ENGN6612/ENGN4612
Digital Signal Processing and Control

DSP Problem Sets & Matlab Scripts
V1.0

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AUSTRALIAN NATIONAL UNIVERSITY
Department of Engineering

ENGN6612/4612 Digital Signal Processing and Control
Problem Set #1 Digital Signals

Q1

Sketch and label carefully each of the following discrete time signals:
(a) $u[n-2]$
(b) $u[-n]$
(c) $u[4-n]$
(d) $2\ \delta[n-5]$
(e) $2\ h[n-1]$ where $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$
(f) $h[1-n]$ where $h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + 2\ \delta[n-3]$
(g) $x[-1]\delta[n+1]$ where $x[n] = +3\ \delta[n+1] - 0.5\ \delta[n] + 2\ \delta[n-1]$
(h) $\cos[n]\ u[n-1]$
(i) $u[k-n]$ when $k > 0$
(j) $u[k-n]$ when $k < 0$
Note that $\delta[n]$ is the discrete-time unit impulse and $u[n]$ is the discrete-time unit step function respectively.

Also plot the signals in Matlab.
(see example digital signals plotted in `L02_CommonSignals.m`)

Q2

Write an equation to describe each of the following discrete time signals:
(a)

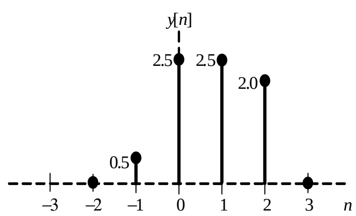

Figure 1: Figure Q2(a)

(b)

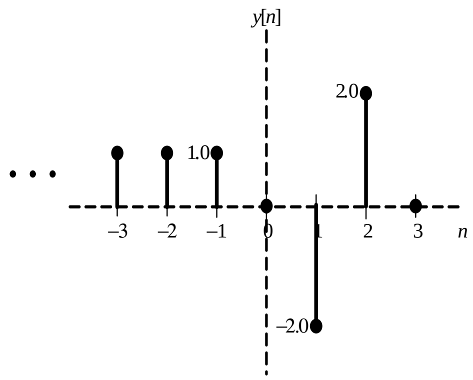

Figure 2: Figure Q2(b)

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Problem Set #1 Solution

Q1

Please see attached pages 3 and 4.

Q2

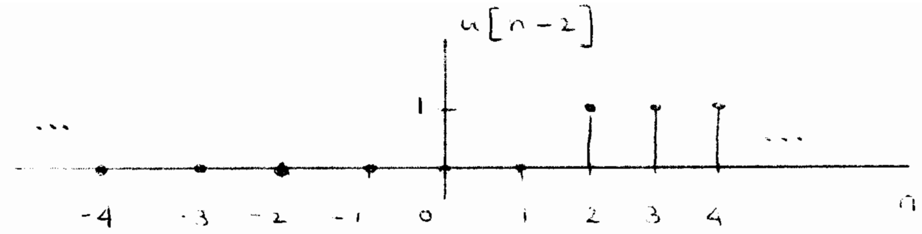
(a)

$y[n] = 0.5\ \delta[n+1] + 2.5\ \delta[n] + 2.5\ \delta[n-1] + 2.0\ \delta[n-2]$

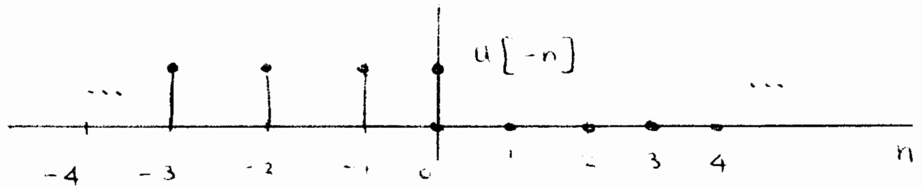
(b)

$y[n] = u[-1-n] - 2.0\ \delta[n-1] + 2.0\ \delta[n-2]$

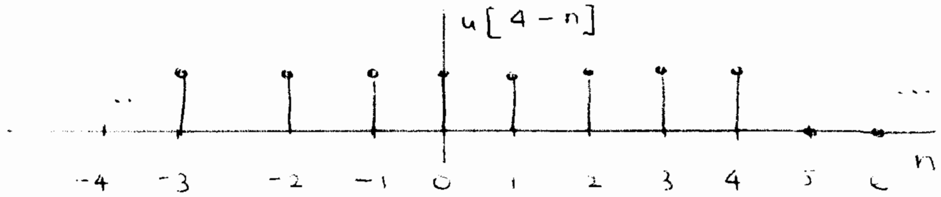
Q1 (a)


u[n-2]
1
...
-4 -3 -2 -1 0 1 2 3 4
n

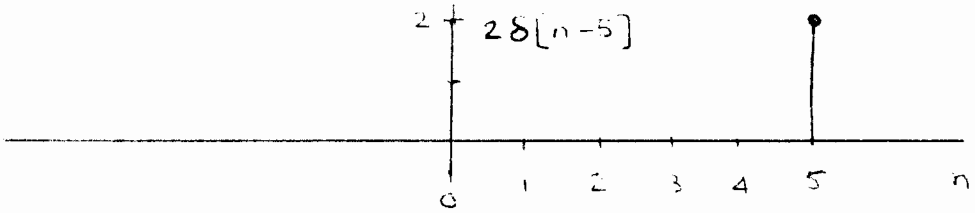
(b)


u[-n]
...
-4 -3 -2 -1 0 1 2 3 4
n

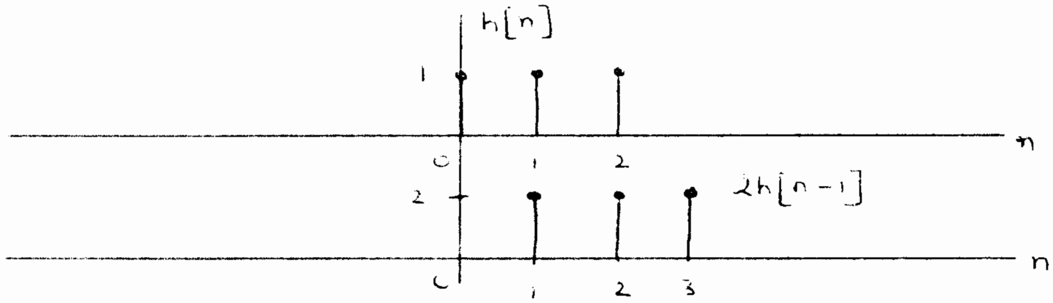
(c)


u[4-n]
...
-4 -3 -2 -1 0 1 2 3 4 5 6
n

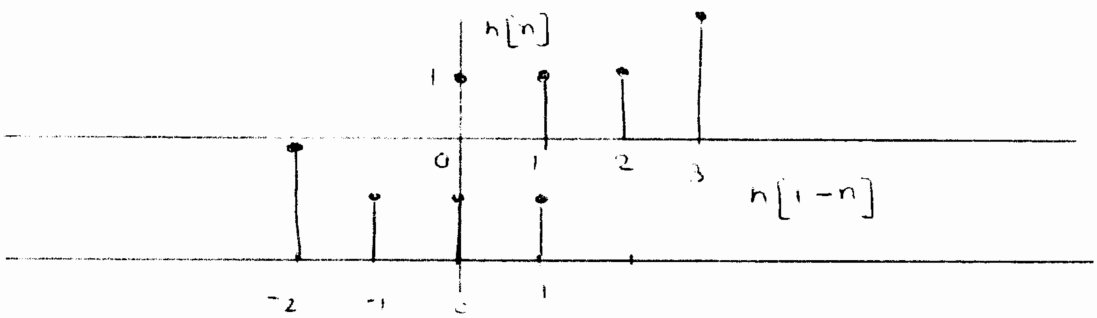
(d)


2
2δ[n-5]
0 1 2 3 4 5
n

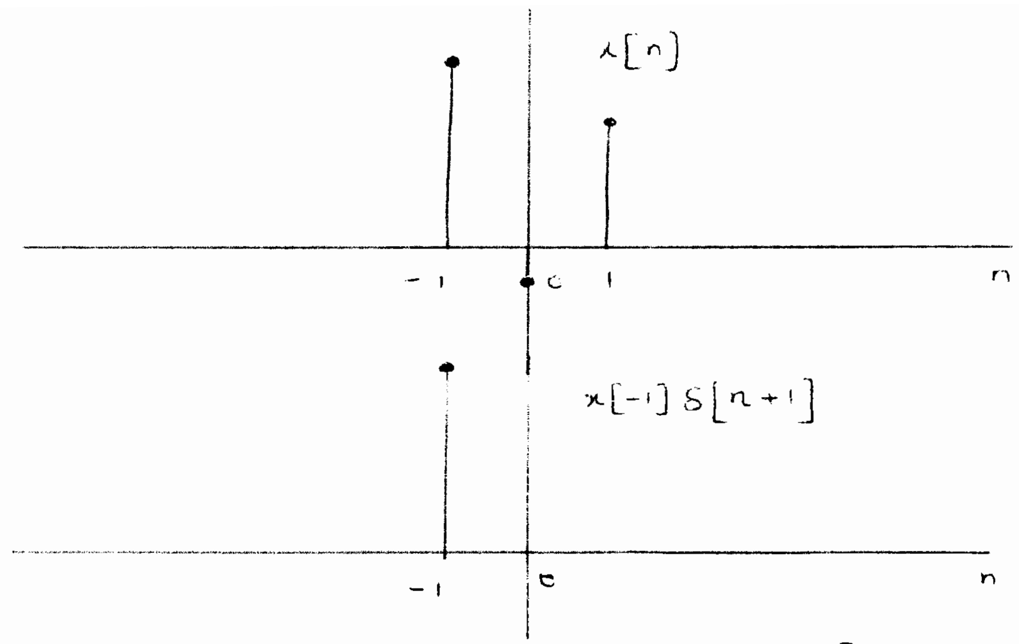
(e)


h[n]
1
0 1 2
n
2
2h[n-1]
0 1 2 3
n

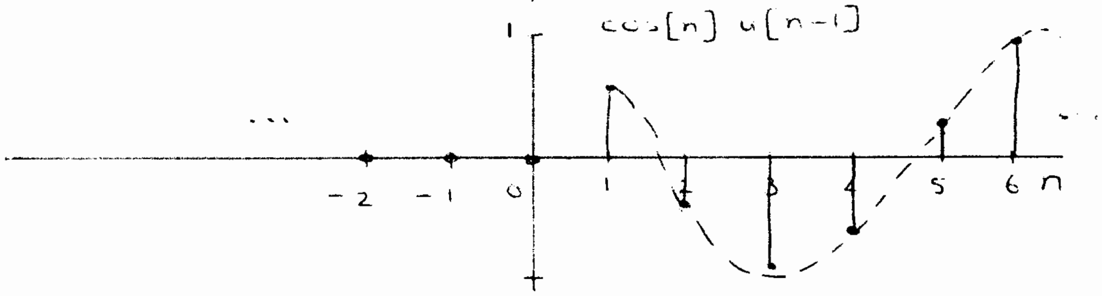
(f)


h[n]
1
0 1 2 3
h[1-n]
-2 -1 0 1

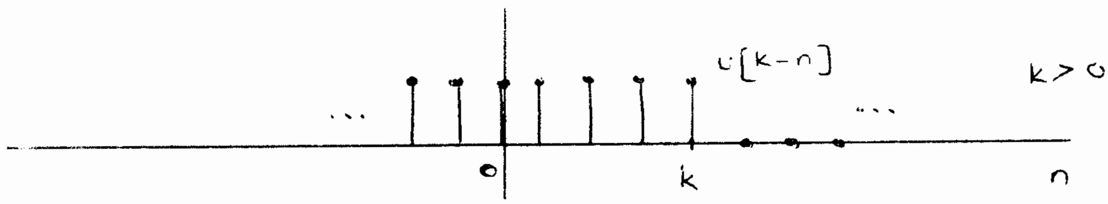
(g)



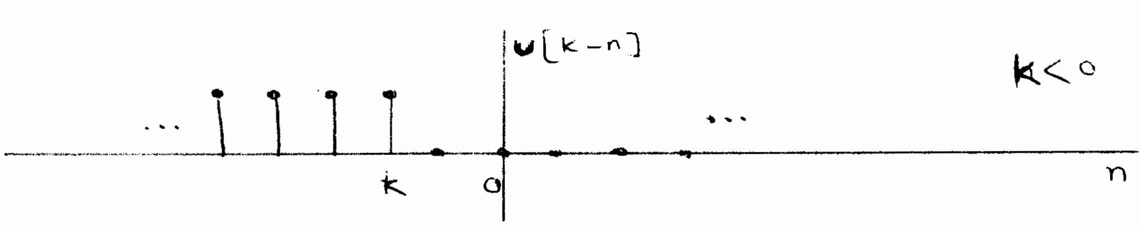
(h)



(i)



(j)



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Problem Set #2 z-Transform

Q1

Using the definition of the z-transform, find the z-transform of the following discrete-time functions:
(a) $\delta[n]$
(b) $\delta[n-k]$
(c) $u[n]$
(d) $c^n\, u[n]$ where c is a complex constant
(e) $\sin[\omega n]u[n]$
(f) $\cos[\omega n]u[n]$
(g) $r^n \sin[\omega n]u[n]$
(h) $r^n \cos[\omega n]u[n]$ (challenge problem)

Q2

Find the z-transform of the following discrete-time functions:
(a) $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$
(b) $x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$
(c) $x[n] = \left\{\frac{5}{12} + \frac{1}{3}(-2)^n - \frac{3}{4}(-3)^n\right\} u[n]$ (challenge problem)

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Problem Set #2 Solution

Q1

(a) Complete Solution

The given function is a discrete-time unit impulse given by

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

Using definition of z-transform, we have

$$\begin{aligned}
\mathcal{Z}\{\delta[n]\} &= \sum_{n=-\infty}^{n=+\infty} \delta[n] z^{-n} \\
&= \ldots + \delta[-2]z^{2} + \delta[-1]z^{1} + \delta[0]z^{0} + \delta[1]z^{-1} + \delta[2]z^{-2} + \ldots \\
&= \ldots + 0 + (1)(z^{0}) + 0 + \ldots \\
&= 1
\end{aligned}$$

Hence,

$$\delta[n] \quad \longleftrightarrow \quad 1$$

(b) Complete Solution

The given function is a time shifted discrete-time unit impulse given by

$$\delta[n-k] = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

Using definition of z-transform, we have

$$\begin{aligned}
\mathcal{Z}\{\delta[n-k]\} &= \sum_{n=-\infty}^{n=+\infty} \delta[n-k] z^{-n} \\
&= \ldots + 0 + (1)(z^{-k}) + 0 + \ldots \\
&= \frac{1}{z^{k}}
\end{aligned}$$

Hence,

$$\delta[n] \quad \longleftrightarrow \quad \frac{1}{z^{k}}$$

(c) Complete Solution

The given function is a discrete-time unit step given by

$$u[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases}$$

Using definition of z-transform, we have

$$\begin{aligned}
\mathcal{Z}\{u[n]\} &= \sum_{n=-\infty}^{n=+\infty} u[n]z^{-n} \\
&= u[0]z^0 + u[1]z^{-1} + u[2]z^{-2} + u[3]z^{-3} + \dots \\
&= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \\
&= \frac{1}{1 - \frac{1}{z}} \\
&= \frac{z}{z-1}
\end{aligned}$$

Hence,

$$u[n] \quad \longleftrightarrow \quad \frac{z}{z-1}$$

(d) Complete Solution

The given function is a discrete-time exponential.

Using definition of z-transform, we have

$$\begin{aligned}
\mathcal{Z}\{c^n u[n]\} &= \sum_{n=-\infty}^{n=+\infty} c^n u[n] z^{-n} \\
&= \sum_{n=-\infty}^{n=+\infty} u[n] \left(\frac{z}{c}\right)^{-n} \\
&= u[0]\left(\frac{z}{c}\right)^0 + u[1]\left(\frac{z}{c}\right)^{-1} + u[2]\left(\frac{z}{c}\right)^{-2} + u[3]\left(\frac{z}{c}\right)^{-3} + \dots \\
&= 1 + \left(\frac{z}{c}\right)^{-1} + \left(\frac{z}{c}\right)^{-2} + \left(\frac{z}{c}\right)^{-3} + \dots \\
&= \frac{1}{1 - \left(\frac{z}{c}\right)^{-1}} \\
&= \frac{z}{z-c}
\end{aligned}$$

Hence,

$$c^n u[n] \quad \longleftrightarrow \quad \frac{z}{z-c}$$

(e) Complete Solution

The given function is a discrete-time sine wave.

