CONJUGATE GRADIENT ALGORITHM FOR EXTRAPOLATION OF SAMPLED BANDLIMITED SIGNALS ON THE 2-SPHERE

Zubair Khalid, Rodney A. Kennedy, Salman Durrani and Parastoo Sadeghi

Research School of Engineering, The Australian National University, Canberra, Australia.
Email: {zubair.khalid,rodney.kennedy, salman.durrani, parastoo.sadeghi }@anu.edu.au

ABSTRACT

In this paper, we consider the problem of signal extrapolation for discrete (i.e., sampled) signals on the sphere. We propose conjugate gradient based algorithm for estimating a signal on the sphere from limited or incomplete measurements in a spatial domain. We prove that the proposed algorithm is guaranteed to converge and show that it has faster convergence compared to the Papoulis algorithm. The results also show that the incomplete measurements distributed in different non-connected spatial regions yield better extrapolation results, compared to the connected region case.

Keywords: unit sphere, signal extrapolation, bandlimited signals, spherical harmonics.

1. INTRODUCTION

Signal extrapolation, i.e., finding an estimate of a signal outside its given observation interval, is an important problem in signal processing [1]. In this regard, because of the difficulty associated with practical implementation of analytic methods such as Slepian's prolate spheroidal wave functions [1], iterative algorithms are generally preferred. For time-frequency analysis, an iterative algorithm was first proposed by Papoulis [2] for continuous signals and later extended to discrete signals [1]. The Papoulis algorithm is based on the principle of reducing the mean-square error between the estimated and the original (limited or incomplete) signal at successive iterations.

Recently, the Papoulis algorithm was revisited for continuous signals on the sphere [3–5]. An analogue of the Papoulis algorithm for continuous signals on the sphere, using the bandlimiting characteristic of a given signal, is proposed in [3,4] and its integral equation formulation is provided in [5]. For discrete (i.e. sampled) signals on the sphere, an iterative gradient algorithm which converges to the minimum norm least square solution is proposed in [4, 6]. However, this algorithm has linear convergence rate and it updates the samples of the extrapolated signal over the complete spatial domain at every iteration which makes it very computationally intensive.

In this work, we consider the problem of signal extrapolation for discrete (i.e., sampled) signals on the sphere. We use the equiangular sampling on the sphere [7], which has the property that an exact quadrature rule can be applied, and formulate the matrix representation of the extrapolation problem. We first present a modified iterative gradient algorithm which only updates the samples of a signal which are known at each iteration, thus reducing the computational complexity compared to the algorithm in [4, 6]. This modified iterative gradient algorithm is used as a benchmark in this paper. We then propose conjugate gradient based algorithm and show that it has faster convergence compared to the modified iterative gradient algorithm. We also show that the extrapolation yields better results when incomplete measurements are distributed in different non-connected spatial regions as opposed to a connected region.

The rest of the paper is organized as follows. The signal model is explained in Section 2. The matrix problem formulation is discussed in Section 3. The modified iterative gradient algorithm and the proposed conjugate gradient algorithm are presented in Section 4. The algorithm performance is illustrated in Section 5. Finally Section 6 concludes the paper. Notation: $\overline{(\cdot)}$, $(\cdot)^T$ and $(\cdot)^H$ denotes the complex conjugate, transpose and Hermitian operations, respectively. Lowercase bold symbols correspond to vectors whereas uppercase bold symbols denote matrices.

2. MATHEMATICAL BACKGROUND

2.1. Signals on the Unit Sphere

Let $f(\theta,\phi)$ be a signal on the sphere, where $\theta \in [0,\pi]$ and $\phi \in [0,2\pi)$ denote the co-latitude and longitude, respectively. The spherical harmonics form basis functions on the sphere. Therefore, any signal $f(\theta,\phi)$ can be expanded as [8]

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\theta,\phi) \tag{1}$$

where f_ℓ^m are the spherical harmonic coefficients given by

$$f_\ell^m \triangleq \langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\theta,\phi)\overline{Y_\ell^m(\theta,\phi)}\, sin\theta d\theta d\phi \tag{2}$$

and the spherical harmonics, $Y_\ell^m(\theta,\phi)$, for degree $\ell \geq 0$ and order $|m| \leq \ell$ are defined as [8]

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)e^{im\phi} \tag{3}$$

where P_ℓ^m are the associated Legendre polynomials [8].

