Connectivity of Three Dimensional Wireless Sensor Networks Using Geometrical Probability

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***Abstract*—This paper investigates the connectivity properties of three dimensional wireless sensor network in which N nodes are independently and uniformly (i.u.d.) distributed either on the surface of a sphere of radius R or inside the volume of a ball of radius R. Our approach utilizes the geometrical probability results for the conditional probability that a random node falls inside a ball centered at an arbitrary sensor node. We obtain exact expressions for the mean node degree and node isolation probability for the two topologies, which can be easily evaluated analytically or numerically. We also illustrate an upper bound for the probability of connectivity. The analysis is validated by comparison with existing results and Monte Carlo simulations.**

***Index Terms*—Wireless sensor networks, connectivity, node degree, node isolation probability, sphere, ball.**

I. Introduction

Recently, there has been a growing interest in three dimensional wireless sensor networks [1], with wide-ranging potential applications including underwater wireless sensor networks [2], space exploration [3], environmental data collection [4], pollution monitoring [5] and tactical surveillance [6]. In modelling three dimensional wireless sensor networks, nodes are assumed to be either randomly distributed on the surface of the sphere or randomly distributed inside the volume of a ball or cube. The sphere and ball topology is relevant for space exploration and planetary monitoring applications where nodes are located on the surface of the planet or in the sky (i.e., inside the sphere). The cube topology is relevant for underwater wireless sensor networks, building networks where nodes are located on different floors and forestry networks with nodes deployed on trees of different heights [7].

The performance of three dimensional wireless sensor networks, assuming sphere or ball topology, has been investigated from different perspectives. In their seminal work, Gupta and Kumar analysed the capacity of three dimensional networks on the sphere [8] and ball [1]. Small world networks on the sphere are investigated in [9]. The optimal deployment of nodes (i.e. topology control) [10, 11] and routing [12] in three dimensions have also been investigated.

The connectivity of three dimensional wireless sensor networks is also an important research problem [13]. Connectivity is a fundamental requirement in any wireless sensor network. A network is said to be connected if there exists a path between any pairs of nodes in the network. Connectivity properties such as mean node degree (average number of neighbours of a node) and node isolation probability (probability that a node has no neighbours) play a vital role in characterizing overall network connectivity. The critical transmission range for connectivity in three dimensional wireless sensor networks is investigated in [7, 14] using percolation theory. Most of the existing connectivity results in the literature consider two dimensional wireless sensor networks [13, 15–18]. Note that is not straight forward to extend these results to finite three dimensional wireless sensor networks. This is due to the border effects (i.e., the reduction in the coverage area for nodes located close to the physical boundaries of the network), which are complicated to characterise in three dimensions [19]. While border effects are absent in the sphere due to its closed geometry, they play a significant role in determining the overall connectivity properties in the ball.

In this paper, we obtain exact expressions for the mean node degree and node isolation probability when N nodes are independently and uniformly (i.u.d.) distributed either on the surface of a sphere of radius R or inside the volume of a ball of radius R. To account for border effects in a tractable manner, we adopt the framework of geometrical probability [20, 21]. The specific contributions of this paper are:

- We present a general formulation to obtain the connectivity properties of three dimensional wireless sensor networks. The framework is based on the conditional probability that a random node falls inside a ball centered at an arbitrary sensor node.
- We obtain exact expressions for the mean node degree and node isolation probability for finite wireless sensor networks on the sphere and ball, which can be easily evaluated analytically or numerically. Using the derived probability of node isolation results, we also illustrate upper bounds for the probability of connectivity. The analysis is validated by comparison with existing results in the literature and Monte Carlo simulations.

