www.ietdl.org

Published in IET Communications
Received on 21st November 2012
Revised on 9th June 2013
Accepted on 14th June 2013
doi: 10.1049/iet-com.2013.0284

ISSN 1751-8628

Error performance analysis of decode-and-forward and amplify-and-forward multi-way relay networks with binary phase shift keying modulation

Shama Naz Islam, Parastoo Sadeghi, Salman Durrani

Research School of Engineering, College of Engineering and Computer Science, The Australian National University, Canberra, Australia

E-mail: shama.islam@anu.edu.au; shamah_islam@yahoo.com

Abstract: In this study, we analyse the error performance of decode and forward (DF) and amplify and forward (AF) multi-way relay networks (MWRNs). The authors consider a MWRN with pair-wise data exchange protocol using binary phase shift keying (BPSK) modulation in both additive white Gaussian noise (AWGN) and Rayleigh fading channels. The authors quantify the possible error events in an L-user DF or AF MWRN and derive accurate asymptotic bounds on the probability for the general case that a user incorrectly decodes the messages of exactly k ($k \in [1, L-1]$) users. They show that at high signal-to-noise ratio (SNR), the higher order error events ($k \geq 3$) are less probable in AF MWRN, but all error events are equally probable in a DF MWRN. They derive the average BER of a user in a DF or AF MWRN in both AWGN and Rayleigh fading channels under high SNR conditions. Simulation results validate the correctness of the derived expressions. The authors results show that at medium to high SNR, DF MWRN provides better error performance than AF MWRN in AWGN channels even with a large number of users (e.g. $L = 100$). Whereas, AF MWRN outperforms DF MWRN in Rayleigh fading channels even for much smaller number of users (e.g. $L > 10$).

1 Introduction

Two-way relay networks (TWRNs), with physical-layer network coding (PNC) protocol, have emerged as a spectrally efficient method for bidirectional communication and exchange of information between two nodes [1–3]. In such systems, the relay utilises the additive nature of physical electromagnetic waves and either amplifies and forwards (AF) [4] or decodes and forwards (DF) [1, 2] the sum of the signals before re-transmission. Both users extract the message of the other user by canceling self-information. Compared with other TWRN protocols, such as digital network coding in which the relay performs XOR operations on bit streams from the two users [1, 5], it has been shown that PNC requires smaller number of time slots for full information exchange [4]. The performance of TWRNs with PNC has been thoroughly analysed from the perspective of capacity [6, 7], bit error rate (BER) [1, 4, 8–13] and practical issues such as channel estimation and synchronisation [14–17].

Recently, TWRNs have been generalised to multi-way relay networks (MWRNs) in which multiple users can exchange information with the help of a relay terminal [18]. Potential applications of MWRNs include file sharing in a peer-to-peer wireless network, local measurement exchange in a sensor network or base station information exchange in a satellite communication network [19]. Different protocols have been proposed for MWRNs, for example, complex field network coding which entails symbol-level operations incorporating complex field coefficients at the physical layer [20] and MWRNs with pair-wise data exchange where the relay decodes or amplifies pair-wise functions of users' messages [21]. In particular, it was shown in [21] that pair-wise DF (at the relay) for binary MWRN is theoretically the optimal strategy since it achieves the common-rate capacity. Optimal user pairing for asymmetric MWRNS, where users have different channel conditions, are studied in [22]. Practical coding schemes, based on low-density parity-check codes, for MWRNs are proposed in [23]. However, a significant practical issue in MWRNs with pair-wise data exchange is error propagation. For example, in a DF MWRN, if a user wrongly decodes another user's message, then this error propagates through the subsequent decoding operations unless another error is made. In an AF MWRN, the mean of the received signal is shifted from its true value because of an earlier error. This can have a significant impact on the average BER for a user in a MWRN.

To the best of our knowledge, an analytical characterisation of the error propagation in a MWRN has not been fully addressed in the literature to date. The probability for the special cases that a user incorrectly decodes the messages of exactly $k = 0$, $k = 1$, $k = 2$ and $k = L–1$ users, respectively, in an L-user DF MWRN is derived in our preliminary work in [24]. The probability for the special case of having at least one error event ($k \geq 1$) for AF MWRN is derived in

[25]. Apart from [24, 25], there has been no attempt to analyse the error performance of MWRNs with pair-wise data exchange. There are two major limitations of these prior works. First, the derived probabilities represent certain special cases of the more general problem of finding the probability of k error events ($k \in [1, L-1]$), that is, where k can take any integer value in the set $[1, L-1]$. The prior works do not address the problem of finding the probability of higher order error events ($k \geq 3$). Secondly, the probabilities of discrete error events offer only a partial view of the overall error performance. From the perspective of the overall system performance, the average BER is a more useful metric since it takes all the error events into account. The prior works leave this as an important open problem [25, page 524].

In this paper, we are concerned with the error performance analysis of DF and AF MWRNs with BPSK Modulation. In particular, we address the following open problems:

(1) How can we characterise the probability of k error events in DF and AF MWRN?
(2) What is the average BER for a user in a DF or AF MWRN?
(3) For a given number of users and operating signal-to-noise ratio (SNR), what is the best relaying strategy (DF or AF) in MWRN?

As an outcome of our analysis, we obtain the following solutions to the above problems:

- We derive accurate asymptotic bounds on the error probability for the general case of k error events in an L-user DF or AF MWRN [cf. (16) and (25)]. These bounds are based on the insights gained from the analysis of the exact probability that a user incorrectly decodes the messages of $k = 1$ and $k = 2$ users. We show that the derived asymptotic bounds are accurate at mid to high SNR range.
- Our analysis of the error probability for the general case of k error events shows that at high SNR (a) the dominant factor in the error propagation in DF MWRN is the probability of consecutive erroneous messages resulting from a single erroneous network coded bit, (b) the dominant factor in the error propagation in AF MWRN is the probability of consecutive errors involving the middle or end users in the transmission protocol and (c) the higher order error events ($k \geq 3$) are less probable in AF MWRN, but all error events are equally probable in a DF MWRN. This affects their BER sensitivity to the number of users in the system, as discussed later.
- We use the asymptotic bounds on the probability of k error events to derive closed-form expressions for the average BER of a user in DF or AF MWRN under high SNR conditions [cf. (17) and (26)]. For both DF and AF MWRN in AWGN channel, the derived BER expressions can accurately predict the average BER of a user in medium to high SNR. For Rayleigh fading channels, the analytical expressions are within 1 dB of the simulation results at high SNR.
- We show that for a given number of users in an AWGN channel, AF MWRN is slightly better than DF MWRN at low SNR, while DF MWRN is better than AF MWRN at medium to high SNRs. This is true even for a large number of users (for example, $L = 100$). For fading channels, AF MWRN begins to outperform DF MWRN for the number of users as low as $L \simeq 10$. We attribute this to the lower probability of high-order error events in AF MWRN, which makes it more robust to the increase in the number of users in terms of average BER.

The rest of the paper is organised in the following manner. The system model is presented in Section 2. The challenges associated with the characterisation of the error performance in MWRNs are discussed in Section 3. The asymptotic bounds on the error probability for the general case of k error events and the average BER for a user in DF and AF MWRNs are derived in Sections 4 and 5, respectively. The analysis is extended to include Rayleigh fading in Section 6. Section 7 provides the simulation results for verification of the analytical solutions. Finally, conclusions are provided in Section 8.

Throughout this paper, we have used the following notation: $\oplus$ denotes XOR operation, $\tilde{(\cdot)}$ and $\widehat{\tilde{(\cdot)}}$ denote decoded values at the relay and users, respectively, $|\cdot|$ denotes absolute value of a complex variable, $\arg(\cdot)$ denotes the argument, $\min(\cdot)$ denotes the minimum value, $E[\cdot]$ denotes the expected value of a random variable and $Q(\cdot)$ is the Gaussian Q-function.

2 System model

Consider a multi-way relay network (MWRN) with L-user nodes and a single relay node R. We assume that (i) there is no direct link between the users and they exchange their information through the relay, (ii) each node has a single antenna and operates in a half-duplex mode, that is, a node cannot transmit and receive simultaneously and (iii) the MWRN operates in time-division duplex (TDD) mode, that is, the uplink and downlink channels are differentiated in time slots but occupy the same frequency slot. We concentrate on a MWRN in which all user transmissions consist of T binary phase shift keying (BPSK) modulated symbols per frame and all the channels are corrupted by additive white Gaussian noise (AWGN) only. Later in Section 6, we extend the model to Rayleigh fading channels.

The communication among the users takes place in two phases, with each phase comprising L–1 time slots [21]. In the first 'multiple access phase', users take turns to simultaneously transmit in a pair-wise manner. Overall, the first and the last user transmit once only while the remaining L–2 users transmit twice. This phase is independent of the transmission protocol used at the relay. In the second 'broadcast phase', depending on the relay transmission protocol, the relay broadcasts the decoded or amplified network coded message to all the users. At the completion of the broadcast phase, all the users have the network coded messages corresponding to each user pair. Then they utilise self information to extract the messages from the other users. This is illustrated in Figs. 1*a* and *b* for an L-user DF MWRN.