2.2. Discrete Signals on the Sphere

We consider discrete (i.e., sampled) signals on the sphere using equiangular sampling across both latitude and longitude [7]. For a discrete bandlimited signal with maximum spherical harmonics degree L, we consider N samples on the sphere consisting of $\sqrt{N}\times\sqrt{N}$ equiangular samples along θ and ϕ. We assume $\sqrt{N}$ is an integer and greater than $2(L+1)$ which is the equivalent of Nyquist-Shannon sampling limit for a bandlimited signal [7].

Let $\hat{\boldsymbol{x}} = [\hat{x}_1,\, \hat{x}_2,\, \cdots,\, \hat{x}_N]$ denote the samples on the sphere, which can be in any order. Each sample $\hat{x}_n = (\theta_j, \phi_k)$ with $\theta_j =$

This work was supported by the Australian Research Council Discovery Grant DP1094350.

$\pi j/\sqrt{N}$, $\phi_k = 2\pi k/\sqrt{N}$ for $1 \leq n \leq N$, $1 \leq j \leq \sqrt{N}$ and $1 \leq k \leq \sqrt{N}$ corresponds to a sample on the equiangular grid. We define the discrete signal $\mathbf{f}$ on the sphere as

$$\mathbf{f} \triangleq [f(\hat{x}_1),\, f(\hat{x}_2),\, \cdots,\, f(\hat{x}_N)]^T \tag{4}$$

and spherical harmonic vector $\mathbf{Y}_\ell^m$ sampled at $(\hat{\boldsymbol{x}})$ as

$$\mathbf{Y}_\ell^m = [Y_\ell^m(\hat{x}_1),\, Y_\ell^m(\hat{x}_2),\, \cdots,\, Y_\ell^m(\hat{x}_N)]^T \tag{5}$$

Since, we have considered the number of samples above the Nyquist-Shannon limit, we can exactly obtain the spherical harmonic coefficient f_ℓ^m in (2) as

$$f_\ell^m = (\mathbf{Y}_\ell^m)^H \,\mathbf{W}\,\mathbf{f} \tag{6}$$

where $\mathbf{W}$ is a diagonal matrix of size $N \times N$ which denotes the weight that must be multiplied with each sample to compensate for the dense sampling due to the use of equiangular sampling on the sphere. Note that the diagonal entries W_{nn} are independent of the longitude component ϕ_k of a sample $\hat{x}_n$ and depend only on the latitude component θ_j and the number of samples N. For $\sqrt{N}$ a power of 2, the entries W_{nn} are given by [7]

$$W_{nn} = \frac{2\sqrt{4\pi}}{N}\sin\theta_j \sum_{q=0}^{\sqrt{N}/2-1} \frac{1}{2q+1}\sin\left([2q+1]\theta_j\right) \tag{7}$$

where $j = 1, \ldots, \sqrt{N}$ and $\hat{x}_n = (\theta_j, \phi_k)$.

3. PROBLEM FORMULATION

In this section we present the matrix formulation of the signal extrapolation problem on the sphere. We also summarise the iterative gradient algorithm in [6].

Let $\mathbf{f}$ denote the bandlimited discrete signal on the sphere with the maximum spherical harmonic degree L, defined in (4) with N the number of samples. Let $\mathbf{g} = [g(\hat{x}_1),\, g(\hat{x}_2),\, \cdots,\, g(\hat{x}_R)]^T$ denote the given spatial-limited signal of R samples on the sphere such that

$$g(\hat{x}_u) \;=\; f(\hat{x}_u) \quad 1 \leq u \leq R. \tag{8}$$

The extrapolation objective is to find the $N - R$ remaining samples on the sphere using the bandlimited characteristic of the signal.