Using definition of z-transform, we have

$$\begin{aligned}\mathcal{Z}\{\sin[\omega n]u[n]\} &= \sum_{n=-\infty}^{n=+\infty} \sin[\omega n]u[n]z^{-n} \\ &= \sum_{n=-\infty}^{n=+\infty} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) u[n]z^{-n}\end{aligned}$$

Let

$$\begin{aligned}c_1 &= e^{j\omega} \\ c_2 &= e^{-j\omega}\end{aligned}$$

Hence we have,

$$\begin{aligned}\mathcal{Z}\{\sin[\omega n]u[n]\} &= \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} (c_1^n - c_2^n)\, u[n]z^{-n} \\ &= \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} u[n]\left(\frac{z}{c_1}\right)^{-n} - \frac{1}{2j}\sum_{n=-\infty}^{n=-\infty} u[n]\left(\frac{z}{c_2}\right)^{-n} \\ &= \frac{1}{2j}\frac{z}{z-c_1} - \frac{1}{2j}\frac{z}{z-c_2} \qquad \text{(using result of Q1 part d)} \\ &= \frac{z(c_1-c_2)}{2j(z-c_1)(z-c_2)} \\ &= \frac{z\left(\frac{c_1-c_2}{2j}\right)}{z^2 - 2z\left(\frac{c_1+c_2}{2}\right)+1} \qquad (\because c_1c_2 = 1) \\ &= \frac{(\sin\omega)z}{z^2-(2\cos\omega)z+1}\end{aligned}$$

Hence,

$$\sin[\omega n]u[n] \longleftrightarrow \frac{(\sin\omega)z}{z^2-(2\cos\omega)z+1}$$

(f) Solution with Hint

Show that the final answer is

$$\cos[\omega n]u[n] \longleftrightarrow \frac{z^2-(\cos(\omega))z}{z^2-(2\cos\omega)z+1}$$

Hint:-
$\cos[\omega n] = \left(\frac{e^{j\omega n}+e^{-j\omega n}}{2}\right)$

(g) Partial Solution

Using definition of z-transform, we have

$$\begin{aligned}
\mathcal{Z}\{r^n \sin[\omega n]u[n]\} &= \sum_{n=-\infty}^{n=+\infty} r^n \sin[\omega n]u[n]z^{-n} \\
&= \sum_{n=-\infty}^{n=+\infty} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) u[n] \left(\frac{z}{r}\right)^{-n}
\end{aligned}$$

Let

$$\begin{aligned}
c_1 &= e^{j\omega} \\
c_2 &= e^{-j\omega}
\end{aligned}$$

Hence we have,

$$\begin{aligned}
\mathcal{Z}\{r^n \sin[\omega n]u[n]\} &= \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} (c_1^n - c_2^n)\, u[n] \left(\frac{z}{r}\right)^{-n} \\
&= \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} u[n]\left(\frac{z}{rc_1}\right)^{-n} - \frac{1}{2j}\sum_{n=-\infty}^{n=+\infty} u[n]\left(\frac{z}{rc_2}\right)^{-n} \\
&= \frac{1}{2j}\frac{z}{z-rc_1} - \frac{1}{2j}\frac{z}{z-rc_2}
\end{aligned}$$

Show that this can be written in the form

$$\begin{aligned}
\mathcal{Z}\{r^n \sin[\omega n]u[n]\} &= \frac{zr\left(\frac{c_1-c_2}{2j}\right)}{z^2 - 2rz\left(\frac{c_1+c_2}{2}\right) + r^2} \\
&= \frac{(r\sin\omega)z}{z^2 - (2r\cos\omega)z + r^2}
\end{aligned}$$

Hence,

$$r^n \sin[\omega n]u[n] \quad \longleftrightarrow \quad \frac{(r\sin\omega)z}{z^2 - (2r\cos\omega)z + r^2}$$

(h) Solution

Show that the final answer is

$$r^n \cos[\omega n]u[n] \quad \longleftrightarrow \quad \frac{z^2 - (r\cos(\omega))z}{z^2 - (2r\cos\omega)z + r^2}$$

Check answer in Matlab using the following commands

```
>> syms w n z
>> f=(r^n)*cos(w*n)*heaviside(n);
>> Ans=maple('ztrans',f,n,z)
>> pretty(Ans)
```

Q2

(a) Complete Solution

Given that,

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

We know that

$$c^n u[n] \longleftrightarrow \frac{z}{z-c}$$

Hence,

$$\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{z}{z-\frac{1}{3}}$$
$$7\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{7z}{z-\frac{1}{3}}$$

Also,

$$\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{z}{z-\frac{1}{2}}$$
$$6\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{6z}{z-\frac{1}{2}}$$

The z-transform of the given function $X(z)$ is thus given by

$$\begin{aligned} X(z) &= \frac{7z}{z-\frac{1}{3}} - \frac{6z}{z-\frac{1}{2}} \\ &= \frac{7z(z-\frac{1}{2}) - 6z(z-\frac{1}{3})}{(z-\frac{1}{3})(z-\frac{1}{2})} \\ &= \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})} \end{aligned}$$

(b) Solution

$$X(z) = \frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$$

(c) Solution

$$X(z) = \frac{z(2z+3)}{(z+2)(z+3)(z-1)}$$

Check answer in Matlab using the following commands

```
>> syms n z
>> f= ((5/12) + ((1/3)*(-2)^n) -((3/4)*(-3)^n))*heaviside(n);
>> Ans=maple('ztrans',f,n,z)
>> pretty(simplify(Ans))
```

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Problem Set #3 Inverse z-Transform

Q1

Using properties of z-transform, find the z-transform of the following discrete-time functions:
(a) $n\,u[n]$
(b) $n^2\,u[n]$
(c) $nc^n u[n]$ `(challenge problem)`
(d) $u[k-2]$

Q2

Using the method based on partial fraction expansion, find x[n] if X(z) equals:

(a) $\frac{z+1}{(z-2)(z+3)}$
(b) $\frac{2z-3}{z(z-0.5)(z+0.3)}$
(c) $\frac{z}{(z-1)(z-4)}$
(d) $\frac{100z^2}{(z-1.1)(z-1)}$
(e) $\frac{0.1z(z+1)}{(z-1)^2(z-0.6)}$ `(challenge problem)`

Also plot $x[n]$ for $0 < n < 4$

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Problem Set #3 Solution

Q1

(a) Complete Solution

We know

$$u[n] \longleftrightarrow \frac{z}{z-1}$$

Hence

$$\begin{aligned}\mathcal{Z}\{nu[n]\} &= -z\frac{d}{dz}\left(\frac{z}{z-1}\right)\\ &= -z\frac{(z-1)(1)-z(1)}{(z-1)^2}\\ &= \frac{z}{(z-1)^2}\end{aligned}$$

(b) Solution with Hint

Show that

$$n^2u[n] \longleftrightarrow \frac{z(z+1)}{(z-1)^3}$$

Hint:-

$$n\,u[n] \longleftrightarrow \frac{z}{(z-1)^2}$$

(c) Solution with Hint

Show that

$$nc^n\,u[n] \longleftrightarrow \frac{zc}{(z-c)^2}$$

Hint:-

$$c^n\,u[n] \longleftrightarrow \frac{z}{z-c}$$

(d) Complete Solution

We know

$$u[n] \longleftrightarrow \frac{z}{z-1}$$

and

$$\mathcal{Z}\{x[n-n_0]\} = \frac{X(z)}{z^{n_0}}$$

Hence using the time shifting property,

$$\begin{aligned}\mathcal{Z}\{u[n-2]\} &= \frac{\frac{z}{z-1}}{z^2}\\ &= \frac{1}{z(z-1)}\end{aligned}$$

Q2

(a) Complete Solution

Given that

$$X(z) = \frac{z+1}{(z-2)(z+3)}$$

Rewriting

$$X(z) = \frac{z(z+1)}{z(z-2)(z+3)} = z\left[\frac{(z+1)}{z(z-2)(z+3)}\right]$$

Using partial fraction expansion, we have

$$\frac{(z+1)}{z(z-2)(z+3)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+3}$$

Evaluating the coefficients,

$$\begin{aligned}
A &= \lim_{z\to 0}\left[\frac{z+1}{(z-2)(z+3)}\right] = -\frac{1}{6} \\
B &= \lim_{z\to 2}\left[\frac{z+1}{(z)(z+3)}\right] = \frac{3}{10} \\
C &= \lim_{z\to -3}\left[\frac{z+1}{(z)(z-2)}\right] = -\frac{2}{15}
\end{aligned}$$

Hence,

$$\begin{aligned}
X(z) &= z\left[\frac{-\frac{1}{6}}{z} + \frac{3}{10}\frac{1}{z-2} - \frac{2}{15}\frac{1}{z+3}\right] \\
&= -\frac{1}{6} + \frac{3}{10}\frac{z}{z-2} - \frac{2}{15}\frac{z}{z+3}
\end{aligned}$$

Taking the inverse z-transform, we have

$$x[n] = -\frac{1}{6}\delta[n] + \frac{3}{10}(2)^n u[n] - \frac{2}{15}(-3)^n u[n]$$

For $0 < n < 4$, we have

$$\begin{aligned}
x[0] &= -\frac{1}{6} + (0.3)(2)^0 - \frac{2}{15}(-3)^0 = 0 \\
x[1] &= 0 + (0.3)(2)^1 - \frac{2}{15}(-3)^1 = 1 \\
x[2] &= 0 + (0.3)(2)^2 - \frac{2}{15}(-3)^2 = 0 \\
x[3] &= 0 + (0.3)(2)^3 - \frac{2}{15}(-3)^3 = 6 \\
x[4] &= 0 + (0.3)(2)^4 - \frac{2}{15}(-3)^4 = -6
\end{aligned}$$

Using the initial value theorem to check the value of $x[0]$, we have

$$\begin{aligned}
x[0] &= \lim_{z\to\infty} X(z) \\
&= \lim_{z\to\infty}\frac{z+1}{(z-2)(z+3)} = \lim_{z\to\infty}\frac{z+1}{z^2+z-6} = \lim_{z\to\infty}\frac{z^{-1}+z^{-2}}{1+z^{-1}-6z^{-2}} = \frac{0+0}{1+0+0} = 0
\end{aligned}$$

The plot of $x[n]$ is shown below:-

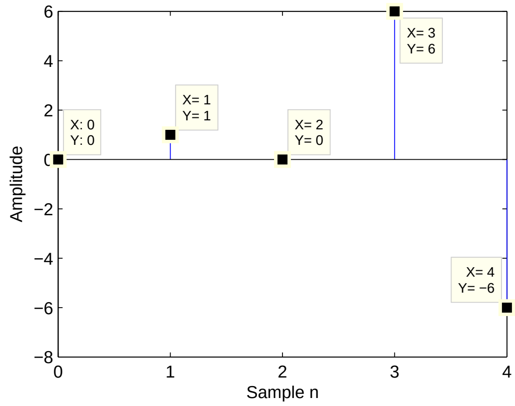

Figure 1: Question 1(a)

(b) Partial Solution

Given that

$$X(z) = \frac{2z-3}{z(z-0.5)(z+0.3)}$$

Rewriting

$$\begin{aligned} X(z) &= \frac{z(2z-3)}{z^2(z-0.5)(z+0.3)} \\ &= z\left[\frac{(2z-3)}{z^2(z-0.5)(z+0.3)}\right] \end{aligned}$$

Using partial fraction expansion, we have

$$\frac{(2z-3)}{z^2(z-0.5)(z+0.3)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-0.5} + \frac{C}{z+0.3}$$

Evaluating the coefficients,

$$\begin{aligned} B &= \lim_{z\to 0}\left[\frac{(2z-3)}{(z-0.5)(z+0.3)}\right] = 20 \\ A &= \lim_{z\to 0}\frac{d}{dz}\left[\frac{(2z-3)}{(z-0.5)(z+0.3)}\right] = -40 \\ C &= \lim_{z\to 0.5}\left[\frac{(2z-3)}{(z^2)(z+0.3)}\right] = -10 \\ D &= \lim_{z\to -0.3}\left[\frac{(2z-3)}{(z^2)(z-0.5)}\right] = 50 \end{aligned}$$

Hence,

$$\begin{aligned} X(z) &= z\left[-\frac{40}{z} + \frac{20}{z^2} - \frac{10}{z-0.5} + \frac{50}{z+0.3}\right] \\ &= -40 + \frac{20}{z} - 10\frac{z}{z-0.5} + 50\frac{z}{z+0.3} \end{aligned}$$

Taking the inverse z-transform, we have

$$x[n] = -40\delta[n] + 20\delta[n-1] - 10(0.5)^n u[n] + 50(-0.3)^n u[n]$$

The plotting is left as an exercise for the students.
Check your answer via Matlab or by comparing your answer with another student.

(c) Solution

Show that

$$x[n] = \frac{1}{3}(4^n - 1)u[n]$$

The plot of $x[n]$ is shown below:-

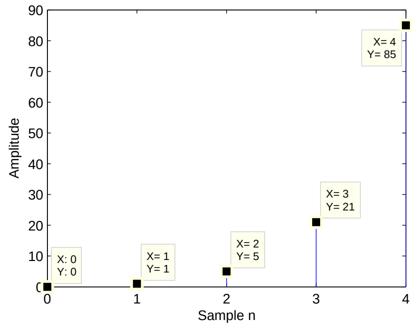

Figure 2: Question 1(c)

(d) Solution

Show that

$$x[n] = 1100(1.1)^n u[n] - 1000u[n]$$

The plot of $x[n]$ is shown below:-

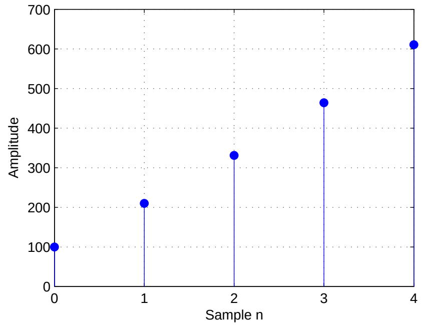

Figure 3: Question 1(d)

(e) Solution

Show that

$$x[n] = \{0.5n - 1 + (0.6)^n\}u[n]$$

The plot of $x[n]$ is shown below:-

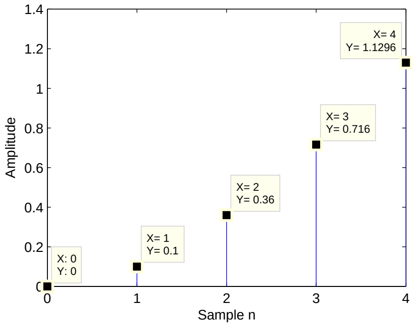

Figure 4: Question 1(e)

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Problem Set #4 Difference Equations

Q1

Find the system transfer function $H(z) = Y(z)/X(z)$ when the LTI system is described by the following difference equation:

(a) $y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$

(b) $y[n] = y[n-1] + y[n-2] + x[n-1]$

(c) $y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n-1] - x[n-2]$ `(challenge problem)`

Also draw the pole-zero plot for $H(z)$ and determine if the system is stable or unstable.

Q2

Consider a discrete time LTI system with following impulse response $h[n]$ and input function $x[n]$:

(a)

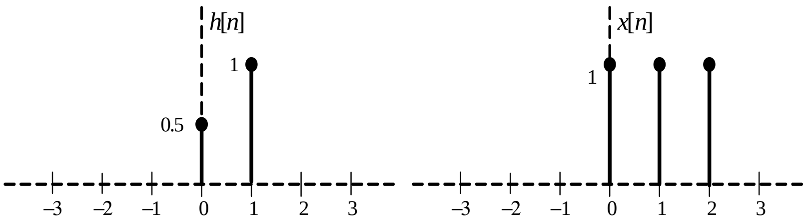

Figure 1: Figure Q2(a)

(b)

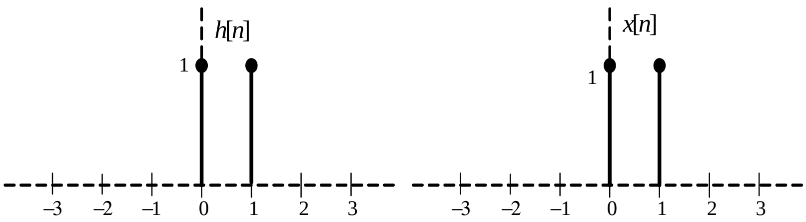

Figure 2: Figure Q2(b)

(c) $x[n] = \delta[n] - \delta[n-1]$ and
$h[n] = \delta[n] + \delta[n-1] + 0.5\delta[n-2] + 0.5\delta[n-3]$ `(challenge problem)`.

Determine the output $y[n]$ using both (i) graphical discrete time convolution and (ii) z-transform method.

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Problem Set #4 Solution

Q1

(a) Complete Solution

The given difference equation is

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

We know

$$x[n-n_0] \quad \longleftrightarrow \quad \frac{X(z)}{z^{n_0}}$$

Using the time shift property, we have

$$\begin{aligned} y[n] \quad &\longleftrightarrow \quad Y(z) \\ y[n-1] \quad &\longleftrightarrow \quad \frac{Y(z)}{z} \\ y[n-2] \quad &\longleftrightarrow \quad \frac{Y(z)}{z^2} \end{aligned}$$

and

$$\begin{aligned} x[n] \quad &\longleftrightarrow \quad X(z) \\ x[n-1] \quad &\longleftrightarrow \quad \frac{X(z)}{z} \end{aligned}$$

Taking the z-transform of both sides of the difference equation, we have

$$Y(z)-\frac{1}{2}\frac{Y(z)}{z} \quad = \quad X(z)+\frac{1}{3}\frac{X(z)}{z}$$

Simplifying, we have

$$\begin{aligned} \left(1-\frac{1}{2z}\right)Y(z) \quad &= \quad \left(1+\frac{1}{3z}\right)X(z) \\ \frac{Y(z)}{X(z)} \quad &= \quad \frac{z+\frac{1}{3}}{z-\frac{1}{2}} \end{aligned}$$

Hence the transfer function in standard form is

$$H(z) \quad = \quad \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

From the transfer function, $H(z)$ has pole at $z=\frac{1}{2}$ and a zero at $z=-\frac{1}{3}$.
As the pole lies within the unit circle $|z|=1$, system is stable.

The pole-zero map is shown in the figure below:

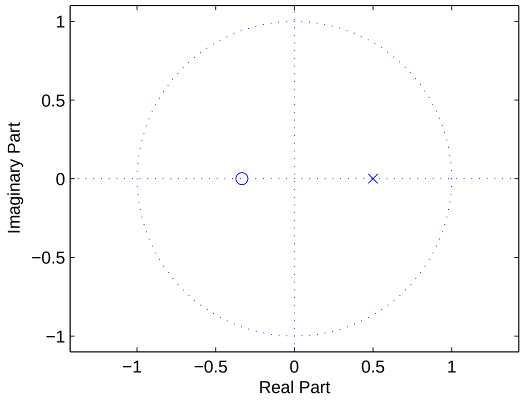


Figure 3: Question 1(a)

Additional student exercise

Take the inverse z-transform of $H(z)$ and show that the corresponding impulse response $h[n]$ is

$$h[n] = -\frac{2}{3}\delta[n] + \frac{5}{3}(1/2)^n u[n]$$

(Hint: use the method based on partial fractions.)

For $0 < n < 4$, the plot of $h[n]$ is shown below:-

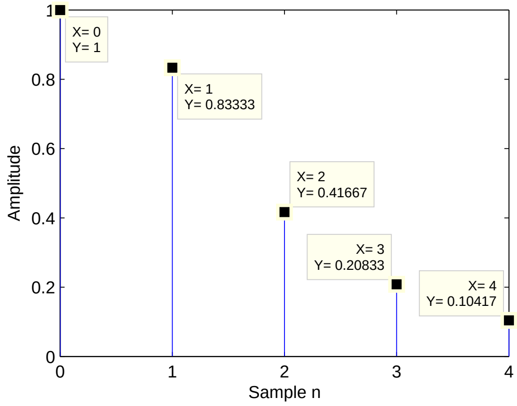


Figure 4: Question 1(a)

(b) Partial Solution

The given difference equation is

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

Taking the z-transform of both sides of the difference equation , we have

$$Y(z) = \frac{Y(z)}{z} + \frac{Y(z)}{z^2} + \frac{X(z)}{z}$$

Simplify and show that the transfer function in standard form is

$$H(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{z}{z^2 - z - 1}$$

From the transfer function, $H(z)$ has poles at $z = -0.618, 1.618$ and a zero at $z = 0$.
As one of the poles lies outside the unit circle $|z| = 1$, system is unstable.
The pole-zero map is shown in the figure below:

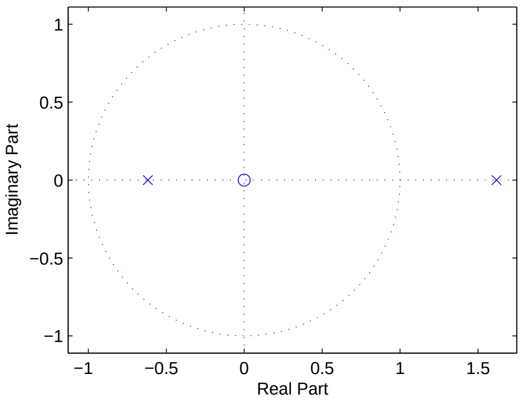


Figure 5: Question 1(b)

(c) Solution

Show that the transfer function is

$$H(z) = \frac{z(-2z^2 + \frac{13}{8}z - \frac{3}{8})}{(z+1)(z+\frac{1}{2})(z-\frac{1}{4})} = \frac{-2 + \frac{13}{8}z^{-1} - \frac{3}{8}z^{-2}}{1 + \frac{5}{4}z^{-1} + \frac{1}{8}z^{-2} - \frac{1}{8}z^{-3}}$$

System is stable (poles at $z = -1, -0.5, 0.25$ and zeros at $z = 0, 0.4063 \pm j0.15$).
The pole-zero map is shown in the figure below:

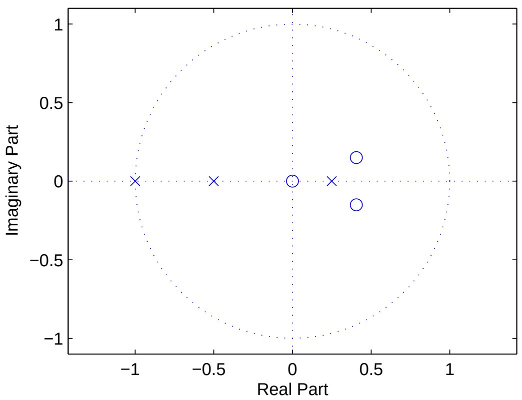


Figure 6: Question 1(c)

Q2

(a) Complete Solution

Please see pages 6-8.