II. System Model and Performance Metrics

Let $\mathbf{u} \equiv \mathbf{u}(r,\theta,\phi) \triangleq [r\sin\theta\cos\phi,\, r\sin\theta\sin\phi,\, r\cos\theta] \in \mathcal{R} \subset \mathbb{R}^3$ denote the position of an arbitrary node in a finite region $\mathcal{R} \subset \mathbb{R}^3$, where $\mathbb{R}^3$ denotes the three dimensional Euclidean domain, $r = \|\mathbf{u}\| \in [0,\infty)$ denotes the radius and $\|\cdot\|$ denotes the Euclidean norm, $\theta \in [0,\pi]$ denotes the co-latitude measured with respect to the positive $z-$axis ($\theta = 0$ corresponds to the north pole) and $\phi \in [0, 2\pi)$ denotes the longitude measured with respect to the positive $x-$axis in the $x-y$ plane. We assume that the region $\mathcal{R}$ is either a sphere $\{\mathbf{u} \in \mathbb{R}^3 : \|\mathbf{u}\| = R\}$ or a ball $\{\mathbf{u} \in \mathbb{R}^3 : \|\mathbf{u}\| \leq R\}$, where

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R denotes the radius of the sphere or ball. We consider N nodes, which are independently and uniformly distributed in $\mathcal{R}$, according to probability density function (PDF) $f_{\mathbf{U}}(\mathbf{u})$. Each node has a fixed wireless transmission range r_{o}.

Let the cumulative density function (CDF) $F(\mathbf{u}; r_{\mathrm{o}})$ denotes the conditional probability that a random node falls inside a ball $\mathcal{B}(\mathbf{u}; r_{\mathrm{o}})$ of radius r_{o} centered at $\mathbf{u}$. Also let the CDF

$$F(r_{\mathrm{o}}) = \int_{\mathcal{R}} F(\mathbf{u}; r_{\mathrm{o}}) f_{\mathbf{U}}(\mathbf{u}) ds(\mathbf{u}), \tag{1}$$

denotes the probability that two nodes i.u.d. in $\mathcal{R}$ are separated by a distance less then or equal to r_{o}, where $ds(\mathbf{u}) = r^2 \sin\theta d\theta d\phi dr$ for ball and $ds(\mathbf{u}) = \sin\theta d\theta d\phi$ for sphere.

The probability of node isolation $P_{\text{iso}}(r_{\mathrm{o}})$ is defined as the average probability that a randomly selected node has no neighbours. Assuming the disc transmission model [1], an arbitrary node will be isolated if there is no node located inside $\mathcal{B}(\mathbf{u}; r_{\mathrm{o}})$. Thus $(1 - F(\mathbf{u}; r_{\mathrm{o}}))$ is the conditional probability of node isolation and the average can be calculated by weighting with the PDF of the distribution of nodes and averaging over all possible locations of the arbitrary node. Hence, $P_{\text{iso}}(r_{\mathrm{o}})$ can be expressed as

$$P_{\text{iso}}(r_{\mathrm{o}}) = \int_{\mathcal{R}} (1 - F(\mathbf{u}; r_{\mathrm{o}}))^{N-1} f_{\mathbf{U}}(\mathbf{u})\, ds(\mathbf{u}), \tag{2}$$

where we have assumed that the probability of node isolation is independent for each node. The use of the CDF $F(\mathbf{u}; r_{\mathrm{o}})$ in (2), for any given range r_{o}, automatically captures the border effects for the calculation of the probability of node isolation.

The probability of overall network connectivity $P_{\text{con}}(r_{\mathrm{o}})$ is defined as the probability that every node pair in the network has at least one path connecting them. Exact expressions for $P_{\text{con}}(r_{\mathrm{o}})$ have been obtained in the literature for simple topologies such as line [22] and circle [23]. Using a cluster expansion approach, [19] recently obtained bounds for two and three dimensional topologies with high node densities. For asymptotically large networks, it is known that we obtain a connected network at the same time as when we obtain a network with no isolated nodes [17]. This property holds with and without border effects [15]. Hence, we use the probability of no isolated node as an upper bound for $P_{\text{con}}(r_{\mathrm{o}})$,

$$P_{\text{con}}(r_{\mathrm{o}}) \le P_{\text{no-iso}} = (1 - P_{\text{iso}}(r_{\mathrm{o}}))^N, \tag{3}$$

where $P_{\text{no-iso}}$ denotes the probability that there is no isolated node in the network and $P_{\text{iso}}(r_{\mathrm{o}})$ is calculated using (2). (3) is asymptotically tight as $P_{\text{con}}(r_{\mathrm{o}}) \approx 1$ [15].