3.1. Matrix Operators for Spatial and Spectral Selection

Define the matrix operator $\mathbf{D} = \{D_{u,v}\}$ of size $R \times N$ given by

$$D_{u,v} = \begin{cases} 1 & ,\; 0 \leq u = v \leq R \\ 0 & ,\; \text{otherwise} \end{cases} \tag{9}$$

which selects the first R samples of a signal consisting of N samples. Note that the matrix operator $\mathbf{D}^T$ appends $N - R$ zeros to the R sample measured signal. Also define the matrix operator $\mathbf{B}_L$ of size $N \times N$ which bandlimits the signal within maximum spherical harmonics degree L as

$$\mathbf{B}_L \;=\; \mathbf{B}\mathbf{W} \tag{10}$$

where $\mathbf{W}$ is the diagonal matrix with entries defined in (7) which compensates for the dense sampling near the poles and $\mathbf{B} = \{B_{u,v}\}$ is a real symmetric matrix of size $N \times N$ [6] as

$$\mathbf{B} \;=\; \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} \mathbf{Y}_\ell^{\mathbf{m}}(\mathbf{Y}_l^m)^H \tag{11}$$

Note that, compared to the case of continuous signals on the sphere, the spectral selection operator $\mathbf{B}_L$ is idempotent and positive definite but non-symmetric for discrete signals because of the weights associated with the choice of equiangular tessellation.

Using the above operators, we can relate the bandlimited signal $\mathbf{f}$ and the known spatial-limited signal $\mathbf{g}$ as

$$\mathbf{g} = \mathbf{D}\mathbf{B_L}\mathbf{f} \tag{12}$$

3.2. Papoulis Algorithm in [4]

Using the matrix operators defined in the previous subsection, the iterative gradient algorithm for discrete signals on the sphere in [6] can be expressed as

$$\mathbf{f}_{d+1}^{+} \;=\; \mathbf{f}_d^{+} + \mathbf{B}_L\mathbf{D}^T\mathbf{g} - \mathbf{B}_L\mathbf{D}^T\mathbf{D}\mathbf{f}_d^{+}, \quad \mathbf{f}_0^{+} = 0 \tag{13}$$

where $\mathbf{f}_d^{+}$ denotes the extrapolated signal obtained after the dth iteration.

The above algorithm iteratively makes the signal more and more bandlimited such that it keeps the known signal as a part of the extrapolated signal. It updates the signal vector during each iteration thus it updates N samples at each iteration. Also it converges to the signal $\mathbf{f}$ in the minimum norm least-squares (MNLS) sense.

4. PROPOSED ALGORITHMS

4.1. Modified Iterative Gradient Algorithm

We first present a modified iterative gradient algorithm as a two step process: the first step is iterative which updates only R samples at each iteration and obtains the extrapolated non-bandlimited signal. The second step is spectral truncation of the extrapolated non-bandlimited signal to obtain the extrapolated signal of N samples. The algorithm is summarized in the Lemma below.

Lemma 1 (Modified Iterative Gradient Algorithm)**.** If $\mathbf{g}_d^{+}$ denote the spatial-limited signal of R samples which is updated at each iteration as

$$\mathbf{g}_{d+1}^{+} = \mathbf{g}_d^{+}\mathbf{g} - \mathbf{D}\mathbf{B}_L\mathbf{D}^T\mathbf{g}_d^{+}, \quad \mathbf{g}_0 = 0 \tag{14}$$

then the extrapolated signal $\mathbf{f}_d^{+}$ after the dth iteration in (13) can be obtained by bandlimiting the signal $\mathbf{g}_d^{+}$ in the spectral domain as