(b) Solution

Solution in time domain

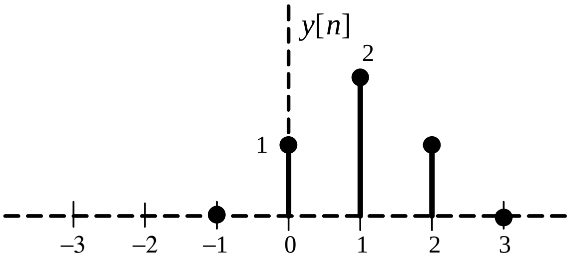


Figure 7: Question 2(b)

Solution in z domain

$$
\begin{aligned}
Y(z) &= \left(\frac{z+1}{z}\right)^2 = 1 + \frac{2}{z} + \frac{1}{z^2} \\
y[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2]
\end{aligned}
$$

(c) Solution

Solution in time domain

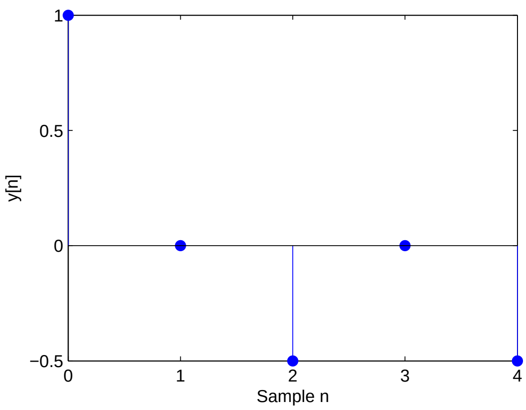


Figure 8: Question 2(c)

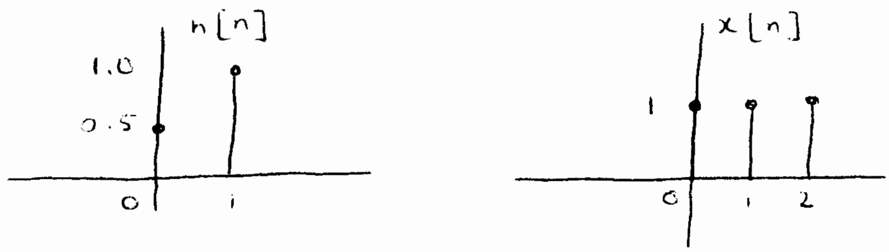
Solution in z domain

$$
\begin{aligned}
X(z) &= 1 - \frac{1}{z} \\
H(z) &= 1 + \frac{1}{z} + \frac{0.5}{z^2} + \frac{0.5}{z^3} \\
Y(z) &= H(z)X(z) = 1 - \frac{0.5}{z^2} - \frac{0.5}{z^4} \\
y[n] &= \delta[n] - 0.5\delta[n-2] - 0.5\delta[n-4]
\end{aligned}
$$

GRAPHICAL DISCRETE TIME CONVOLUTION

We know $$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

The given waveforms are

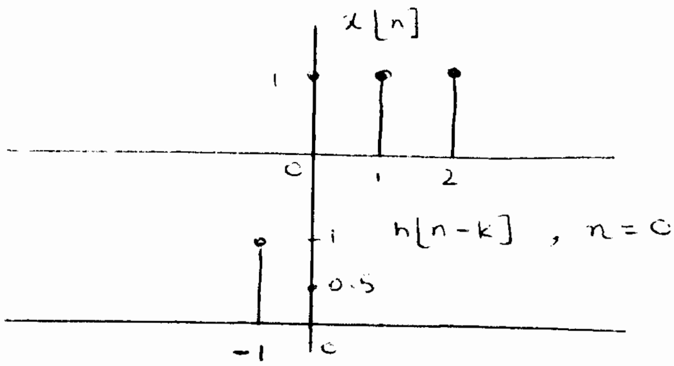


n < 0

When n is less than 0, there is no overlap.

Hence $y[n] = 0$

n = 0

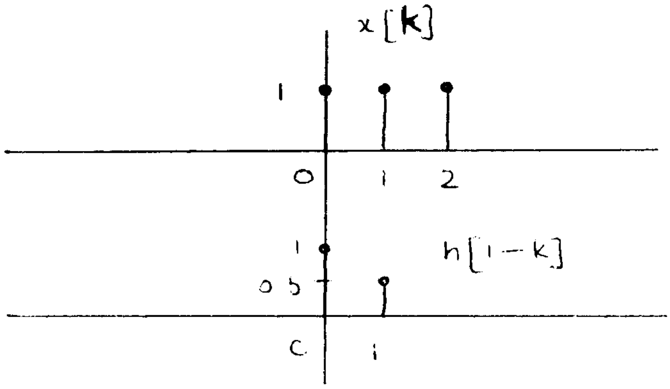


$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[0-k]$$

$$= (0.5)(1)$$

$$= 0.5$$

$\underline{n=1}$

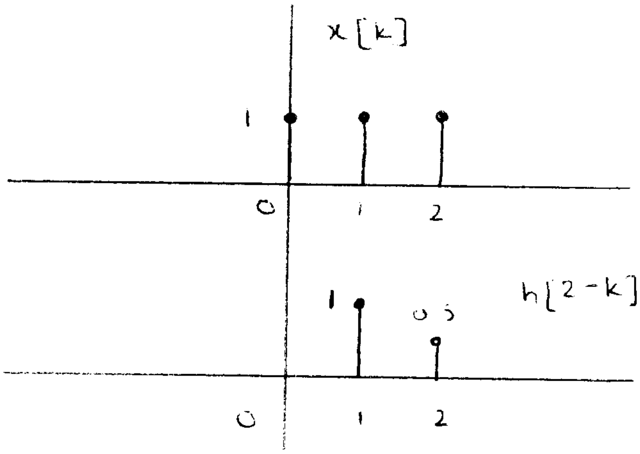


$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\,h[1-k]$$

$$= (1)(1) + (1)(0.5) + (2)(0)$$

$$= 1 + 0.5 = 1.5$$

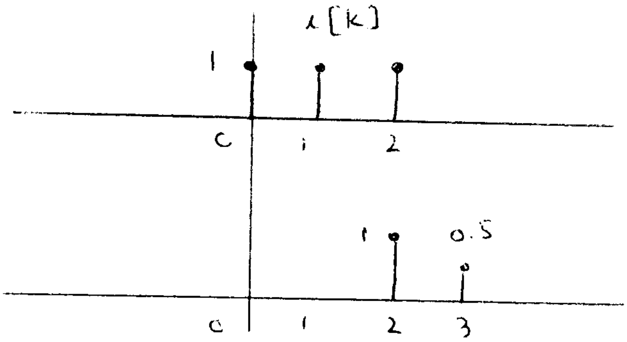
$\underline{n=2}$



$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\,h[2-k]$$

$$= (1)(0) + (1)(1) + (1)(0.5) = 1.5$$

$\underline{n=3}$



$$y[3] = \sum_{k=-\infty}^{\infty} x[k]\,h[3-k]$$

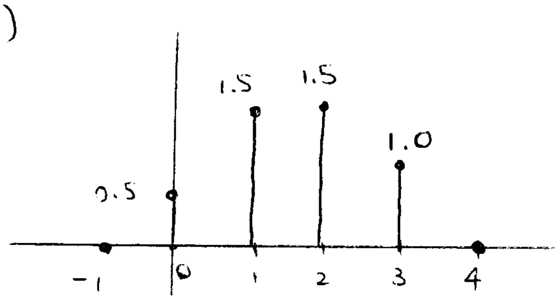
$$= (1)(0) + (1)(0) + (1)(1) + (0)(0.5)$$

$$= 1$$

$\underline{n=4}$ There is no overlap

Hence $y[n] = 0$ for $n > 4$.

Plot of $y[n]$ is shown opposite



SOLUTION IN Z-DOMAIN

The equation for impulse response $h[n]$ is

$$h[n] = 0.5\,\delta[n] + \delta[n-1]$$

The equation for input $x[n]$ is

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

Taking the z-transform, we have

$$H(z) = 0.5 + \frac{1}{z} \qquad \left(\because \ \delta[n] \longleftrightarrow 1, \quad \delta[n-k] \longleftrightarrow \frac{1}{z^k} \right)$$

$$X(z) = 1 + \frac{1}{z} + \frac{1}{z^2}$$

For a discrete-time LTI system,

$$Y(z) = H(z)\,X(z)$$

$$= \left[0.5 + \frac{1}{z}\right]\left[1 + \frac{1}{z} + \frac{1}{z^2}\right]$$

$$= 0.5 + \frac{0.5}{z} + \frac{0.5}{z^2} + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}$$

$$= 0.5 + \frac{1.5}{z} + \frac{1.5}{z^2} + \frac{1}{z^3}$$

Taking the inverse z-transform,

$$y[n] = 0.5\,\delta[n] + 1.5\,\delta[n-1] + 1.5\,\delta[n-2] + \delta[n-3]$$

Check :- This is the same output as found using discrete-time convolution

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Problem Set #5 Discrete Time Fourier Transform (DTFT)

Q1

Find $X(e^{j\omega})$ and sketch $|X(e^{j\omega})|$ and $\angle X(e^{j\omega})$ when $x[n]$ is given by the following:
(a) $a^n u[n], \quad a = -0.6$
(b) $\delta[n-3]$ `(challenge problem)`

Q2

Consider a discrete time filter described by the following difference equations:
(a) $y[n] = \frac{1}{2}(x[n] + x[n-1])$ (This is called a two-point moving-average filter)
(b) $y[n] = x[n] + 0.6y[n-1]$
(c) $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$ `(challenge problem)`.

For each filter:-

- Find the transfer function $H(z)$.
- Find whether the filter is FIR or IIR.
- Determine the frequency response $H(e^{j\omega})$.
- Determine and roughly sketch magnitude of the frequency response of the filter for $-\pi \leq \omega \leq \pi$.

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Problem Set #5 Solution

Q1

(a) Partial Solution

Given that

$$x[n] = a^n u[n], \qquad a = -0.6$$

Taking the z-transform , we have

$$X(z) = \frac{z}{z-a}$$

Let

$$z = e^{j\omega}$$

Hence the frequency response is given by

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}$$

Magnitude Response

We have

$$X(e^{j\omega}) = \frac{\cos\omega + j\sin\omega}{(\cos\omega - a) + j\sin\omega}$$

The magnitude response is given by

$$|X(e^{j\omega})| = \frac{|\cos\omega + j\sin\omega|}{|(\cos\omega - a) + j\sin\omega|} = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

Evaluating $|X(e^{j\omega})|$ for $-\pi \leq \omega \leq \pi$, we have

ω (rad/s)	$\lvert X(e^{j\omega})\rvert$
$-\pi$	2.5000
-3	2.4112
-2	1.0779
$-\pi/2$	0.8575
-1	0.7056
0	0.6250
1	0.7056
$\pi/2$	0.8575
2	1.0779
3	2.4112
π	2.5000

The plot of $|X(e^{j\omega})|$ is shown in the figure below:

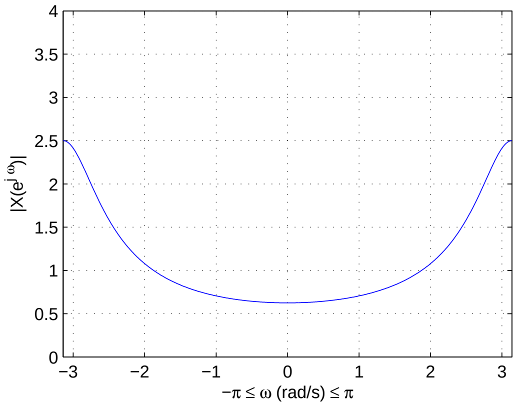


Figure 1: Question 1(a)

Phase Response

We have

$$\begin{aligned} X(e^{j\omega}) &= \frac{\cos\omega + j\sin\omega}{(\cos\omega - a) + j\sin\omega} \\ &= \frac{\cos\omega + j\sin\omega}{(\cos\omega - a) + j\sin\omega} \times \frac{(\cos\omega - a) - j\sin\omega}{(\cos\omega - a) - j\sin\omega} \end{aligned}$$

Show that the above expression simplifies to

$$X(e^{j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} + j\frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega}$$

The phase response is given by

$$\begin{aligned} \angle X(e^{j\omega}) &= \tan^{-1}\left(\frac{\Im\{H(e^{j\omega})\}}{\Re\{H(e^{j\omega})\}}\right) \\ &= \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) \end{aligned}$$

Evaluating $\angle X(e^{j\omega})$ for $-\pi \leq \omega \leq \pi$, we have

ω (rad/s)	$\angle X(e^{j\omega})$ (degs)
$-\pi$	0
-3	-11.7802
-2	-36.0222
$-\pi/2$	-30.9638
-1	-20.8708
0	0
1	20.8708
$\pi/2$	30.9638
2	36.0222
3	11.7802
π	0

The plot of $\angle X(e^{j\omega})$ is shown in the figure below:

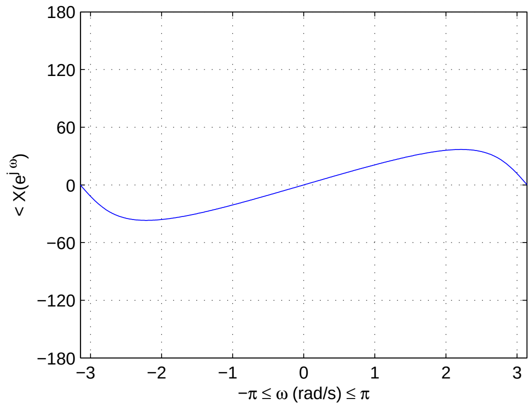


Figure 2: Question 1(a)

(b) Solution

The frequency response is given by

$$X(e^{j\omega}) = e^{-j3\omega}$$

The plot of $|X(e^{j\omega})|$ is shown in the figure below:

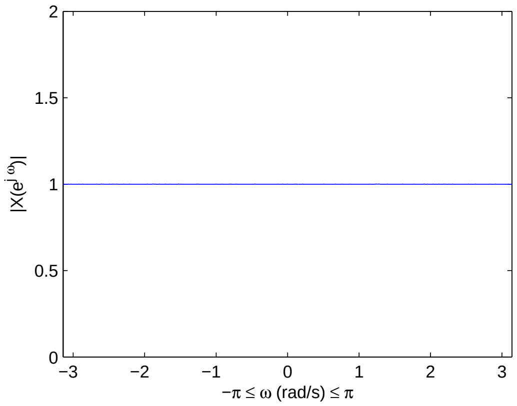


Figure 3: Question 1(b)

The plot of $\angle X(e^{j\omega})$ is shown in the figure below:

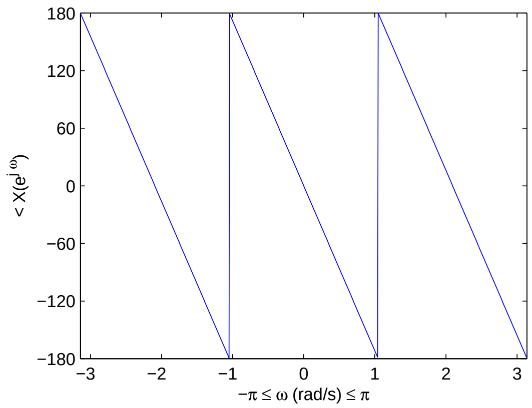


Figure 4: Question 1(b)

Q2

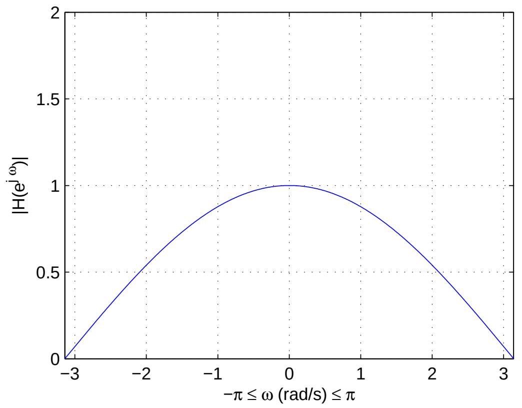
(a) Partial Solution

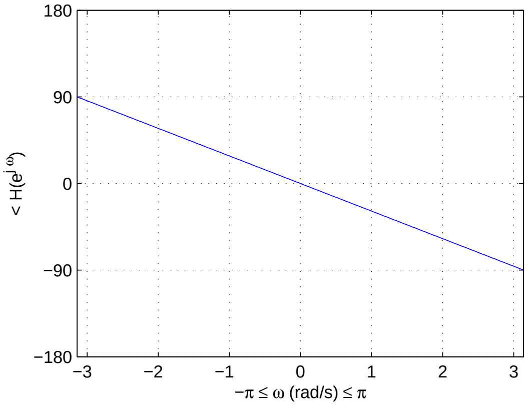
$$
\begin{aligned}
H(z) &= \frac{1}{2} + \frac{1}{2}z^{-1} \\
h[n] &= \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1] \qquad \text{(FIR filter)} \\
H(e^{j\omega}) &= \frac{1}{2}(1 + e^{-j\omega}) \\
&= \frac{1}{2}e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2}) \\
&= e^{-j\omega/2}\cos(\omega/2)
\end{aligned}
$$

Evaluating $|H(e^{j\omega})|$ for $-\pi \leq \omega \leq \pi$, we have

ω (rad/s)	$\lvert H(e^{j\omega})\rvert$
$-\pi$	0.0000
-3	0.0707
-2	0.5403
-1	0.8776
0	1.0000
1	0.8776
2	0.5403
3	0.0707
π	0.0000

