A Tractable Framework for Exact Probability of Node Isolation in Finite Wireless Sensor Networks

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Abstract

We present a tractable analytical framework for the exact calculation of probability of node isolation when N sensor nodes are independently and uniformly distributed inside a finite square region. The proposed framework can accurately account for the boundary effects by partitioning the square into subregions, based on the transmission range and the node location. We show that for each subregion, the probability that a random node falls inside a disk centered at an arbitrary node located in that subregion can be expressed analytically in closed-form. Using the results for the different subregions, we obtain the exact probability of node isolation. The proposed framework is validated by comparison with simulation results.

System Model

Consider N nodes which are uniformly and independently distributed inside a square region $\mathcal{R} \in \mathbb{R}^2$. Each sensor node has a fixed transmission range r_o. An arbitrary node will be isolated if there is no node located inside a disk $\mathcal{D}(\boldsymbol{u}; r_o)$ of radius r_o centered at that node. Let the cumulative density function (CDF) $F(\boldsymbol{u}; r_o)$ denote the probability that a random node falls inside the disk $\mathcal{D}(\boldsymbol{u}; r_o)$. It can be expressed as

$$F(\boldsymbol{u}; r_o) = \frac{|\mathcal{D}(\boldsymbol{u}; r_o) \cap \mathcal{R}|}{|\mathcal{R}|}. \tag{1}$$

The conditional probability of node isolation (conditioned on knowing the location of the arbitrary node) can be expressed as

$$P_{\text{iso}}(\boldsymbol{u}; r_o) = \left(1 - F(\boldsymbol{u}; r_o)\right)^{N-1}. \tag{2}$$

By averaging over all possible locations of the arbitrary node, the probability of node isolation is

$$P_{\text{iso}}(r_o) = \int_{\mathcal{R}} P_{\text{iso}}(\boldsymbol{u}; r_o) f_{\mathbf{U}}(\boldsymbol{u}) ds(\boldsymbol{u}). \tag{3}$$

The probability of no isolated node provides an upper bound for the probability of overall network connectivity as

$$P_{\text{con}}(r_o) \leq P_{\text{no-iso}}(r_o) = \left(1 - P_{\text{iso}}(r_o)\right)^N. \tag{4}$$

Problem Formulation

The basic building blocks in our approach to characterise the *boundary effects* are (i) the circular segment areas formed outside each side (*border effects*) and (ii) the corner overlap areas between two circular segments formed at each vertex (*corner effects*).

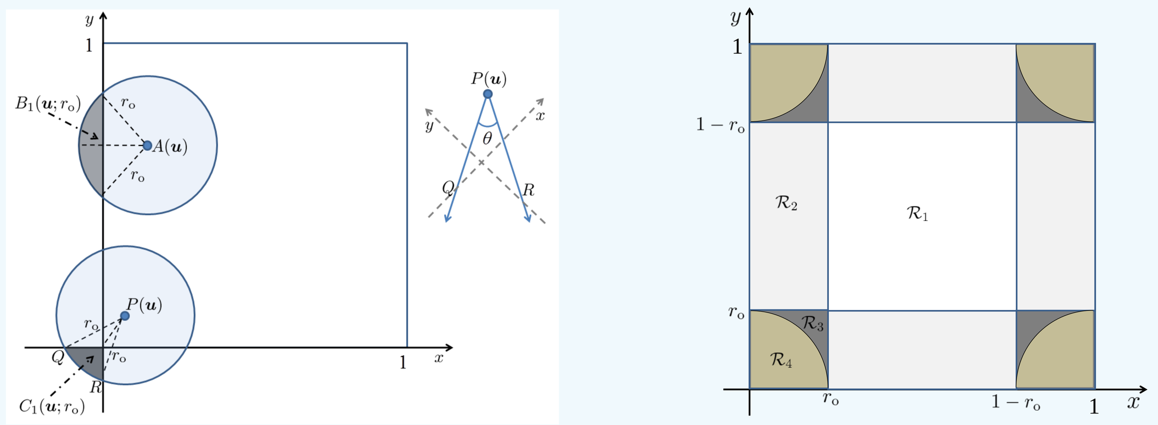

Figure 1: Illustration of border and corner effects.