The mean node degree D is the average number of neighbours of a node. For finite number of nodes i.u.d. in a finite region, the mean node degree obeys the Binomial distribution with parameters $N-1$ and $F(r_{\mathrm{o}})$ [15, 24]. Using this result, the mean node degree is given by

$$D = (N-1) F(r_{\mathrm{o}}). \tag{4}$$

The use of the CDF $F(r_{\mathrm{o}})$ in (4), for any given range r_{o}, automatically captures the border effects for the calculation of the mean node degree.

III. Connectivity of wireless sensor networks in a ball

For a ball, all nodes located at least r_{o} away from the border do not experience any border effects while nodes located closer than r_{o} to the border experience border effects. The PDF $f_{\mathbf{U}}(\mathbf{u})$ is given by

$$f_{\mathbf{U}}(\mathbf{u}) = \begin{cases} \frac{1}{\frac{4}{3}\pi R^3}, & \{\mathbf{u} \in \mathbb{R}^3 : \|\mathbf{u}\| \le R\} \\ 0, & \text{elsewhere} \end{cases} \tag{5}$$

Using the result for the sphere-sphere intersection in [25] the CDF $F(\mathbf{u}; r_{\mathrm{o}})$, which captures the border effects, can be expressed as

$$F(\mathbf{u}; r_{\mathrm{o}}) = \frac{1}{\frac{4}{3}\pi R^3} \begin{cases} \frac{4}{3}\pi r_{\mathrm{o}}^3, & 0 \le r \le |R - r_{\mathrm{o}}|, 0 \le r_{\mathrm{o}} < R \\ \frac{\zeta(r)}{12r}, & |R - r_{\mathrm{o}}| \le r \le R, 0 \le r_{\mathrm{o}} < R \\ \frac{4}{3}\pi R^3, & 0 \le r \le |R - r_{\mathrm{o}}|, R \le r_{\mathrm{o}} \le 2R \\ \frac{\zeta(r)}{12r}, & |R - r_{\mathrm{o}}| \le r \le R, R \le r_{\mathrm{o}} < 2R \\ \frac{4}{3}\pi R^3, & 0 \le r \le R, 2R \le r_{\mathrm{o}} \end{cases} \tag{6}$$

where $\zeta(r) = \pi(R + r_{\mathrm{o}} - r)^2 (r^2 + 2r_{\mathrm{o}} r - 3r_{\mathrm{o}}^2 + 2rR + 6r_{\mathrm{o}} R - 3R^2)$ and $|\cdot|$ denotes the absolute value. Substituting (6) in (2) and simplifying, $P_{\text{iso}}(r_{\mathrm{o}})$ can be expressed as

$$P_{\text{iso}}(r_{\mathrm{o}}) = \begin{cases} \frac{3}{R^3} \int_0^{|R-r_{\mathrm{o}}|} \left(1 - \frac{r_{\mathrm{o}}^3}{R^3}\right)^{N-1} r^2 dr \\ \quad + \frac{3}{R^3} \int_{|R-r_{\mathrm{o}}|}^{R} \left(1 - \frac{\frac{\zeta(r)}{12r}}{\frac{1}{\frac{4}{3}\pi R^3}}\right)^{N-1} r^2 dr, & 0 \le r_{\mathrm{o}} < R \\ \frac{3}{R^3} \int_{|R-r_{\mathrm{o}}|}^{R} \left(1 - \frac{\frac{\zeta(r)}{12r}}{\frac{1}{\frac{4}{3}\pi R^3}}\right)^{N-1} r^2 dr, & R \le r_{\mathrm{o}} < 2R \\ 0 & 2R \le r_{\mathrm{o}}, \end{cases} \tag{7}$$

where we have used the result $\int_0^{2\pi} \int_0^{\pi} \frac{1}{\frac{4}{3}\pi R^3} \sin\theta d\theta d\phi = 3/R^3$ in simplification. (7) does not have a closed form result, due to the $N-1$ terms in the exponent. However, it can be easily and accurately evaluated numerically. Note that numerical calculation of integrals is widely practiced in the literature [15]. Substituting (7) in (3) allows the probability of connectivity to be numerically evaluated.