The plots are shown below:-





(b) Solution

$$\begin{aligned} H(z) &= \frac{1}{1-0.6z^{-1}} \\ h[n] &= (0.6)^n u[n] \qquad \text{(IIR filter)} \\ H(e^{j\omega}) &= \frac{1}{1-0.6e^{-j\omega}} \end{aligned}$$

Evaluating $|H(e^{j\omega})|$ for $-\pi \leq \omega \leq \pi$, we have

ω (rad/s)	$\lvert H(e^{j\omega})\rvert$
$-\pi$	0.6250
-3	0.6265
-2	0.7334
-1	1.1854
0	2.5000
1	1.1854
2	0.7334
3	0.6265
π	0.6250

The plot is shown below:-

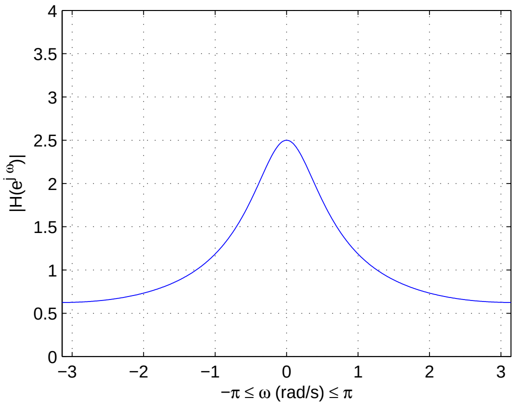


Figure 5: Question 2(b)

(c) Solution

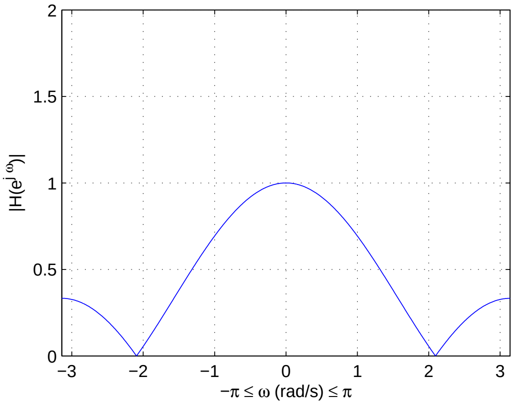
This is a three-point moving-average filter.

$$\begin{aligned} H(z) &= \frac{1}{3}(z+1+z^{-1}) = \frac{1}{3}\frac{1+z^{-1}+z^{-2}}{z^{-1}} \\ h[n] &= \frac{1}{3}(\delta[n+1]+\delta[n]+\delta[n-1]) \qquad \text{(FIR filter)} \\ H(e^{j\omega}) &= \frac{1}{3}(e^{j\omega}+1+e^{-j\omega}) = \frac{1}{3}(1+2\cos\omega) \end{aligned}$$

Evaluating $|H(e^{j\omega})|$ for $-\pi \leq \omega \leq \pi$, we have

ω (rad/s)	$\lvert H(e^{j\omega})\rvert$
$-\pi$	0.3333
-3	0.3267
-2	0.0559
-1	0.6935
0	1.0000
1	0.6935
2	0.0559
3	0.3267
π	0.3333

The plots are shown below:-



Check answer in Matlab using the following commands

```
>> num=[1/3 1/3 1/3];
>> den=[0 1 0];
>> w=[-pi:0.01:pi];
>> [H,W]=freqz(num,den,w)
>> plot(W,abs(H))
```

Challenge Question:
Why does `fvtool` give an error for 3 point moving-average FIR filter coefficients defined above?

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Problem Set #6 Discrete Fourier Transform (DFT)

Q1

The periodic function $x[n]$ is defined as:

(a)

$$x[n] = \begin{cases} 1 & \text{for } n = 4l + 2 \\ 0 & \text{elsewhere} \end{cases}$$

(b)

$$x[n] = \begin{cases} 1 & \text{for } n = 4l \text{ and } n = 4l + 3 \\ 2 & \text{for } n = 4l + 1 \text{ and } n = 4l + 2 \end{cases}$$

(c)`(challenge problem)`

$$x[n] = \begin{cases} 0 & \text{for } n = 4l \\ 1 & \text{for } n = 4l + 1 \text{ and } n = 4l + 3 \\ 2 & \text{for } n = 4l + 2 \end{cases}$$

with $l = 0, \pm 1, \pm 2, \cdots$.

For each $x[n]$:

- Plot the fundamental interval for $x[n]$.
- Calculate the N-point DFT of $x[n]$.
- Calculate and plot the magnitude and phase of DFT.
- Calculate and plot the real and imaginary parts of DFT..

Q2

The N-point DFT $X[k]$ is defined as:

(a) $N = 4$

$$X[k] = \begin{cases} 1 & \text{for } k = 4l \text{ and } k = 4l + 1 \text{ and } k = 4l + 2 \text{ and } k = 4l + 3 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $N = 4$

$$X[k] = \begin{cases} 2 & \text{for } k = 4l + 1 \\ 2 & \text{for } k = 4l + 3 \\ 0 & \text{elsewhere} \end{cases}$$

(c) $N = 16$ `(challenge problem)`

$$X[k] = \begin{cases} 2 & \text{for } k = 16l + 1 \text{ and } k = 16l + 15 \\ 1 & \text{for } k = 16l + 3 \text{ and } k = 16l + 13 \\ 0 & \text{elsewhere} \end{cases}$$

with $l = 0, \pm 1, \pm 2, \cdots$.

For each $X[k]$:

- Plot the fundamental interval for $X[k]$.
- Calculate the N-point IDFT of $X[k]$.
- Plot the fundamental interval for $x[n]$.

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Problem Set #6 Solution

Q1

(a) Complete Solution

Given that

$$x[n] = \begin{cases} 1 & \text{for } n = 4l+2 \\ 0 & \text{elsewhere} \end{cases}$$

For fundamental interval $l = 0$. Hence

$$x[n] = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{for } n = 0,1,3 \end{cases}$$

The plot of fundamental interval of $x[n]$ is shown in the figure below:

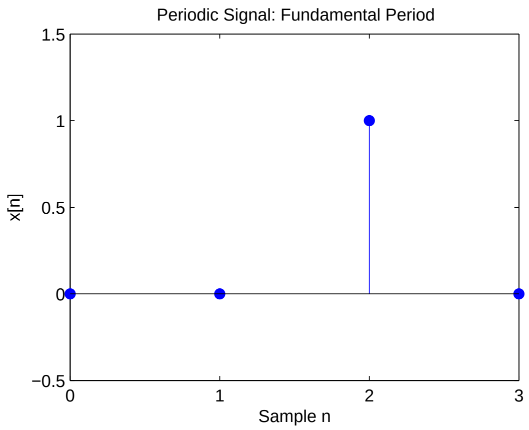


Figure 1: Question 1(a)

The 4-point DFT of $x[n]$ is

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{3} x[n]e^{-j\frac{\pi}{2}kn} \\ &= x[2]e^{-j\pi k} \\ &= e^{-j\pi k} \end{aligned}$$

Hence

$$\begin{aligned} X[0] &= e^{-j0} = 1 = 1\angle 0^\circ \\ X[1] &= e^{-j\pi} = -1 = 1\angle 180^\circ \\ X[2] &= e^{-j2\pi} = 1 = 1\angle 0^\circ \\ X[3] &= e^{-j3\pi} = -1 = 1\angle 180^\circ \end{aligned}$$

The plot of magnitude $|X[k]|$ and phase $\angle X[k]$ is shown in the figures below:

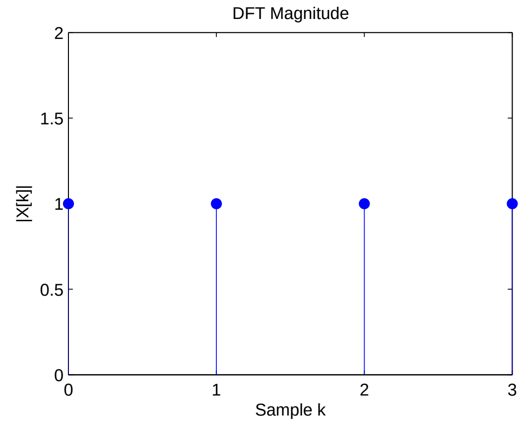


Figure 2: Question 1(a): Magnitude of DFT

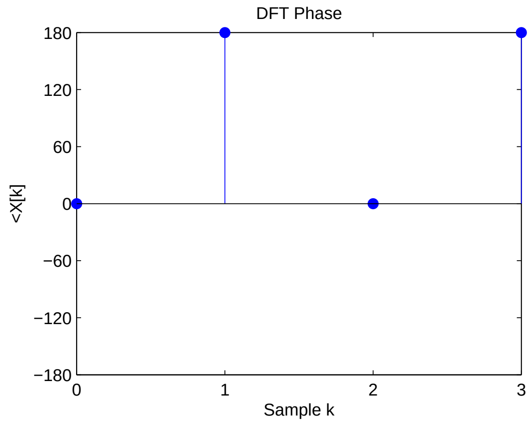


Figure 3: Question 1(a): Phase of DFT

The plot of real part $\Re\{X[k]\}$ and imaginary part $\Im\{X[k]\}$ is shown in the figures below:

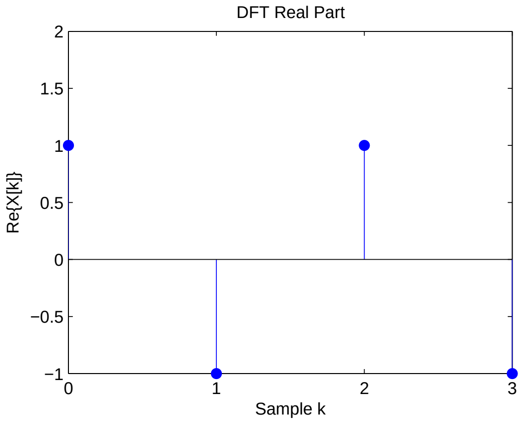


Figure 4: Question 1(a): Real part of DFT

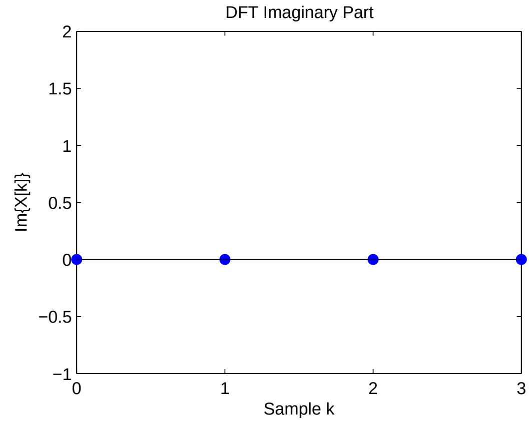


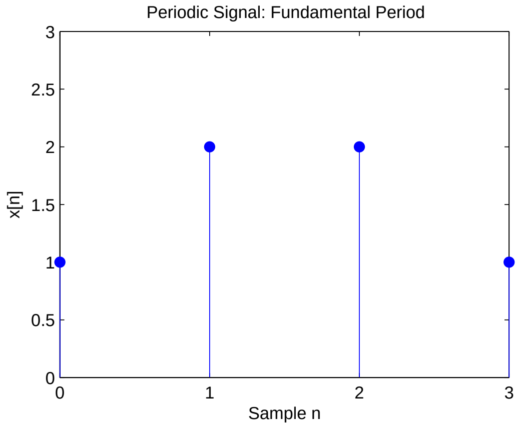
Figure 5: Question 1(a): Imaginary part of DFT

Check answer in Matlab using the following commands

```
>> n=[0 1 2 3];
>> x=[0 0 1 0];
>> X=fft(x);
>> MagX=abs(X);
>> PhaseX=angle(X)*180/pi;
>> RealX=real(X);
>> ImagX=imag(X)
```

(b) Solution

The plot of fundamental interval of $x[n]$ is shown in the figure below:



The plot of magnitude $|X[k]|$, phase $\angle X[k]$, real part $\Re\{X[k]\}$ and imaginary part $\Im\{X[k]\}$are:

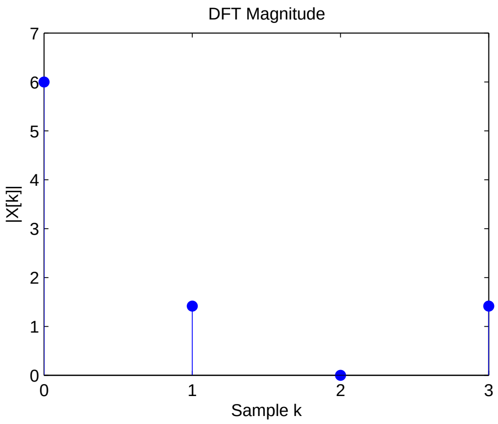


Figure 6: Question 1(b): Magnitude of DFT

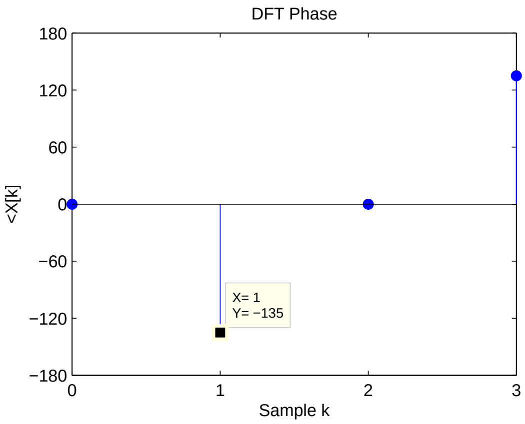


Figure 7: Question 1(b): Phase of DFT

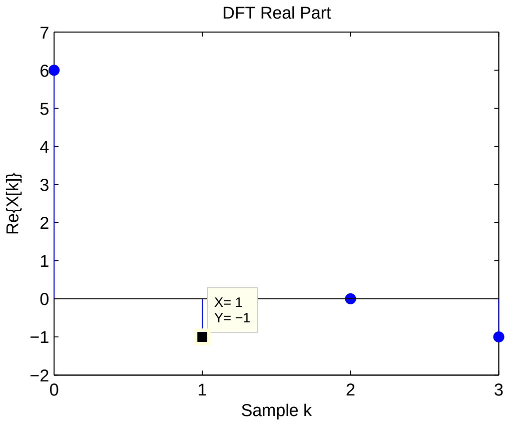


Figure 8: Question 1(b): Real part of DFT

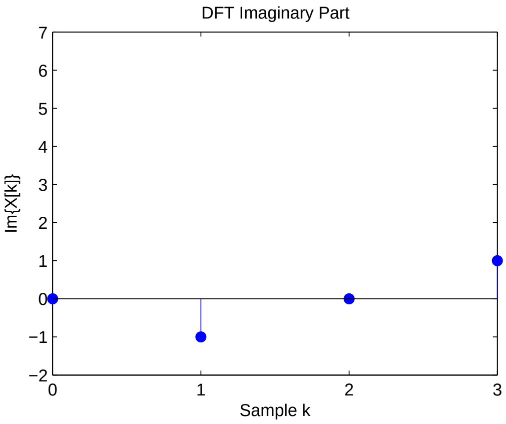
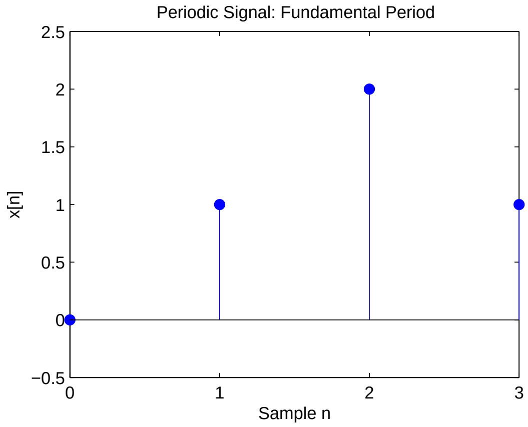


Figure 9: Question 1(b): Imaginary part of DFT

(c) Solution

The plot of fundamental interval of $x[n]$ is shown in the figure below:



The plot of magnitude $|X[k]|$, phase $\angle X[k]$, real part $\Re\{X[k]\}$ and imaginary part $\Im\{X[k]\}$are:

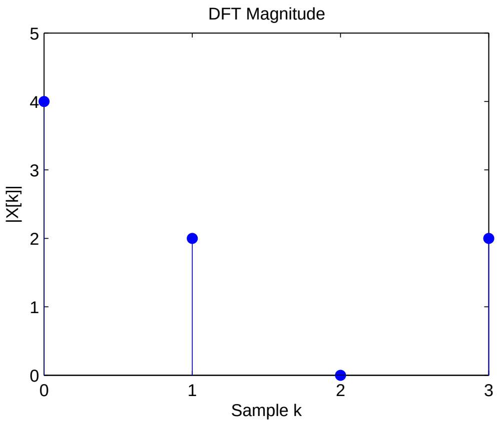


Figure 10: Question 1(c): Magnitude of DFT

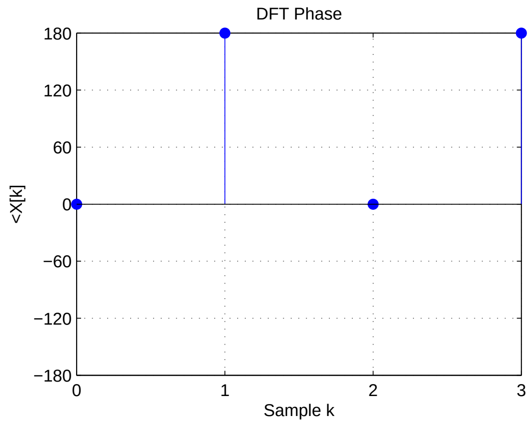


Figure 11: Question 1(c): Phase of DFT

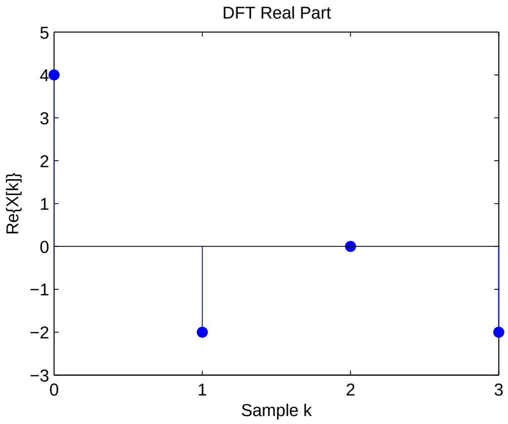


Figure 12: Question 1(c): Real part of DFT

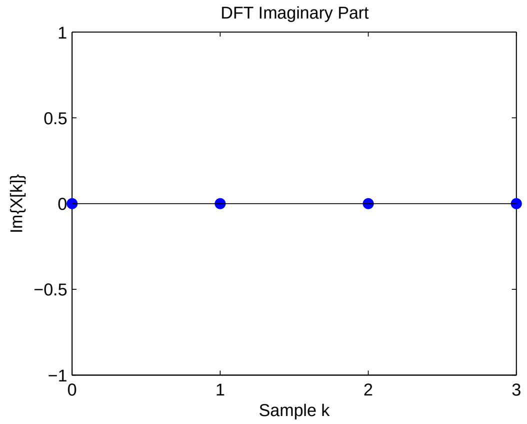


Figure 13: Question 1(c): Imaginary part of DFT

Q2

(a) Complete Solution

Given that

$$X[k] = \begin{cases} 1 & \text{for } k = 4l \text{ and } k = 4l+1 \text{ and } k = 4l+2 \text{ and } k = 4l+3 \\ 0 & \text{elsewhere} \end{cases}$$