2.1 Transmission protocol at the users (for both DF and AF)

Let the ith and $(i+1)$th user transmit binary symbols, W_i and W_{i+1}, which are BPSK modulated to X_i and X_{i+1} respectively, where $W_i \in \{0, 1\}$, $X_i \in \{\pm 1\}$ and $i = 1, 2, L-1$. The relay receives the signal

$$r_{i,i+1} = X_i + X_{i+1} + n_1 \quad (1)$$

where n_1 is the zero mean AWGN in the user-relay link with noise variance $\sigma^2_{n_1}$. For a fair comparison between TWRNs

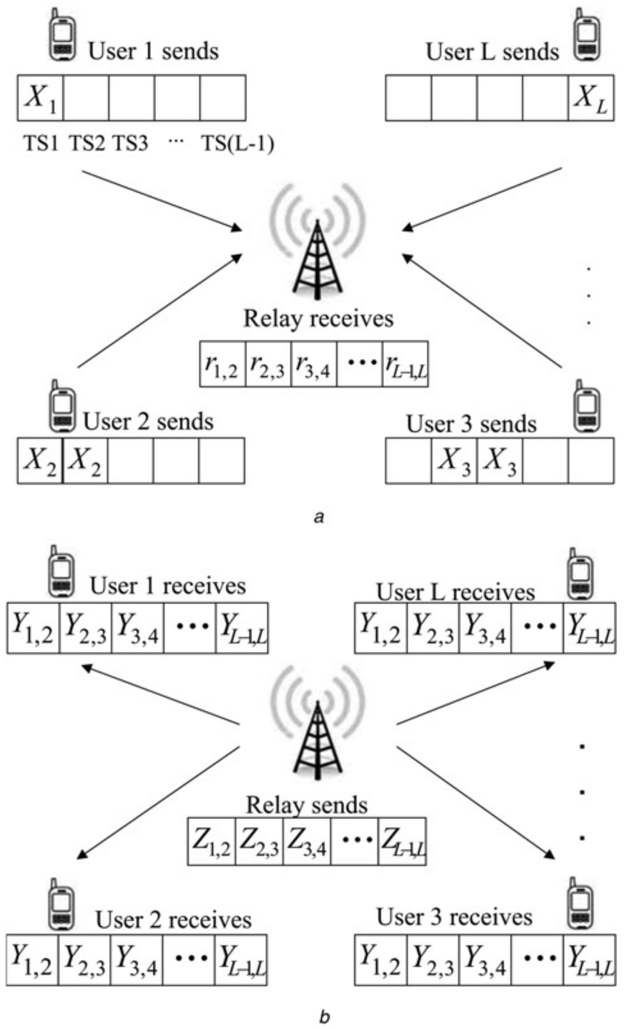


Fig. 1 *System model for an L-user DF MWRN, where the users exchange information with each other via the relay R. Here, TS means time slot and the other mathematical symbols are explained in Sections 2.1 and 2.2*

a MAC phase
b BC phase

and MWRNs later in our simulations, we maintain the same average power per user in a MWRN as that in a TWRN and set $\sigma_{n_1}^2 = [(2L-2)/L](N_0/2)$, where $(N_0/2)$ is the noise variance in TWRN. In addition, we assume equal power at the users and the relay, which are normalised to one unit. Thus, the SNR per bit per user can be defined as

$$\rho = \frac{1}{([2L-2]/L)N_0} \tag{2}$$

Depending on the relay protocol (i.e. DF or AF), the relay makes use of the received signal $r_{i,i+1}$ in different ways, which is discussed in the next two subsections.

2.2 Transmission protocol at the relay for decode-and-forward

The relay first decodes the superimposed received signal $r_{i,i+1}$ (as illustrated in Fig. 1*b*), using the maximum a posteriori (MAP) criterion, to obtain $\hat{V}_{i,i+1}$, which is an estimate of the true network coded symbol, $V_{i,i+1} = W_i \oplus W_{i+1}$, transmitted by the users. The optimum threshold, γ_r, for MAP detection at the relay is derived in [1] and is defined later in Section 3 after (12). The relay then performs BPSK modulation on the recovered network coded symbol and retransmits to all the users, which receive a noisy version of the signal as

$$Y_{i,i+1} = Z_{i,i+1} + n_2 \tag{3}$$

where $Z_{i,i+1} \in \{\pm 1\}$ and n_2 is the zero mean AWGN in the relay-user link with noise variance $\sigma_2^2 = [(2L-2)/L](N_0/2)$.

Each user receives and decodes the signal $Y_{i,i+1}$ (illustrated in Fig. 1*b*) using MAP criterion to obtain the network coded symbol $\hat{\hat{V}}_{i,i+1}$. The optimum threshold, γ, for MAP detection at the users is derived in [24] and is defined later in Section 3 after (12). After decoding the network coded information of all the user pairs, the ith user performs XOR operation between its own information symbols W_i and the decoded symbols $\hat{\hat{V}}_{i,i+1}$ to extract the information of the $(i+1)$th user as

$$\hat{\hat{W}}_{i+1} = \hat{\hat{V}}_{i,i+1} \oplus W_i \tag{4}$$

Then the ith user utilises this extracted information of the $(i+1)$th user to obtain the information of the $(i+2)$th user in the same manner. This process is continued until all the users' transmitted information is recovered. The sequential downward information extraction process can be expressed as

$$\hat{\hat{W}}_{i+2} = \hat{\hat{V}}_{i+1,i+2} \oplus \hat{\hat{W}}_{i+1}, \quad \hat{\hat{W}}_L = \hat{\hat{V}}_{L-1,L} \oplus \hat{\hat{W}}_{L-1} \tag{5}$$

Note that for all users other than the first user, the sequential upward information extraction process is also performed, that is, $\hat{\hat{W}}_{i-1} = \hat{\hat{V}}_{i-1,i} \oplus W_i$, $\hat{\hat{W}}_{i-2} = \hat{\hat{V}}_{i-2,i-1} \oplus \hat{\hat{W}}_{i-1}, \ldots, \hat{\hat{W}}_1 = \hat{\hat{V}}_{1,2} \oplus \hat{\hat{W}}_2$.

2.3 Transmission protocol at the relay for amplify-and-forward

The relay amplifies the superimposed received signal $r_{i,i+1}$ with an amplification factor α and then retransmits to all the users, which receive a noisy version of this retransmitted signal as

$$Y_{i,i+1} = \alpha(X_i + X_{i+1} + n_1) + n_2 \tag{6}$$

where $\alpha = \sqrt{\frac{1}{2 + [(2L-2)/L](N_0/2)}}$ is chosen to maintain unity power at the users and the relay.

The ith user subtracts its own signal multiplied by α from the received signal $Y_{i,i+1}$ and then performs maximum-likelihood (ML) detection on the resulting signal to estimate the message of the $(i+1)$th user as

$$\hat{\hat{W}}_{i+1} = \arg \min_{X_i \in \{\pm 1\}} |Y_{i,i+1} - \alpha X_i|^2 \tag{7}$$

Then, the ith user utilises the BPSK modulated version of this extracted information, that is, $\hat{\hat{X}}_{i+1}$ to obtain the information of the $(i+2)$th user in the same manner. This process is continued until all the users' transmitted information is recovered. The sequential downward information extraction process can be expressed as

$$\begin{aligned} \hat{\hat{W}}_{i+2} &= \arg \min_{\hat{\hat{X}}_{i+1} \in \{\pm 1\}} \left|Y_{i+1,i+2} - \alpha \hat{\hat{X}}_{i+1}\right|^2, \\ \hat{\hat{W}}_L &= \arg \min_{\hat{\hat{X}}_{L-1} \in \{\pm 1\}} \left|Y_{L-1,L} - \alpha \hat{\hat{X}}_{L-1}\right|^2 \end{aligned} \tag{8}$$

Note that for all users other than the first user, the sequential upward information extraction process can similarly be performed.

3 Characterising the error performance in a MWRN

In this section, we discuss the different metrics used to characterise the error performance in a MWRN. We also highlight the challenges associated with calculating these metrics.

For an error-free communication, each user in a MWRN must correctly decode the information from 'all' other users. Depending on the number of users whose information is incorrectly decoded by a certain user, different error events can occur. As highlighted earlier in Section 1, previous works have focused on characterising the special cases of error events $k = 0, 1, 2, L–1$ [24] for DF and $k \geq 1$ [25] for AF. The error probability for the general case of k error events in an L-user DF or AF MWRN has not been addressed. In addition, these discrete error events offer only a partial view of the overall error performance. For a complete characterisation of the error performance, we need a metric that takes into account all the error events, as well as their relative impacts. Hence, in this paper, we also consider the average BER as the error performance metric for a MWRN.