Based on (6), the corresponding CDF $F(r_{\mathrm{o}})$ can be expressed using (1) as

$$F(r_{\mathrm{o}}) = \frac{3}{R^3} \begin{cases} \int_0^{|R-r_{\mathrm{o}}|} \frac{r_{\mathrm{o}}^3}{R^3} r^2 dr + \int_{|R-r_{\mathrm{o}}|}^{R} \frac{\frac{\zeta(r)}{12r}}{\frac{1}{\frac{4}{3}\pi R^3}} r^2 dr, & 0 \le r_{\mathrm{o}} < R \\ \int_0^{|R-r_{\mathrm{o}}|} r^2 dr + \int_{|R-r_{\mathrm{o}}|}^{R} \frac{\frac{\zeta(r)}{12r}}{\frac{1}{\frac{4}{3}\pi R^3}} r^2 dr, & R \le r_{\mathrm{o}} < 2R \\ \frac{R^3}{3}, & 2R \le r_{\mathrm{o}}, \end{cases} \tag{8}$$

which simplifies to

$$F(r_o) = \begin{cases} \frac{r_o^6}{32R^6} - \frac{9}{16}\frac{r_o^4}{R^4} + \frac{r_o^3}{R^3}, & 0 \le r_o < 2R \\ 1, & 2R \le r_o. \end{cases} \quad (9)$$

Note that in the simplification of the expression for $F(r_o)$ in (8), the integral for the first two ranges of r_o evaluates to the same value. (8) is also derived in [21] using a different approach. Substituting (8) in (4), the mean node degree is given by

$$D = (N-1)\left(\frac{r_o^6}{32R^6} - \frac{9}{16}\frac{r_o^4}{R^4} + \frac{r_o^3}{R^3}\right). \quad (10)$$

IV. Connectivity of wireless sensor networks on a sphere

There are no border effects in a sphere due to its closed geometry. Without loss of generality, we can assume that the reference node is located at the north pole since this is always possible by reorienting the coordinate axes. The PDF $f_{\mathbf{U}}(\mathbf{u})$ is given by

$$f_{\mathbf{U}}(\mathbf{u}) = f(\theta,\phi) = \begin{cases} \left(\frac{1}{2\pi}\right)\left(\frac{1}{2}\sin\theta\right), & \{\mathbf{u} \in \mathbb{R}^3 : \|\mathbf{u}\| = R\} \\ 0, & \text{elsewhere} \end{cases} \quad (11)$$

where the $\sin\theta$ factor is due to the curvature of the sphere and is necessary to avoid bunching of the nodes at the poles.

For the sphere, it is convenient to express the distance distributions in terms of central angle distributions, since the distance between any two nodes on the sphere can be expressed in terms of the central angle Δ subtended by the smaller arc length of great circle joining them. Due to the symmetry and absence of any border effects, the CDF $F(\mathbf{u}; r_o)$ can be expressed as [26]

$$F(\mathbf{u}; r_o) \equiv F(r_o) = 1 - \cos^2\left(\frac{\Delta}{2}\right) = \left(\frac{r_o}{2R}\right)^2, \quad (12)$$

where $\Delta = 2\arcsin(\frac{r_o}{2R})$ and the simplification arises from the identity $\cos^2(\arcsin(a)) = 1 - a^2$. Substituting (12) in (2) and simplifying, we get

$$P_{\text{iso}}(r_o) = (1 - F(r_o))^{N-1}\left(\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin\theta d\theta d\phi\right) = \left[1 - \left(\frac{r_o}{2R}\right)^2\right]^{N-1} \quad (13)$$

Substituting (13) in (3), we obtain

$$P_{\text{con}}(r_o) \le P_{\text{no-iso}} = \left[1 - \left(1 - \left(\frac{r_o}{2R}\right)^2\right)^{N-1}\right]^N. \quad (14)$$

For the node degree, substituting (12) in (4) and simplifying, we obtain

$$D = (N-1)\left(\frac{r_o}{2R}\right)^2. \quad (15)$$

V. Results

In order to verify the analytical and numerical results, we compare with simulation results obtained by averaging over 5000 Monte Carlo simulation runs.

