \circ \gamma_{k+1}^{k+2} \circ \cdots \circ \gamma_{i-2}^{i-1} \circ \gamma_{i-1}^{i} \circ \gamma_i^k < \mathrm{Id}. \tag{43}$$

Applying this reasoning repeatedly, the e-system satisfies the cyclic-small-gain condition in [6], [12].

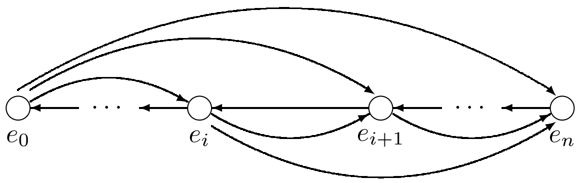

Fig. 3. The system graph of the e-system.

An ISS-Lyapunov function is constructed as:

$$V(e) = \max_{0 \le i \le n} \{\sigma_i(V_i(e_i))\} \tag{44}$$

with $\sigma_1(s) = s$, $\sigma_i(s) = \hat{\gamma}_1^2 \circ \cdots \circ \hat{\gamma}_{i-1}^i(s)$ ($2 \le i \le n$) and $\sigma_0(s) = \max_{1 \le i \le n}\{\sigma_i \circ \hat{\gamma}_i^0(s)\}$ for $s \in \mathbb{R}_+$, where the $\hat{\gamma}_{(\cdot)}^{(\cdot)}$'s are $\mathcal{K}_\infty$ functions smooth on $(0, \infty)$ and slightly larger than the corresponding $\gamma_{(\cdot)}^{(\cdot)}$'s and still satisfy the cyclic-small-gain condition.

Recall $|d| \le \bar{d}$. Denote $\epsilon_0 = 0$. We represent the maximal influence of d, μ and ϵ_i ($1 \le i \le n$) as

$$\theta = \max_{0 \le i \le n} \left\{\sigma_i \circ \chi_i^d(\bar{d}), \sigma_i \circ \chi_i^\mu(\mu), \sigma_i(\epsilon_i)\right\}. \tag{45}$$

Using the Lyapunov-based cyclic-small-gain theorem in [12], we achieve that if $|y| \le M\mu$, then the e-system with quantized control law (40)–(42) satisfies

$$V(e) \ge \theta \Rightarrow \nabla V(e)\dot{e} \le -\alpha(V(e)) \tag{46}$$

wherever $\nabla V(e)$ exists, with α positive definite. Note that $\nabla V(e)$ exists almost everywhere [12].

In the recursive design approach, we can make the $\gamma_{(\cdot)}^{(\cdot)}$'s (and thus the $\hat{\gamma}_{(\cdot)}^{(\cdot)}$'s) arbitrarily small to get arbitrarily small σ_i's ($0 \le i \le n,\ i \ne 1$). We can also select the χ_i^d's ($0 \le i \le n$), the ϵ_i's ($1 \le i \le n$) and the χ_i^μ's ($0 \le i \le n,\ i \ne 1$) arbitrarily small. In this way, for arbitrarily small $\theta_0 > 0$, we can design the gains such that $\max_{0 \le i \le n}\left\{\sigma_i \circ \chi_i^d(\bar{d}), \sigma_i(\epsilon_i)\right\} \le \theta_0$ and $\max_{0 \le i \le n,\ i \ne 1}\left\{\sigma_i \circ \chi_i^\mu(\mu)\right\} \le \theta_0$.

Recall $\chi_1^\mu(s) = \alpha_V(s/c_1)$ for $s \in \mathbb{R}_+$ defined in (28). If $|y| \le M\mu$, then quantized control law (40)–(42) guarantees

$$V(e) \ge \max\left\{\alpha_V(\mu/c_1), \theta_0\right\} \Rightarrow \nabla V(e)\dot{e} \le -\alpha(V(e)) \tag{47}$$

wherever $\nabla V(e)$ exists.

E. Dynamic Quantization

Define $\Theta = \alpha_V(M\mu)$. Then, $\mu = \alpha_V^{-1}(\Theta)/M$. Dynamic quantization can be determined by designing an update law for Θ. Denote $x = [x_1, \ldots, x_n]^T$ and $\xi = [\xi_2, \ldots, \xi_n]^T$. Recall $e = [e_0, \ldots, e_n]^T$ and the definition of e_i for $i = 0, \ldots, n$. The transformed state variable e can be considered as a continuous function of (x, ξ, μ). Note that

$\mu = \alpha_V^{-1}(\Theta)/M$. Denote $e = e(x, \xi, \Theta)$. In dynamic quantization, Θ is piece-wise constant on the time-line and denoted as $\Theta(t)$. Clearly, the piece-wise constant adjustment of Θ leads to jumps of e.

