Revisiting Slepian Concentration Problem on the Sphere for Azimuthally Non-Symmetric Regions

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***Abstract*—The problems of filtering, spectral analysis and spectral estimation have been investigated on the sphere using azimuthally symmetric functions as kernels which treat all the directions uniformly. In this work, we extend the concentration problem on the sphere for an azimuthally non-symmetric spatial region on the sphere. Our approach is different in a sense that we obtain the family of spatially concentrated bandlimited mutually orthogonal functions by maximizing the contribution of spherical harmonics components of all degrees and orders within the spectral bandwidth. We also provide analysis of the eigenfunctions for different bandwidths and non-symmetric regions and illustrate the concentration of eigenfunctions with the help of examples. Also we formulate the definition of filtering using azimuthally non-symmetric functions. The proposed eigenfunctions can be used to revisit the problems of estimation, localized spectral analysis, smoothing and filter design on the sphere.**

I. Introduction

Signals processing on the unit sphere has direct applications in many diverse fields such as geophysics [1], cosmology [2], 3D beamforming [3], image processing [4], computer graphics [5], electromagnetic inverse problems [6] and medical imaging [7]. Extending the well formulated signal processing techniques in Euclidean domain such as convolution, filtering, smoothing, estimation, prediction to the unit sphere domain is a natural way to analyze the signal inherently defined on the sphere.

Convolutional smoothing is the low pass filtering of signals in time domain to reduce the effect of noise and the removal of high frequency components and is carried out using a mean filter that averages the signal at any time instant by taking into account the values in the neighborhood. This type of filtering is accomplished by the convolution of filter response and the signal. Such type of filtering has been investigated for the signals defined on the unit sphere in [4], where azimuthally symmetric Gaussian functions as filter are used for spherical diffusion. The azimuthally symmetric functions vary only with the co-latitude in the spatial domain and are defined only for zero order spherical harmonics, thus, the spherical harmonics reduce to Legendre polynomials and are called zonal harmonics. We also presented the design of low pass filters using the weighted sum of azimuthally symmetric eigenfunctions obtained from concentration problem on the sphere [8].

The Slepian concentration problem [9, 10] on the sphere to find the family of orthogonal eigenfunctions which are optimally concentrated in both spatial and spectral domains have been presented in [11], and rigorously investigated by Simons [1]. The azimuthally symmetric eigenfunctions obtained as a solution of concentration problem are applied for spectral estimation [12] and localized spectral analysis to study the admittance and coherence between two windowed functions [13]. The non-trivial equivalence between various definitions of convolution is shown in [14] for azimuthally symmetric kernels. Filtering using azimuthally symmetric filter functions can be thought as non-directional convolution as the output at any spatial point on the sphere is the weighted average of the signal over the polar cap centered at that point, where the filter function defines the weights.

In this work, we revisit the concentration problem for azimuthally non-symmetric region and introduce a family of bandlimited functions on the unit sphere, whose spatial response is concentrated in the strip region around the north pole, and are orthonormal over the span of whole spectral domain. To the best of our knowledge, the concentration problem on the sphere has not been explored for non-symmetric regions. These eigenfunctions can be used to construct the bandlimited azimuthally non-symmetric spatially concentrated filter function to perform directional smoothing on the sphere. We first define the non-symmetric region around the north pole and formulate filtering as convolution for directional smoothing using the bandlimited and optimally spatially concentrated filter function. We briefly discuss the existing literature on the concentration problem for symmetric case [1, 11] and extend the work to the non-symmetric region. We also provide analysis of the eigenfunctions for different bandwidths and non-symmetric regions. We illustrate the work with the help of examples and formulate a realization of the filter as a weighted sum of concentrated eigenfunctions.

The rest of the paper is organized as follows. We briefly review mathematical background on spherical harmonics and rotation operations in Section II. We define the non-symmetric region and formulate the filtering in Section III. In Section IV, we pose and solve the concentration problem, analyze the eigenfunctions and illustrate with the help of numerical examples. Finally, Section V concludes the paper and indicates the future directions.

Notations and terms: $\overline{(\cdot)}$ denotes the complex conjugate operation. Lowercase bold symbols correspond to vectors

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whereas uppercase bold symbols denote matrices. $|(\,\cdot\,)|$ and $\|(\,\cdot\,)\|$ denote the magnitude and ℓ_2-norm respectively. $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and hermitian operations respectively.