A. Probability of node isolation

Figs. 1 and 2 plot the probability of node isolation for ball and sphere as a function of normalized transmission range r_o/R for $N = 10, 50$ nodes respectively. For ball, the probability of node isolation is also plotted assuming nodes are distributed according to an infinite homogeneous Poisson process in $\mathbb{R}^3$ with corresponding node density $\rho = \frac{N}{\frac{4}{3}\pi R^3}$ (i.e. without border effects). The figures show that there is complete agreement between the theoretical and simulation results, which verifies (7) and (13). For ball, the gap between the curves for with and without border effects becomes larger as N decreases. This highlights the importance of characterising border effects in analysing the connectivity of three dimensional wireless sensor networks with finite number of nodes.

B. Probability of connectivity

Figs. 3 and 4 plot the probability of connectivity for ball and sphere as a function of normalized transmission range r_o/R for $N = 10, 50, 100$ nodes respectively. The theoretical bound on $P_{\text{con}}(r_o)$ given in (3) is plotted using (7) and (3) for ball and (14) for sphere. For sphere, we also plot the high density approximation for $P_{\text{con}}(r_o)$ presented in [19] as

$$P_{\text{con}}(r_o) \approx 1 - Ne^{-\rho\frac{\pi}{\beta}} - \frac{N}{\pi}e^{-\rho\frac{3\pi}{2\beta}}, \quad (16)$$

where ρ denotes the density and $\beta = (r_o/R)^{-2}$.

Comparing with simulation results, we can see that the results in this paper provide an upper bound for $P_{\text{con}}(r_o)$ which gets tight if $P_{\text{con}}(r_o) > 0.95$. This is important since it is often of interest to predict the critical range r_o which corresponds to a probability of connectivity of 95% or 99% [13]. For the sphere, we can see that the approximation in (16) acts as a lower bound, which is quite loose at low number of nodes but gets tighter for higher number of nodes.

C. Critical transmission range

We have seen in the illustration in the preceding subsection that the bound on the probability of connectivity given by (3) gets tighter as $P_{\text{con}}(r_o)$ approaches 1. Therefore, the bound on $P_{\text{con}}(r_o)$ given in (3) can be used to predict the critical transmission range which ensures connectivity near 1. Here, we analyse the accuracy of the analytical prediction. Using (14) or (3) in conjunction with (7), we analytically predict the critical value of normalized transmission range r_o/R which ensures connectivity of 99%. The predicted transmission range and the simulated transmission range for a 99% connected network is plotted for different number of nodes uniformly distributed inside a ball. The simulated transmission range which ensures no isolated node in the network is also plotted to support the analytical prediction. The transmission range

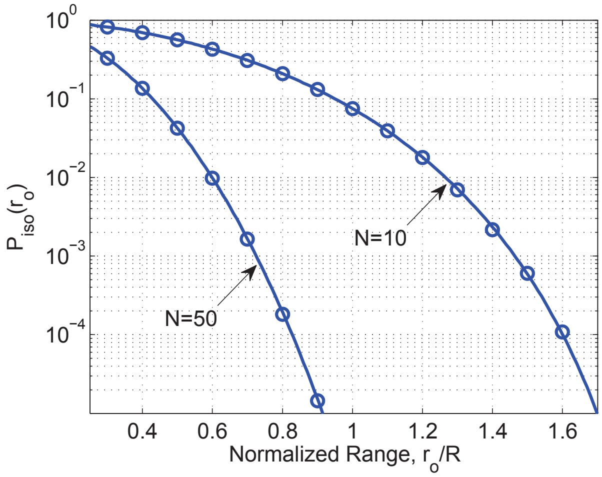

Fig. 1. Probability of node isolation $P_{\text{iso}}(r_{\text{o}})$ versus normalized transmission range r_{o}/R for $N = 10, 50$ nodes i.u.d. on the sphere. The lines denote theoretical results and markers indicate simulation results.

