# AMBIGUITY FUNCTION AND WIGNER DISTRIBUTION ON THE SPHERE

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## ABSTRACT

The ambiguity function and the Wigner distribution are fundamental tools in the time-frequency analysis. In this paper, we present an analog of the ambiguity function and the Wigner distribution for signals on the sphere. First, we formulate the ambiguity function for signals on the sphere which represents the signals in joint spatio-spectral domain and derive an inversion operation to obtain the signal from its ambiguity function. Next, we formulate the Wigner distribution for azimuthally symmetric signals on the sphere as a two dimensional spherical harmonics transform of the ambiguity function. We provide the matrix formulation of the Wigner distribution and discuss some of its useful properties. Finally, we illustrate the use of Wigner distribution for spatial and/or spectral localization of a signal in joint spatio-spectral domain. The obtained results provide the first step in designing more sophisticated transforms on the sphere.

**Keywords:** unit sphere, spherical harmonics, Wigner distribution, ambiguity function.

## 1. INTRODUCTION

The ambiguity function and the Wigner distribution are two fundamental tools in the time-frequency analysis, with applications in many diverse fields including optics, music and radar [1–4]. The relationship between the ambiguity function and the Wigner distribution in (cf. (3)) was first derived in [5] and has been studied in detail in [1]. Among the broader time-frequency analysis techniques, it is well known that the Wigner distribution of multi-component signals exhibits cross-terms, while the short time Fourier transform (STFT) technique inherently involves smearing in both time and frequency domains. However more sophisticated distributions based on ambiguity function have been proposed which solve the cross-term drawback [1]. Note that the wavelet transform is another popular time-scale representation technique [1, 6]

A number of the time-frequency analysis techniques have been extended for signals defined on the unit sphere to enable spatio-spectral analysis techniques for various fields of science and engineering [7–10]. The windowed spherical harmonics transform was first proposed in [7] as an analog of the STFT, where the windowing is performed in the spatial domain. The windowed spherical harmonics transform method is extended for localized spectral analysis to study the admittance and coherence between functions on the sphere in [8]. Numerous spherical wavelet transform techniques have also been developed and applied for signal analysis on the sphere in geodesy [9]. A Wigner distribution on the sphere is defined in [10] in the context of optics, however, the authors use Sherman-Volobuyev plane-wave basis functions to represent the spectral domain coefficients.

In this paper, we present an analog of the ambiguity function and the Wigner distribution for signals on the sphere with spherical harmonics as the basis functions. Since the rotations on the sphere are defined using three Euler angles, we formulate the ambiguity function for a signal in joint spatio-spectral domain where the spatial domain is SO(3) instead of unit sphere. We present the expression of the ambiguity function for a signal in spectral domain and derive an inversion operation to obtain the signal from its ambiguity function. Using the ambiguity function, we define the Wigner distribution for azimuthally symmetric signals. We also provide the matrix formulation of the Wigner distribution and discuss some of its useful properties. Finally, we provide illustrations of the use of the Wigner distribution to study the spatially and/or spectrally localized signals in spatio-spectral domain. The obtained results provide the first step in designing more sophisticated transforms to deal with cross-terms on the sphere.

The rest of the paper is organized as follows. The mathematical background is provided in Section 2. The ambiguity function on the sphere is presented in Section 3 and the Wigner distribution for azimuthally symmetric signals is formulated in Section 4. The simulation examples are provided in Section 5 and Section 6 concludes the paper. Notations and terms: $\overline{(\cdot)}$ denotes the complex conjugate operation. Lowercase bold symbols correspond to vectors whereas uppercase bold symbols denote matrices. $\delta$ denotes the Kronecker delta function defined as $\delta_{ab} = 1$ if $a = b$ and $\delta_{ab} = 0$ if $a \neq b$.

