Connectivity of Wireless Ad Hoc Networks with Random Beamforming: An Analytical Approach

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***Abstract*— Random beamforming, where each node selects a main beam direction without any coordination with other nodes, has been proposed as a simple technique to improve connectivity in wireless ad hoc networks. This paper presents an analytical model for evaluating the impact of random beamforming on the connectivity of wireless ad hoc networks in the presence of path loss and shadowing effects. We investigate the connectivity with random beamforming from the view points of a single node and the entire network. The correctness of our analytical approach is validated by comparing the analytical results with simulations. We show that for a path loss exponent of $\alpha < 3$, irrespective of shadowing effects, random beamforming improves both the local and overall connectivity compared to omnidirectional antennas.**

I. Introduction

One of the fundamental issues in wireless ad hoc networks is the requirement that all nodes maintain connectivity with all other nodes in the network [1]. There is generally no centralized coordination in wireless ad hoc networks and a connection between any two nodes is usually established via multiple direct links between intermediate nodes. Due to the multi-hop, dynamic nature of such networks, maintenance of connectivity is a crucial issue.

Most studies on connectivity (e.g. probability of achieving a fully connected network) of wireless ad hoc networks assume omnidirectional antennas [2]–[8]. In [2], a probability density function of the distance between two nodes in a rectangular or hexagonal region is analytically derived using a space decomposition method and is used to calculate the average number of neighbours of a node (i.e. node degree) with a simple path loss model. A semi-analytical procedure for determination of the critical node density for an almost surely connected network for the case of path loss channels is considered in [3]. The results are extended to a shadowing environment in [4] and it is shown that the channel randomness caused by shadowing can improve network connectivity by reducing the number of isolated nodes. An alternative analytical method, based on the concept of effective coverage area, is proposed in [5] to analyse the effect of path loss and shadowing on connectivity of wireless ad hoc networks. The connectivity metrics in [2]–[6] assume an efficient medium access control protocol so that inter-node interference is negligible. Some recent papers have also used Signal to Interference plus Noise Ratio [7] and Symbol Error Rate [8] to study network connectivity.

Recently different beamforming schemes have been proposed in literature to improve the connectivity of wireless ad hoc networks [9]–[13]. It must be noted that there is an inherent trade-off in the use of beamforming in wireless ad hoc networks, i.e. hardware complexity versus performance improvements offered by the beamforming schemes. A very simple strategy called random beamforming is proposed in [9]. In this scheme, each node in the network randomly selects a main beam direction without any coordination with other nodes. Using simulations, it is shown that while random beamforming may decrease the number of neighbours of a node, it leads to overall improvement in the network connectivity [9]. Similar conclusion are drawn in [11] where performance of random beamforming is compared to centre directed beamforming via simulations. In [12], an analytical study is used to show that random beamforming can both increase or decrease the number of isolated nodes and network connectivity, but no insight is given into when (i.e. under what channel conditions) this occurs. In [13], an analytical approach is used to analyse the probability of node isolation with random beamforming for path loss model.

In this paper, we extend our previous work in [13] to include both path loss and shadowing. We investigate both the local and overall connectivities with random beamforming, i.e., (i) connectivity from the viewpoint of a single node (probability of node isolation and average node degree, respectively) and (ii) connectivity from the viewpoint of the entire network (probability of connectivity). Our work is a combination and extension of the approaches in [5] and [4] to include the effect of random beamforming. The contributions of this paper are as follows:-

- In Section III, we present an analytical model for evaluating the impact of random beamforming on the connectivity of wireless ad hoc networks in the presence of path loss and shadowing effects.
- In Section IV, we show that for a path loss exponent of $\alpha < 3$, irrespective of the presence of shadowing, random beamforming improves the connectivity compared to the use of omni-directional antennas. In addition, for a given path loss exponent and a given number of antennas, an increase in shadowing effect helps to improve connectivity. Our results provide analytical insights that confirm and explain the simulation results in [9], [11].

The antenna and network connectivity definitions used in this work are described in Section II.

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II. System Model

We consider N nodes uniformly distributed over a two-dimensional square region with area $L \times L$ m^2. We assume that the node density ρ is a homogenous Poisson point process [3]. A homogeneous Poisson process can be regarded as the limiting case of a uniform distribution of N nodes on an area L^2, as the area of the network approaches ∞ while keeping ρ constant.