The average BER for the ith user in a MWRN can be defined as the expected probability of all the error events, that is

$$P_{i,\text{avg}} = \frac{1}{L-1}\sum_{k=1}^{L-1} kP_i(k) \tag{9}$$

where $P_i(k)$, for $k \in [1, L-1]$, represents the probability of exactly k errors at the ith user, the factor k represents number of errors in kth error event and L–1 denotes the number of possible error events. Note that the average BER in (9) is the average across the information bits of all the users decoded by a user.

The average BER depends on the probability of exactly k error events, which is given by (see (10))

It is not straightforward to characterise the error probability $P_i(k)$ for the general case of k error events and consequently the average BER for a user in a MWRN because of following two main reasons. First, in a DF or AF MWRN, the decision about each user depends on the decision about previous users. For example, according to (4) and (5) in a DF MWRN, if an error occurs in the message extraction process, the error propagates through to the following messages, until another error is made. Also according to (7) and (8) in an AF MWRN, the mean of the next signal is shifted from its true value by the previous error. These dependencies will be explained in detail in Sections 4 and 5, respectively. Secondly, while a TWRN has only one possible error event, that is, only one user's message can be incorrectly decoded, an L-user MWRN consists of (L–1) user pairs and so (L–1) error events are possible. This can be quite large, depending on the number of users.

In the next two sections, we address these challenges and characterise the error probability $P_i(k)$ for the general case of k error events and the average BER for a user in DF and AF MWRN.

4 Probability of *k* error events and average BER for a user in DF MWRN

In this section, we first derive exact closed-form expressions for the probability of $k = 1$ and $k = 2$ error events in an L-user DF MWRN. Based on the insights provided by this analysis, we then obtain an approximate expression for the probability of $k \geq 3$ error events $P_i(k)$, which we use to obtain the average BER for a user.

4.1 Probability of k = 1 error event

A single error event in a DF MWRN occurs from:

- error case A_1: two consecutive erroneous network coded bits or,
- error case B_1: an error in the network coded bits involving one of the end users.

For example, as illustrated in Table 1, error case A_1 can occur when user 1 wrongly decodes the message of user 2 by making consecutive errors in the detection of $\hat{\hat{V}}_{1,2}$ and $\hat{\hat{V}}_{2,3}$. Similarly, error case B_1 can occur if there is an error in the decoding of $\hat{\hat{V}}_{1,2}$ at any user $i \neq 1$ (or $\hat{\hat{V}}_{L-1,L}$ at any user $i \neq L$). Note that the error examples shown in Table 1 are not unique and other combinations of errors are also possible.

Let P_{A_1} and P_{B_1} denote the probability of occurrence of error cases A_1 and B_1, respectively. We have

$$P_{A_1} = \left(1 - P_{\text{DF}}\right)^{L-3} P_{\text{DF}}^2 \tag{11a}$$

$$P_{B_1} = \left(1 - P_{\text{DF}}\right)^{L-2} P_{\text{DF}} \tag{11b}$$

where P_{DF} is the probability that the network coded message of any one user pair is incorrectly decoded, which is the same as the average BER in a TWRN and is given by Islam and Sadeghi [24] (see (12))

where ρ is the average SNR per bit per user defined in (2), $\gamma_r = 1 + (1/4\rho)\ln\left(1 + \sqrt{1 - \text{e}^{-8\rho}}\right)$ [1] and $\gamma = \frac{1}{4\rho} \ln\left(4\left(\text{erfc}\left(\left[(\gamma_r + 2)/(\sqrt{1/\rho})\right]\right) + \text{erfc}\left(\left[(\gamma_r - 2)/(\sqrt{1/\rho})\right]\right) + 2\text{erfc}\left(\left[(\gamma_r)/(\sqrt{1/\rho})\right]\right)\right)^{-1} - 1\right)$ [24] are the optimum

$$P_i(k) = \frac{\text{Number of events where } i\text{th user incorrectly decodes messages of exactly } k \text{ users}}{\text{Packet length, } T} \tag{10}$$

$$\begin{aligned} P_{\text{DF}} = \frac{1}{8}\Bigg[& \text{erfc}\left(\frac{-\gamma-1}{\sqrt{1/\rho}}\right)\left\{\text{erf}\left(\frac{\gamma_r+2}{\sqrt{1/\rho}}\right) + \text{erf}\left(\frac{\gamma_r-2}{\sqrt{1/\rho}}\right) + 2\text{erfc}\left(\frac{\gamma_r}{\sqrt{1/\rho}}\right)\text{erfc}\left(\frac{\gamma-1}{\sqrt{1/\rho}}\right)\right\} \\ & + \text{erfc}\left(\frac{1-\gamma}{\sqrt{1/\rho}}\right)\left\{\text{erfc}\left(\frac{\gamma_r+2}{\sqrt{1/\rho}}\right) + \text{erfc}\left(\frac{\gamma_r-2}{\sqrt{1/\rho}}\right) + 2\text{erf}\left(\frac{\gamma_r}{\sqrt{1/\rho}}\right)\text{erfc}\left(\frac{\gamma+1}{\sqrt{1/\rho}}\right)\right\}\Bigg] \end{aligned} \tag{12}$$

IET Commun., 2013, Vol. 7, Iss. 15, pp. 1605–1616
doi: 10.1049/iet-com.2013.0284

thresholds for MAP detection at the relay and user, respectively, and $\text{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2}dt$ and $\text{erfc}(x) = (2/\sqrt{\pi})\int_x^\infty e^{-t^2}dt$ are the error function and complementary error function, respectively.

Note that in (11a), the factor P_{DF}^2 represents the probability of incorrectly decoding two consecutive erroneous network coded bits from two user pairs while the factor $(1 - P_{\text{DF}})^{L-3}$ represents the probability that the network coded messages of the remaining L–3 user pairs are correctly decoded. Similarly in (11b), the factor P_{DF} represents the probability of incorrectly decoding network coded bit involving an end user while the factor $(1 - P_{\text{DF}})^{L-2}$ represents the probability that the network coded messages of the remaining L–2 user pairs are correctly decoded. Recall that there are $L-1$ user pairs in an L-user MWRN.

Using (11), the exact probability of one error event in a DF MWRN can be expressed as

$$P_{i,\text{DF}}(1) = \begin{cases} (L-3)P_{A_1} + 2P_{B_1} & i \neq 1 \text{ and } i \neq L \\ (L-2)P_{A_1} + P_{B_1} & i = 1 \text{ or } i = L \end{cases} \tag{13}$$

where the two cases arise from the consideration of the two end users and the remaining users.

Remark 1: Equation (13) represents the probability that a user incorrectly decodes the message of exactly 1 user in an L-user DF MWRN.

4.2 Probability of k = 2 error events

Two error events in a DF MWRN can occur from:

- error case C_1: if two wrong network coded bits are separated by one correct network coded bit or,
- error case D_1: if the network coded bit involving one end user is correct but the following (or preceding) bit is incorrect or,
- error case E_1: if there are two pairs of consecutive erroneous network coded bits or,
- error case F_1: if the network coded bit involving one end user, as well as two other consecutive network coded bits, are incorrect or,
- error case G_1: if the network coded bits involving both the end users are incorrect.

These error cases are illustrated in Table 1. For example, error case C_1 can occur if user 1 incorrectly decodes user 2 and 3's messages by making errors in detecting $\hat{\hat{V}}_{1,2}$ and $\hat{\hat{V}}_{3,4}$. Other error cases can similarly be explained.

Let P_{C_1}, P_{D_1}, P_{E_1}, P_{F_1} and P_{G_1} denote the probability of occurrence of these five error cases. Using similar logic as before, we can express these probabilities as

$$P_{C_1} = (1 - P_{\text{DF}})^{L-3} P_{\text{DF}}^2 \tag{14a}$$

$$P_{D_1} = (1 - P_{\text{DF}})^{L-2} P_{\text{DF}} \tag{14b}$$

$$P_{E_1} = (1 - P_{\text{DF}})^{L-5} P_{\text{DF}}^4 \tag{14c}$$

$$P_{F_1} = (1 - P_{\text{DF}})^{L-4} P_{\text{DF}}^3 \tag{14d}$$

$$P_{G_1} = (1 - P_{\text{DF}})^{L-3} P_{\text{DF}}^2 = P_{C_1} \tag{14e}$$

where P_{DF} is given in (12). Then, using (14), the exact probability of two error events in a DF MWRN can be expressed as (see (15)) where m is the decoding order difference between the two users that are incorrectly decoded and $(L-3-m)$ indicates the number of such user pairs. For example, if $i = 2$, $L = 10$ and $m = 2$, then user 2 can make error about (10–3–2) = 5 user pairs (i.e. user pair (3,5), (4,6), (5,7), (6,8) or (7,9). In this case, messages of users 3 and 5 can be incorrectly decoded by wrong detection of $\hat{\hat{V}}_{2,3}$, $\hat{\hat{V}}_{3,4}$, $\hat{\hat{V}}_{4,5}$ and $\hat{\hat{V}}_{5,6}$.

Remark 2: Equation (15) represents the probability that a user incorrectly decodes the messages of exactly 2 users in an L-user DF MWRN.