## 2. MATHEMATICAL BACKGROUND

### 2.1. Ambiguity Function and Wigner Distribution

In time-frequency analysis, the ambiguity function $A(t, \omega)$ of a signal $f(t)$ is given by [1, 3]

$$A(t,\omega) = \int f(\tau + \frac{t}{2})\overline{f(\tau - \frac{t}{2})}e^{-j\omega\tau}d\tau \qquad (1)$$

and the Wigner distribution is given by [1, 2]

$$W(t,\omega) = \int f(t + \frac{\tau}{2})\overline{f(t - \frac{\tau}{2})}e^{-j\omega\tau}d\tau \qquad (2)$$

The Wigner distribution $W(t, w)$ in (2) can also be formulated as two dimensional Fourier transform of the ambiguity function as [1, 2, 5]

$$W(t,w) = \int A(t',\omega')e^{-jt'\omega}e^{jt\omega'}dt'dw' \qquad (3)$$

We will use the above definition to formulate the Wigner distribution for signals on the sphere in Section 4.

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### 2.2. Signals on the Unit Sphere

We consider a function $f(\theta,\phi)$ defined on the unit sphere $\mathbb{S}^2 \triangleq \{\mathbf{r} \in \mathbb{R}^3 : \|\mathbf{r}\| = 1\}$, i.e., if $\mathbf{r} \in \mathbb{S}^2$ then $\mathbf{r}$ is a unit vector, $\theta \in [0,\pi]$ denotes the co-latitude and $\phi \in [0, 2\pi)$ denotes the longitude. The inner product of two functions $f$ and $h$ on $\mathbb{S}^2$ is defined as $\langle f, h\rangle \triangleq \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})\overline{h(\hat{\boldsymbol{x}})}ds(\hat{\boldsymbol{x}})$ where $\hat{\boldsymbol{x}} = (\theta, \phi)$ parameterize a point on the unit sphere, $ds(\hat{\boldsymbol{x}}) = \sin\theta d\theta d\phi$ and integration is carried out over the whole unit sphere. All finite energy signals such that $\|f\| \triangleq \langle f, f\rangle^{1/2} < \infty$, form complex Hilbert space $L^2(\mathbb{S}^2)$.

The spherical harmonics $Y_\ell^m(\hat{\boldsymbol{x}}) = Y_\ell^m(\theta,\phi)$ for degree $\ell \geq 0$ and order $|m| \leq \ell$ are defined as [11]

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)e^{im\phi} \tag{4}$$

and $P_\ell^m$ are the associated Legendre polynomials [11]. Spherical harmonics form an orthonormal set of basis functions on the sphere and by completeness, any signal $f \in L^2(\mathbb{S}^2)$ can be expanded as

$$f(\hat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\hat{\boldsymbol{x}}) = \sum_{\ell,m}^{\infty} f_\ell^m Y_\ell^m(\hat{\boldsymbol{x}}) \tag{5}$$

where we have used the shorthand $\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \triangleq \sum_{\ell,m}^{\infty}$ to express two summations as one summation in succinct form and $f_\ell^m \triangleq \langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})\overline{Y_\ell^m(\hat{\boldsymbol{x}})}\,ds(\hat{\boldsymbol{x}})$ are the spherical harmonic Fourier coefficient of degree $\ell$ and order $m$. In this work, we only consider the real signals on the sphere. For an azimuthally symmetric signal $f(\theta,\phi) = f(\theta)$, $f_\ell^m = 0$ for $m \neq 0$.

### 2.3. Rotation on the Sphere

A proper rotation of a signal on the sphere can be parameterized in terms of the Euler angles, $\alpha \in [0, 2\pi)$, $\beta \in [0,\pi]$, $\gamma \in [0, 2\pi)$. Define the rotation operator $\mathscr{D}_\rho$ with $\rho = (\alpha, \beta, \gamma)$ which rotates a signal first as a $\gamma$ rotation about $z-$axis, followed by a $\beta$ rotation about $y-$axis, and then an $\alpha$ rotation about $z-$axis. The spherical harmonics coefficient of the rotated signal $[\mathscr{D}_\rho f](\hat{\boldsymbol{x}})$ is given by [11]

$$\left\{\mathscr{D}_\rho f\right\}_\ell^m = \langle \mathscr{D}_\rho f, Y_\ell^m\rangle = \sum_{m'=-\ell}^{\ell} D_\ell^{m,m'}(\rho) f_\ell^{m'} \tag{6}$$

where $D_\ell^{m,m'}(\rho) = D_\ell^{m,m'}(\alpha,\beta,\gamma)$ is the Wigner-$D$ function [11]. Note that for an azimuthally symmetric signal, rotation of $\gamma$ around $z$-axis does not change the signal.