II. Mathematical Preliminaries

Let $f(\hat{\boldsymbol{x}})$ be a square integrable function, defined on unit sphere $\mathbb{S}^2 \triangleq \{\mathbf{r} \in \mathbb{R}^3 : \|\mathbf{r}\| = 1\}$ in complex Hilbert space $L^2(\mathbb{S}^2)$, where $\hat{\boldsymbol{x}} \equiv \hat{\boldsymbol{x}}(\theta,\phi) \triangleq (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)' \in \mathbb{R}^3$, $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi)$. Note that $\theta = 0$ correspond to the north pole. The inner product of two functions $f(\hat{\boldsymbol{x}})$ and $h(\hat{\boldsymbol{x}})$ on $\mathbb{S}^2$ is defined as [14]

$$\langle f, h\rangle \triangleq \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})\overline{h(\hat{\boldsymbol{x}})}ds(\hat{\boldsymbol{x}}) \tag{1}$$

where $ds(\hat{\boldsymbol{x}}) = \sin\theta d\theta d\phi$ and integration is performed over the whole unit sphere.

A. Spherical Harmonics

The spherical harmonics, $Y_\ell^m(\theta,\phi)$, for degree $\ell \geq 0$ and order $|m| \leq \ell$ are defined as [15]

$$Y_\ell^m(\theta,\phi) = N_\ell^m P_\ell^m(\cos\theta)e^{im\phi} \tag{2}$$

where $i = \sqrt{-1}$ is the imaginary unit and N_ℓ^m is the normalization constant defined as

$$N_\ell^m = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \tag{3}$$

and P_ℓ^m are the associated Legendre polynomials defined as

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}\sqrt{(1-x^2)^m}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \tag{4}$$

$$P_\ell^{-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x) \tag{5}$$

for $|x| \leq 1$ and $m \geq 0$. The term $(-1)^m$ in the definition of associated Legendre polynomials is the Condon-Shortley phase factor. Using the above definitions, spherical harmonic functions form an orthonormal set of basis functions for $L^2(\mathbb{S}^2)$, therefore, any function $f(\theta,\phi)$ defined on unit sphere can be expanded in terms of spherical harmonics as

$$f(\hat{\boldsymbol{x}}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m Y_\ell^m(\hat{\boldsymbol{x}}) \tag{6}$$

where f_ℓ^m is the spherical harmonics coefficient, which is obtained by projecting the function $f(\hat{\boldsymbol{x}})$ onto $Y_\ell^m(\hat{\boldsymbol{x}})$ as

$$f_\ell^m \triangleq \langle f, Y_\ell^m\rangle = \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}})\overline{Y_\ell^m(\hat{\boldsymbol{x}})}ds(\hat{\boldsymbol{x}}) \tag{7}$$

With the definition of spherical harmonics in (2), the following relation holds [15]

$$\overline{Y_\ell^m(\hat{\boldsymbol{x}})} = (-1)^m Y_\ell^{-m}(\hat{\boldsymbol{x}}) \tag{8}$$

and spherical harmonic coefficients of a real signal follow

$$\overline{f_\ell^m} = (-1)^m f_\ell^{-m} \tag{9}$$

We also use the unique mapping

$$(\ell, m) \leftrightarrow c, \quad c = \ell^2 + \ell + m \tag{10}$$

to express the spherical harmonics Y_ℓ^m as Y_c in terms of only one variable c instead of two variables ℓ and m.

B. Rotation on the Sphere in Spherical Harmonics Domain

Rotations on the sphere serve as counterpart of translations on the Euclidean domain. Rotation of a function on the sphere is parameterized in terms of Euler angles, $\alpha \in [0, 2\pi)$, $\beta \in [0,\pi]$, $\gamma \in [0, 2\pi)$. The rotation operator $D(\alpha,\beta,\gamma)$ rotates the function on the unit sphere by the γ rotation about $z-$axis followed by the β rotation about $y-$axis and then α rotation about $z-$axis. Each spherical harmonic coefficient f_ℓ^m of degree ℓ and order m of the rotated function $\{[D(\alpha,\beta,\gamma)f](\theta,\phi)\}_\ell^m$ is transformed into a linear combination of different order spherical harmonics of the same degree as

$$\{[D(\alpha,\beta,\gamma)f](\theta,\phi)\}_\ell^m = \sum_{m'=-\ell}^{\ell} D_\ell^{m,m'}(\alpha,\beta,\gamma)f_\ell^{m'} \tag{11}$$

where $D(\alpha,\beta,\gamma)$ is a rotation operator and $D_\ell^{m,m'}(\alpha,\beta,\gamma)$ is the Wigner-D function given by [15, 16]

$$D_\ell^{m,m'}(\alpha,\beta,\gamma) = e^{-im\alpha}d_\ell^{m,m'}(\beta)e^{-im'\gamma} \tag{12}$$

where $d_\ell^{mm'}(\beta)$ is the Wigner-d function [15, 16].