Due to space limitation, some results in this subsection are presented without proofs. However, the proofs are available from the authors upon request.

1) Zooming-out Stage: The design of the zooming-out stage is motivated by [11].

In this stage, the control input u and the state ξ of the observer are set to be zero. The small-time norm-observability assumed in Assumption 1 guarantees that for $d_t > 0$, there exists a $\varphi \in \mathcal{K}_\infty$ such that

$$|x(t_k + d_t)| \leq \varphi(\|y\|_{[t_k, t_k+d_t]}) \tag{48}$$

for all $k \in \mathbb{Z}_+$. Considering the definitions of V and e, for $d_t > 0$, there exists a $\bar{\varphi} \in \mathcal{K}_\infty$ such that

$$|V(e(x(t_k + d_t), 0, 0))| \leq \bar{\varphi}(\|y\|_{[t_k, t_k+d_t]}) \tag{49}$$

for all $k \in \mathbb{Z}_+$.

The forward completeness assumed in Assumption 1 guarantees that we can increase Θ fast enough to dominate the growth rate of $\bar{\varphi}(|y|)$. Thus, we can design the zooming-out logic to increase Θ (and thus μ) fast enough such that at some time $t_{k^*} > 0$ with $k^* \in \mathbb{Z}_+$, it holds that

$$\Theta(t_{k^*}) \geq \bar{\varphi}(\|y\|_{[t_{k^*}-d_t, t_{k^*}]}) \geq \max\{V(e(x(t_{k^*}), 0, 0)), \theta_0\}. \tag{50}$$

From the definition of S_i in (26) and (37), it can be observed that increase of μ (and thus Θ) leads to increase of $\max S_i$ and decrease of $\min S_i$. Using the definition of e_{i+1}, increase of Θ leads to decrease or hold of $|e_{i+1}|$ (and thus decrease or hold of $V(e)$). Note that $\xi(t_{k^*}) = 0$. From (50), we achieve

$$\Theta(t_{k^*}) \geq \max\{V(e(x(t_{k^*}), \xi(t_{k^*}), \Theta(t_{k^*}))), \theta_0\}. \tag{51}$$

2) Zooming-in Stage: With the help of Assumption 4, in the constructive control design procedure, we can choose c_1 satisfying $1/M < c_1 < 1$. Then, from the definition $\Theta = \alpha_V(M\mu)$, one can find a positive definite ρ_1^z such that

$$\alpha_V(\mu/c_1) \leq (\mathrm{Id} - \rho_1^z)(\Theta). \tag{52}$$

Suppose that at some time $t_k > 0$ with $k \in \mathbb{Z}_+$, it holds that

$$\Theta(t_k) \geq \max\{V(e(x(t_k), \xi(t_k), \Theta(t_k))), \theta_0\}. \tag{53}$$

We want to find a $Q_{\mathrm{in}}^\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Theta(t_{k+1}) = Q_{\mathrm{in}}^\Theta(\Theta(t_k))$ satisfies

$$\Theta(t_{k+1}) \geq \max\{V(e(x(t_{k+1}), \xi(t_{k+1}), \Theta(t_{k+1}))), \theta_0\} \tag{54}$$

where $t_{k+1} - t_k = d_t$. One can find a positive definite ρ_2^z such that $(\mathrm{Id} - \rho_2^z) \in \mathcal{K}_\infty$ and $(\mathrm{Id} - \rho_2^z)(s) \geq \max\{(\mathrm{Id} - \rho_1^z)(s), s - d_t \cdot \min_{(\mathrm{Id}-\rho_1^z)(s) \leq v \leq s} \alpha(V)\}$ for $s \in \mathbb{R}_+$. Define

$$\Xi = \mathrm{Id} - \rho_2^z. \tag{55}$$

Condition (53) implies $V(e(x(t_k),\xi(t_k),\Theta(t_k))) \leq \alpha_V(M\mu(t_k))$. From (47) and (52), if (53) holds, then

$$V(e(x(t_{k+1}),\xi(t_{k+1}),\Theta(t_k))) \leq \max\{\Xi(\Theta(t_k)),\theta_0\}. \tag{56}$$

Using the property of continuous functions, we can find a positive definite $\rho_3^z < \mathrm{Id}$ such that

$$|V(e(x,\xi,\Theta-\rho_3^z(h))) - V(e(x,\xi,\Theta))| \leq h. \tag{57}$$

holds for all x,ξ,Θ satisfying $0 \leq \Theta \leq \Theta(t_{k^*})$ and $V(e(x,\xi,\Theta)) \leq \max\{\Theta(t_{k^*}),\theta_0\}$ and all $h \geq 0$.