## 3. AMBIGUITY FUNCTION ON THE SPHERE

In this section, we present an analog of the the ambiguity function in (1) for the signals on the sphere. The ambiguity function in (1) can be thought as the Fourier transform of the product of the signal and its translated version. On the sphere, the rotations correspond to translations. Since, the rotation is parameterized using three Euler angles, we define the ambiguity function on SO(3) instead of $\mathbb{S}^2$. The domain of the ambiguity function is jointly spatial (parameterized by $\rho = (\alpha,\beta,\gamma)$) and spectral (parameterized by spherical harmonics degree $\ell$ and order $m$).

**Definition 1.** The ambiguity function $A(\rho, \ell, m)$ of a signal $f$ is given by the spherical harmonics transform of the signal $f$, with maximum spherical harmonics degree $L_f$, and its rotated version $\mathscr{D}_\rho f$ as

$$A(\rho,\ell,m) \triangleq \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})[\mathscr{D}_\rho f](\hat{\boldsymbol{x}})\overline{Y_\ell^m(\hat{\boldsymbol{x}})}ds(\hat{\boldsymbol{x}}) \tag{7}$$

for $\rho = (\alpha,\beta,\gamma)$ with $\alpha \in [0, 2\pi)$, $\beta \in [0,\pi]$, $\gamma \in [0, 2\pi)$ and $0 \leq \ell \leq L_A$, $-\ell \leq m \leq \ell$ with $L_A = 2L_f$.

Note that since the ambiguity function is bilinear in nature, i.e., it is the spherical harmonics transform of the product of the signal and its rotated version, the maximum spherical harmonics degree $L_A$ is $2L_f$.

### 3.1. Spectral Domain Representation

Using (5) and (6), we can express the ambiguity function in (7) as

$$A(\rho,\ell,m) = \sum_{r,s}^{L_f}\sum_{p,q}^{L_f}\sum_{q'=-p}^{p} f_r^s f_p^{q'}\, y(p,q,r,s,\ell,m)\, D_p^{q,q'}(\rho) \tag{8}$$

where $y(p,q,r,s,\ell,m) = \int_{\mathbb{S}^2} Y_p^q(\hat{\boldsymbol{x}})Y_r^s(\hat{\boldsymbol{x}})\overline{Y_\ell^m(\hat{\boldsymbol{x}})}ds(\hat{\boldsymbol{x}})$ is the spherical harmonics triple product and can be calculated using Wigner 3-j symbols [11].

### 3.2. Signal Inversion

We can obtain the signal in spectral domain from its ambiguity function by averaging the signal over all possible rotations. We call this operation the spherical harmonics marginal due to similarity in nature with the frequency marginal of the time-frequency distribution.

**Lemma 1.** Given the ambiguity function $A(\rho,\ell,m)$ of a signal $f$ in (7), we can obtain the spherical harmonic coefficient $f_\ell^m$ of a signal up to a constant by averaging over SO(3) as

$$f_\ell^m = \frac{1}{\sqrt{4\pi} f_0^0}\int_{\mathrm{SO}(3)} A(\rho,\ell,m)d\rho \tag{9}$$

considering all the rotations $\rho = (\alpha,\beta,\gamma)$ and $d\rho = d\alpha\sin\beta d\beta d\gamma$.