C. Slepian's Concentration Problem on the Sphere

Concentration problem on the sphere analogous to Slepian concentration problem in time-frequency domain was first investigated in [11] for bandlimited functions, and revisited in detail by Simons [1, 13]. Here we discuss their work briefly to lay the mathematical background. In general, to maximize the spatial concentration of a bandlimited signal $f(\hat{\boldsymbol{x}})$ having maximum spherical harmonic degree L within the region R, we maximize the spatial concentration ratio [1]

$$\lambda = \frac{\int_R f(\hat{\boldsymbol{x}})\overline{f(\hat{\boldsymbol{x}})}ds(\hat{\boldsymbol{x}})}{\langle f, f\rangle} \tag{13}$$

where $0<\lambda<1$ is a measure of spatial concentration. Using spherical harmonics representation of $f(\hat{\boldsymbol{x}})$ in (6), the concentration problem in (13) can be expressed as

$$\lambda = \frac{\sum\limits_{\ell=0}^{L}\sum\limits_{m=-\ell}^{\ell}\sum\limits_{\ell'=0}^{L}\sum\limits_{m'=-\ell'}^{\ell'} f_\ell^m\overline{f_{\ell'}^{m'}}\mathcal{E}_{\ell\ell',mm'}}{\sum\limits_{\ell=0}^{L}\sum\limits_{m=-\ell}^{\ell}\|f_\ell^m\|^2} \tag{14}$$

where

$$\mathcal{E}_{\ell\ell',mm'} = \int_R Y_\ell^m(\hat{\boldsymbol{x}})\overline{Y_{\ell'}^{m'}(\hat{\boldsymbol{x}})}ds(\hat{\boldsymbol{x}}) \tag{15}$$

The concentration problem in (14) is solved as an eigenvalue problem and the solution of this eigenvalue problem gives $(L+1)^2$ orthonormal eigenvectors. Eigenvalue associated with eigenvector is a measure of concentration of corresponding spectral limited spatial eigenfunction in the desired region, R.

The number of optimally concentrated eigenfunctions N with eigenvalues close to unity is related to area A of the region R under consideration

$$N \approx (L+1)^2 \frac{A}{4\pi} \tag{16}$$

III. Problem Formulation

Convolutional smoothing on the sphere has been presented as spherical diffusion using azimuthally symmetric Gaussian kernels [4] and low pass filtering using weighted sum of eigenfunctions as a filter function [8]. Filtering or localization using azimuthally symmetric kernels treat all the directions uniformly in the neighborhood, whereas, if we use the non-symmetric functions, we can attain the directional smoothing *i.e.*, the averaging of the values in the neighborhood which has larger width in one direction than the other. Here, we revisit the concentration problem for azimuthally non-symmetric case on the unit sphere such that the resulting eigenfunctions are spatially concentrated in the strip region around the north pole. We first parameterize the strip region around the north pole and then formulate the filtering using non-symmetric filter functions, which can be constructed using our proposed eigenfunctions.

A. Problem Parametrization

We seek the family of bandlimited eigenfunctions such that each function $f(\hat{\boldsymbol{x}})$ is bandlimited in spectral domain with maximum spherical harmonic degree L and optimally concentrated in the spatial strip region at the north pole bounded by the maximum colatitude θ_c and a maximum absolute value in the y coordinate equal to $\sin^{-1}\phi_c$. Thus, the maximum colatitude θ_c and the longitude ϕ_c parameterize the region. We can think of this region as two planes at $y=\sin^{-1}\phi_c$ and $y=-\sin^{-1}\phi_c$ cutting through the polar cap region of central angle θ_c. The plane $y=\sin^{-1}\phi_c$ intersects the equatorial latitude $\theta=\pi/2$ at $\phi=\phi_c$ in the first quadrant. This region is shown in Fig. 1 as shaded region. Mathematically, we are seeking a signal $f(\hat{\boldsymbol{x}})$ such that

(A1) Spherical harmonic coefficients should satisfy

$$f_\ell^m = 0; \quad L<\ell,\, -\ell \le m \le \ell \tag{17}$$

and all of the eigenfunctions are orthorormal over the whole spectral domain within maximum spherical harmonic degree.

(A2) The spatial response is optimally concentrated within the strip region $\mathcal{R} \in \mathbb{S}^2$ defined as

$$\mathcal{R} = \begin{cases} 0 \le \theta \le \phi_c, & 0 \le \phi < 2\pi \\ \phi_c < \theta \le \theta_c, & 0 \le \phi < \phi_0 \\ & \pi - \phi_0 \le \phi < \pi + \phi_0 \\ & 2\pi - \phi_0 \le \phi < 2\pi \end{cases} \tag{18}$$

with $\phi_0 = \sin^{-1}\big(\sin\phi_c / \sin\theta\big)$, $0 \le \theta_c \le \pi$, $0 \le \phi_c \le \pi/2$ and $\theta_c > \phi_c$.