Define

$$\Theta_{\text{in}}^{\Theta}(\Theta) = \Theta - \rho_3^z\left(\frac{\Theta - \max\{\Xi(\Theta),\theta_0\}}{2}\right). \tag{58}$$

Then, (56), (57) and (58) imply

$$\begin{aligned} & V(e(x(t_{k+1}),\xi(t_{k+1}),\Theta(t_{k+1}))) \\ \leq \quad & \frac{\Theta(t_k) + \max\{\Xi(\Theta(t_k)),\theta_0\}}{2}, \end{aligned} \tag{59}$$

and (53) and (58) imply

$$\Theta(t_{k+1}) \geq \frac{\Theta(t_k) + \max\{\Xi(\Theta(t_k)),\theta_0\}}{2} \geq \theta_0. \tag{60}$$

Properties (59) and (60) together guarantee (54).

Lemma 3.3: Suppose that $\Theta(t_{k^*}) \geq \theta_0$ with $k^* \in \mathbb{Z}_+$. Then, the zooming-in logic $\Theta(t_{k+1}) = Q_{\text{in}}^{\Theta}(\Theta(t_k))$ for $k \in \mathbb{Z}_+$ with Q_{in}^{Θ} defined in (58) guarantees

$$\lim_{k\to\infty} \Theta(t_k) = \theta_0. \tag{61}$$

The proof of Lemma 3.3 is in Subsection IV-C. The motions of $\Theta(t)$ and $V(e(x(t),\xi(t),\Theta(t)))$ in the zooming-in stage are shown in Fig. 4.

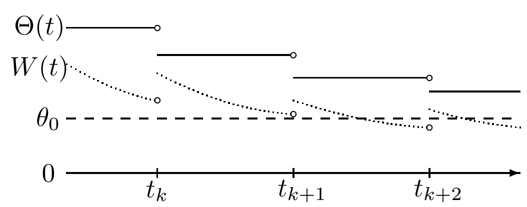

Fig. 4. Motions of $\Theta(t)$ and $W(t) = V(e(x(t),\xi(t),\Theta(t)))$ in the zooming-in stage.

With the appropriately designed zooming-in logic in (58), it always holds that $V(e(x(t),\xi(t),\Theta(t))) \leq \Theta(t)$ in the zooming-in stage. Thus, the closed-loop signals are bounded. By using (61), we have $\overline{\lim}_{t\to\infty} V(e(x(t),\xi(t),\Theta(t))) \leq \theta_0$. Recall the definition of V in (44). It can be observed that $y = x_1 = e_1$ ultimately converges to within the region $|y| \leq \alpha_V^{-1}(\theta_0)$. By choosing θ_0 arbitrarily small, output y can be steered to within an arbitrarily small neighborhood of the origin.

Recall $\Theta = \alpha_V(M\mu)$. With $Q_{\rm in}^{\Theta}$ defined in (58), the zooming-in logic for μ is designed as

$$Q_{\rm in}(\mu) = \frac{1}{M}\alpha_V^{-1} \circ Q_{\rm in}^{\Theta} \circ \alpha_V(M\mu). \tag{62}$$

F. Main Result

The main result of this paper is summarized in Theorem 1.

Theorem 1: Consider system (1)–(4) with output quantization satisfying (5). Under Assumptions 1–4, the closed-loop signals are bounded, and in particular, the output y can be steered to within an arbitrarily small neighborhood of the origin with the quantized output-feedback controller composed of reduced-order observer (9)–(10), control law (40)–(42), and dynamic quantization of form (6) with zooming-in dynamics $Q = Q_{\rm in}$ defined in (62).

IV. PROOFS OF LEMMAS 3.1–3.3

A. Proof of Lemma 3.1

Using (16), one can find $\psi_\phi^\eta, \psi_\phi^{\delta_1}, \dots, \psi_\phi^{\delta_m} \in \mathcal{K}_\infty$ such that

$$|\phi(\eta, \delta_1, \dots, \delta_m)| \le \psi_\phi^\eta(|\eta|) + \sum_{k=1}^{m} \psi_\phi^{\delta_k}(|\delta_k|). \tag{63}$$

Note that $\psi_\phi^\eta(s) + \sum_{k=1}^{m} \psi_\phi^{\delta_k} \circ \left(\chi_\eta^k\right)^{-1} \circ \alpha_V(s) + \frac{\iota}{2}s$ is a $\mathcal{K}_\infty$ function of s. With Lemma 1 in [2], for any $0 < c < 1$ and $\epsilon > 0$, one can find a $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ positive, nondecreasing and smooth on $(0, \infty)$, such that

$$\begin{aligned} & (1-c)\rho((1-c)^2 s^2)s \\ \ge\ & \psi_\phi^\eta(s) + \sum_{k=1}^{m} \psi_\phi^{\delta_k} \circ \left(\chi_\eta^k\right)^{-1} \circ \alpha_V(s) + \frac{\iota}{2}s \end{aligned} \tag{64}$$

for all $s \ge \sqrt{2\epsilon}$. Define $\kappa(r) = -\rho(r^2)r$ for $r \in \mathbb{R}$. Then, κ is smooth, odd, strictly decreasing and radially unbounded.