*Proof.* The integral of the wigner-D function over SO(3) is

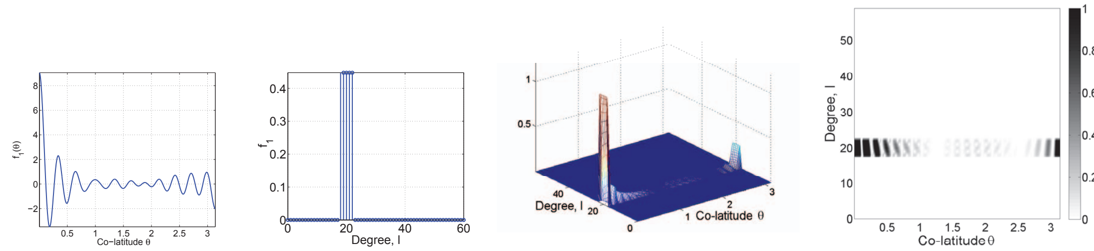
$$\int_{\mathrm{SO}(3)} D_p^{q,q'}(\rho) = \delta_{p0}\delta_{p'0}\delta_{q'0} \tag{10}$$

Using (8), along with (10), and $y(0,0,r,s,\ell,m) = \delta_{r\ell}\delta_{sm}/\sqrt{4\pi}$ which is obtained by using the orthonormal properties of spherical harmonics, we obtain the stated result in (9). ☐

**Remark 1.** We can exactly recover a signal from its ambiguity function, if the dc-component $f_0^0$ of a signal is known.

## 4. WIGNER DISTRIBUTION ON THE SPHERE FOR AZIMUTHALLY SYMMETRIC SIGNALS

In this section, we follow the analogy in (3) to define the Wigner distribution on the sphere as the two dimensional spherical harmonics transform of the ambiguity function in (7). Note that the domain of ambiguity function in (7) is jointly spatio-spectral SO(3) $\times\ \mathbb{Z}^2$. Since the Wigner-$D$ function in the formulation of ambiguity function is three dimensional in spectral domain, the two dimensional




(a) (b) (c) (d)

**Fig. 1**. (a) $f_1(\hat{\boldsymbol{x}})$, (b) $\mathbf{f_1}$ (spectrum of $f_1(\hat{\boldsymbol{x}})$ ), (c) $\mathbf{w}_1(\hat{\boldsymbol{x}})$ as surface plot and (d) $\mathbf{w}_1(\hat{\boldsymbol{x}})$ as shaded plot.

transform operation which converts the spatial domain ($\rho$) to spectral domain and the spatial domain to spectral domain projects the signal to $\mathbb{S}^2 \times \mathbb{Z}^3$.

In this work, we consider azimuthally symmetric signals on the sphere such that the spatial domain of the ambiguity function is $\mathbb{S}^2$ instead of SO(3). Using the relation $D_p^{q,0}(\phi,\theta,0) = K_p\overline{Y_p^q(\theta,\phi)}$ with $K_p = \sqrt{\frac{4\pi}{2p+1}}$ and judiciously choosing the rotations $\rho = (\alpha,\beta,\gamma) = (\phi,\theta,0)$ which characterize the spatial domain of the ambiguity function on $\mathbb{S}^2$, we obtain the ambiguity function $A(\rho,\ell,m)$ in (8) for azimuthally symmetric signals as

$$A(\rho,\ell,m) = \sum_{p,q}^{L_f}\sum_{r=0}^{L_f} K_p f_p^0 f_r^0\, y(p,q,r,0,\ell,m)\, Y_p^q(\theta,\phi) \quad (11)$$

for $0 \le \ell \le L_A$ where $L_A = 2L_f$, $\rho = (\alpha,\beta,\gamma) = (\phi,\theta,0)$ and $\hat{\boldsymbol{x}} = (\theta,\phi)$. We only seek the contribution of zero-order spherical harmonics in the spatio-spectral domain and define the Wigner distribution component as two dimensional spherical harmonics transform of ambiguity function in (11).