$$\mathbf{f}_d^{+} = \mathbf{B}_L\mathbf{D}^T\mathbf{g}_d^{+} \tag{15}$$

Proof. Multiplying both sides of (14) by $\mathbf{B}_L\mathbf{D}^T$ yields

$$\mathbf{B}_L\mathbf{D}^T\mathbf{g}_{d+1}^{+} = \mathbf{B}_L\mathbf{D}^T\mathbf{g}_d^{+} + \mathbf{B}_L\mathbf{D}^T\mathbf{g} - \mathbf{B}_L\mathbf{D}^T\mathbf{D}\mathbf{B}_L\mathbf{D}^T\mathbf{g}_d^{+} \tag{16}$$

which can be shown to be equivalent to the iterative algorithm in (13) using (15). □

Remark 1. Using the modified iterative gradient algorithm, we can obtain the extrapolated signal $\mathbf{f}_d^{+}$ by updating only R samples of a signal at each iteration instead of N samples as proposed in (13). Thus the computational complexity at each iteration is reduced.

4.2. Conjugate Gradient Algorithm

The iterative gradient algorithms in (13) and Lemma 1 have linear convergence rate and a large number of iterations are required in order to obtain an extrapolated signal close to the original signal in the MNLS sense [9]. In order to improve the convergence rate, we present conjugate gradient algorithm, which is summarized below.

Theorem 1 (Conjugate Gradient Algorithm)**.** *The spatial-limited signal $\mathbf{g}_d^+$ composed of R samples is updated at each iteration using the conjugate gradient algorithm as*

$$\mathbf{g}_{d+1}^+ = \mathbf{g}_d^+ + \alpha_d \mathbf{s}_d, \quad \mathbf{g}_0 = 0 \tag{17}$$

$$\mathbf{h}_d = \mathbf{D}\mathbf{B}_L\mathbf{D}^T\mathbf{g}_d^+ - \mathbf{g} \tag{18}$$

$$\beta_d = \mathbf{h}_{d+1}^T \mathbf{s}_d / (\mathbf{s}_d^T \mathbf{D}\mathbf{B}_L\mathbf{D}^T\mathbf{s}_d) \tag{19}$$

$$\mathbf{s}_{d+1} = -\mathbf{h}_{d+1} + \beta_d \mathbf{s}_d \tag{20}$$

$$\alpha_d = -\mathbf{h}_d^T \mathbf{s}_d / (\mathbf{s}_d^T \mathbf{D}\mathbf{B}_L\mathbf{D}^T\mathbf{s}_d) \tag{21}$$

where $\mathbf{h}_d$ and $\mathbf{s}_d$ are spatial-limited signals consisting of R samples for $d = 0, 1, \cdots, R-1$, with initial values $\mathbf{h}_0 = -\mathbf{g}$ and $\mathbf{s}_0 = \mathbf{g}$. The extrapolated signal $\mathbf{f}_d^+$ is obtained as

$$\mathbf{f}_d^+ = \mathbf{B}_L\mathbf{D}^T\mathbf{g}_d^+ \tag{22}$$

which converges to the signal $\mathbf{f}$ in MNLS sense in at most R iterations.

Proof. The iterative algorithm in (17)-(21) corresponds to the conjugate gradient algorithm [10] associated with the minimization problem

$$\min_{\mathbf{g}^+} \|(\mathbf{g}^+)^T\mathbf{D}\mathbf{B}_L\mathbf{D}^T\mathbf{g}^+ - (\mathbf{g}^+)^T\mathbf{g}\| \tag{23}$$

which has a unique solution $\mathbf{g}^+ = (\mathbf{D}\mathbf{B}_L\mathbf{D}^T)^{-1}\mathbf{g}$. This means that $\mathbf{g}_d^+$ converges to $\mathbf{g}^+$ in R iterations and the extrapolated signal in (22) converges to $\mathbf{B}_L\mathbf{D}^T(\mathbf{D}\mathbf{B}_L\mathbf{D}^T)^{-1}\mathbf{g}$ which is closest to the signal $\mathbf{f}$ in MNLS sense. □