A. Antenna Model

Without loss of generality, we assume that each node in the network is equipped with a uniform circular array (UCA) of M omni-directional antenna elements, placed evenly on a circle in the xy-plane with neighbouring antenna elements separated by half a wavelength. Beamforming is achieved by phase shifting the response of each antenna in the array such that the array main beam points towards the desired direction. The gain of the array antenna can be expressed as [14]

$$G = \frac{| E(\theta, \phi) |^2}{\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} | E(\theta, \phi) |^2 \sin(\theta)\, d\theta\, d\phi}, \quad (1)$$

where $\phi \in [0, 2\pi)$ is the angle from the x-axis in the xy-plane and $\theta \in [0, \pi)$ is the angle from the z-axis. For a UCA, the electric field can be expressed as [15]

$$E(\theta, \phi) = \sum_{m=1}^{M} E_0 \exp[jka \sin(\theta) \cos(\phi - \phi_m) + j\alpha_m], \quad (2)$$

where $E_0 = 1$ is the electric field pattern of the omnidirectional antennas (i.e., a constant), a is the radius of the circular array, $k = \frac{2\pi}{\lambda}$, $\phi_m = \frac{2\pi m}{M}$, and α_m is the phase shift of the mth antenna. For conventional co-phasal excitation, [15]

$$\alpha_m = -ka \sin(\Theta_0) \cos(\Phi_0 - \phi_m), \quad (3)$$

where $\Theta_0 = \pi/2$ (i.e., the xy plane) and Φ_0 are the angles of the desired main beam.

Substituting (2) and (3) into (1), we can calculate the antenna gain for any azimuthal angle ϕ. Note that the resulting antenna gain G from (1) is a function of ϕ, Φ_0 and M. Thus, we denote it as $G(\phi, \Phi_0, M)$.

B. Connectivity Metrics

The following connectivity metrics are used in the discussion of the analytical and simulation results [3], [9]:

- *Probability of Node Isolation* (P(iso)) is defined as the probability that a randomly selected node in an ad hoc network has no connected neighbours.
- *Average Node Degree* is defined as the average number of direct links any given node has to other nodes.
- *Probability of Connectivity* (P(con)) is defined as the probability that every node pair in the network has at least one path connecting them.
- *Critical Node Density* (ρ_c) is defined as the density of nodes that is needed to achieve, with high probability, a connected network (following [3], we use $P(\text{con}) = 0.99$).
- *Path Probability* (P(path)) is defined as the probability that two randomly chosen nodes in an ad hoc network are connected via a direct link or a multi-hop path. It is known that $P(\text{path}) \geq P(\text{con})$ [3].

III. Analytical Model

We assume that each node transmits a signal with power P_T. We consider a wireless channel with path loss and log-normal shadowing effects. The received signal power P_R is given by [16]

$$P_R = 10^{\frac{w}{10}} \frac{1}{d^\alpha} C\, G_T\, G_R\, P_T, \quad (4)$$

where d is the distance between the transmitting and receiving nodes, $C = (\lambda/(4\pi))^2$ is a constant, G_T and G_R are the antenna gains of the transmitting and receiving nodes, respectively, w is a Gaussian random variable with zero mean and standard deviation σ (hence $10^{\frac{w}{10}}$ is normal in dB) and α is the path loss exponent. Note that in typical wireless communication scenarios $\sigma/\alpha > 1$ [17].

Without loss of generality, we can normalise (4) with respect to constant C, so that the power attenuation is expressed as

$$\beta(d) = \frac{P_T}{P_R} = \frac{d^\alpha}{10^{\frac{w}{10}} G_T G_R}. \quad (5)$$

Assuming identical node hardware and negligible inter-node interference, two nodes at a distance d are connected if $\beta(d) < \beta_{\text{th}}$, where β_{th} is the threshold signal power attenuation. From (5), the probability of having no direct connection between two nodes at a distance d is given by

$$\begin{aligned} P(\beta \geq \beta_{\text{th}}) &= P\left(\frac{d^\alpha}{10^{\frac{w}{10}} G_T G_R} \geq \beta_{\text{th}}\right), \\ &= P((\beta_{\text{th}} 10^{\frac{w}{10}} G_T G_R)^{\frac{1}{\alpha}} \leq d). \end{aligned} \quad (6)$$