- $B_1(\boldsymbol{u}; r_o)$ is the area of the circular segment formed outside the side S_1, given by

$$B_1(\boldsymbol{u}; r_o) = r_o^2 \arccos\left(\frac{x}{r_o}\right) - x\sqrt{r_o^2 - x^2}. \tag{5}$$

- $C_1(\boldsymbol{u}; r_o)$ denotes the area of the corner overlap region between two circular segments at vertex V_1, given by

$$C_1(\boldsymbol{u}; r_o) = r_o^2 \arcsin\left(\frac{\text{abs}\left(\sqrt{\theta_1}\right)}{2r_o}\right) - \frac{1}{2}\left(\sqrt{r_o^2 - y^2} - x\right) y - \frac{1}{2}\left(\sqrt{r_o^2 - x^2} - y\right) x, \tag{6}$$

where $\theta_1 = 2r_o^2 - 2x\sqrt{r_o^2 - y^2} - 2y\sqrt{r_o^2 - x^2}$.

- Similarly, the areas of the circular segments formed outside the sides S_2, S_3, S_4 and the areas of the corner overlap region formed at vertex V_2, V_3 and V_4 can be calculated.

Proposed Framework

Let $\mathcal{R}_1, \mathcal{R}_2, \ldots \mathcal{R}_M$ denotes the type of subregions and n_i, $i \in \{1, 2 \ldots, M\}$ denotes the number of subregion of type $\mathcal{R}_i$. (3) can be written as

$$P_{\text{iso}}(r_o) = \sum_{i=1}^{M} n_i \int_{\mathcal{R}_i} \left(1 - F_i(\boldsymbol{u}; r_o)\right)^{N-1} ds(\boldsymbol{u}). \tag{7}$$

where $F_i(\boldsymbol{u}; r_o)$ denotes the probability that a random node falls inside the disk $\mathcal{D}(\boldsymbol{u}; r_o)$ of radius r_o centered at $\boldsymbol{u} \in \mathcal{R}_i$. $\mathcal{R}_i$ and $F_i(\boldsymbol{u}; r_o)$ are defined in Table 1 and Figure 2 for different r_o.

Proposed Framework (Continued)

Subregion $\mathcal{R}_i$	n_i	$F_i(\boldsymbol{u}; r_o)$
$\mathcal{R}_1 = \{x \in (r_o, 1 - r_o), y \in (r_o, 1 - r_o)\}$	1	πr_o^2
$\mathcal{R}_2 = \{x \in (0, r_o), y \in (r_o, 1 - r_o)\}$	4	$\pi r_o^2 - (B_1)$
$\mathcal{R}_3 = \{x \in (0, r_o), y \in (\sqrt{r_o^2 - x^2}, 1)\}$	4	$\pi r_o^2 - (B_1 + B_2)$
$\mathcal{R}_4 = \{x \in (0, r_o), y \in (0, \sqrt{r_o^2 - x^2})\}$	4	$\pi r_o^2 - (B_1 + B_2 - C_1)$

Table 1: Subregions and conditional probabilities for calculation of $P_{\text{iso}}(r_o)$ $(0 < r_o \leq \frac{1}{2})$.