For fundamental interval $l = 0$. Hence

$$X[k] = 1 \text{ for } k = 0, 1, 2, 3$$

The 4-point IDFT of $X[k]$ is

$$\begin{aligned} x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn} \\ &= \frac{1}{4}\sum_{k=0}^{3} x[n]e^{j\frac{\pi}{2}kn} \\ &= \frac{1}{4}\left\{x[0]e^{j0} + x[1]e^{j\frac{\pi}{2}n} + x[2]e^{j\pi n} + x[3]e^{j\frac{3\pi}{2}n}\right\} \\ &= \frac{1}{4}\left\{1 + e^{j\frac{\pi}{2}n} + e^{j\pi n} + e^{j\frac{3\pi}{2}n}\right\} \end{aligned}$$

Hence

$$\begin{aligned} x[0] &= \frac{1}{4}\left\{1 + e^{j\frac{\pi}{2}0} + e^{j\pi 0} + e^{j\frac{3\pi}{2}0}\right\} = \frac{1}{4}(1+1+1+1) = 1 \\ x[1] &= \frac{1}{4}\left\{1 + e^{j\frac{\pi}{2}} + e^{j\pi} + e^{j\frac{3\pi}{2}}\right\} = \frac{1}{4}(1+j-1-j) = 0 \\ x[2] &= \frac{1}{4}\left\{1 + e^{j\pi} + e^{j2\pi} + e^{j3\pi}\right\} = \frac{1}{4}(1-1+1-1) = 0 \\ x[3] &= \frac{1}{4}\left\{1 + e^{j\frac{3\pi}{2}n} + e^{j3\pi} + e^{j\frac{9\pi}{2}n}\right\} = \frac{1}{4}(1-j-1+j) = 0 \end{aligned}$$

The plot of fundamental interval of $X[k]$ and $x[n]$ is shown in the figures below:

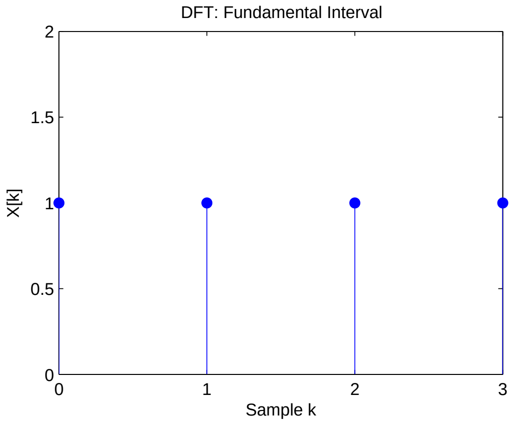


Figure 14: Question 2(a): DFT Fundamental Interval

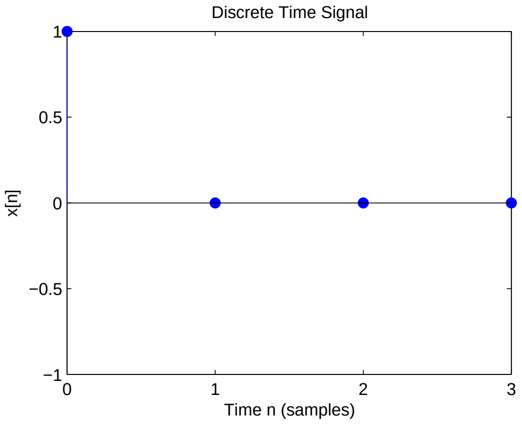


Figure 15: Question 2(a): Periodic signal

Check answer in Matlab using the following commands

```
>> k=[0 1 2 3];
>> X=[1 1 1 1];
>> x=ifft(X);
```

Q2

(b) Partial Solution

Given that

$$X[k] = \begin{cases} 2 & \text{for } k = 4l+1 \\ 2 & \text{for } k = 4l+3 \\ 0 & \text{elsewhere} \end{cases}$$

For fundamental interval $l = 0$. Hence

$$X[k] = \begin{cases} 2 & \text{for } k = 1,3 \\ 0 & \text{for } k = 0,2 \end{cases}$$

The 4-point IDFT of $X[k]$ is

$$\begin{aligned} x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn} \\ &= \frac{1}{4}\sum_{k=0}^{3} x[n]e^{j\frac{\pi}{2}kn} \\ &= \frac{1}{4}\left\{x[1]e^{j\frac{\pi}{2}n} + x[3]e^{j\frac{3\pi}{2}n}\right\} \\ &= \frac{1}{2}\left\{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right\} \\ &= \cos\left(\frac{n\pi}{2}\right) \\ &= \cos\left(\frac{2n\pi}{4}\right) \end{aligned}$$

The plot of fundamental interval of $X[k]$ and $x[n]$ is shown in the figures below:

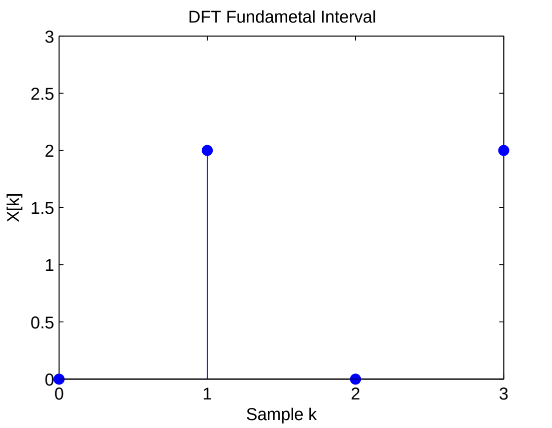


Figure 16: Question 2(b): DFT Fundamental Interval

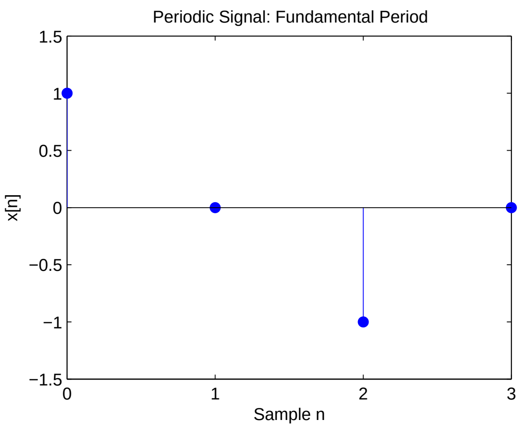


Figure 17: Question 2(b): Periodic signal

Q3

(c) Solution

Show that 4-point IDFT of $X[k]$ is

$$x[n] = \frac{1}{4}\cos(\pi n/8) + \frac{1}{8}\cos(3\pi n/8)$$

The plot of fundamental interval of $X[k]$ and $x[n]$ is shown in the figures below:

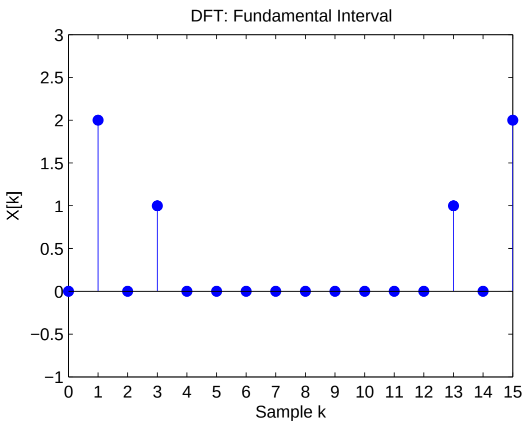


Figure 18: Question 2(c): DFT Fundamental Interval

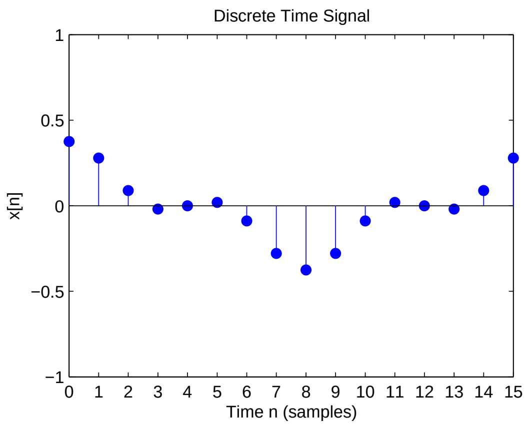


Figure 19: Question 2(c): Periodic signal

Check answer in Matlab using `ifft` command.
See also `L10_DFT.m`

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ENGN6612/4612 Digital Signal Processing and Control
Problem Set #7 Fast Fourier Transform (FFT)

Q1

A discrete signal $x[n]$ is defined as:

(a)

$$x[n] = \begin{cases} 1 & \text{for } n = 4l+2 \\ 0 & \text{elsewhere} \end{cases}$$

with $l = 0, \pm 1, \pm 2, \cdots$.

(b)

$$x[n] = \begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1,3 \\ 2 & \text{for } n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

For each $x[n]$:

- State whether the signal is periodic, (non-periodic) finite or (non-periodic) finite duration.
- Calculate the 8-point DFT of $x[n]$.
- Assuming $x[n]$ is a finite duration signal (that exists only for $0 \leq n \leq 8$), calculate the DTFT of $x[n]$.
- Show that DFT is sampled version of DTFT (consider both real and imaginary parts).

Q2

Show that the FFT shown schematically in the figure below corresponds to a 4-point DFT. `(challenge problem)`

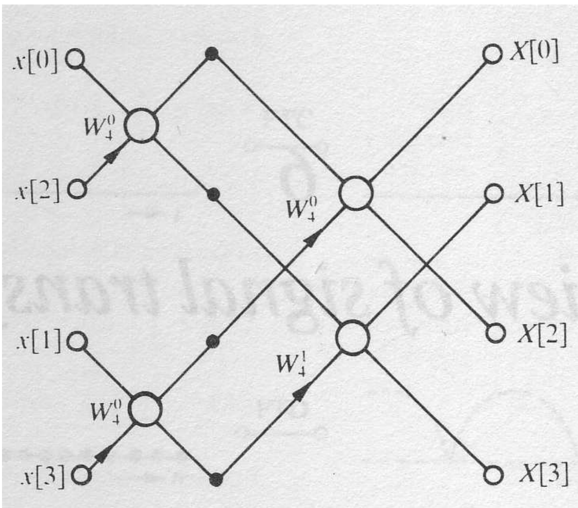

Figure 1: Question 2

Q3

A discrete signal $x[n]$ is defined as:
(a)

$$x[n] = \begin{cases} 1 & \text{for } n = 4l + 2 \\ 0 & \text{elsewhere} \end{cases}$$

with $l = 0, \pm 1, \pm 2, \cdots$.

(b)

$$x[n] = \begin{cases} n + 1 & \text{for } 0 \leq n < 4 \\ 0 & \text{elsewhere} \end{cases}$$

For this signal:

- Calculate $X[k]$ using definition of DFT (take $N = 4$).
- Calculate $X[k]$ by making use of the diagram shown in Question 2.

Q4

Consider the periodic sequences $x_p[n]$ and $h_p[n]$ (with period $N = 4$):

(a)

$$h_p[n] = \begin{cases} n + 1 & \text{for } 0 \leq n \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$x_p[n] = \begin{cases} 1 & \text{for } n = 1, 2 \\ 0 & \text{elsewhere} \end{cases}$$

(b)

$$h_p[n] = \begin{cases} n & \text{for } 0 \leq n \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$x_p[n] = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

Determine the output $z_p[n] = x_p[n] \circledast h_p[n]$ using both (i) graphical discrete-time circular convolution and (ii) DFT method.

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ENGN6612/4612 Digital Signal Processing and Control
Problem Set #7 Solution

Q1

(a) Complete Solution

The given signal is periodic with period $N = 4$.
The plot of first 8 samples of the signal is shown below:-

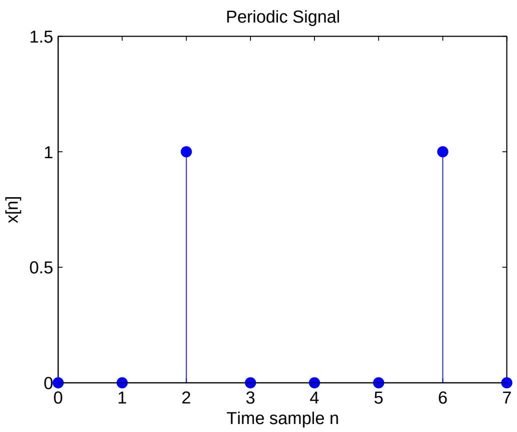


Figure 2: Question 1(a)

DFT

The 8-point DFT of $x[n]$ is

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \\ &= \sum_{n=0}^{7} x[n]e^{-j\frac{\pi}{4}kn} \\ &= x[2]e^{-j\frac{\pi}{2}k} + x[6]e^{-j\frac{3\pi}{2}k} \\ &= e^{-j\frac{\pi}{2}k} + e^{-j\frac{3\pi}{2}k} \end{aligned}$$

Hence $X[0] = 2$, $X[1] = 0$, $X[2] = -2$, $X[3] = 0$, $X[4] = 2$, $X[5] = 0$, $X[6] = -2$, $X[7] = 0$.

The result is summarised in the table below:-

Frequency Sample k	Discrete Frequency $\omega_k = \frac{2\pi k}{N}$ (rad/s)	$\Re\{X[k]\}$	$\Im\{X[k]\}$
0	0	2	0
1	$\frac{\pi}{4}$	0	0
2	$\frac{\pi}{2}$	-2	0
3	$\frac{3\pi}{4}$	0	0
4	π	2	0
5	$\frac{5\pi}{4}$	0	0
6	$\frac{3\pi}{2}$	-2	0
7	2π	0	0

DTFT

Assuming $x[n]$ is a finite duration signal (that exists only for $0 \leq n \leq 8$), we have

$$\begin{aligned} x[n] &= \delta[n-2] + \delta[n-6] \\ X(z) &= \frac{1}{z^2} + \frac{1}{z^6} \\ X(e^{j\omega}) &= e^{-j2\omega} + e^{-j6\omega} = \{\cos(2\omega) + \cos(6\omega)\} + j\{-\sin(2\omega) - \sin(6\omega)\} \end{aligned}$$

Evaluating $\Re\{X(e^{j\omega})\}$ and $\Im\{X(e^{j\omega})\}$ for the fundamental interval $0 \leq \omega \leq 2\pi$, we have

Frequency ω (rad/s)	$\Re\{X(e^{j\omega})\} = \cos(2\omega) + \cos(6\omega)$	$\Im\{X(e^{j\omega})\} = -\sin(2\omega) - \sin(6\omega)$
0	2	0
$\frac{\pi}{4}$	0	0
$\frac{\pi}{2}$	-2	0
$\frac{3\pi}{4}$	0	0
π	2	0
$\frac{5\pi}{4}$	0	0
$\frac{3\pi}{2}$	-2	0
2π	0	0

Comparing the results in the two tables, we see that DFT is sampled version of DTFT.

The plots are shown in the figures below:

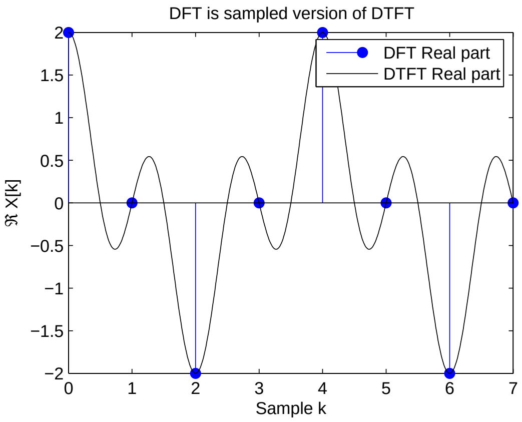


Figure 3: Question 1(a): DFT and DTFT Real part

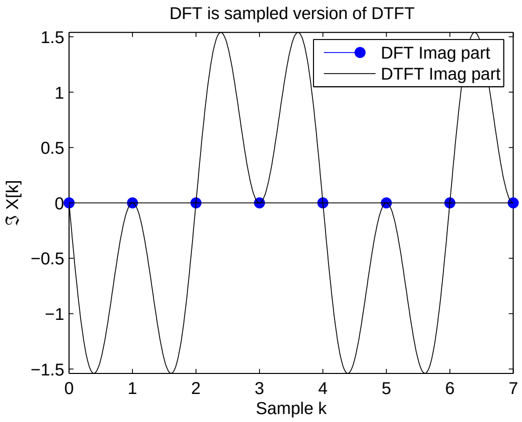


Figure 4: Question 1(a): DFT and DTFT Imaginary part

Compare with 4-point DFT evaluated in `Problem Set 6: Q1a.`

(b) Partial Solution

The given signal is (non-periodic) finite duration.
The plot of first 8 samples of the signal is shown below:-

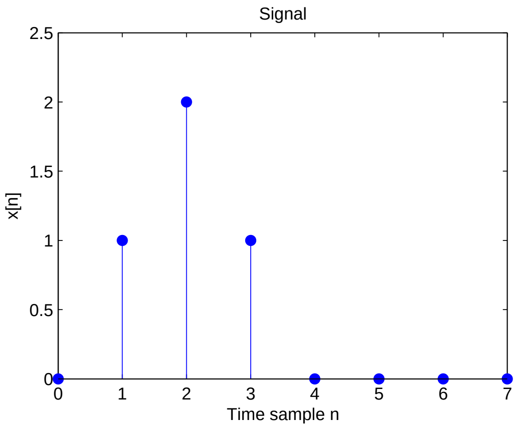


Figure 5: Question 1(b)

DFT

The 8-point DFT of $x[n]$ is

$$X[k] = e^{-j\frac{\pi}{4}k} + 2e^{-j\frac{\pi}{2}k} + e^{-j\frac{3\pi}{4}k}$$

DTFT

$$X(e^{j\omega}) = e^{-j\omega} + 2e^{-j2\omega} + e^{-j3\omega}$$

The results are summarised in the tables below:-

Frequency ω (rad/s)	$\Re\{X(e^{j\omega})\}$	$\Im\{X(e^{j\omega})\}$
0	4	0
$\frac{\pi}{4}$	0	-3.4142
$\frac{\pi}{2}$	-2	0
$\frac{3\pi}{4}$	0	0.5858
π	0	0
$\frac{5\pi}{4}$	0	-0.5858
$\frac{3\pi}{2}$	-2	0
2π	0	3.4142

Frequency Sample k	Discrete Frequency $\omega_k = \frac{2\pi k}{N}$ (rad/s)	$\Re\{X[k]\}$	$\Im\{X[k]\}$
0	0	4	0
1	$\frac{\pi}{4}$	0	-3.4142
2	$\frac{\pi}{2}$	-2	0
3	$\frac{3\pi}{4}$	0	0.5858
4	π	0	0
5	$\frac{5\pi}{4}$	0	-0.5858
6	$\frac{3\pi}{2}$	-2	0
7	2π	0	3.4142