We define a random variable R as

$$R = (\beta_{\text{th}} 10^{\frac{w}{10}} G_T G_R)^{\frac{1}{\alpha}}. \quad (7)$$

Substituting in (6), we get $P(\beta \geq \beta_{\text{th}}) = P(R \leq d)$. Hence the random variable R can be referred to as the effective communication range, i.e., the node is able to communicate with all nodes lying within a distance of R m. The effective coverage area of a node can thus be considered as a disk with radius R, centered at the node. Therefore, the effective coverage area is given by πR^2. Assuming shadowing and beamforming are independent, we have

$$\begin{aligned} E[R^2] &= E\left[(\beta_{\text{th}} 10^{\frac{w}{10}} G_T G_R)^{\frac{2}{\alpha}}\right], \\ &= (\beta_{\text{th}})^{\frac{2}{\alpha}} E[10^{\frac{w}{5\alpha}}] E\left[(G_T G_R)^{\frac{2}{\alpha}}\right], \end{aligned} \quad (8)$$

where $E[\cdot]$ denotes statistical expectation. The expectation involving log-normal shadowing can be calculated as follows. We denote the expectation involving log-normal shadowing by $E[X] = E[10^{\frac{w}{5\alpha}}]$. To calculate $E[X]$, we first take the natural logarithm of X, given by

$$\ln X = \ln 10^{\frac{w}{5\alpha}} = \frac{\ln 10}{5\alpha} w. \quad (9)$$

From the property of log-normal distributed random variables [18], we get $E[X] = e^{\left(\frac{\ln 10}{5\alpha}\sigma\right)^2/2}$. Hence we get,

$$E\left[10^{\frac{w}{5\alpha}}\right] = e^{\left(\frac{\sigma \ln 10}{5\sqrt{2}\alpha}\right)^2}. \quad (10)$$

The expectation involving the beamforming gains can be calculated using the procedure outlined in our previous work in [13]. The detailed derivation steps are omitted here for brevity. The expectation can be expressed as [13]

$$E\left[(G_T G_R)^{\frac{2}{\alpha}}\right] = \quad (11)$$
$$\frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}(G(\phi,\Phi_T,M)G(\pi+\phi,\Phi_R,M))^{\frac{2}{\alpha}}d\Phi_R d\Phi_T d\phi,$$

where M is the number of antennas, ϕ is the relative angle of the receiver node from the transmitter node with respect to the x-axis, Φ_T is the main beam direction of the transmitter node, Φ_R is the main beam direction of the receiver node and $G(\phi, \Phi_T, M)$ and $G(\pi+\phi, \Phi_T, M)$ are the antenna gains which can be determined from (1).

We refer to $E\left[(G_T G_R)^{\frac{2}{\alpha}}\right]$ as the effective beamforming gain. It can be seen that the effective beamforming gain depends on the transmit and receive antenna gains, as well as the path loss exponent α. As there is no closed-form solution to (11), we evaluate it numerically. Table I summarizes the results for different α and M. Note that for omnidirectional antennas, the effective beamforming gain is unity.

TABLE I
EFFECTIVE BEAMFORMING GAIN FOR RANDOM BEAMFORMING

Number of antenna elements M	4	6	8	10
Path loss exponent $\alpha = 2$	1.48	1.51	1.60	1.84
Path loss exponent $\alpha = 2.5$	1.12	1.12	1.14	1.24
Path loss exponent $\alpha = 3$	0.95	0.96	0.96	1.01
Path loss exponent $\alpha = 3.5$	0.87	0.89	0.87	0.90
Path loss exponent $\alpha = 4$	0.82	0.85	0.82	0.85

Substituting the values from Table I and (10) in (8), we can find the expected value of the effective coverage area, i.e., $\pi E[R^2]$. This can then be used to analyse the connectivity properties with random beamforming, as discussed below.