4.3 Probability of k error events

The preceding subsections help to illustrate the point that finding an exact general expression for the probability of k error events, where $k \geq 3$, is difficult because of the many different ways k error events can occur. Hence, in this subsection, we focus on finding an approximate expression for the probability of k error events using high SNR assumption. This will be useful in deriving the average

$$P_{i,\text{DF}}(2) = \begin{cases} (L-3)P_{C_1} + P_{D_1} + \sum_{m=2}^{L-3}(L-2-m)P_{E_1} + \\ (L-3)P_{F_1}, & i = 1 \text{ or } i = L \\ (L-4)P_{C_1} + P_{D_1} + \sum_{m=2}^{L-4}(L-3-m)P_{E_1} + \\ 2(L-4)P_{F_1} + P_{C_1}, & i = 2 \text{ or } i = L-1 \\ (L-5)P_{C_1} + 2P_{D_1} + \sum_{m=2}^{L-4}(L-3-m)P_{E_1} + \\ 2(L-4)P_{F_1} + P_{C_1}, & i = 3 \text{ or } i = L-2 \\ (L-5)P_{C_1} + 2P_{D_1} + \sum_{m=2}^{i-2}(L-4-m)P_{E_1} + \\ \sum_{m=i-1}^{L-i-1}(L-3-m)P_{E_1} + \sum_{m=L-i}^{L-3}(L-2-m)P_{E_1} + \\ 2(L-4)P_{F_1} + P_{C_1}, & i \notin \{1, 2, 3, L-2, L-1, L\} \end{cases} \tag{15}$$

Table 1 Illustration of the error cases for one and two error events in a 10-user DF MWRN. Here, ✓ and × represent correct and incorrect detection, respectively

Error case	Decoding user i	Network coded message									Error event
		$\hat{V}_{1,2}$	$\hat{V}_{2,3}$	$\hat{V}_{3,4}$	$\hat{V}_{4,5}$	$\hat{V}_{5,6}$	$\hat{V}_{6,7}$	$\hat{V}_{7,8}$	$\hat{V}_{8,9}$	$\hat{V}_{9,10}$	
A_1	$i \in \{1, L\}$	×	×	✓	✓	✓	✓	✓	✓	✓	1
B_1	$i \neq 1$	×	✓	✓	✓	✓	✓	✓	✓	✓	1
B_1	$i \neq L$	✓	✓	✓	✓	✓	✓	✓	✓	×	1
C_1	$i \in \{1, L\}$	×	✓	×	✓	✓	✓	✓	✓	✓	2
D_1	$i \neq 1, 2$	✓	×	✓	✓	✓	✓	✓	✓	✓	2
D_1	$i \neq L-1, L$	✓	✓	✓	✓	✓	✓	✓	×	✓	2
E_1	$i \in \{1, L\}$	✓	×	×	✓	×	×	✓	✓	✓	2
F_1	$i \neq 1$	×	✓	✓	✓	×	×	✓	✓	✓	2
F_1	$i \neq L$	✓	✓	✓	✓	×	×	✓	✓	×	2
G_1	$i \neq 1, L$	×	✓	✓	✓	✓	✓	✓	✓	×	2

BER in the next subsection. Note that the use of the high SNR assumption to facilitate closed-form results is commonly used in two-way [4, 10, 16] and other types of relay networks [26, 27].

Comparing (11) and (14), we can see that $P_{C_1} = P_{A_1}$ and $P_{D_1} = P_{B_1}$. At high SNR, the higher order terms involving $P_{\rm DF}^2$ and higher powers can be neglected and only the terms P_{B_1} and P_{D_1} effectively contribute to the probability of one and two error events in (13) and (15), respectively. Recall that P_{B_1} is the probability of one error about the network coded message of an end user and P_{D_1} is the probability of one erroneous network coded bit involving users just following (or preceding) the end user. Extending this analogy to the case of k error events, the dominating factor at high SNR would represent the scenario when the network coded bit involving the kth and $(k+1)$th (or $(L-k+1)$th and $(L-k)$th) users is incorrectly decoded, resulting in error about k users' messages. Thus, the probability of k error events can be asymptotically approximated as

$$P_{i,{\rm DF}}(k) \simeq \left(1 - P_{\rm DF}\right)^{L-2} P_{\rm DF} \simeq P_{\rm DF} \tag{16}$$

where in the last step we have used the fact that at high SNR $P_{\rm DF} \ll 1$ and hence $(1 - P_{\rm DF}) \simeq 1$. It will be shown in Section 7 that for medium to high SNRs, (16) can accurately predict the probability of k error events in a DF MWRN.

Remark 3: Equation (16) shows that at high SNR in an L-user DF MWRN, all the error events are equally probable and their probability can be asymptotically approximated as $P_{\rm DF}$, in (12), that is, the average BER in a TWRN.

4.4 Average BER

Substituting (16) in (9) and simplifying, the average BER for a user in DF MWRN is

$$\begin{aligned} P_{i,{\rm avg,DF}} &= \left(\sum_{k=1}^{L-1} k\right) \frac{P_{\rm DF}}{L-1} = \frac{L(L-1)}{2} \frac{P_{\rm DF}}{L-1} \\ &= \frac{L}{2} P_{\rm DF} \end{aligned} \tag{17}$$

Remark 4: Equation (17) shows that at high SNR, the average BER in an L-user DF MWRN can be asymptotically approximated as the average BER in a TWRN scaled by a factor of $L/2$. Although (17) is obtained using a high SNR assumption, it will be shown later in Section 7 that the average BER is well approximated even at medium to high SNRs.

5 Probability of k error events and average BER for a user in AF MWRN

In this section, we characterise the average BER for a user in an L-user AF MWRN. The general approach in our analysis is similar to the case of DF MWRN, with some important differences which are highlighted in the following subsections.

5.1 Probability of $k = 1$ error event

A single error event in an AF MWRN occurs from:

- error case A_2: a middle user's message is wrongly estimated with correct decision about the following user or,
- error case B_2: an error in the estimated signal of one of the end users.

These error cases are illustrated in Table 2.

Let P_{A_2} and P_{B_2} denote the probability of occurrence of error cases A_2 and B_2, respectively. We have

$$P_{A_2} = \left(1 - P_{\rm AF}\right)^{L-3} P_{\rm AF}\left(1 - P'_{\rm AF}\right) \tag{18a}$$

$$P_{B_2} = \left(1 - P_{\rm AF}\right)^{L-2} P_{\rm AF} \tag{18b}$$

where $P_{\rm AF}$ is the probability that the message of any one user is incorrectly decoded, which is the same as the average BER in an AF TWRN and is given by Cui and Kliewer [11]

$$P_{\rm AF} = \frac{1}{2} {\rm erfc}\left(\frac{\alpha}{\sqrt{\left(\alpha^2 + 1\right)(1/\rho)}}\right) \tag{19}$$

where ρ is the average SNR per bit per user defined in (2), α is the amplification factor defined below (6) and $P'_{\rm AF}$ is the probability of wrongly detecting the message of a user given that the previous user's message is also incorrect. This can be easily found as follows. To find $P'_{\rm AF}$, we need

IET Commun., 2013, Vol. 7, Iss. 15, pp. 1605–1616
doi: 10.1049/iet-com.2013.0284

Table 2 Illustration of the error cases for one and two error events in a 10-user AF MWRN. Here, ✓ and × represent correct and incorrect detection, respectively

Error case	Decoding user i	Extracted messages										Error event
		$\hat{\hat{X}}_1$	$\hat{\hat{X}}_2$	$\hat{\hat{X}}_3$	$\hat{\hat{X}}_4$	$\hat{\hat{X}}_5$	$\hat{\hat{X}}_6$	$\hat{\hat{X}}_7$	$\hat{\hat{X}}_8$	$\hat{\hat{X}}_9$	$\hat{\hat{X}}_{10}$	
A_2	$I\in\{1, L\}$	✓	×	✓	✓	✓	✓	✓	✓	✓	✓	1
B_2	$i\neq 1$	×	✓	✓	✓	✓	✓	✓	✓	✓	✓	1
B_2	$i\neq L$	✓	✓	✓	✓	✓	✓	✓	✓	✓	×	1
C_2	$i\in\{1, L\}$	✓	✓	×	×	✓	✓	✓	✓	✓	✓	2
D_2	$i\neq 1, 2$	×	×	✓	✓	✓	✓	✓	✓	✓	✓	2
D_2	$i\neq L-1, L$	✓	✓	✓	✓	✓	✓	✓	✓	×	×	2
E_2	$i\in\{1, L\}$	✓	×	✓	✓	×	✓	✓	✓	✓	✓	2
F_2	$i\neq 1$	×	✓	✓	✓	×	✓	✓	✓	✓	✓	2
F_2	$i\neq L$	✓	✓	✓	✓	×	✓	✓	✓	✓	×	2
G_2	$i\neq 1, L$	×	✓	✓	✓	✓	✓	✓	✓	✓	×	2

to find the probability $P\left(\hat{\hat{W}}_{i+2} \neq W_{i+2}|\hat{\hat{W}}_{i+1} \neq W_{i+1}\right)$. If $\hat{\hat{X}}_{i+1} \neq X_{i+1}$, then $\hat{\hat{X}}_{i+2} = \alpha X_{i+1} + \alpha X_{i+2} + \alpha n_1 + n_2 - \alpha \hat{\hat{X}}_{i+1} = \alpha X_{i+2} + \alpha n_1 + n_2 + 2\alpha X_{i+1}$. Thus, the mean of the received signal is shifted by either 2α or -2α. Using this fact and (19), we have

$$P'_{\text{AF}} = \frac{1}{4}\left[\text{erfc}\left(\frac{3\alpha}{\sqrt{(\alpha^2+1)(1/\rho)}}\right) + \text{erfc}\left(\frac{-\alpha}{\sqrt{(\alpha^2+1)(1/\rho)}}\right)\right] \tag{20}$$

Finally, using (18), the exact probability of one error event in an AF MWRN can be expressed as

$$P_{i,\text{AF}}(1) = \begin{cases} (L-3)P_{A_2} + 2P_{B_2} & i \neq 1 \text{ and } i \neq L \\ (L-2)P_{A_2} + P_{B_2} & i = 1 \text{ or } i = L \end{cases} \tag{21}$$

where the two cases arise from the consideration of the two end users and the remaining users.