The plots are shown in the figures below:

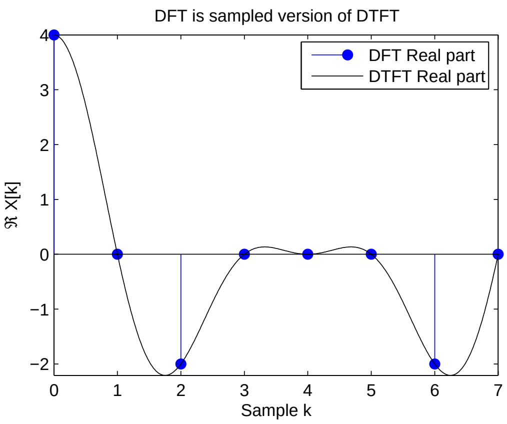


Figure 6: Question 1(b): DFT and DTFT Real part

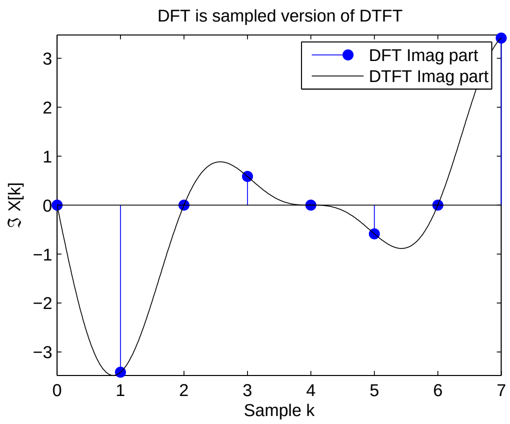


Figure 7: Question 1(b): DFT and DTFT Imaginary part

Q2

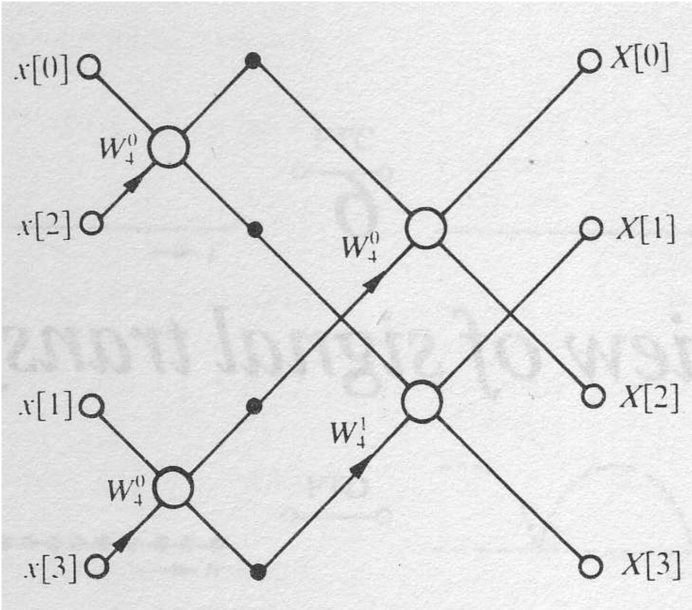


Figure 8: Question 2

The proof is left as an exercise for the students.

Hint

By tracing the paths in the flow graph of Fig. 8, show that each input sample contributes the proper amount to the output of the DFT sample, i.e. verify that

$$X[0] = \sum_{n=0}^{N-1} x[n]$$

$$X[1] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}n}$$

$$X[2] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}2n}$$

$$X[3] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}3n}$$

Reference

Please see Chapter 9 in “Discrete-Time Signal Processing” by Oppenheim and Schafer for comprehensive discussion of FFT.

Q3

(a) Partial Solution

Using DFT Definition

$$
\begin{aligned}
X[k] &= e^{-j\pi k} \\
X[0] &= e^{-j0} = 1 = 1 + j0 \\
X[1] &= e^{-j\pi} = -1 = -1 + j0 \\
X[2] &= e^{-j2\pi} = 1 = 1 + j0 \\
X[3] &= e^{-j3\pi} = -1 = -1 + j0
\end{aligned}
$$

For details, see `Problem Set 06: Q1 (a)`.

Using FFT butterfly

We have the twiddle factor

$$
\begin{aligned}
W_N^p &= e^{-j\frac{2\pi}{N}p} \\
&= e^{-j\frac{\pi}{2}p}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
W_4^0 &= e^{-j0} = 1 \\
W_4^1 &= e^{-j\frac{\pi}{2}} = -j
\end{aligned}
$$

Writing the equations for the intermediate terms in the diagram, we have

$$
\begin{aligned}
a &= x[0] + W_4^0 x[2] = x[0] + x[2] \\
b &= x[0] - W_4^0 x[2] = x[0] - x[2] \\
c &= x[1] + W_4^0 x[3] = x[1] + x[3] \\
d &= x[1] - W_4^0 x[3] = x[1] - x[3]
\end{aligned}
$$

Writing the equations for the output terms in the diagram, we have

$$
\begin{aligned}
X[0] &= a + W_4^0 c = a + c \\
X[2] &= a - W_4^0 c = a - c \\
X[1] &= b + W_4^1 d = b - jd \\
X[3] &= b - W_4^1 d = b + jd
\end{aligned}
$$

The output samples $X[k]$ can be expressed in terms of input samples $x[n]$ as

$$
\begin{aligned}
X[0] &= \{x[0] + x[2]\} + \{x[1] + x[3]\} \\
X[2] &= \{x[0] + x[2]\} - \{x[1] + x[3]\} \\
X[1] &= \{x[0] - x[2]\} - j\{x[1] - x[3]\} \\
X[3] &= \{x[0] - x[2]\} + j\{x[1] - x[3]\}
\end{aligned}
$$

Substituting the values,
$X[0] = 1$, $X[1] = -1$, $X[2] = 1$, $X[3] = -1$.

(b) Solution

$X[0] = 10$, $X[1] = -2 + j2$, $X[2] = -2$, $X[3] = -2 - j2$.

Check answer in Matlab using the following commands

```
>> n=[0 1 2 3];
>> x=[1 2 3 4];
>> X=fft(x);
```

Q4

(a) Complete Solution

Please see attached pages 10 – 12.

(b) Solution

The input sequences are shown in the figure below:

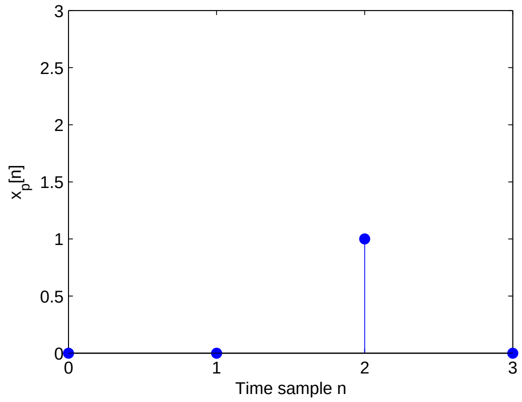


Figure 9: Question 4(b): Periodic sequence $x_p[n]$.

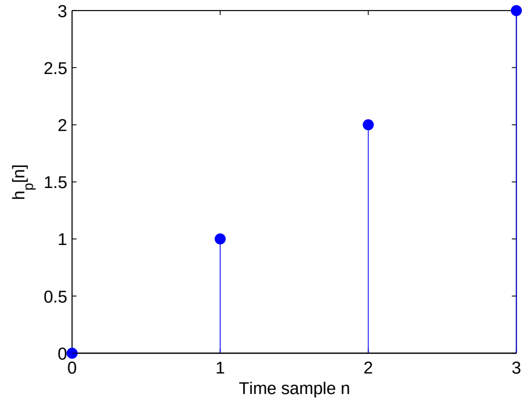


Figure 10: Question 4(b): Periodic sequence $h_p[n]$.

The output $z_p[n] = x_p[n] \circledast h_p[n]$ is shown in the figure below:

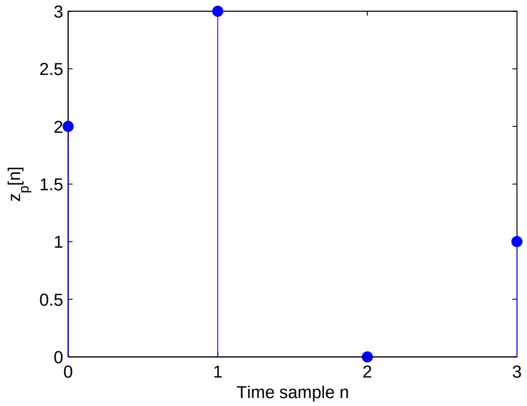


Figure 11: Question 4(b): Output periodic sequence $z_p[n]$.

Check answer in Matlab using the following commands

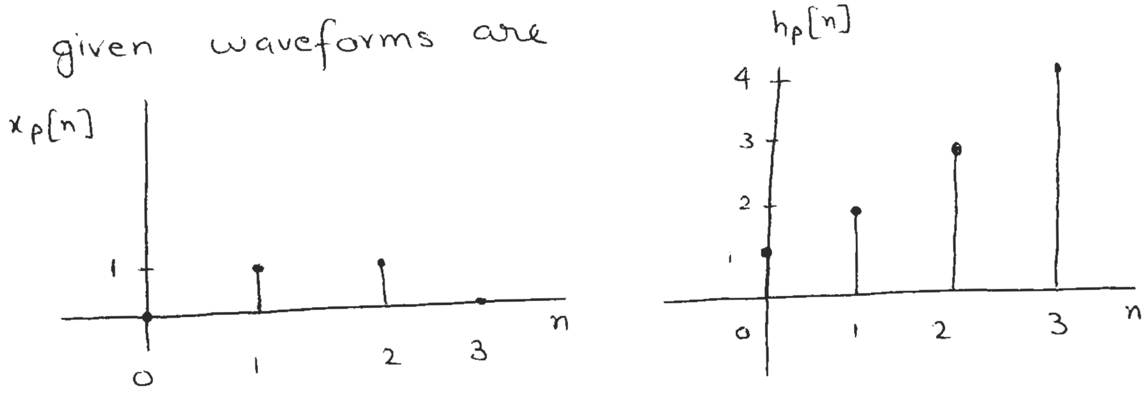
```
>> xp=[0 0 1 0];
>> hp=[0 1 2 3];
>> zp=ifft(fft(xp).*fft(hp))
```

GRAPHICAL DISCRETE-TIME CIRCULAR CONVOLUTION

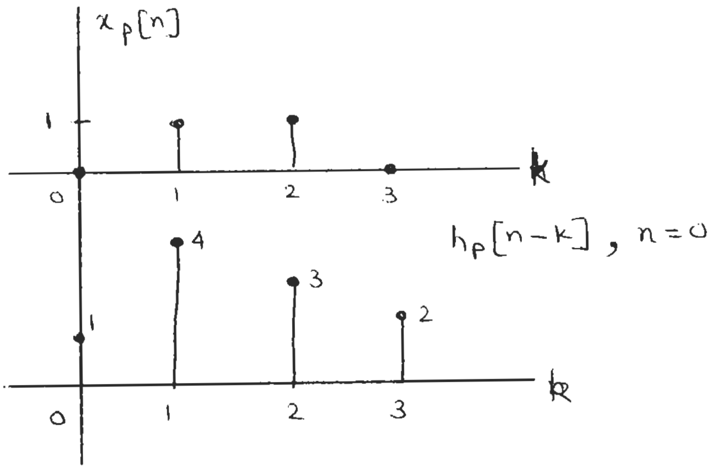
We know the circular convolution of two periodic signals with Period N is defined as

$$y_p[n] = \sum_{k=0}^{N-1} x_p[k]\, h_p[n-k]$$

The given waveforms are



<u>n = 0</u>



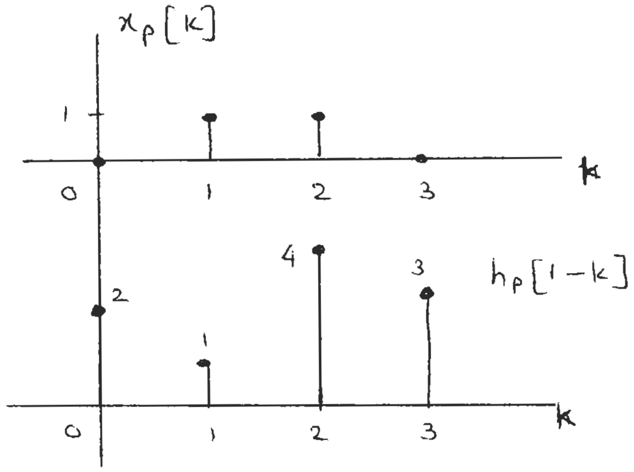
$$y_p[0] = \sum_{k=0}^{3} x_p[k]\, h_p[0-k]$$

$$= (4)(1) + (3)(1)$$

$$= 7.$$

(Please note technique for circular reversal shown in the figure).

$\underline{n = 1}$



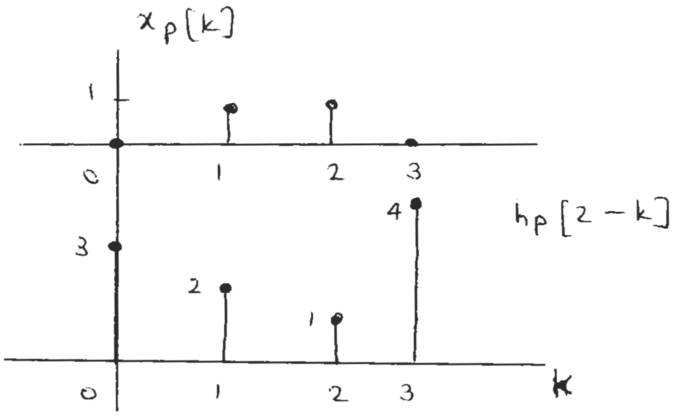
$$z_p[1] = \sum_{k=0}^{3} x_p[k]\, h_p[1-k]$$

$$= (1)(1) + (4)(1)$$

$$= 5$$

(Please note technique of circular time shift shown in figures)

$\underline{n = 2}$

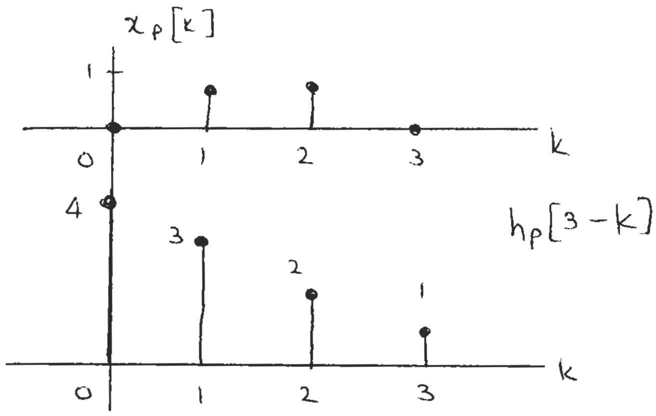


$$z_p[2] = \sum_{k=0}^{3} x_p[k]\, h_p[2-k]$$

$$= (2)(1) + (1)(1)$$

$$= 3$$

$\underline{n = 3}$

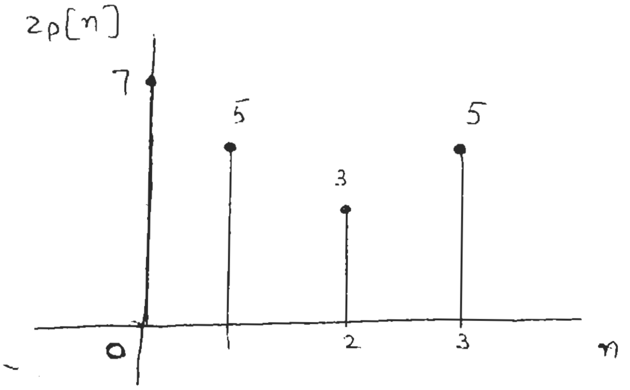


$$z_p[3] = \sum_{k=0}^{4} x_p[k]\, h_p[3-k]$$

$$= (3)(1) + (2)(1)$$

$$= 5$$

The output waveform is shown below



FREQUENCY DOMAIN

Show that 4-point DFT of $x_p[n]$ is

$$X_p[k] = e^{-j\pi k}$$

Hence

$X_p[0] = 1$

$X_p[1] = -1$

$X_p[2] = 1$

$X_p[3] = -1$

Show that 4-point DFT of $h_p[n]$ is

$H_p[0] = 6$

$H_p[1] = -2 + j2$

$H_p[2] = -2$

$H_p[3] = -2 - j2$

(Hint :- 4-point FFT with FFT butterfly equations also can be used to calculate this result)

For output

$$Z_p[k] = X_p[k]\, H_p[k]$$

Hence $Z_p[0] = 6$

$Z_p[1] = 2 - j2$

$Z_p[2] = -2$

$Z_p[3] = 2 + j2$.

Take 4-point IDFT of $Z_p[k]$ and show that $Z_p[n]$ is

$Z_p[0] = 7$, $z_p[1] = 5$, $Z_p[2] = 3$, $Z_p[3] = 5$

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ENGN6612/4612 Digital Signal Processing and Control
Problem Set #8 Filter Structures

Q1

A digital filter defined by the transfer function:-

(a) $H(z) = \frac{1}{3}\left\{1 + z^{-1} + z^{-2}\right\}$

(b) $H(z) = \frac{2}{1 - 3z^{-1} + z^{-2}}$

For each $H(z)$:
- Identify the filter type (FIR or IIR).
- Find the difference equation.
- Draw the block diagram representation of the filter in Direct-Form I.

Q2

A digital filter defined by the transfer function:-

$$H(z) = \frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

Draw the block diagram representation of the filter in
- Direct-Form I.
- Cascade Form.
- Parallel Form.

Q3

A digital filter defined by the transfer function:-

(a) $H(z) = \frac{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$

(b) $H(z) = \frac{1 - 6z^{-1} + 8z^{-2}}{1 - \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2}}$

For each $H(z)$, draw the block diagram representation of the filter in Direct-Form II.