A. Average Node Degree

For a node deployment following a homogeneous Poisson point process with density ρ, the node degree D has a Poisson distribution with parameter $\rho\pi E[R^2]$ [5]. Therefore, the average node degree $E[D]$ is given by

$$E[D] = \rho\pi E[R^2]. \quad (12)$$

Substituting from (8), the normalised average node degree $E[D]/N$ with random beamforming is given by

$$\frac{E[D]}{N} = \frac{\pi}{L^2}(\beta_{\text{th}})^{\frac{2}{\alpha}} e^{\left(\frac{\sigma \ln 10}{5\sqrt{2}\alpha}\right)^2} E[(G_T G_R)^{\frac{2}{\alpha}}], \quad (13)$$

where N is the number of nodes, and L^2 is the area of the square region.

B. Probability of Isolation

The probability of node isolation is given by [4]

$$P(\text{iso}) = \exp\{-E[D]\}. \quad (14)$$

Substituting from (8) and (12) into (14), we can determine the probability of isolation with random beamforming from

$$P(\text{iso}) = \exp\left\{-\rho\pi(\beta_{\text{th}})^{\frac{2}{\alpha}} e^{\left(\frac{\sigma \ln 10}{5\sqrt{2}\alpha}\right)^2} E\left[(G_T G_R)^{\frac{2}{\alpha}}\right]\right\}. \quad (15)$$

C. Probability of Connectivity

It has been shown that the probability of no isolated nodes provides a tight upper bound for the probability of connectivity at high probabilities [4]. Hence an upper bound for the probability of connectivity is given by [4]

$$P(\text{con}) = \exp\{-\rho L^2 P(\text{iso})\}. \quad (16)$$

Substituting from (15) into (16), we can calculate the upper bound for the probability of connectivity with random beamforming. Note that in (16), $P(\text{iso})$ is also a function of ρ as given in (15).

IV. RESULTS

In this section, we compare our analytical results with simulation results and investigate the effect of random beamforming on (i) average node degree, (ii) probability of isolation, (iii) probability of connectivity and (iv) path probability. The simulations are carried out using Matlab. In the simulations, nodes are uniformly distributed on a square of area $A \times A$ m^2. To eliminate border effects, we use the sub-area simulation method [3], i.e., we only compute the connectivity measures for nodes located on an inner square of smaller area L^2 ($L^2 \ll A^2$). The results are then calculated by averaging over 5000 Monte Carlo simulation trials.

A. Model Validation: Average Node Degree

We compare our analytical results for normalised average node degree $E\{D\}/N$ with:-

1) our simulation results which *exclude* border effects and
2) simulation results presented in [9] which *include* border effects.

The system parameters used are: $\beta_{\text{th}} = 40, 50$ dB, $L = 500, 1000$ m, $M = 1, 4, 6$, $\alpha = 2, 3$ and $\sigma = 0$ (i.e., no shadowing). The analytical results are calculated from Eq. (13). The results are shown in Table II. We can see that the analytical results are in excellent agreement with our simulation results. This confirms the validity of our analytical approach. The analytical results also match well with the trend of the simulation results in [9]. This is in keeping with the general observation that simulation results with border effects tend to underestimate node degree.

From Table II, we can see that random beamforming slightly decreases the average node degree compared to omnidirectional antennas for $\alpha = 3$ and significantly increases the average node degree for $\alpha = 2$. This can be explained using

TABLE II

AVERAGE NODE DEGREE

α	β_{th} (dB)	L (m)	Antenna	M	$E\{D\}/N$ Simulations [9] (with border effects)	$E\{D\}/N$ Simulations (without border effects)	$E\{D\}/N$ Analytical results with Eq. (13)
3	50	500	omnidirectional	1	0.0250	0.0271	0.0271
3	50	500	UCA	4	0.0227	0.0256	0.0258
3	50	500	UCA	6	0.0230	0.0261	0.0261
3	40	500	omnidirectional	1	0.00561	0.0059	0.0058
3	40	500	UCA	4	0.00522	0.0056	0.0056
2	40	1000	omnidirectional	1	0.028	0.0312	0.0314
2	40	1000	UCA	4	0.0375	0.0467	0.0466