**Definition 2.** The Wigner distribution component $w(\hat{\boldsymbol{x}},\ell)$ of an azimuthally symmetric signal with maximum spectral degree $L_f$ using the ambiguity function $A(\rho,\ell,0)$ of a signal in (11) is given as

$$w(\hat{\boldsymbol{x}},\ell) = \sum_{\ell'=0}^{L_A}\int_{\mathbb{S}^2} A(\rho,\ell',0)\, Y_\ell^0(\hat{\boldsymbol{z}})\, ds(\hat{\boldsymbol{z}}) \quad (12)$$

for $0 \le \ell \le L_w$, $\rho = (\phi',\theta',0)$, $\hat{\boldsymbol{z}} = (\theta,\phi)$, $L_A = 2L_f$ and $L_w = L_f$.

**Remark 2.** The Wigner distribution component $w(\hat{\boldsymbol{x}},\ell)$ represents the contribution of the zero order spherical harmonic of degree $\ell$. Thus it reveals the spatially varying contribution of $Y_\ell^0$, which may not be visible if we see the signal only in spectral domain.

### 4.1. Spectral Domain Representation and Matrix Formulation

The Wigner distribution component $w(\hat{\boldsymbol{x}},\ell)$ can be expressed in terms of the signal $f$ using (11) as

$$w(\hat{\boldsymbol{x}},\ell) = K_\ell f_\ell^0 \sum_{r=0}^{L_f} f_r^0 \sum_{\ell'=0}^{2L_f} y(\ell,0,r,0,\ell',0) Y_{\ell'}^0(\hat{\boldsymbol{x}}) \quad (13)$$

We define the complete spatio-spectral Wigner distribution $\mathbf{w}(\hat{\boldsymbol{x}})$ as an indexed vector of $w(\hat{\boldsymbol{x}},\ell)$ for $0 \le \ell \le L_f$ as

$$\mathbf{w}(\hat{\boldsymbol{x}}) \triangleq [w(\hat{\boldsymbol{x}},0),\, w(\hat{\boldsymbol{x}},1),\ldots,\, w(\hat{\boldsymbol{x}},L_f)], \quad (14)$$

which can be expressed in matrix form as

$$\mathbf{w}(\hat{\boldsymbol{x}}) = \mathbf{f}\,\boldsymbol{\Upsilon}(\hat{\boldsymbol{x}})\,\mathbf{F} \quad (15)$$

where $f = [f_0^0,\, f_1^0,\ldots,\, f_{L_f}^0]$, $F$ is a diagonal matrix of size $(L_f+1)\times(L_f+1)$ with diagonal entries $F_{\ell\ell} = K_\ell f_\ell^0$ and $\boldsymbol{\Upsilon}(\hat{\boldsymbol{x}})$ is a matrix of size $(L_f+1)\times(L_f+1)$ with entries $\Upsilon_{r\ell}(\hat{\boldsymbol{x}}) = \sum_{\ell'=0}^{2L_f} y(\ell,0,r,0,\ell',0)Y_{\ell'}^0(\hat{\boldsymbol{x}})$. Using the matrix formulation in (15), the Wigner distribution of a signal in the desired spatio-spectral domain can be determined. It should be noted that the entries of the matrix $\boldsymbol{\Upsilon}(\hat{\boldsymbol{x}})$ are independent of the signal.

### 4.2. Properties of the Wigner Distribution on the Sphere

In time-frequency analysis, the Wigner distribution of a signal has some useful properties, e.g., it satisfies the time and frequency marginals and it has the finite supports in joint time-frequency domain. The Wigner distribution on the sphere exhibits similar but not the same properties. The expression in (13) reveals that the Wigner distribution follows the finite support property in the spectral domain, i.e.,, $w(\hat{\boldsymbol{x}},\ell) = 0$ only when $f_\ell^0 = 0$. The spherical harmonics marginal of the Wigner distribution in (12) is obtained by integrating over the spatial domain as

$$\int_{\mathbb{S}^2} w(\hat{\boldsymbol{x}},\ell)\, ds(\hat{\boldsymbol{x}}) = K_\ell |f_l^0|^2 \quad (16)$$

which indicates that the the Wigner distribution satisfies spherical harmonics marginal property. Note that the spherical harmonics marginal, as obtained in (16), is mathematically equivalent to the convolution of an azimuthally symmetric signal with itself in the spectral domain.