Q4

A digital filter is shown in the block diagram shown below:

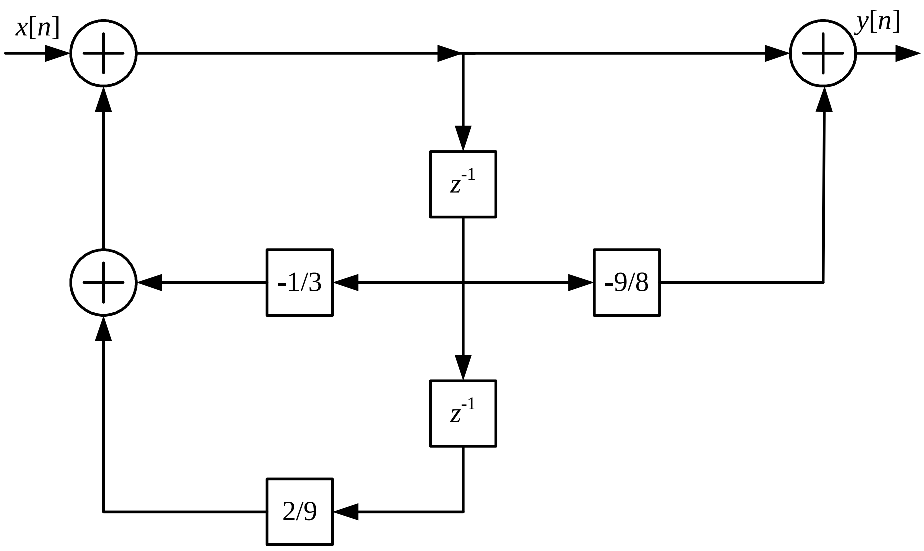

Figure 1: Question 4.

For the given filter, find the difference equation. `(challenge problem)`

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Problem Set #8 Solution

Q1

(a) Complete Solution

Given that

$$H(z) = \frac{1}{3}\left\{1+z^{-1}+z^{-2}\right\}$$

This is a FIR filter (3-point moving-average FIR filter).

Re-writing the transfer function, we have

$$\begin{aligned}\frac{Y(z)}{X(z)} &= \frac{1}{3}\left\{1+z^{-1}+z^{-2}\right\} \\ Y(z) &= \frac{1}{3}X(z)+\frac{1}{3}\frac{X(z)}{z}+\frac{1}{3}\frac{X(z)}{z^2}\end{aligned}$$

Taking the inverse z-transform, we have

$$y[n] = \frac{1}{3}x[n]+\frac{1}{3}x[n-1]+\frac{1}{3}x[n-2]$$

The block diagram representation of the filter in Direct-Form I is shown below:

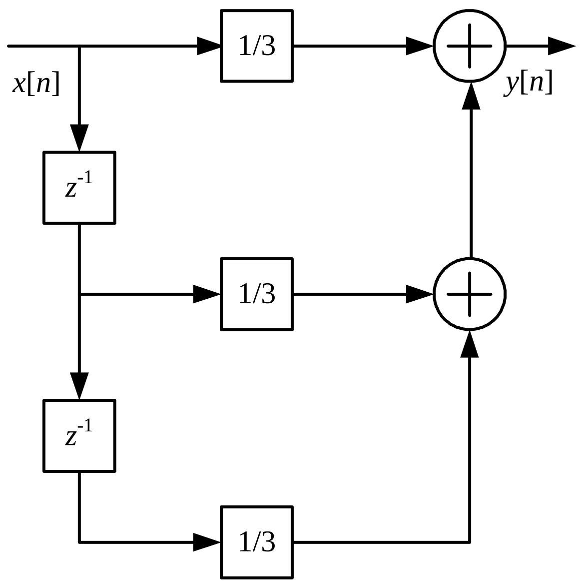

Figure 2: Direct-Form I implementation (FIR filter) for Question 1(a).

(b) Complete Solution

Given that

$$H(z) = \frac{2}{1 - 3z^{-1} + z^{-2}}$$

This is an IIR filter.

Re-writing the transfer function, we have

$$\begin{aligned} \frac{Y(z)}{X(z)} &= \frac{2}{1 - 3z^{-1} + z^{-2}} \\ Y(z) &= 2X(z) + 3\frac{Y(z)}{z} - \frac{Y(z)}{z^2} \end{aligned}$$

Taking the inverse z-transform, we have

$$y[n] = 3y[n-1] - y[n-2] + 2x[n]$$

The block diagram representation of the filter in Direct-Form I is shown below:

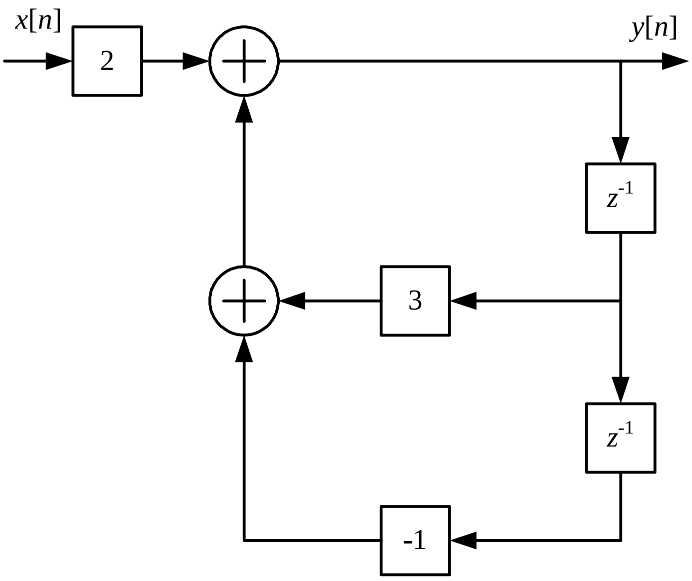

Figure 3: Direct-Form I implementation (IIR filter) for Question 1(b).

Q2

Partial Solution

The given transfer function is

$$H(z) = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

The corresponding difference equation is

$$y[n] = x[n]-\frac{1}{4}y[n-1]+\frac{1}{8}y[n-2]$$

The block diagram representation of the filter in Direct-Form I is shown below:

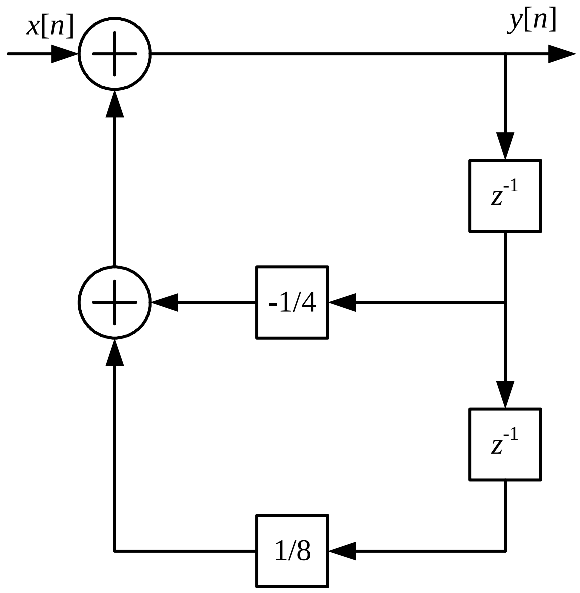

Figure 4: Direct-Form I implementation for Question 2.

Cascade Form

Factorising, the transfer function can be written as

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = H_1(z)H_2(z)$$

where

$$H_1(z) = \frac{1}{(1+\frac{1}{2}z^{-1})}$$

$$H_2(z) = \frac{1}{(1-\frac{1}{4}z^{-1})}$$

Using $H_1(z)$ and $H_2(z)$, the cascade form implementation of $H(z)$ can be drawn.

Parallel Form

Using partial fractions,

$$H(z) = \frac{2/3}{1+\frac{1}{2}z^{-1}}+\frac{1/3}{1-\frac{1}{4}z^{-1}} = H_3(z)+H_4(z)$$

Using $H_3(z)$ and $H_4(z)$, the parallel form implementation of $H(z)$ can be drawn.

The block diagram representation of the filter in Cascade Form is shown below:

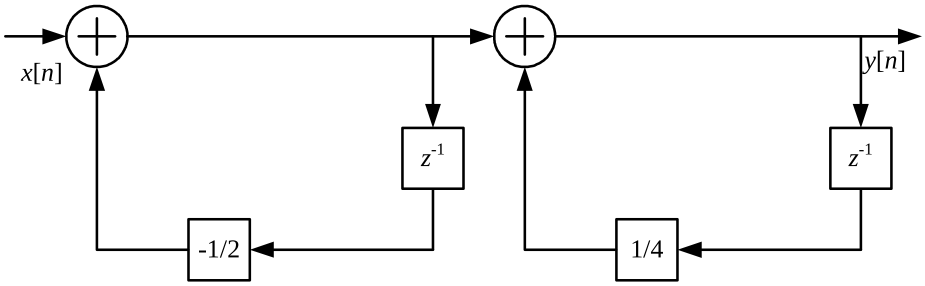

Figure 5: Cascade Form implementation for Question 2.

The block diagram representation of the filter in Parallel Form is shown below:

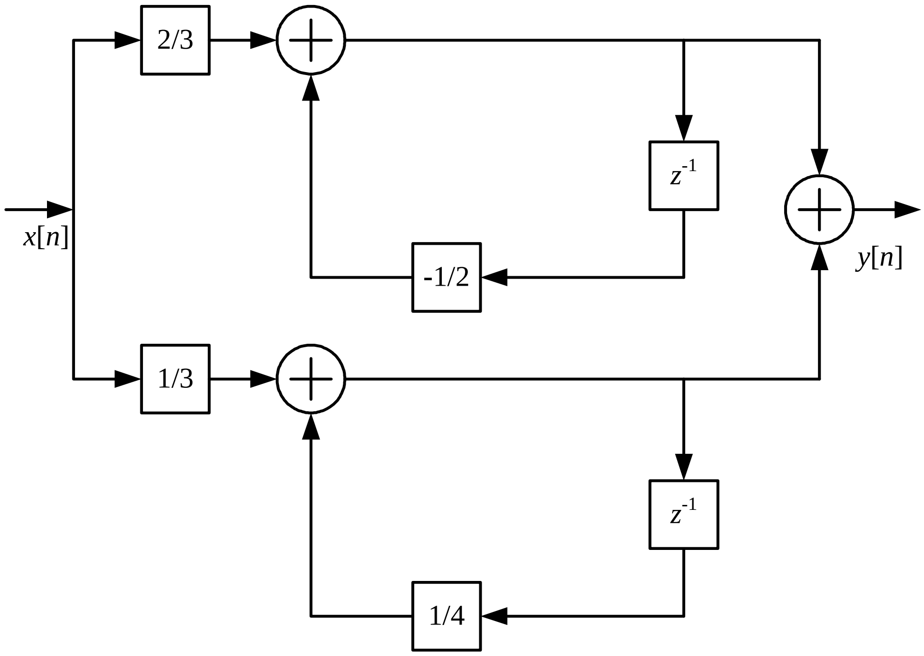

Figure 6: Parallel Form implementation for Question 2.

Q3

(a) Complete Solution

The given transfer function can be written as

$$H(z) = \frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} = H_1(z)\,H_2(z)$$

where

$$H_1(z) = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

$$H_2(z) = 1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}$$

Implementation of $H_1(z)$

Re-writing $H_1(z)$, we have

$$\frac{Y_1(z)}{X_1(z)} = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

$$Y_1(z) = X_1(z)-\frac{1}{4}\frac{Y_1(z)}{z}+\frac{1}{8}\frac{Y_1(z)}{z^2}$$

Taking the inverse z-transform, we have

$$y_1[n] = x_1[n]-\frac{1}{4}y_1[n-1]+\frac{1}{8}y_1[n-2]$$

The block diagram representation of $H_1(z)$ in Direct-Form I is shown below:

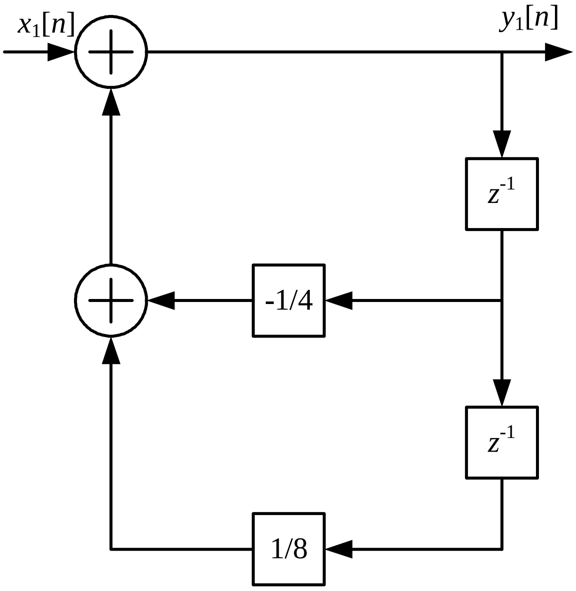

Figure 7: Direct-Form I implementation of $H_1(z)$ for Question 3(a).

Implementation of $H_2(z)$

Re-writing $H_2(z)$, we have

$$\frac{Y_2(z)}{X_2(z)} = 1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}$$

$$Y_2(z) = X_2(z)-\frac{7}{4}\frac{X_2(z)}{z}-\frac{1}{2}\frac{X_2(z)}{z^2}$$

Taking the inverse z-transform, we have

$$y_2[n] \quad = \quad x_2[n] - \frac{7}{4}x_2[n-1] - \frac{1}{2}x_2[n-2]$$

The block diagram representation of $H_2(z)$ in Direct-Form I is shown below:

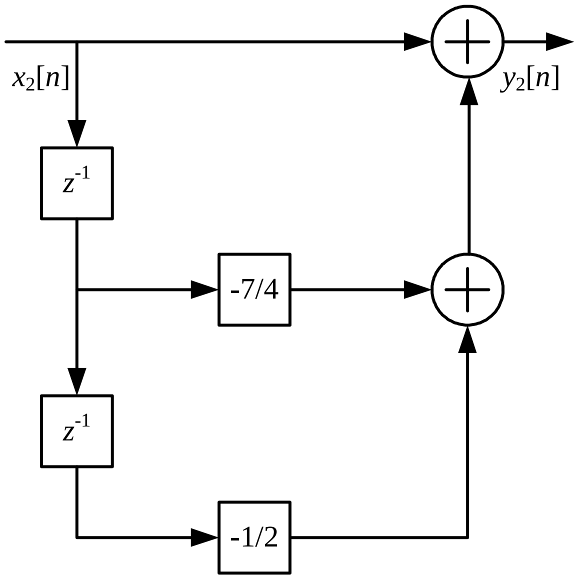

Figure 8: Direct-Form I implementation of $H_2(z)$ for Question 3(a).

Cascading the blocks, we have

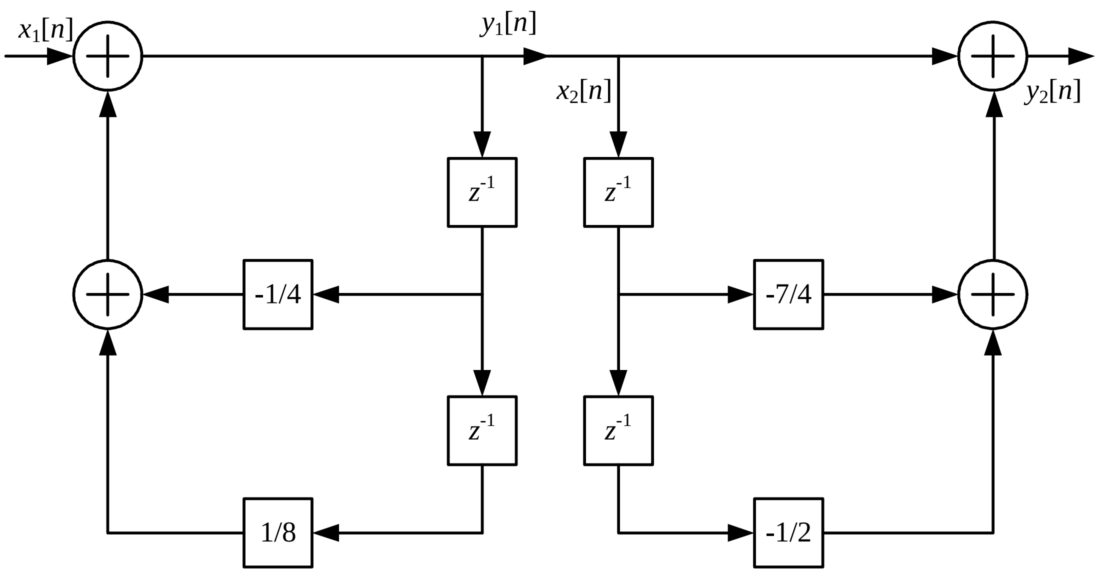

Figure 9: Direct-Form I implementation of $H(z)$ for Question 3(a).

Eliminating the common delay elements, we have

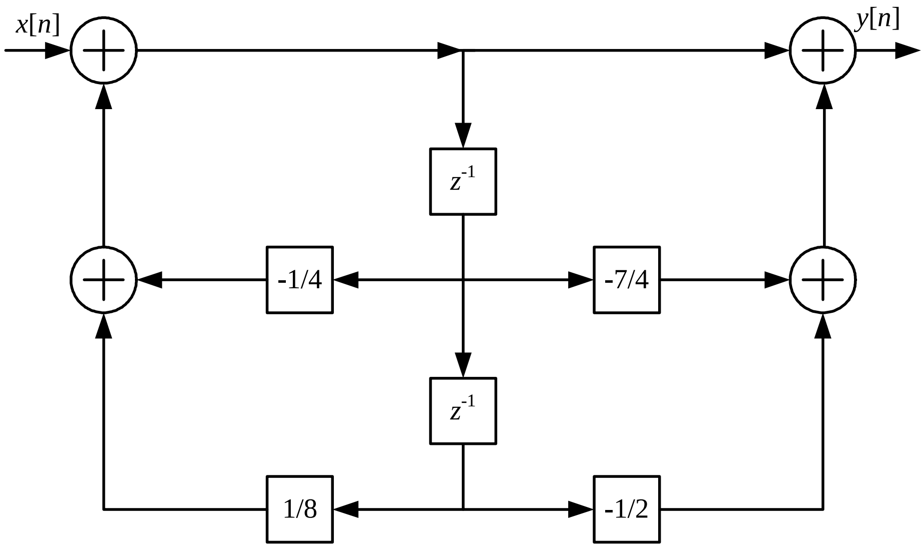

Figure 10: Direct-Form II implementation of $H(z)$ for Question 3(a).

(b) Solution

The difference equation representing the filter is

$$y[n]-\frac{2}{3}y[n-1]+\frac{1}{9}y[n-2] \quad = \quad x[n]-6x[n-1]+8[n-2]$$

The block diagram representation of the filter in Direct-Form II is shown below:

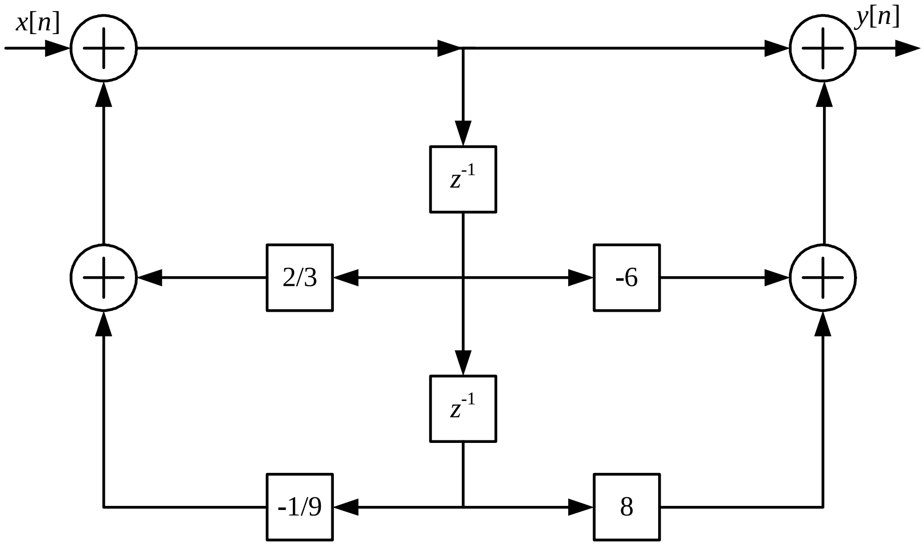

Figure 11: Direct-Form II implementation for Question 3(b).