Remark 2. While it is possible to find the unique MNLS solution $\mathbf{g}^+ = (\mathbf{D}\mathbf{B}_L\mathbf{D}^T)^{-1}\mathbf{g}$ directly, it is not feasible practically because $\mathbf{D}\mathbf{B}_L\mathbf{D}^T$ becomes increasingly ill-conditioned as R increases, despite the fact that $\mathbf{B}_L$ is positive definite. Hence an iterative approach is preferred.

Remark 3. Note that the above formulation assumes that the signal is bandlimited between spherical harmonic degree 0 and the maximum spherical harmonic degree L. If the signal is bandlimited between spherical harmonic degree L_1 and L_2, the spectral selection operator $\mathbf{B}_L$ in (9) can be properly modified to account for this case.

5. SIMULATION EXAMPLES

In this section, we present simulation examples to illustrate the performance of the proposed conjugate gradient algorithm. The modified iterative gradient algorithm is adopted as the benchmark for comparison. To illustrate the convergence performance of the proposed algorithm, we study the percentage absolute error E_d between the extrapolation and the actual signal after successive iterations, defined as

$$E_d = \frac{|\mathbf{f}_d^+ - \mathbf{f}|}{|\mathbf{f}|} \tag{24}$$

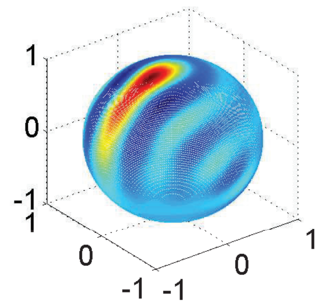

Fig. 1. The global signal $\mathbf{f}$ defined in Section 5.

where $|(\,\cdot\,)|$ denotes the l_1 norm of a vector.

Consider the bandlimited signal $\mathbf{f} = \sum_{\ell=0}^{10}\sum_{m=-\ell}^{\ell} \mathbf{Y}_\ell^m$, comprising of $N = 65536$ samples on the sphere as shown in Fig. 1. We study the extrapolation problem for this signal from the incomplete spatial-limited signal $\mathbf{g}$ known only at all the samples in a region $\mathscr{R}$ for the following three cases:

- *Example 1:* $\mathscr{R} = \{0 \le \theta \le \pi/4, 0 \le \phi \le 2\pi\}$ with $R/N = 0.25$.
- *Example 2:* $\mathscr{R} = \{0 \le \theta \le \pi/8, 0 \le \phi \le 2\pi \wedge \pi/4 \le \theta \le \pi/2, \pi \le \phi \le 5.07\}$ with $R/N = 0.20$.
- *Example 3:* $\mathscr{R} = \{0 \le \theta \le \pi/8, 0 \le \phi \le 2\pi \wedge \pi/4 \le \theta \le \pi/2, 2.35 \le \phi \le 3.32 \wedge \pi/4 \le \theta \le \pi/2, 3.93 \le \phi \le 4.89\}$ with $R/N = 0.20$.

Note that the area of the region $\mathscr{R}$ in all three examples is the same, which means that the ratio of the known signal information to the information in the global signal is same. However, the region R in Examples 2 and 3 is the union of two and three non-connected regions, respectively.

For the examples 1, 2 and 3, the extrapolated signal $\mathbf{f}_d^+$ is obtained using the modified iterative gradient algorithm and the proposed conjugate gradient algorithm and the error is calculated using (24). The results are plotted in Figs, 1, 2 and 3 respectively. It can be seen from the figures that the conjugate gradient algorithm has faster convergence rate and smaller error than the modified iterative gradient algorithm. Comparing the results in Figs. 2(d), 3(d) and 4(d), it is evident that the error E_d for either of the modified iterative gradient algorithm or conjugate gradient algorithm is smaller if the signal is known over different spatial regions. In addition, the error decreases as the number of non-connected spatial regions increases from 2 to 3.