Remark 5: Assume that $X_{i+2} = 1$. Although the shift of the mean of the signal by 2α (when $X_{i+1} = 1$) is helpful in reducing the probability of error in detecting $X_{i+2} = 1$, the shift in the mean by -2α (when $X_{i+1} = -1$) would be seriously detrimental for its detection. We will use this fact later in our high SNR BER analysis by setting $P'_{\text{AF}} \simeq 1/2$.

Remark 6: Equation (21) represents the probability that a user incorrectly decodes the message of exactly 1 user in an L-user AF MWRN. Equation (21) is different from (13) because of the presence of (20), which is large even at moderate to high SNRs.

5.2 Probability of $k = 2$ error events

Two error events in an AF MWRN can occur from:

- error case C_2: if messages of two consecutive users are incorrectly decoded but the message of the user next to them is correct or,
- error case D_2: if the estimated message of the end user and that of the following (or preceding) user are incorrect or,
- error case E_2: if two middle users' messages are incorrectly estimated provided that the message of the users adjacent to each of them are correct or,
- error case F_2: if there is error about the message of one end user and any other user, provided that the messages of the users in between them are correctly estimated or,
- error case G_2: if both the end users' messages are incorrectly estimated.

These error cases are illustrated in Table 2.

Let P_{C_2}, P_{D_2}, P_{E_2}, P_{F_2} and P_{G_2} denote the probability of occurrence of these five error cases. Using similar logic as before, we can express these probabilities as

$$P_{C_2} = \left(1 - P_{\text{AF}}\right)^{L-4} P_{\text{AF}}\left(1 - P'_{\text{AF}}\right)P'_{\text{AF}} \tag{22a}$$

$$P_{D_2} = \left(1 - P_{\text{AF}}\right)^{L-3} P_{\text{AF}} P'_{\text{AF}} \tag{22b}$$

$$P_{E_2} = \left(1 - P_{\text{AF}}\right)^{L-5} P^2_{\text{AF}}\left(1 - P'_{\text{AF}}\right)^2 \tag{22c}$$

$$P_{F_2} = \left(1 - P_{\text{AF}}\right)^{L-4} P^2_{\text{AF}}\left(1 - P'_{\text{AF}}\right) \tag{22d}$$

$$P_{G_2} = \left(1 - P_{\text{AF}}\right)^{L-3} P^2_{\text{AF}} \neq P_{C_2} \tag{22e}$$

where P_{AF} and P'_{AF} are given in (19) and (20), respectively. Note that the expressions for error cases C_2 to E_2 are different from the error cases C_1 to E_1. This is because of the different relay processing in AF and DF MWRNs.

Using (22), the exact probability of two error events in an AF MWRN can be expressed as (see (23)) where m is the decoding order difference between the two users that are incorrectly decoded.

Remark 7: Equation (23) represents the probability that a user incorrectly decodes the message of exactly 2 users in an L-user AF MWRN.

5.3 Probability of k error events

As for the case of DF MWRN, it is very hard to find an exact general expression for the probability of k error events in AF MWRN. Hence, in this subsection, we focus on finding an approximate expression for the probability of k error events using high SNR assumption.

At high SNR, we can neglect P_{E_2}, P_{F_2} and P_{G_2} in (22) since they involve higher order product terms of probabilities.

Comparing (18) and (22), we can see that the relationship between the dominating terms in the probability of one and two error events at high SNR is $C_2 = [(P'_{AF})/(1 - P_{AF})]A_2$, $D_2 = [(P'_{AF})/(1 - P_{AF})]B_2$. Recall that C_2 and D_2 correspond to the cases of two consecutive errors involving middle users and two consecutive errors involving one of the end users, respectively. Extending this analogy to the case of k error events, the dominating terms at high SNR would represent the cases of k consecutive errors in the middle users and k consecutive errors involving one end user and $k-1$ following (or preceding) users. Thus, the probability of k error events can be asymptotically approximated as

$$P_{i,\mathrm{AF}}(k) \simeq \left(\frac{P'_{\mathrm{AF}}}{1 - P_{\mathrm{AF}}}\right)^{k-1} \times \left\{(L-k-1)(1-P_{\mathrm{AF}})^{L-3}P_{\mathrm{AF}}(1-P'_{\mathrm{AF}}) + (1-P_{\mathrm{AF}})^{L-2}P_{\mathrm{AF}}\right\} \tag{24}$$

$$\simeq \frac{L-k+1}{2^k}P_{\mathrm{AF}} \tag{25}$$

where in the last step we have used the fact that at high SNR $P'_{\mathrm{AF}} \simeq 1/2$ and $1 - P_{\mathrm{AF}} \simeq 1$. It will be shown in Section 7 that for medium to high SNRs, (25) can accurately predict the probability of k error events in an AF MWRN.

Remark 8: Equation (25) shows that at high SNR the probability of k error events in an AF MWRN can be asymptotically approximated as the average BER of an AF TWRN scaled by a factor $(L-k+1)/2^k$, which depends on both L and k. Comparing (25) and (16), we can see that, at high SNR, the higher order error events are less probable in an L-user AF MWRN, but all error events are equally probable in an L-user DF MWRN.

5.4 Average BER

Substituting (25) in (9) and simplifying, the average BER for a user in AF MWRN is

$$P_{i,\mathrm{avg,AF}} = P_{\mathrm{AF}}\sum_{k=1}^{L-1}\frac{L-k+1}{2^k} = \left(\frac{L+1}{L-1}\left(2 - \frac{L}{2^{L-2}}\right) - \frac{3}{L-1}\left(2 - \frac{L^2-3}{2^{L-2}}\right)\right)P_{\mathrm{AF}} \tag{26}$$

Remark 9: Equation (26) shows that at high SNR the average BER in an L-user AF MWRN can be asymptotically approximated as the average BER in a TWRN scaled by a factor $([(L+1)/(L-1)](2 - [L/(2^{L-2})]) - [(3/(L-1))](2 - (L^2-3)/(2^{L-2})))$. Comparing (17) and (26), we can see that, the larger number of error events have a smaller contribution in the average BER for a user in AF MWRN, whereas they have the same contribution as the small number of error events in a DF MWRN.

6 Average BER for a user in MWRN with Rayleigh fading

In this section, we demonstrate that the preceding analysis is also applicable for the case of DF or AF MWRN with Rayleigh fading channels. Following [8], we assume that (i) all the channels are reciprocal, which is typical in TDD systems, (ii) the channel coefficients are modelled as independent zero-mean and unit-variance complex-valued Gaussian random variables, (iii) the channel coefficients are independent during the multiple access and broadcast phases and (iv) perfect channel state information is available at the relay and the user nodes. In practice, accurate channel state information can be obtained by sending pilot symbols [8, 25, 28], the consideration of which is outside the scope of this paper. Taking Rayleigh fading into account, (1) modifies to

$$r_{i,i+1} = h_iX_i + h_{i+1}X_{i+1} + n_1 \tag{27}$$

$$P_{i,\mathrm{AF}}(2) = \begin{cases} (L-3)P_{C_2} + P_{D_2} + \sum_{m=2}^{L-3}(L-2-m)P_{E_2} + (L-3)P_{F_2}, & i = 1 \text{ or } i = L \\ (L-4)P_{C_2} + P_{D_2} + \sum_{m=2}^{L-4}(L-3-m)P_{E_2} + 2(L-4)P_{F_2} + P_{G_2}, & i = 2 \text{ or } i = L-1 \\ (L-5)P_{C_2} + 2P_{D_2} + \sum_{m=2}^{L-4}(L-3-m)P_{E_2} + 2(L-4)P_{F_2} + P_{G_2}, & i = 3 \text{ or } i = L-2 \\ (L-5)P_{C_2} + 2P_{D_2} + \sum_{m=2}^{i-2}(L-4-m)P_{E_2} + \sum_{m=i-1}^{L-i-1}(L-3-m)P_{E_2} + \sum_{m=L-i}^{L-3}(L-2-m)P_{E_2} + 2(L-4)P_{F_2} + P_{G_2}, & i \notin \{1,\ 2,\ 3,\ L-2,\ L-1,\ L\} \end{cases} \tag{23}$$

IET Commun., 2013, Vol. 7, Iss. 15, pp. 1605–1616
doi: 10.1049/iet-com.2013.0284

where h_i and h_{i+1} are the complex channel coefficients for the ith and $(i+1)$th user, respectively.