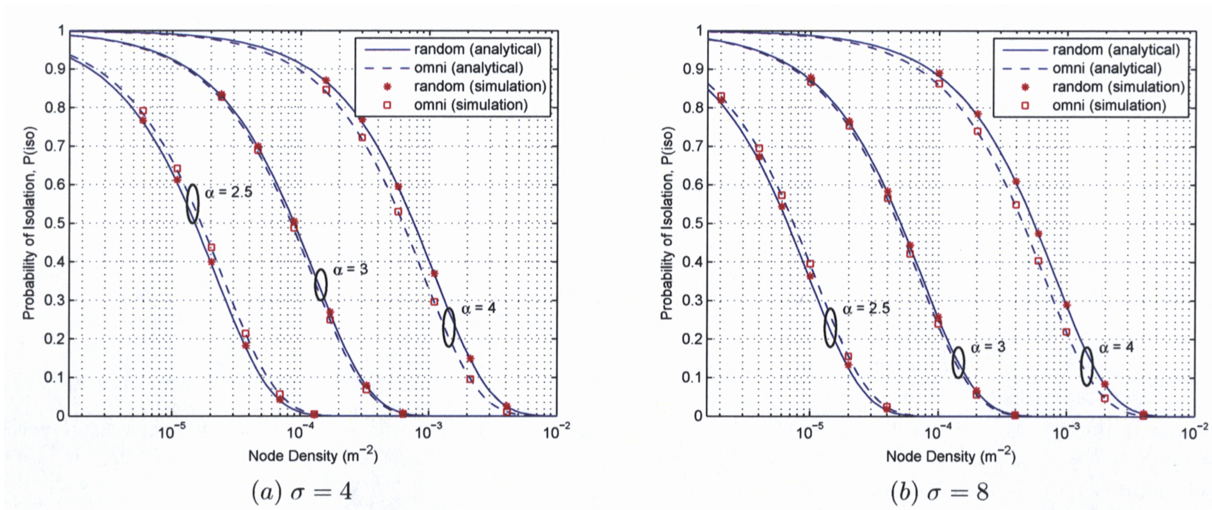

Fig. 1. Probability of node isolation vs. node density with system parameters: $\beta_{\text{th}} = 50$ dB, $M = 4$, $\alpha = 2.5, 3, 4$ and (a) $\sigma = 4$ and (b) $\sigma = 8$ (lines = analytical results from Eq. (15), markers = simulation results).

our analytical model as follows. We see from Table I that the effective beamforming gain for random beamforming is less than or equal to 1 for $\alpha \geq 3$ and greater than 1 for $\alpha < 3$. From Eq. (13), we can see that this increases the average node degree for $\alpha < 3$. We also see from Table I that increasing the number of antennas from $M=4$ to $M=6$ or 8 provides no or marginal improvement in the effective beamforming gain.

B. Probability of Isolation

Fig. 1 shows the probability of isolation versus node density for $\beta_{\text{th}} = 50$dB, $M = 4$, $\alpha = 2.5, 3, 4$ and (a) $\sigma = 4$ and (b) $\sigma = 8$. In the figures, the analytical results are indicated with lines while the simulation results are indicated with markers. The analytical results are calculated from Eq. (15). The results for omnidirectional antennas are provided as a reference. It can be seen that the simulation results are in excellent agreement with the analytical results in all cases. The use of random beamforming results in a lower probability of isolation when $\alpha < 3$ (e.g., up to around 17% for $\alpha = 2.5$) and a higher probability of isolation when $\alpha > 3$. Comparing the plots for (a) $\sigma = 4$ and (b) $\sigma = 8$, we can see that a higher σ results in lower probability of isolation for both omnidirectional and random beamforming antennas. This trend is in agreement with the findings reported in [4], [5] concerning the effect of shadowing with omnidirectional antennas.

C. Probability of Connectivity

Fig. 2 shows the changes in the probability of connectivity with node density for $\beta_{\text{th}} = 50$dB, $L = 400$ m, $\alpha = 3$, $\sigma = 0$, 4 and (a) $M = 4$ and (b) $M = 8$. The analytical bounds are calculated from Eq. (16). The results show that the agreement between the analytical results and simulation results is generally good, improving as σ increases. Comparing the plots for (a) $M = 4$ and (b) $M = 8$, we observe that the analytical bounds get tighter as M increases. We can also see that when the network is almost surely connected, i.e., $P(\text{con}) = 0.99$, the analytical results provide a very tight bound for the simulation results. We carried out simulations for different sets of parameters and confirmed that both connectivity curves converge for higher probabilities. We can thus use the analytical results to investigate the critical node densities. This approach is also used in [3] for studying critical node densities with omnidirectional antennas.