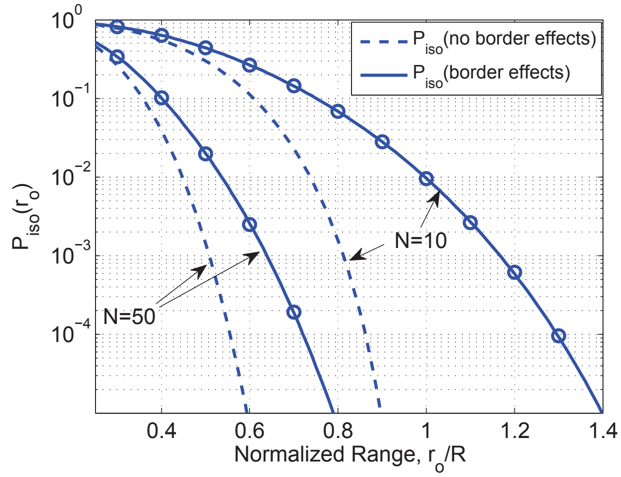

Fig. 2. Probability of node isolation $P_{\text{iso}}(r_{\text{o}})$ versus normalized transmission range r_{o}/R for $N = 10, 50$ nodes i.u.d. in the ball. The lines (solid and dash) denote theoretical results and markers indicate simulation results.

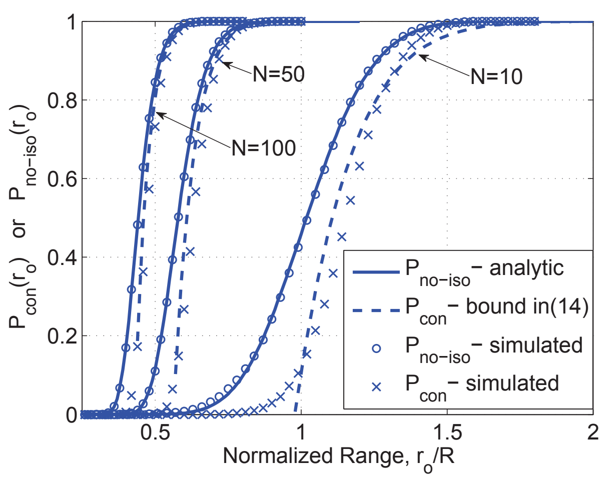

Fig. 3. Probability of connectivity $P_{\text{con}}(r_{\text{o}})$, probability of no-isolated node $P_{\text{no-iso}}$ and the bound on $P_{\text{con}}(r_{\text{o}})$ given in (16) are plotted versus normalized transmission range r_{o}/R for $N = 10, 50, 100$ nodes i.u.d. on the sphere. The lines denote theoretical results and markers indicate simulation results.

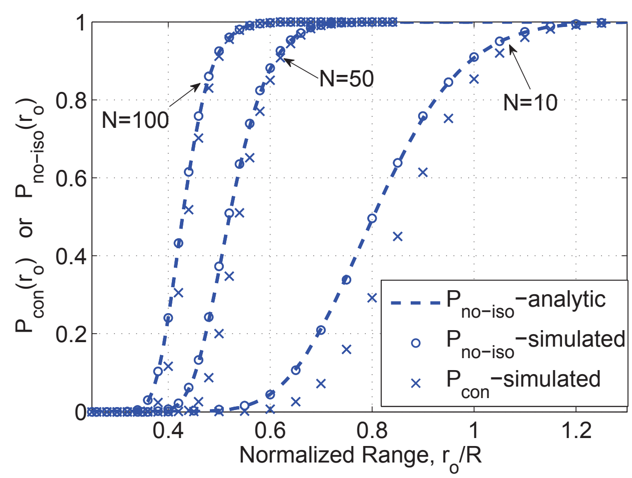

Fig. 4. Probability of connectivity $P_{\text{con}}(r_{\text{o}})$ and probability of no-isolated node $P_{\text{no-iso}}$ versus normalized transmission range r_{o}/R for $N = 10, 50, 100$ nodes i.u.d. in the ball. The lines denote theoretical results and markers indicate simulation results.

predicted for nodes distributed according to homogeneous Poisson process is also plotted, which does not provide a good approximation of the transmission range to ensure the desired connectivity level. However, our predicted value of transmission yields fairly good approximation. This is due to the fact that we have taken border effects into account in the evaluation of probability of isolation.