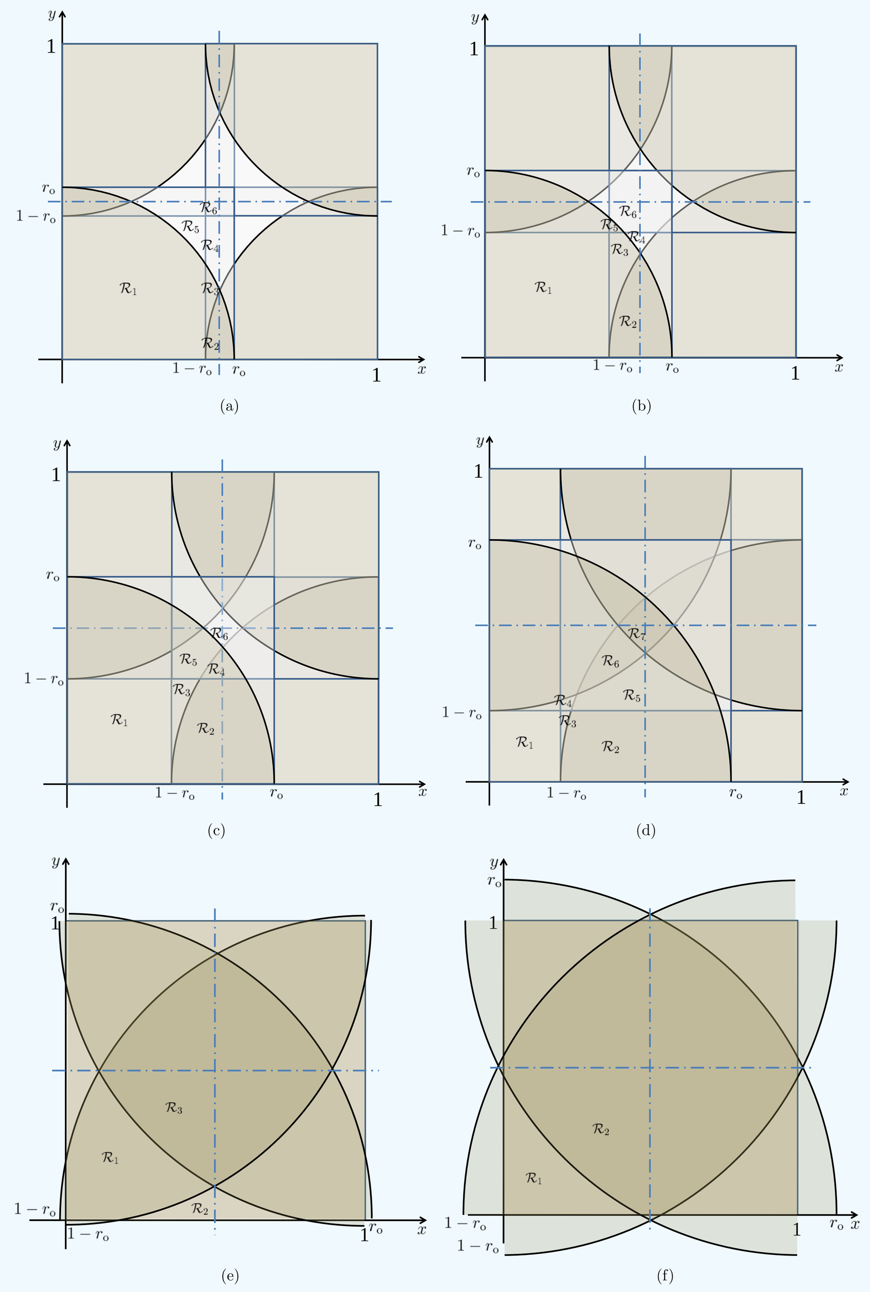

Figure 2: Subregions for the transmission range (a) $0.5 \leq r_o \leq (2 - \sqrt{2})$, (b) $(2 - \sqrt{2}) \leq r_o \leq 0.625$, (c) $0.625 \leq r_o \leq \sqrt{2}/2$, (d) $\sqrt{2}/2 \leq r_o \leq 1$, (e) $1 \leq r_o \leq \sqrt{5}/2$ and (f) $\sqrt{5}/2 \leq r_o \leq \sqrt{2}$.

Results

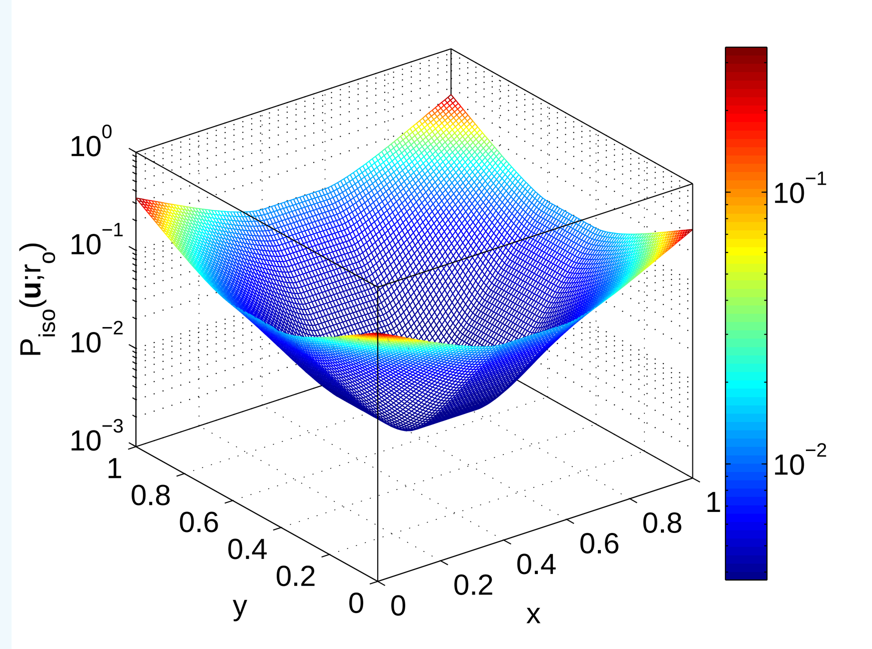

Figure 3: $P_{\text{iso}}(\boldsymbol{u}; r_o)$ versus position of arbitrary node with $r_o = 0.4$, $N = 10$.