Our results show that random beamforming reduces the critical node density for $\alpha < 3$, e.g., for $\beta_{\text{th}} = 50$dB, $\alpha = 2.5$, $\sigma = 4$, a node density of $\rho_c = 0.1951$ is required with omnidirectional antennas, while a node density of $\rho_c = 0.1719$ is required for random beamforming with $M = 4$ antenna elements. The relative improvement for $\alpha < 3$ decreases with higher σ.

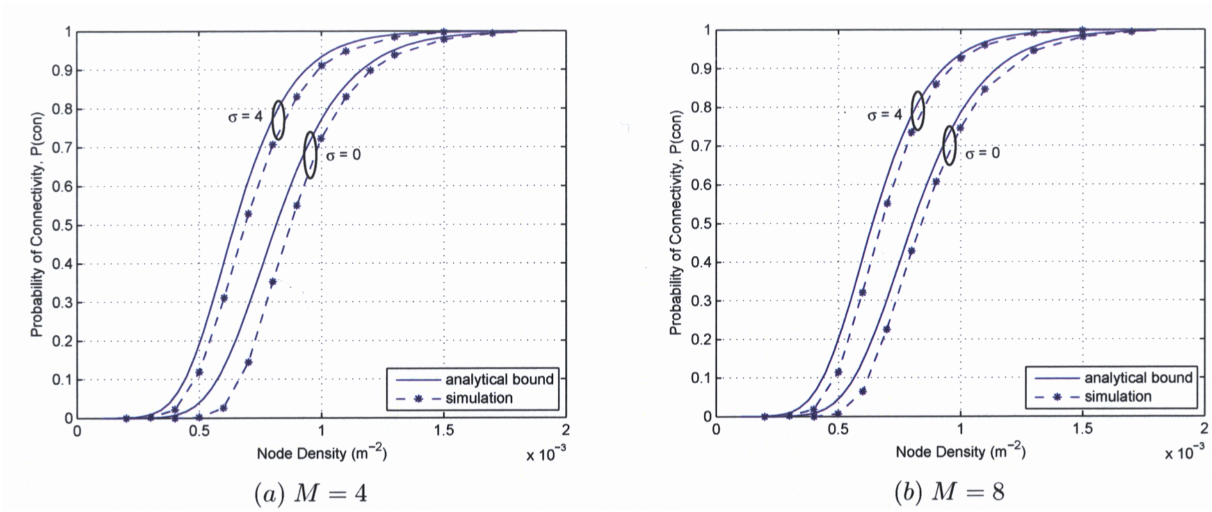

Fig. 2. Probability of connectivity vs. node density for random beamforming with following system parameters: $\beta_{\text{th}} = 50$ dB, $L = 400$ m, $\alpha = 3$, $\sigma = 0, 4$ and (a) $M = 4$ and (b) $M = 8$. (lines = analytical results from Eq. (16), dashed line with markers = simulation results).

D. Path Probability

An analytical expression for the path probability with random beamforming is still an open problem [9]. Hence simulation based studies have been used to show that random beamforming improves the path probability for $\alpha < 3$ [9]. This effect of a lower α leading to higher node degree is explained intuitively in [9] with the argument that for low α the longest links in the beamforming scenario are much longer than the links for the case of omnidirectional antenna. However for higher α the longest links are only slightly longer [9]. The results in Table I provide an analytical justification for the above argument. We can see that the effective beamforming gain for random beamforming is greater than 1 for $\alpha < 3$. Hence this improves the path probability for random beamforming for $\alpha < 3$ compared to the case of omnidirectional antennas.

V. Conclusion

In this paper, we have proposed an analytical approach to investigate the effect of random beamforming on the connectivity of wireless ad hoc networks. We calculated the effective coverage area of the node taking into account path loss, shadowing and random beamforming. The effect of random beamforming is characterised by the effective beamforming gain. The correctness of the analytical approach is verified by comparison with simulation results. It has been shown that for relatively few antennas ($M = 4$) and path loss $\alpha < 3$, both local and overall connectivity improvements can be made by utilising random beamforming, irrespective of shadowing effects. Our future research will be focused on the extension of our analytical results for path probability with random beamforming.

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