6.1 DF MWRN with Rayleigh fading

The relay decodes the received signal using ML criterion [8] and obtains an estimate of the corresponding network coded message. The relay then broadcasts the estimated signal. Thus, (3) modifies to

$$Y_{i,i+1} = h_i Z_{i,i+1} + n_2 \tag{28}$$

The users then detect the received signal through ML criterion [8].

With the modified signal model, the error propagation in DF MWRN is almost similar to the case as before. Thus, it can be shown that the probability of large number of errors is asymptotically the same as that of small number of errors, even in the presence of fading. Hence, we can use (17) to find the average BER for a user. In order to do this, we need an expression for the BER in a DF TWRN, $P_{\rm DF}$. No exact expression is available in the literature for the average BER in a TWRN with Rayleigh fading. However, upper and lower bounds have been derived in [8]. In this work, we use the upper bound for $P_{\rm DF}$, which is given by Ju and Kim [8]

$$P_{\rm DF} = 2\Phi_1(\bar{\gamma}) + \frac{1}{2}\Xi(\bar{\gamma}) \tag{29}$$

where $\bar{\gamma}$, $\boldsymbol{\Phi}_1(\bar{\gamma})$ and $\Xi(\bar{\gamma})$ are given at the bottom of the page

6.2 AF MWRN with Rayleigh fading

For AF MWRN, the amplified and retransmitted signal in (6) modifies to

$$Y_{i,i+1} = h_i\alpha\big(h_i X_i + h_{i+1}X_{i+1} + n_1\big) + n_2 \tag{30}$$

After subtracting self information, user i performs ML detection to estimate the other user's message. The sequential downward and upward message extraction process is the same as before.

With the modified signal model, the error propagation in AF MWRN is different from the AWGN case. This is because the primary cause of error propagation in AF MWRN is the shifting of the mean of the received signal when the previous message has been incorrectly detected. For example, if $\hat{\hat{X}}_{i+1} \neq X_{i+1}$, then $\hat{\hat{X}}_{i+2} = \alpha h_i h_{i+2} X_{i+2} + \alpha h_i n_1 + n_2 + 2\alpha h_i h_{i+1} X_{i+1}$. Thus we can see that the mean of the received signal is affected by the channel coefficients. That is why, we cannot ignore $P'_{\rm AF}$ and obtain (25) from (24). So, instead of (25), we will use (24) to provide the analytical expression of average BER for a user, where the exact average BER for an AF TWRN in Rayleigh fading is given by the authors [9, 28]

$$P_{\rm AF} = Q\left(\sqrt{\frac{|h_i|^2|h_{i+1}|^2}{2|h_i|^2(1/\rho) + |h_{i+1}|^2(1/\rho) + (1/\rho)^2}}\right) \tag{31}$$

and the expression for $P'_{\rm AF}$ is similarly derived as

$$P'_{\rm AF} = Q\left(\sqrt{\frac{|h_i|^2|h_{i+2}|^2}{4|h_i|^2|h_{i+1}|^2 + 2|h_i|^2(1/\rho) + |h_{i+2}|^2(1/\rho) + (1/\rho)^2}}\right) \tag{32}$$

where $Q(x) = 1/\sqrt{2\pi}\int_x^\infty {\rm e}^{-[t^2/2]}\,{\rm d}t$ is the Gaussian Q-function.

7 Results

In this section, we compare the BER expressions obtained by our analysis with the BER results obtained by Monte Carlo simulations. We consider three cases $L = 10$, $L = 50$ and $L = 100$ users and each user transmits a packet of $T = 10\,000$ bits. The SNR is assumed to be SNR per bit per user and user 1 is assumed to be decoding the messages of all other users. The simulation results are averaged over 1000 Monte Carlo trials per SNR point.

7.1 Probability of different error events in an AWGN DF MWRN

Fig. 2 plots the probability of k error events $P_{i,\rm DF}(k)$ in an $L = 10$ user DF MWRN in the case of AWGN. The simulation results are plotted for $k = 1, 2, 3, 5, 7$ and compared with the asymptotic bound in (16). For $k = 1, 2$ the exact probabilities are also plotted using (13) and (15), respectively. As highlighted in Remark 3, in an L-user DF MWRN, all the error events are equally probable and their probability can be asymptotically approximated as (16). This is confirmed by the results in Fig. 2. We can see that for medium to high SNRs (SNR > 5 dB), the asymptotic expression in (16) is very accurate in predicting the probability of k error events, for all the considered values of k. This verifies the accuracy of (16).

7.2 Probability of different error events in an AWGN AF MWRN

Figs. 3*a* and *b* plot the probability of k error events $P_{i,\rm AF}(k)$ in an $L = 10$ user AF MWRN corrupted by AWGN for $k = 1,2$

$$\bar{\gamma} = \rho,\ \Phi_1(\bar{\gamma}) = \left[1 - \sqrt{\left[\bar{\gamma}/(1+\bar{\gamma})\right]}\right]/2,$$

$$\Xi(\bar{\gamma}) = 2\Phi_1(\bar{\gamma}) - 4\{\Phi_1(\bar{\gamma})\}^2 - 2\Phi_2(\bar{\gamma}) - 2\sqrt{\bar{\gamma}/(1+\bar{\gamma})}\,\Phi_3(\bar{\gamma}),$$

$$\Phi_2(\bar{\gamma}) = 1/2\pi\left[\pi/2 - 2\sqrt{\bar{\gamma}/(1+\bar{\gamma})}\left(\pi/2 - \tan^{-1}\sqrt{\bar{\gamma}/(1+\bar{\gamma})}\right)\right],$$

$$\Phi_3(\bar{\gamma}) = 1/2\pi\left[\pi/2 - \delta_1\left(\pi/2 + \tan^{-1}\zeta_1\right) - \delta_2\left(\pi/2 + \tan^{-1}\zeta_2\right)\right],$$

$$\delta_1 = \sqrt{(1+\bar{\gamma})/(3+\bar{\gamma})},\ \ \delta_2 = \sqrt{\bar{\gamma}/(2+\bar{\gamma})}\ \text{and}\ \zeta_j = -\delta_j\cot\left(\sqrt{\bar{\gamma}/(1+\bar{\gamma})}\right)\ \text{for}\ j = 1, 2$$

doi: 10.1049/iet-com.2013.0284

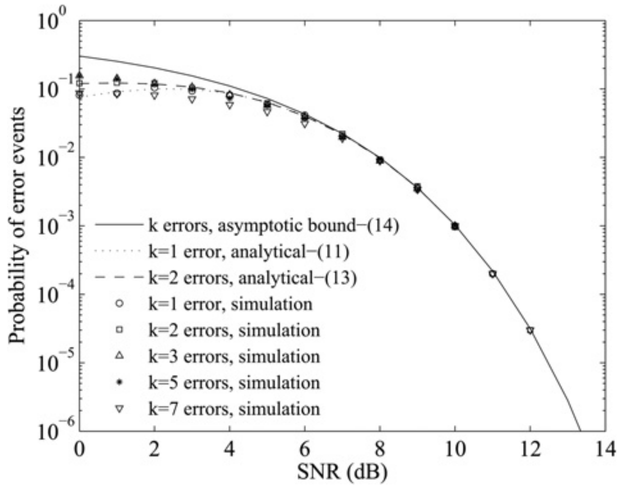


Fig. 2 *Probability of k = 1, 2 ,3 ,5, 7 error events in an L = 10 user DF MWRN with AWGN*

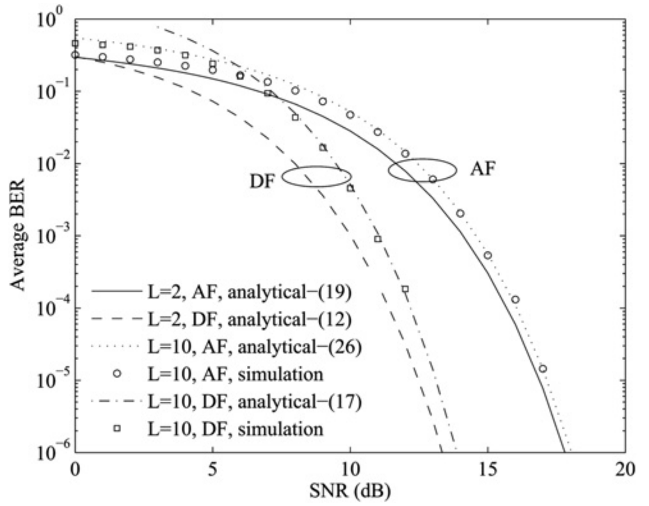


Fig. 4 *Average BER for a user in an L = 10 user DF or AF MWRN with AWGN*

error events and $k = 3$, 5, 7 error events, respectively. The simulation results are plotted for $k = 1$, 2, 3, 5, 7 and compared with the asymptotic bound in (25). For $k = 1,2$ the exact probabilities are also plotted using (21) and (23), respectively. As highlighted in Remark 8, in an L-user AF MWRN, the probability of error events depends on the value of k, with the higher order error events being less probable. This is confirmed by the results in Figs. 3*a* and *b*. We can see that for medium to high SNR (SNR > 10 dB), the asymptotic expression in (25) for k error events matches very well with the simulation results. This verifies the accuracy of (25).