Q4

Solution with Hint

The given block diagram is in Direct-Form II.
Show the following steps:

- Draw Direct-Form I implementation.
- Identify the cascaded blocks $H(z) = H_1(z)H_2(z)$.
- Find the transfer function for $H_1(z)$ and $H_2(z)$ respectively.
- Find the overall transfer function $H(z)$.
- From $H(z)$, find the difference equation.

The difference equation representing the filter is

$$y[n] + \frac{1}{3}y[n-1] - \frac{2}{9}y[n-2] \quad = \quad x[n] - \frac{9}{8}x[n-1]$$

ENGN6612/ENGN4612
Digital Signal Processing and Control

DSP Matlab Scripts
V1.0

Dr. Salman Durrani

August 2005

```
%% L02_Fourier
%% ==============
%% Function to find Fourier-transform using Matlab symbolic toolbox.
%%
%% Created  : 15 July 2005
%% Modified : 15 July 2005
%%
%% Copyright (c) 2005 Salman Durrani.
%% ------------------------------------------------------------

%% clear matlab memory
clear all;
clc

syms f x w
syms a b w0 real

%% Take Fourier Transform

%% Example 1: Unit step u[n]
f = heaviside(x);
Ans1=maple('fourier',f,x,w)
pretty(Ans1)

%% Example 2:
f = exp(-x^2);
Ans2=maple('fourier',f,x,w)
pretty(Ans2)

%% Example 3:
f = sin(w0*x);
Ans3=maple('fourier',f,x,w)
pretty(Ans3)

%% Take Inverse Fourier Transform

%% Example 1:
F = 1/(a+i*w);
Ans1=maple('invfourier',F,w,x)
pretty(Ans1)

%% Example 2:
F = 1/(a+i*w)^2;
Ans2=maple('invfourier',F,w,x)
pretty(Ans2)
```

```
%% L02_Laplace
%% ==============
%% Function to find laplace-transform using Matlab symbolic toolbox.
%%
%% Created  : 15 July 2005
%% Modified : 15 July 2005
%%
%% Copyright (c) 2005 Salman Durrani .
%% ------------------------------------------------------------

%% clear matlab memory
clear all;
clc

syms f b w t s

%% Take Laplace Transform

%% Example 1: Unit step u[n]
f = heaviside(t);
Ans1=maple('laplace',f,t,s)

%% Example 2:
f = (t^4);
Ans2=maple('laplace',f,t,s)

%% Example 3:
f= t*sin(w*t);
Ans3=maple('laplace',f,t,s)
pretty(Ans3)

%% Example 4:
f= exp(t)*cos(w*t);
Ans4=maple('laplace',f,t,s)
pretty(Ans4)

%% Take Inverse Laplace Transform

%% Example 1:
F = 1/(s^2+s);
Ans1=maple('invlaplace',F,s,t)
pretty(Ans1)

%% Example 2:
F = s/((s^2+b^2))^2;
Ans2=maple('invlaplace',F,s,t)
pretty(Ans2)
```

```
%% Example 3:
F = (3*s)/(s^2+2*s-8);
Ans3=maple('invlaplace',F,s,t)
pretty(Ans3)
```

```
%% L03_Ltransform
%% ===============
%% Script to find analyse RC circuit.
%%
%% Created  : 15 July 2005
%% Modified : 15 July 2005
%%
%% Copyright (c) 2005 Salman Durrani.
%% ----------------------------------------------------------------

%% clear matlab memory
clear all;
clc

%% RC components
R1 = 9e3;
R2 = 1e3;
C=0.3193e-6;

fc = 1/(2*pi*(R1+R2)*C);

%% find transfer function
a = R2*C;
b = (R1+R2)*C;

num =[a 1];
den =[b 1];
H = tf(num,den);

%% impulse response
figure
impulse(H)

%% pole-zero plot
figure
pzmap(num,den)

%% step response
figure
step(H)

%% Bode Plot
%% generate 100 point freq vector from 1Hz to 1kHz
f = logspace(1,3,100);
bode(H,f);
grid on
```

```
%% L03_ztransform
%% ================
%% Function to find z-transform using symbolic toolbox (maple).
%%
%% Created  : 15 July 2005
%% Modified : 15 July 2005
%%
%% Copyright (c) 2005 Salman Durrani .
%% -------------------------------------------------------------------

%% clear matlab memory
clear all;
clc

syms f w c n z

%% Take z transform

%% Unit step u[n]
f = heaviside(n);
Ans1=maple('ztrans',f,n,z)

%% Discrete Exponential c^n u[n]
%% Folowing two expressions give the same answer
f = (c^n);
Ans2=maple('ztrans',f,n,z)

f= (c^n)*heaviside(n);
Ans3=maple('ztrans',f,n,z)

%% Discrete Sine wave
f= sin(w*n);
Ans4=maple('ztrans',f,n,z)
pretty(Ans4)
```

```
%% L04_Invztransform
%% ================
%% Script to find inverse z-transform using symbolic toolbox (maple).
%% Refer to: Lecture04, Example 1, slide 16-18
%%
%% Created  : 27 July 2005
%% Modified : 27 July 2005
%%
%% Copyright (c) 2005 Salman Durrani.
%% -----------------------------------------------------------------

%% clear matlab memory
clear all;
clc

syms w a c n z

%% take inverse z transform
H = (z+ 0.4)/((z-0.2)*(z-0.5));
Ans8 = maple('invztrans',H,z,n)
pretty(Ans8)

%% plot h[n]
n=[0:1:10];
h = -10.*((1/5).^n) + 6.*((1/2).^n);
h(1) = h(1)+4;
stem(n,h,'filled')
xlabel('Sample n')
ylabel('Amplitude')
```

```
%% L05_Hz
%% ================
%% Script to analyse tranfer function in matlab.
%% See Problem Set 04: Q1 (a)
%%
%% Created  : 31 July 2005
%% Modified : 31 July 2005
%%
%% Copyright (c) 2005 Salman Durrani .
%% ---------------------------------------------------------------------

clc
clear all

%% Specify transfer function H(z) coefficients in standard DSP form
num=[1 1/3];
den=[1 -1/2];

%% pole zero map
figure
zplane(num,den)

%% impulse response h[n]
figure
impz(num,den)

%% step response
figure
stepz(num,den)
```

```
%% L06_FIR
%% ==============
%% Script to analyze FIR filters
%%
%% Created  : 28 July 2005
%% Modified : 28 July 2005
%%
%% Copyright (c) 2005 Salman Durrani .
%% ------------------------------------------------------------

clc
clear all

%% FIR Filter
%% change the vlaue of b1 to see effect on FIR filter
b1=0.5;
num=[1 b1];
den=[1];

%% pole-zero plot
figure
zplane(num,den)

%% impulse response
figure
impz(num,den)
grid on

%% step response
figure
stepz(num,den)
grid on
```

```
%% L06_IIR
%% ================
%% Script to analyze IIR filters
%%
%% Created  : 28 July 2005
%% Modified : 28 July 2005
%%
%% Copyright (c) 2005 Salman Durrani .
%% ---------------------------------------------------------------------

clc
clear all

%% IIR Filter
%% change the vlaue of a1 to see effect on IIR filter stability
a1=0.5;
num=[1];
den=[1 a1];

%% pole-zero plot
figure
zplane(num,den)

%% impulse response
figure
impz(num,den)
grid on

%% step response
figure
stepz(num,den)
grid on
```

```matlab
%% L07_DTFT
%% ===============
%% Script to illustrate DTFT plots: See lecture07, DTFT Example 2
%%
%% Created  : 07 Aug 2005
%% Modified : 07 Aug 2005
%%
%% Copyright (c) 2005 Salman Durrani.
%% ------------------------------------------------------------

clc
clear all
close all

%% Define transfer function
%% h[n]=a^n u[n] , H(z)=z/(z-a)
a=0.5;
num=[1];
den=[1 -a];

%% Define the frequency vector
w = [-pi:0.1:pi];
%% Calculate frequency response
[H, W]= freqz(num,den,w);

%% Magnitude response
figure
plot(W,abs(H))
axis([-pi pi -Inf Inf])
%% Phase response
figure
plot(W,angle(H)*180/pi)
axis([-pi pi -180 180])

%% Bode plot
freqz(num,den,w);

%% Alternative: use Filter Vizualization Tool
fvtool(num,den)
```

```
%% L09_Recursion
%% ==============
%% Script to illustrate Recursive (IIR) filter implementation.
%% See Lecture09
%%
%% Created  : 15 Aug 2005
%% Modified : 15 Aug 2005
%%
%% Copyright (c) 2005 Salman Durrani .
%% --------------------------------------------------------------

clc
clear all

%% Sampling frequency
Fs = 500;

%% Time vector of 0.5second
t = 0:1/Fs:0.5;

%% Create a sine wave of 10 Hz corrupted by sine wave of 100 Hz
x = sin(2*pi*t*10) +sin(2*pi*t*100);
L=length(x);

%% Recursive/IIR filter (low pass) fc = 0.05*Fs/2 = 12.5 Hz
a0=0.15;
b1=0.85;
y(1)=0;

for k=2:L
    y(k)=a0*x(k)+b1*y(k-1);
end

%% Generate the plot, title and labels.
figure
plot(t,sin(2*pi*t*10),'b');
hold on
plot(t,sin(2*pi*t*100),'g')
xlabel('Time (s)');
ylabel('Amplitude')
legend('10 Hz','100 Hz')
axis([0 0.5 -2 2])

figure
plot(t,x,'k');
hold on
plot(t,y,'r','Linewidth',2)
xlabel('Time (s)');
```

```
ylabel('Amplitude')
legend('Filter Input','Filter Output')
axis([0 0.5 -2 2])

figure
plot(t,sin(2*pi*t*10),'b');
hold on
plot(t,y,'r','Linewidth',2)
xlabel('Time (s)');
ylabel('Amplitude')
legend('10 Hz','Filter Output')
axis([0 0.5 -2 2])

%% Normalised Frequency Response
figure
num=[a0];
den=[1 -b1];
freqz(num,den)

%% Scaled Frequency Response of Filter using knowledge of sampling
%% frequency
figure
num=[a0];
den=[1 -b1];
freqz(num,den,512,Fs)
```

```
%% L10_DFT
%% ==========================
%% Script to illustrate how to:-
%% (i) calculate DFT and
%% (ii)investigate relationship between DFT and DTFT.
%%
%% Created  : 17 Aug 2005
%% Modified : 17 Aug 2005
%%
%% Copyright (c) 2005 Salman Durrani.
%% ---------------------------------------------------------------------------

clc
clear all
close all

n=[0 1 2 3];
x=[0 1 0 0];

%% Find FFT
N=4;
X = fft(x);

%% Find DTFT
num=[0 1];
den=1;
w = [0:0.01:2*pi];
[H,W]=freqz(num,den,w);

%% Plot Mags (without scaling x axis)
subplot(2,2,1)
stem(n,abs(X),'filled')
hold on
plot(W,abs(H),'k-')

xlabel('Sample k')
ylabel('|X[k]|')
legend('DFT Magnitude','DTFT Magnitude')

%% Plot Phases (without scaling x axis)
subplot(2,2,3)
stem(n,angle(X)*180/pi,'filled')
hold on
plot(W,angle(H)*180/pi,'k-')

xlabel('Sample k')
ylabel('< X[k] (degs)')
legend('DFT Phase','DTFT Phase')
```

```
%% Plot Mags (scaling x axis)
subplot(2,2,2)
stem(n,abs(X),'filled')
hold on
plot(W*N/(2*pi),abs(H),'k-')

xlabel('Sample k')
ylabel('|X[k]|')
legend('DFT Magnitude','DTFT Magnitude')

%% Plot Phases (scaling x axis)
subplot(2,2,4)
stem(n,angle(X)*180/pi,'filled')
hold on
plot(W*N/(2*pi),angle(H)*180/pi,'k-')

xlabel('Sample k')
ylabel('< X[k] (degs)')
legend('DFT Phase','DTFT Phase')
```

```
%% L11_Windowing
%% ====================
%% Script to illustrate application of windows to reduce
%% effect of frequency leakage in spectrum.
%%
%% Created  : 19 Aug 2005
%% Modified : 19 Aug 2005
%%
%% Copyright (c) 2005 Salman Durrani.
%% ---------------------------------------------------------------

clc
clear all
close all

%% N-point DFT: Change value of N here
N=64;

%% periodic signal
n = [0:1:N-1];
x = cos(2*pi.*n./6);

%% Hamming window
w = 0.5.*(1- cos(2*pi.*n./N));

%% modified periodic signal
x_mod = x.*w;

%% Calculate DFT
X = fft(x_mod,N);
magX =  abs(X)

%% Plot w[n]
figure
stem(n,w,'filled')
xlabel('Sample n')
ylabel('w[n]')
title('Hamming Window')

%% Plot x[n] and x_mod[n]
figure
stem(n,x,'filled')
xlabel('Sample n')
ylabel('x[n]')
title('Original Signal: Fundamental Period')

figure
stem(n,x_mod,'filled')
```

```
xlabel('Sample n')
ylabel('x[n]')
title('Windowed Signal: Fundamental Period')

%% Plot DFT magnitude
figure
stem(n,magX,'filled')
xlabel('Sample k')
ylabel('|X[k]|')
title('DFT Magnitude')
```

```
function PlotFFT(x, Fs);
%% PlotFFT
%% ==============
%% Function to plot (scaled) fft magnitude of a signal.
%%
%% Code fragment adapted from:-
%% http://www.mathworks.com/support/tech-notes/1700/1702.html
%%
%% x is input signal
%% Fs is sampling frequency
%%
%% Created  : 23 Aug 2005
%% Modified : 23 Aug 2005
%%
%% 2005 Salman Durrani .
%% ------------------------------------------------------------------------

Fn=Fs/2;
NFFT=2.^(ceil(log(length(x))/log(2)));
FFTX=fft(x,NFFT);
NumUniquePts = ceil((NFFT+1)/2);
FFTX=FFTX(1:NumUniquePts);
MX=abs(FFTX);
MX=MX*2;
MX(1)=MX(1)/2;
MX(length(MX))=MX(length(MX))/2;
MX=MX/length(x);
f=(0:NumUniquePts-1)*2*Fn/NFFT;
plot(f,MX);
xlabel('Frequency (Hz)')
ylabel('Magnitude')
```

```
%% L12_Audio_Filtering
%% ======================
%% Script to illustrate application of Butterworth filters (lowpass,
%% highpass, bandpass and bandstop) to a digital audio signal.
%%
%% Created  : 24 Aug 2005
%% Modified : 24 Aug 2005
%%
%% Copyright (c) 2005 Salman Durrani.
%% ------------------------------------------------------------------

clc
clear all
close all

%% wav file
[Ysignal,Fs]=wavread('speech_dft');
% [Ysignal,Fs]=wavread('wchimes30');

%% Chip.mat is available inside Matlab
%% Fs=8192 Hz
% load chirp
% Ysignal=y;

%% Nyquist Freq
Fn = Fs/2;
%% Filter order = 2*n
n=8;

%% Low pass filter
Fc=1000;
[num,den] = butter(n,Fc/Fn,'low');

%% High pass filter
% Fc=1000;
% [num,den] = butter(n,Fc/Fn,'high');

%% Bandstop filter
% Fc= [300 1500];
% [num,den] = butter(2,Fc./Fn,'stop');

%% Bandpass filter
% Fc= [3000 3200];
% [num,den] = butter(n,Fc./Fn);

%% Analyse filter
fvtool(num,den)
```

```
%% Apply the filter
Yfiltered = filter(num,den, Ysignal);

%% Compare time domain signal
figure
subplot(2,1,1)
plot(Ysignal);
title('Original␣Signal')
xlabel('Time␣Sample␣n')
ylabel('Amplitude')
subplot(2,1,2)
plot(Yfiltered);
title('Filtered␣Signal')
xlabel('Time␣Sample␣n')
ylabel('Amplitude')

%% Compare spectrum
figure
subplot(2,1,1)
PlotFFT(Ysignal,Fs);
title('Original␣Signal')

subplot(2,1,2)
PlotFFT(Yfiltered,Fs);
title('Filtered␣Signal')

%% Play sound
wavplay(Ysignal,Fs)
pause(2);
wavplay(Yfiltered,Fs)
```

```
%% L14_Filter_Structure
%% ==================================
%% Script to illustrate and compare filter implementation with
%% Matlab and Simulink.
%%
%% Compare with output from demo Simulink file "fxpdemo_direct_form2",
%% which shows the direct-form II implementation.
%%
%% Created  : 29 Aug 2005
%% Modified : 29 Aug 2005
%%
%% Copyright (c) 2005 Salman Durrani .
%% --------------------------------------------------------------------------

clc
clear all
close all

%% Filter Coeffs (See Lecture 14)
num=[1 2.2 1.85 0.5];
den=[1 -0.5 0.84 0.09];

%% Square wave frequency (Hz)
f = 0.005;

%% Time vector
t = [0:1:200];

%% Create a square wave of unit amplitude and frequency f
x = -1.*square(2*pi*f*t);

%% Filtered signal
y=filter(num,den,x);

%% Plot results
figure
plot(t,x,'k')
hold on
plot(t,y,'b')
xlabel('Time (s)');
ylabel('Amplitude')
axis([0 Inf -8 10])
legend('input','output')
grid on
```

```
%% L14_SOS
%% ========
%% Script to illustrate second order sections, which can be imported in
%% fdatool.
%%
%% Created  : 31 Aug 2005
%% Modified : 31 Aug 2005
%%
%% Copyright (c) 2005 Salman Durrani.
%% ---------------------------------------------------------------

clc
clear all
close all

%% Define filter coefficients [See DSP Guide (pg 336)]
%% Low pass Chebyshev filter (0.5% passband ripple: type I)
%% fc = 0.5
b0 = 2.858110e-01;
b1 = 5.716221e-01;
b2 = 2.858110e-01;
a1 = 5.423258e-02;
a2 = -1.974768e-01;

%% H(z)
num=[b0 b1 b2];
den=[1 -a1 -a2];

%% Analyse filter
fvtool(num,den)
% axis([0 1 -40 0])

%% Create two second order sections
SOS = [b0 b1 b2 1 -a1 -a2
       b0 b1 b2 1 -a1 -a2];

%% Import into fdatool using "Import Filter From Workspace"
fdatool
```

Bibliography

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