Further exploring this fact, we extrapolate the signal $\mathbf{f}$ in Fig. 1 from a spatial-limited signal $\mathbf{g}$ obtained by selecting $R = 128$ random samples of the signal $\mathbf{f}$ of $N = 1024$ samples over the complete sphere i.e., overall we use lesser number of samples compared to examples $1-3$. It is found that the error reduces to below 10^{-11} in only $d = 20$ iterations for the conjugate gradient algorithm. This is an interesting observation which is of practical importance that the incomplete measurements distributed in different non-connected spatial regions yield better and faster extrapolation results.

6. CONCLUSIONS AND FUTURE WORK

In this work, we have studied the extrapolation problem for discrete signals on the sphere. Using the formulation of spatial and spectral selection operators on the sphere for the discrete signals, we have presented a modified iterative gradient algorithm which only updates the known number of samples of a signal and thus has reduced computational complexity compared to the algorithm in [6]. We have

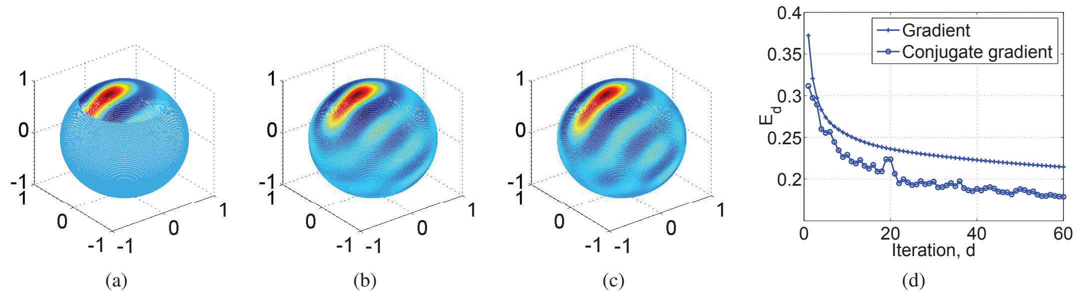

Fig. 2. (a) Known spatial-limited signal $\mathbf{D}^T\mathbf{g}$ in region $\mathscr{R}$ defined in example 1, Extrapolated signal $\mathbf{f}_d^+$ for $d = 60$ using (b) gradient algorithm and (c) conjugate gradient algorithm and (d) percentage absolute error E_d.

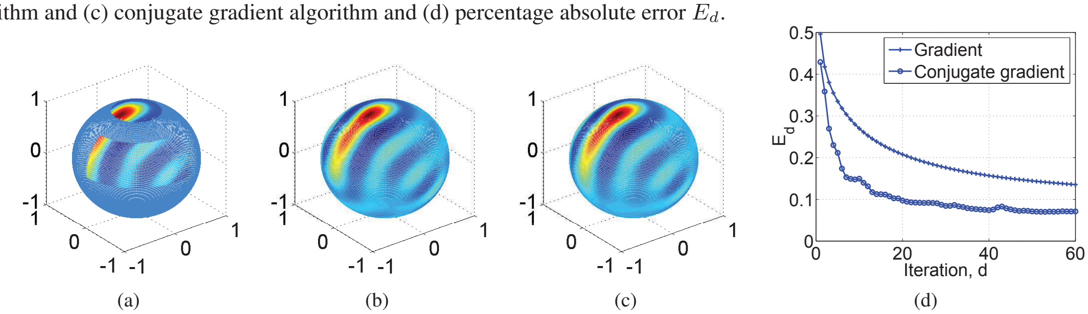

Fig. 3. (a) Known spatial-limited signal $\mathbf{D}^T\mathbf{g}$ in region $\mathscr{R}$ defined in example 2, Extrapolated signal $\mathbf{f}_d^+$ for $d = 60$ using (b) gradient algorithm and (c) conjugate gradient algorithm and (d) percentage absolute error E_d.