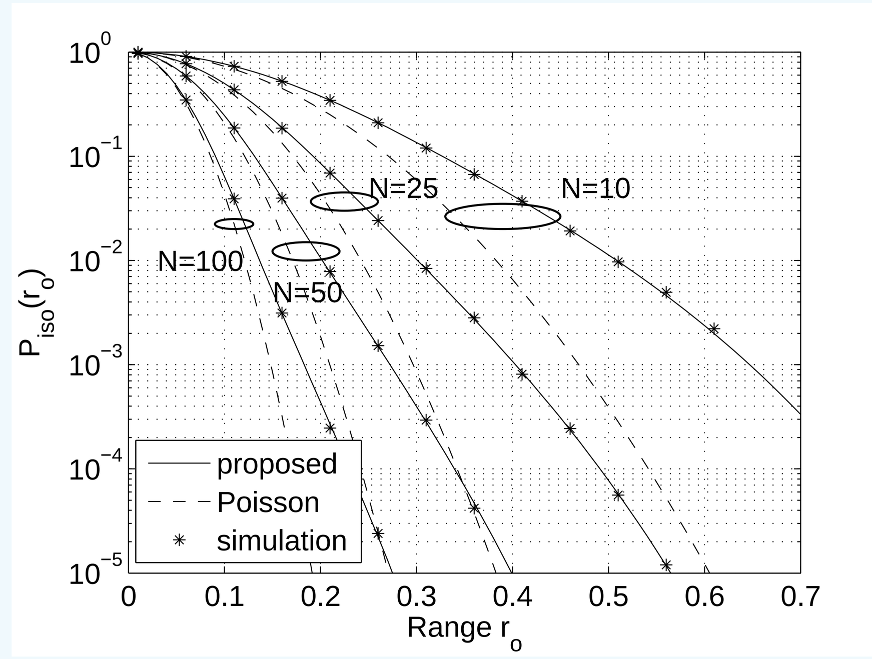

Figure 4: $P_{\text{iso}}(r_o)$ versus r_o.

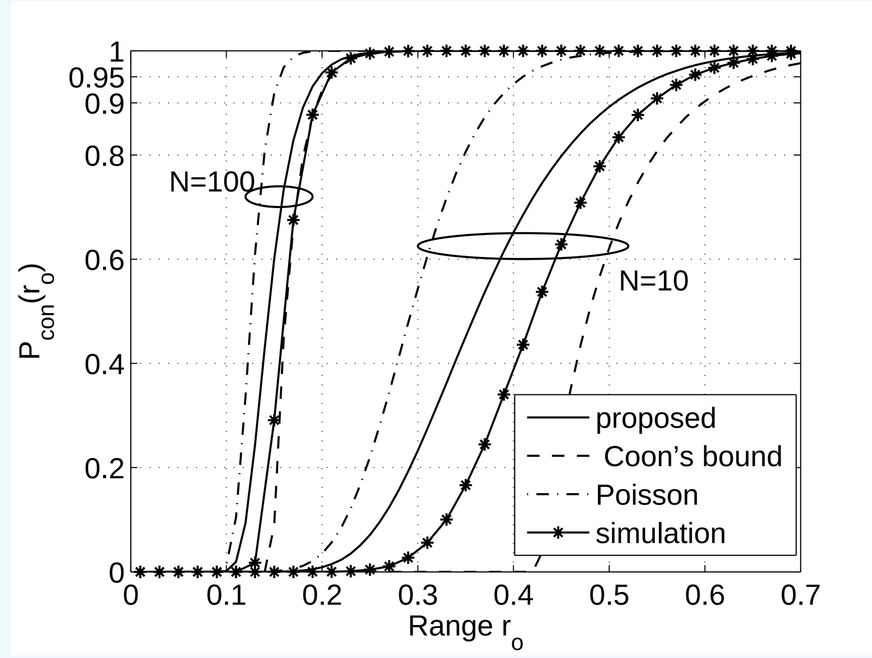

Figure 5: $P_{\text{con}}(r_o)$ versus r_o.

Conclusions

We have presented a tractable analytical framework for the calculation of probability of node isolation in finite wireless sensor networks. The proposed framework can accurately account for the boundary effects.

• S. Durrani, Z. Khalid, and J. Guo, “A Tractable Framework for Exact Probability of Node Isolation in Finite Wireless Sensor Networks,” submitted to *IEEE Transactions on Vehicular Technology*, 2012. [Online]. Available: http://arxiv.org/abs/1212.1283