7.3 Average BER for a user in AWGN DF or AF MWRN

Figs. 4 and 5 plot the average BER for a user in an AWGN DF or AF MWRN with $L = 10$ and $L = 100$ users, respectively. The average BER of DF or AF TWRN, from (12) or (19), respectively, is plotted as a reference. The average BER of

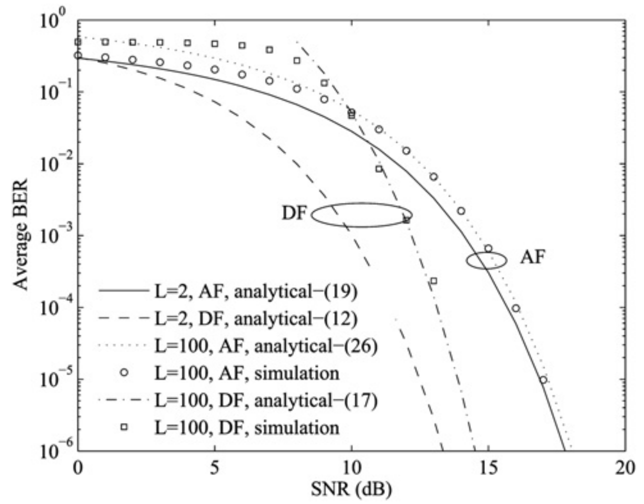
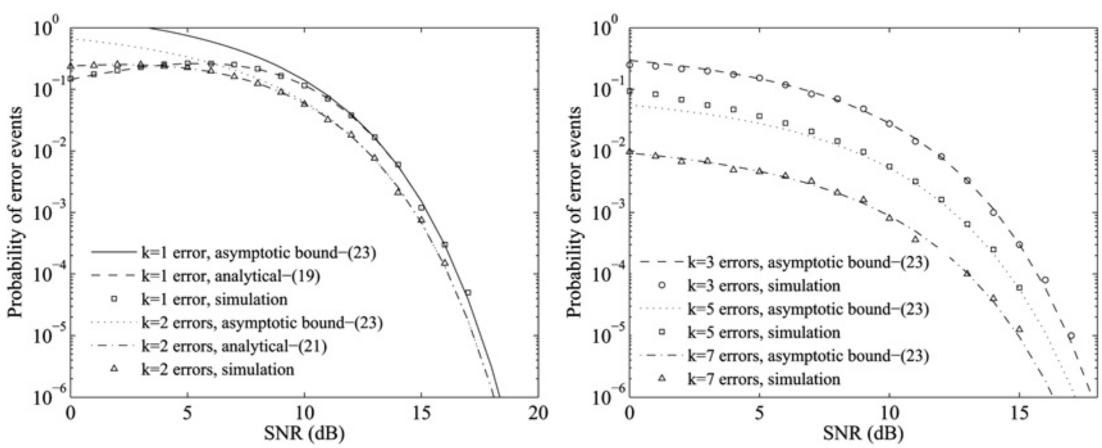


Fig. 5 *Average BER for a user in an L = 100 user DF or AF MWRN with AWGN*



a　　*b*

Fig. 3 *Probability of k error events in an L = 10 user AF MWRN with AWGN*

a $k = 1$, 2
b $k = 3$, 5, 7

doi: 10.1049/iet-com.2013.0284

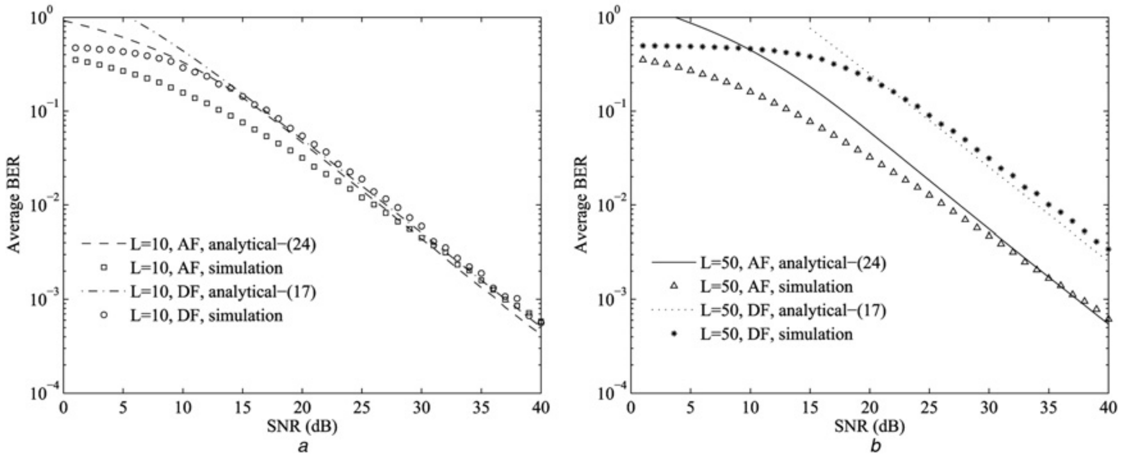


Fig. 6 *Average BER for a user in DF or AF MWRN with Rayleigh fading and L = 10, 50 users*
a $L = 10$
b $L = 50$

DF and AF MWRN is plotted using (17) and (26), respectively. From the figures, we can see that as the number of users increases ($L = 2$, 10, 100), the average BER increases for both DF or AF MWRN, which is intuitive. For DF MWRN, (17) can predict the average BER for a user accurately in medium to high SNR (approximately SNR > 7 dB for $L = 10$ users and SNR > 10 dB for $L = 100$ users). Also for AF MWRN, (26) can accurately predict the average BER for a user in medium to high SNR (approximately SNR > 10 dB).

Comparing DF and AF MWRNs, we can see that for low SNR, AF MWRN is slightly better than DF MWRN. However, at medium to high SNRs, DF MWRN is better than AF MWRN. For TWRN, it can be easily shown that the high SNR penalty for using AF, compared with DF, is 4.77 dB (see the Appendix). In MWRN, this high SNR penalty decreases as the number of users increases, for example, from Figs. 4 and 5, it is about 4 dB for $L = 10$ users and about 3.5 dB for 100 users. This can be explained using our analysis as follows. From (17), we can see that for DF MWRN the effective number of error terms in the average BER equation increases in proportion to the number of users. However, for AF MWRN, (26) shows that the probability of larger number of error events is very small, hence, the increase in the effective number of error terms for larger number of users is smaller. This results in a smaller SNR penalty for AF MWRN when larger number of users are involved, which agrees with the observations from Figs. 4 and 5.

7.4 Rayleigh fading

Figs. 6*a* and *b* plot the average BER for a user in DF or AF MWRN in Rayleigh fading channels and $L = 10$ and $L = 50$ users, respectively. The analytical result for DF MWRN is plotted using (17) and (29) and the analytical result for AF MWRN is plotted using (9), (24), (31) and (32). We can see that for both DF and AF MWRN the analytical results are within 1 dB of the simulation results for high SNR. Comparing the curves for $L = 10$ and $L = 50$ users, we can see that the average BER for a user in DF MWRN degrades significantly as the number of users increases. However, the average BER for a user in AF MWRN is more robust to the increase in the number of users. As explained before, this is due to the fact that the probability of larger number of error events in AF is much smaller compared with DF MWRN.

8 Acknowledgments

This work was supported under Australian Research Council Discovery Projects funding scheme (project no. DP0984950).

9 Conclusions

In this paper, we presented a method for analysing (i) the probability of k error events and (ii) the average BER for a user in both DF and AF MWRNs. The method is based on insights provided by the exact analysis of $k = 1$ and $k = 2$ error events, which leads to an accurate asymptotic expression for k error events in such systems. For both DF and AF MWRN in AWGN channel, the derived expression can accurately predict the BER of a user in medium to high SNR. For Rayleigh fading channel, the derived expressions match with simulations within 1 dB in high SNR. Using our analysis, we showed that DF MWRN outperforms AF MWRN in AWGN channels even with a larger number of users, whereas AF MWRN outperforms DF MWRN in Rayleigh fading channels even for a much smaller number of users.