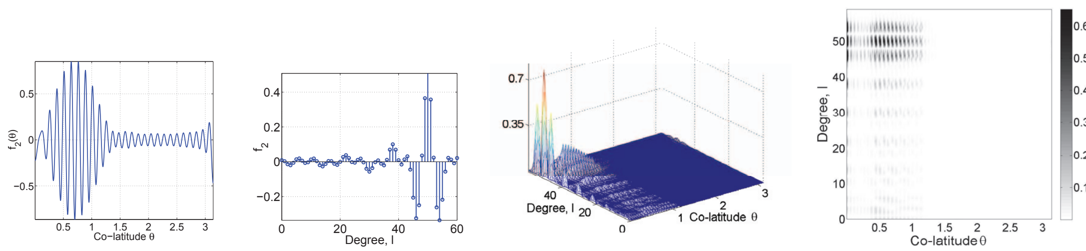
Similar to the time-frequency Wigner distribution, the Wigner distribution on the sphere is not linear : if we have two signals $\mathbf{f}_1$ and $\mathbf{f}_2$ with respective Wigner distributions $\mathbf{w}_1(\hat{\boldsymbol{x}})$ and $\mathbf{w}_2(\hat{\boldsymbol{x}})$, the Wigner distribution $\mathbf{w}(\hat{\boldsymbol{x}})$ of a signal $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ can be obtained using the matrix formulation as

$$\mathbf{w}(\hat{\boldsymbol{x}}) = \mathbf{w}_1(\hat{\boldsymbol{x}}) + \mathbf{w}_2(\hat{\boldsymbol{x}}) + \mathbf{f_1}\,\boldsymbol{\Upsilon}(\hat{\boldsymbol{x}})\,\mathbf{F_2} + \mathbf{f_2}\,\boldsymbol{\Upsilon}(\hat{\boldsymbol{x}})\,\mathbf{F_1} \quad (17)$$

where $\mathbf{F_1}$ and $\mathbf{F_2}$ are diagonal matrix with elements of $\mathbf{f_1}$ and $\mathbf{f_2}$ as diagonal entries. The last two terms in (17) are the analog of cross-terms which also appear in the time-frequency Wigner distribution.

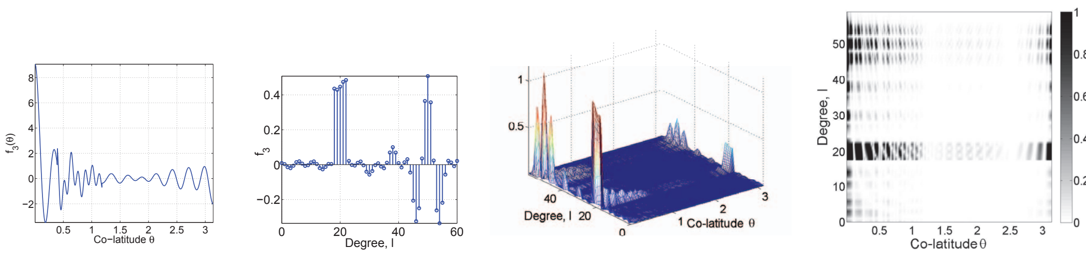
## 5. EXAMPLES

In this section, we provide examples of the Wigner distribution of spectrally localized and the spatio-spectral localized signals. We consider two azimuthally symmetric, unit energy normalized, and bandlimited signals with maximum spherical harmonics degree $L_f = 60$. We define a spectrally localized signal $f_1(\hat{\boldsymbol{x}})$ which consists of equal contribution of spherical harmonics $Y_{18}^0$, $Y_{19}^0$, $Y_{20}^0$, $Y_{21}^0$ and $Y_{22}^0$ in the complete spatial domain as shown in Fig 1(a) and (b) in spatial and spectral domains respectively. Also define a spatio-spectral localized signal $f_2(\hat{\boldsymbol{x}})$ which contains the contribution of spherical harmonics $Y_{49}^0$, $Y_{50}^0$ and $Y_{51}^0$ spatially truncated in a region




(a) (b) (c) (d)

**Fig. 2**. (a) $f_2(\hat{\boldsymbol{x}})$, (b) $\mathbf{f_2}$ (spectrum of $f_2(\hat{\boldsymbol{x}})$ ), (c) $\mathbf{w}_2(\hat{\boldsymbol{x}})$ as surface plot and (d) $\mathbf{w}_2(\hat{\boldsymbol{x}})$ as shaded plot.