Recall $V_\eta(\eta) = \alpha_V(|\eta|) = \frac{1}{2}\eta^2$. Consider the case of $V_\eta(\eta) \ge \max_{k=1,\dots,m}\left\{\chi_\eta^k(|\delta_k|), \alpha_V\left(\frac{\delta_{m+1}}{c}\right), \epsilon\right\}$. We have

$$|\delta_k| \le \left(\chi_\eta^k\right)^{-1} \circ \alpha_V(|\eta|), \quad 1 \le k \le m \tag{65}$$

$$\delta_{m+1} \le c\alpha_V^{-1}(V_\eta(\eta)) = c|\eta| \tag{66}$$

$$|\eta| \ge \sqrt{2\epsilon} \tag{67}$$

The v satisfying (17) can be represented as $v = \kappa(\eta') = -\rho(\eta'^2)\eta'$ where $\eta' = \eta + \mathrm{sgn}(\eta)|\delta_{m+2}| + a\delta_{m+1}$ with $|a| \le 1$. With $0 < c < 1$ and (66), when $\eta \ne 0$, we have $\mathrm{sgn}(\eta') = \mathrm{sgn}(\eta)$, $|\eta'| \ge |\eta + a\delta_{m+1}| \ge (1-c)|\eta|$ and thus $\rho(\eta'^2)|\eta'| \ge (1-c)\rho((1-c)^2\eta^2)|\eta|$.

From (63)–(67) and the discussion above, we obtain

$$
\begin{aligned}
& \nabla V_\eta(\eta)(\phi(\eta, \delta_1, \ldots, \delta_m) + v) \\
= & \ \eta(\phi(\eta, \delta_1, \ldots, \delta_m) - \rho(\eta'^2)\eta') \\
\leq & \ |\eta||\phi(\eta, \delta_1, \ldots, \delta_m)| - |\eta|\rho(\eta'^2)|\eta'| \\
\leq & \ |\eta|\Big(\psi_\phi^\eta(|\eta|) + \sum_{k=1}^{m} \psi_\phi^{\delta_k} \circ \left(\chi_\eta^k\right)^{-1} \circ \alpha_V(|\eta|) \\
& \qquad\qquad - (1-c)\rho((1-c)^2\eta^2)|\eta|\Big) \\
\leq & \ -\frac{\iota}{2}|\eta|^2 = -\iota V_\eta(\eta). \qquad (68)
\end{aligned}
$$

This ends the proof.

B. Proof of Lemma 3.2

We simply use S_k instead of $S_k(e_1, \bar{\xi}_k, \mu)$ for $1 \leq k \leq i-1$. We only consider the time instants at which $e_i > 0$. The proof for the time instants at which $e_i < 0$ is similar.

Consider the definition of S_1 in (26) and the iteration-type definitions of S_k's in (31). Recall (29), (30), (33) and (34). The smoothness of the $\kappa_{(\cdot)}$'s implies the smoothness of $\max S_1$ with respect to e_1 and the smoothness of $\max S_k$ with respect to ξ_k and $\max S_{k-1}$ for $2 \leq k \leq i-1$. Repeatedly using the property of the composition of smooth functions, we can see $\max S_{i-1}$ is smooth with respect to $(e_1, \bar{\xi}_{i-1})$ and thus $\partial \max S_{i-1}/\partial[e_1, \bar{\xi}_{i-1}^T]^T$ is continuous with respect to $[(e_1, \bar{\xi}_{i-1})$. In the case of $e_i > 0$, the dynamics of e_i can be derived as

$$
\begin{aligned}
\dot{e}_i &= \dot{\xi}_i - \frac{\partial \max S_{i-1}}{\partial[e_1, \bar{\xi}_{i-1}^T]^T}[\dot{e}_1, \dot{\bar{\xi}}_{i-1}^T]^T \\
&= \xi_{i+1} + \phi_i(e_1, \xi_2, w) - \frac{\partial \max S_{i-1}}{\partial[e_1, \bar{\xi}_{i-1}^T]^T}[\dot{e}_1, \dot{\bar{\xi}}_{i-1}^T]^T \\
&:= \xi_{i+1} - e_{i+1} + \phi_i^*(\bar{e}_{i+1}, d, \mu, w, \bar{\xi}_i) \qquad (69)
\end{aligned}
$$

Specifically, $\xi_{n+1} = u$. We used (20), (21) and (22) to get the last equality above.