10 References

1 Zhang, S., Liew, S.C., Lam, P.P.: 'Hot topic: physical-layer network coding'. Proc. ACM Mobicom, 2006, pp. 358–365
2 Katti, S., Gollakota, S., Katabi, D.: 'Embracing wireless interference: analog network coding'. Proc. ACM SIGCOMM, 2007, pp. 397–408
3 Rankov, B., Wittneben, A.: 'Spectral efficient signaling for half-duplex relay channels'. Proc. Asilomar Conf. Signals, Systems and Computers, 2005, pp. 1066–1071
4 Louie, R.H.Y., Li, Y., Vucetic, B.: 'Practical physical layer network coding for two-way relay channels: performance analysis and comparison', *IEEE Trans. Wirel. Commun.*, 2010, **9**, (2), pp. 764–777
5 Katti, S., Rahul, H., Hu, W., Katabi, D., Médard, M., Crowcroft, J.: 'XORs in the air: practical wireless network coding'. Proc. ACM SIGCOMM, 2006, pp. 243–254
6 Rankov, B., Wittneben, A.: 'Achievable rate regions for the two-way relay channel'. Proc. IEEE Int. Symp. on Information Theory (ISIT), 2006, pp. 1668–1672
7 Gündüz, D., Tuncel, E., Nayak, J.: 'Rate regions for the separated two-way relay channel'. Proc. Allerton Conf. Communication, Control, and Computing, 2008, pp. 1333–1340

8 Ju, M.C., Kim, I.M.: 'Error performance analysis of BPSK modulation in physical layer network coded bidirectional relay networks', *IEEE Trans. Commun.*, 2010, **58**, (10), pp. 2770–2775
9 Hwang, K.S., Ko, Y.C., Alouini, M.S.: 'Performance analysis of two-way amplify and forward relaying with adaptive modulation over multiple relay network', *IEEE Trans. Commun.*, 2011, **59**, (2), pp. 402–406
10 Song, L.: 'Relay selection for two-way relaying with amplify-and-forward protocols', *IEEE Trans. Veh. Technol.*, 2011, **60**, (4), pp. 1954–1959
11 Cui, T., Kliewer, J.: 'Memoryless relay strategies for two-way relay channels: performance analysis and optimization'. Proc. IEEE Int. Conf. Communications (ICC), 2008, pp. 1139–1143
12 You, Q., Li, Y., Chen, Z.: 'Joint relay selection and network coding using decode-and-forward protocol in two-way relay channels'. Proc. IEEE Global Telecommunications Conf. (GLOBECOM), 2010, pp. 1–6
13 Zhao, M., Zhou, Y., Ren, D., Yang, Y.: 'A minimum power consumption scheme for two-way relay with physical-layer network coding'. Proc. 2nd IEEE Int. Conf. Network Infrastructure and Digital Content, 2010, pp. 704–708
14 Lu, L., Liew, S.C.: 'Asynchronous physical-layer network coding', *IEEE Trans. Wirel. Commun.*', 2012, **11**, (2), pp. 819–831
15 Abdallah, S., Psaromiligkos, I.N.: 'Blind channel estimation for amplify-and-forward two-way relay networks employing *M*-PSK modulation', *IEEE Trans. Signal Process.*, 2012, **60**, (7), pp. 3604–3615
16 Wang, C., Liu, T.C.K., Dong, X.: 'Impact of channel estimation error on the performance of amplify-and-forward two-way relaying', *IEEE Trans. Veh. Technol.*, 2012, **61**, (3), pp. 1197–1207
17 Jiang, B., Gao, F., Gao, X., Nallanathan, A.: 'Channel estimation and training design for two-way relay networks with power allocation', *IEEE Trans. Wirel. Commun.*, 2010, **9**, (6), pp. 2022–2032
18 Gündüz, D., Yener, A., Goldsmith, A., Poor, H.V.: 'The multi-way relay channel'. Proc. IEEE Int. Symp. on Information Theory (ISIT), 2009, pp. 339–343
19 Gündüz, D., Yener, A., Goldsmith, A., Poor, H.V.: 'The multi-way relay channel', *IEEE Trans. Inf. Theory*, 2013, **59**, (1), pp. 51–63
20 Wang, T., Giannakis, G.B.: 'Complex field network coding for multiuser cooperative communications', *IEEE J. Sel. Areas Commun.*, 2008, **26**, (3), pp. 561–571
21 Ong, L., Johnson, S.J., Kellett, C.M.: 'An optimal coding strategy for the binary multi-way relay channel', *IEEE Commun. Lett.*, 2010, **14**, (4), pp. 330–332
22 Noori, M., Ardakani, M.: 'Optimal user pairing for asymmetric multi-way relay channels with pairwise relaying', *IEEE Commun. Lett.*, 2012, **16**, (11), pp. 1852–1855
23 Timo, R., Lechner, G., Ong, L., Johnson, S.: 'Multi-way relay networks: orthogonal uplink, source-channel separation and code design', *IEEE Trans. Commun.*, 2013, **61**, (2), pp. 753–768
24 Islam, S.N., Sadeghi, p.p.: 'Error propagation in a multi-way relay channel'. Proc. Int. Conference on Signal Processing and Communication Systems (ICSPCS), 2011, pp. 1–8
25 Amarasuriya, G., Tellambura, C., Ardakani, M.: 'Performance analysis of pairwise amplify-and-forward multi-way relay networks', *IEEE Wirel. Commun. Lett.*, 2012, **1**, (5), pp. 524–527
26 Zhao, Y., Adve, R., Lim, T.J.: 'Symbol error rate of selection amplify-and-forward relay systems', *IEEE Commun. Lett.*, 2006, **10**, (11), pp. 757–759
27 Su, W., Sadek, A., Ray Liu, K.: 'Cooperative communication protocols in wireless networks: performance analysis and optimum power allocation', *Wirel. Personal Commun.*, 2008, **44**, pp. 181–217
28 Cui, T., Ho, T., Kliewer, J.: 'Memoryless relay strategies for two-way relay channels', *IEEE Trans. Commun.*, 2009, **57**, (10), pp. 3132–3143

11 Appendix: Proof of SNR penalty for using AF

At high SNR, γ_r and γ can be approximated to 1 and 0, respectively. Substituting these values in (12), we obtain $\mathrm{erfc}\left(\left[\gamma_r+2\right]/\left[\sqrt{1/\rho}\right]\right)\simeq 0$ and $\mathrm{erf}\left(\left[\gamma_r+2\right]/\left[\sqrt{1/\rho}\right]\right)\simeq 1$. Based on this, the asymptotic error probability of a DF TWRN can be given as (see (33))
Putting $\mathrm{erfc}(-x)=2-\mathrm{erfc}(x)$ and $\mathrm{erf}(x)=1-\mathrm{erfc}(x)$ and after some simplifications, the above equation can be written as (see (34))
At high SNR, neglecting the higher order terms, the error probability of a DF TWRN can be approximated as

$$P_{\mathrm{DF},\infty}\simeq \mathrm{erfc}\left(\sqrt{\rho}\right) \tag{35}$$

Similarly, for an AF TWRN, after substituting the value of α in (19), the error probability can be approximated at high SNR as

$$P_{\mathrm{AF},\infty}\simeq \mathrm{erfc}\left(\sqrt{\frac{\rho}{3}}\right) \tag{36}$$

Comparing (35) and (36)

$$\text{SNR penalty in AF}=\frac{\rho}{\rho/3}=3=4.77\,\mathrm{dB} \tag{37}$$

$$\begin{aligned}P_{\mathrm{DF}}=\frac{1}{8}\Bigg[&\mathrm{erfc}\left(\frac{-1}{\sqrt{1/\rho}}\right)\left(1+\mathrm{erf}\left(\frac{-1}{\sqrt{1/\rho}}\right)+2\mathrm{erfc}\left(\frac{1}{\sqrt{1/\rho}}\right)\mathrm{erfc}\left(\frac{-1}{\sqrt{1/\rho}}\right)\right)\\&+\mathrm{erfc}\left(\frac{1}{\sqrt{1/\rho}}\right)\left(\mathrm{erfc}\left(\frac{-1}{\sqrt{1/\rho}}\right)+2\mathrm{erf}\left(\frac{1}{\sqrt{\rho}}\right)\mathrm{erfc}\left(\frac{1}{\sqrt{\rho}}\right)\right)\Bigg]\end{aligned} \tag{33}$$

$$\begin{aligned}P_{\mathrm{DF}}=\frac{1}{8}\Bigg[&12\mathrm{erfc}\left(\frac{1}{\sqrt{1/\rho}}\right)+2\left(\mathrm{erfc}\left(\frac{1}{\sqrt{1/\rho}}\right)\right)^3-10\left(\mathrm{erfc}\left(\frac{1}{\sqrt{1/\rho}}\right)\right)^2\\&+2\left(\mathrm{erfc}\left(\frac{1}{\sqrt{1/\rho}}\right)\right)^2\left(\mathrm{erf}\left(\frac{1}{\sqrt{1/\rho}}\right)\right)\Bigg]\end{aligned} \tag{34}$$

doi: 10.1049/iet-com.2013.0284