(a) (b) (c) (d)

**Fig. 3**. (a) $f_3(\hat{\boldsymbol{x}})$, (b) $\mathbf{f_3}$ (spectrum of $f_3(\hat{\boldsymbol{x}})$ ), (c) $\mathbf{w}_3(\hat{\boldsymbol{x}})$ as surface plot and (d) $\mathbf{w}_3(\hat{\boldsymbol{x}})$ as shaded plot.

$R = \{\theta, \phi \in \mathbb{S}^2 : \pi/8 \leq \theta \leq 3\pi/8, 0 \leq \phi < 2\pi\}$ as shown in Fig 2(a) and (b) in spatial and spectral domains respectively. Due to truncation in the spectral domain, the spectrum of $f_1(\hat{\boldsymbol{x}})$ is no longer bandlimited but we consider a maximum spectral degree $L_f = 60$, which produces the spreading of a signal in the spatial domain outside the region $R$. The Wigner distributions $\mathbf{w}_1(\hat{\boldsymbol{x}})$ and $\mathbf{w}_2(\hat{\boldsymbol{x}})$ which respectively correspond to the signals $f_1(\hat{\boldsymbol{x}})$ and $f_2(\hat{\boldsymbol{x}})$ are shown in Fig. 1(c),(d) and Fig. 2(c),(d) as a function of colatitude $\theta$, which reflects the spatial and spectral localization of signals in the spatial and spectral domain.

Finally, we illustrate the use of the Wigner distribution in revealing the information about the localization of signal in different spatio-spectral regions. Consider a signal $f_3(\hat{\boldsymbol{x}}) = f_1(\hat{\boldsymbol{x}}) + f_2(\hat{\boldsymbol{x}})$ whose spatial and spectral domains are shown in Fig. 3(a) and (b) and the Wigner distribution $\mathbf{w}_3(\hat{\boldsymbol{x}})$ is shown in Fig. 3(c),(d). The information about the spatio-spectral localization of the signal component $f_2(\hat{\boldsymbol{x}})$ in $f_3(\hat{\boldsymbol{x}})$ is not available in either spatial or spectral domain, but it is reflected in the joint spatio-spectral domain. We also observe that the Wigner distribution $\mathbf{w}_2(\hat{\boldsymbol{x}})$ is not the sum of $\mathbf{w}_1(\hat{\boldsymbol{x}})$ and $\mathbf{w}_2(\hat{\boldsymbol{x}})$ and there exist some artifacts due to the appearance of cross-terms (see (17)).

## 6. CONCLUSIONS AND FUTURE WORK

In this paper, we have presented an analog of the ambiguity function and the Wigner distribution for signals on the sphere. The ambiguity function represents the signals on the sphere in joint spatio-spectral domain, where the spatial domain is SO(3) instead of $\mathbb{S}^2$ because the rotations are defined on $\mathrm{SO}(3)$. We have proved the inversion operation to obtain the signal from its ambiguity function. Using the ambiguity function, the Wigner distribution, its matrix formulation and properties has been presented for azimuthally symmetric signals. Finally, we illustrated that the Wigner distribution reflects the spatial and spectral localization of a signal in the spatio-spectral domain. The obtained results provide the first step in designing more sophisticated transforms on the sphere, e.g., the ambiguity function can be used to formulate a general class of spatio-spectral distributions for signals on the sphere similar to the Cohen class of time-frequency distributions [1]. Furthermore, the comparison of the Wigner distribution with existing spatio-spectral techniques [7, 9] needs to be explored.

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