D. Mean node degree

Fig. 6 plots the mean node degree for ball and sphere as a function of normalized transmission range r_{o}/R for $N = 10, 50, 100$ nodes respectively. The analytical results match perfectly with the simulation results which verifies (10) and (15). We can see that for the same normalized transmission range r_{o}/R, the mean node degree is higher for a ball than a sphere, which follows from (10) and (15).

E. Connectivity of wireless sensor networks in a cube

The framework presented in this paper is also applicable to a cube. For a cube, result for $F(r)$ is known in the literature [21]. The result for $F(\mathbf{u}; r_{\text{o}})$ for cube may perhaps be known, though we have been unable to find it in the literature. Recently in [27], a framework was proposed for analytically calculating $F(\mathbf{u}; r_{\text{o}})$ in two-dimensional regular polygons. By extending the result for $F(\mathbf{u}; r_{\text{o}})$ for square (which involves square-disk interaction) it is possible to obtain the result for $F(\mathbf{u}; r_{\text{o}})$ for cube (which would involve cube-sphere interaction). The challenge then is to obtain $P_{\text{iso}}(r_{\text{o}})$ by averaging over the cube. This is an important open problem for future research work.

VI. Conclusions

In this paper, we have proposed a general framework to analyse the connectivity of finite three dimensional wireless

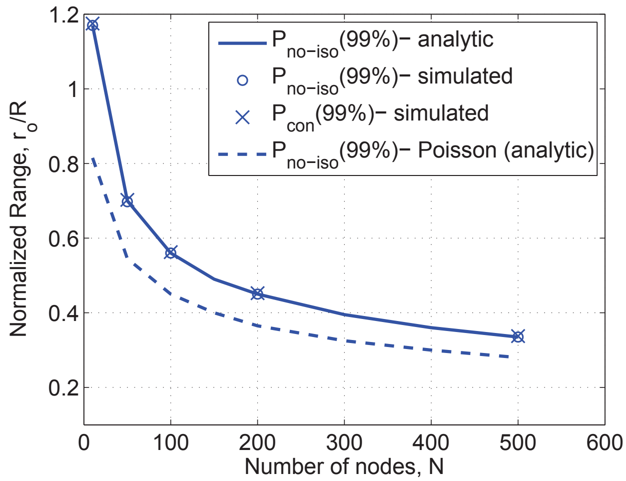

Fig. 5. Normalized range r_o/R versus number of nodes i.u.d. in the ball. The normalized range is plotted for which probability of no-isolated node or probability of connectivity is 0.99.

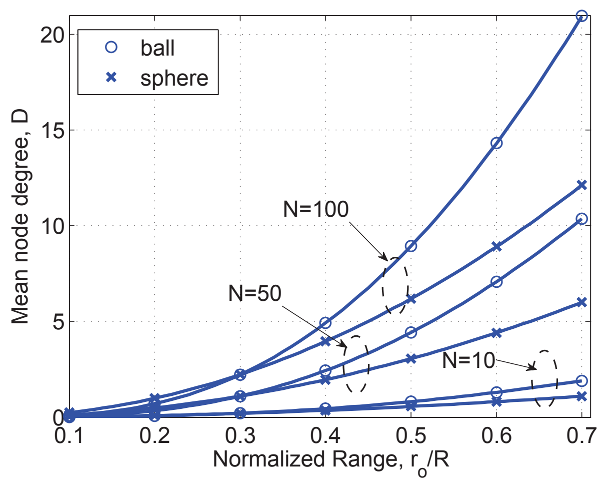

Fig. 6. Mean node degree D versus normalized transmission range r_o/R for $N = 10, 50, 100$ nodes i.u.d. on the sphere and ball respectively. The lines denote theoretical results and markers indicate simulation results.

sensor networks. We have obtained the expressions for probability of node isolation and mean node degree when N nodes are independently and uniformly (i.u.d.) distributed either on the surface of a sphere of radius R or inside the volume of a ball of radius R. The accuracy of the theoretical results has been verified with simulation results.

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