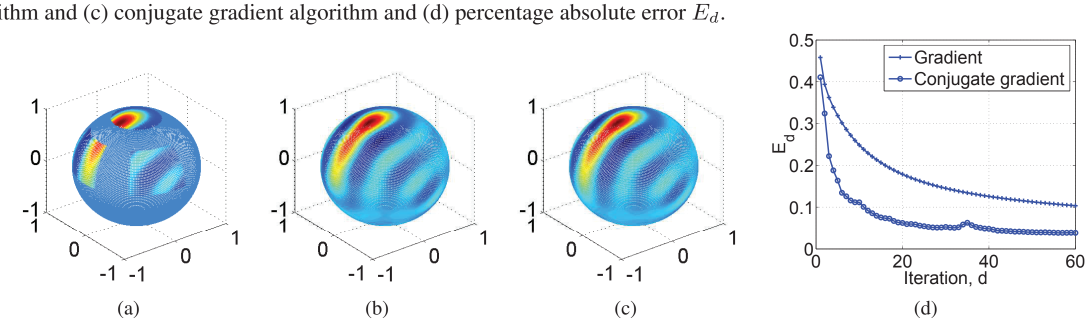

Fig. 4. (a) Known spatial-limited signal $\mathbf{D}^T\mathbf{g}$ in region $\mathscr{R}$ defined in example 3, Extrapolated signal $\mathbf{f}_d^+$ for $d = 60$ using (b) gradient algorithm and (c) conjugate gradient algorithm and (d) percentage absolute error E_d.

also proposed conjugate gradient algorithm to improve the convergence rate. Both algorithms converge to the MNLS solution of the extrapolation problem. Finally, we have provided examples to illustrate the improved performance of proposed algorithm. We remark here that we have considered the problem where the spatial-limited components of a signal are not corrupted with noise. We indicate the more practical consideration of the problem in the presence of noise as future work.

7. REFERENCES

[1] A. Jain and S. Ranganath, "Extrapolation algorithms for discrete signals with application in spectral estimation," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 29, no. 4, pp. 830–845, Aug 1981.

[2] A. Papoulis, "A new algorithm in spectral analysis and band-limited extrapolation," *IEEE Transactions on Circuits and Systems*, vol. 22, no. 9, pp. 735–742, Sep 1975.

[3] W. Zhang, R. A. Kennedy, and T. D. Abhayapala, "Signal estimation from incomplete data on the sphere," in *Australian Communications Theory Workshop, AusCTW 2008.*, Feb. 2008, pp. 39–44.

[4] ——, "Iterative extrapolation algorithm for data reconstruction over sphere," in *IEEE International Conference on Acoustics, Speech and Signal Processing, ICASSP 2008.*, April 2008, pp. 3733–3736.

[5] R. A. Kennedy, W. Zhang, and T. D. Abhayapala, "Spherical harmonic analysis and model-limited extrapolation on the sphere: Integral equation formulation," in *2nd International Conference on Signal Processing and Communication Systems, 2008. ICSPCS 2008.*, Dec. 2008, pp. 1–6.

[6] W. Zhang, "Measurement and modelling of head-related transfer function for spatial audio synthesis," Ph.D. dissertation, CECS, The Australian National University, Aug. 2010.

[7] J. R. Driscoll and D. M. Healy, "Computing Fourier transforms and convolutions on the 2-sphere," *Advances in Applied Mathematics*, vol. 15, no. 2, pp. 202–250, 1994.

[8] J. J. Sakurai, *Modern Quantum Mechanics*, 2nd, Ed. Reading, MA: Addison Wesley, 1994.

[9] E. Issacson and H. Keller, *Analysis of Numerical Methods*. Dover Publications, 1994.

[10] M. Hanke, *Conjugate Gradient Type Methods for Ill-Posed Problems*. Chapman and Hall., 1995.