Recall $|e_1| \leq M\mu \Rightarrow |w| \leq \mu$. From (20), (21) and (22), we can see $|\phi_i(e_1, \xi_2, w)|$ is bounded by a $\mathcal{K}_\infty$ function of $|(e_1, \xi_2, \mu)|$, and $|[\dot{e}_1, \dot{\bar{\xi}}_{i-1}^T]^T|$ is bounded by a $\mathcal{K}_\infty$ function of $|(\bar{e}_1, d, \mu, \bar{\xi}_i)|$. Thus, we achieve that $|\phi_i^*(\bar{e}_{i+1}, d, \mu, w, \bar{\xi}_i)|$ is bounded by a $\mathcal{K}_\infty$ function of $|(\bar{e}_{i+1}, d, \mu, \bar{\xi}_i)|$. To prove (36), we show that $|\bar{\xi}_i|$ is bounded by a $\mathcal{K}_\infty$ function of $|(\bar{e}_i, \mu)|$. As pointed out in [14], this can be achieved by proving $[\bar{e}_i^T, \mu]^T = 0 \Rightarrow \bar{\xi}_i = 0$. By using the definitions of S_1 and e_2 in (26) and (27) and the definitions of S_k and e_{k+1} in (31) and (32), we have

$$
\begin{aligned}
& S_1 = \{0\} \Rightarrow \xi_2 - e_2 = 0 \Rightarrow \xi_2 = 0 \Rightarrow \cdots \Rightarrow \\
& S_k = \{0\} \Rightarrow \xi_{k+1} - e_{k+1} = 0 \Rightarrow \xi_{k+1} = 0, \quad 2 \leq k \leq i-1.
\end{aligned}
$$

This ends the proof.

C. Proof of Lemma 3.3

Consider the following two cases.

(a) $\Xi(\Theta(t_k)) \geq \theta_0$. From the definition of Ξ in (55), one can find a positive definite function ρ_1^* such that $\rho_3^z\left(\frac{s-\Xi(s)}{2}\right) = \rho_3^z\left(\frac{\rho_2^z(s)}{2}\right) \geq \rho_1^*(s)$ for $s \in \mathbb{R}_+$. In the case of $\Xi(\Theta(t_k)) \geq \theta_0$, we have

$$\begin{aligned}\Theta(t_{k+1}) &= \Theta(t_k) - \rho_3^z\left(\frac{\Theta(t_k) - \Xi(\Theta(t_k))}{2}\right) \\ &\leq \Theta(t_k) - \rho_1^*(\Theta(t_k)) \end{aligned} \tag{70}$$

which guarantees that there exists a $t_{k^o} > 0$ with $k^o \in \mathbb{Z}_+$ such that $\Xi(\Theta(t_{k^o})) < \theta_0$, and equivalently, $\Theta(t_{k^o}) < \Xi^{-1}(\theta_0)$.

(b) $\Xi(\Theta(t_k)) < \theta_0$. Define $\Theta'(t_k) = \Theta(t_k) - \theta_0$ for $k \in \mathbb{Z}_+$. Then, we obtain

$$\Theta'(t_{k+1}) = \Theta'(t_k) - \rho_3^z\left(\frac{\Theta'(t_k)}{2}\right) \tag{71}$$

which is a asymptotically stable first-order discrete-time system [5], and implies $\lim_{k\to\infty} \Theta'(t_k) = 0$.

Recall the definition of Ξ in (55). We can see $\Xi^{-1} > \mathrm{Id}$ and $\Xi^{-1}(\theta_0)$ is larger than θ_0. Considering both cases (a) and (b), we have $\lim_{k\to\infty} \Theta(t_k) = \theta_0$. This ends the proof.

V. CONCLUSIONS

This paper makes significant progress on the challenging, yet important, problem of quantized output-feedback control of nonlinear systems, by developing a new tool based on recent cyclic-small-gain techniques [6], [12]. The result shows that dynamic quantization can be implemented even with a 1-bit uniform quantizer. Furthermore, the influence of the external disturbance can be attenuated to an arbitrarily small level. It is our firm belief that the proposed design tool will prove helpful for quantized feedback control of other important classes of nonlinear systems.

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