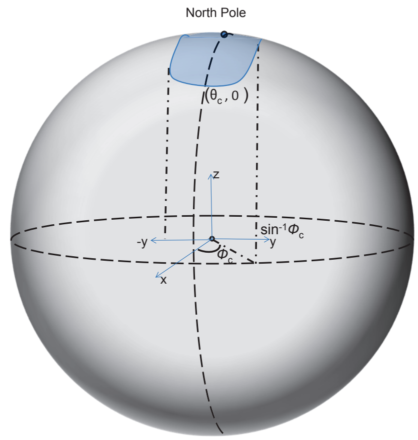


Fig. 1. Azimuthally non-symmetric region on the Sphere defined in (18) in terms of θ_c and ϕ_c.

B. Filtering Operation

Filtering on the unit sphere may be defined as spherical convolution. If $g(\theta,\phi)$ represents the signal which is filtered using the filter function $h(\theta,\phi)$, the filtered output $y(\theta,\phi)$ is given by

$$y(\theta,\phi) = h(\theta,\phi) * g(\theta,\phi) \tag{19}$$

where '*' denotes the convolution operation. There are different notions of spherical convolution available in the literature [14]. We define the output of the filter to be the integral sum of the signal and the rotated filter function, where the rotations are performed using the rotation operator $D(\alpha,\beta,\gamma)$ which projects the output of the filter on SO(3) [8]. We use the function that meets the specifications (A1) and (A2) as a filter to perform smoothing operation. For fixed rotation γ, the output of filter will be a function on $\mathbb{S}^2$ parameterized by $\phi=\alpha$ and $\theta=\beta$. For fixed $\gamma=\psi$ rotation, the output of filter $y(\theta,\phi)$ is given by

$$y(\theta,\phi) = \int_{\mathbb{S}^2} [D(\phi,\theta,\psi)h](\theta',\phi') g(\theta',\phi') \sin\theta' d\theta' d\phi' \tag{20}$$

The rotation ψ is the orientation of the filter function with respect to north pole and defines the direction of larger width θ_c of the strip region $\mathcal{R}$ in (18), where the filter function is concentrated. Due to the symmetry of the region $\mathcal{R}$ around x-axis, the range of orientation is $0 \le \psi < \pi$. Using the effect of rotation operator on spherical harmonics coefficients as defined in (11), we express the filter output $y(\theta,\phi)$ in terms of spherical harmonic coefficients of the filter and the signal as

$$y(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{p=0}^{L} \sum_{q=-p}^{p} \sum_{q'=-p}^{p} g_\ell^m D_p^{q,q'}(\phi,\theta,\psi) h_p^{q'} \int_{\mathbb{S}^2} Y_\ell^m(\hat{\boldsymbol{x}'}) Y_p^q(\hat{\boldsymbol{x}'})\, ds(\hat{\boldsymbol{x}'}) \tag{21}$$

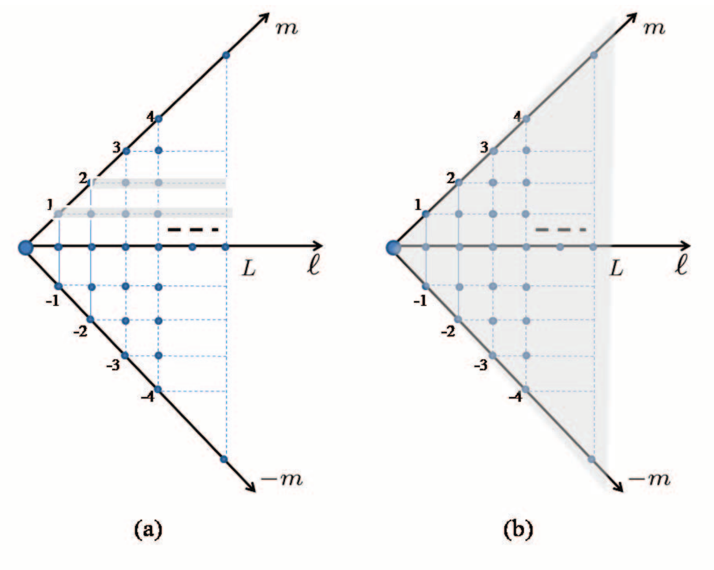


Fig. 2. Spectral domain for signals defined on the unit sphere (a) Shaded region shows the region of fixed spherical harmonics order $m = 1$ and $m = 2$ within maximum spherical harmonics degree L. (b) Shaded region shows the whole spectral domain within the maximum spherical harmonic degree L.

which can be simplified using orthonormal property of spherical harmonics as

$$y(\theta,\phi) = \sum_{p=0}^{L} \sum_{q=-p}^{p} (-1)^q \, g_p^{-q} \Big(\sum_{q'=-p}^{p} h_p^{q'} \Big) D_p^{q,q'}(\phi,\theta,\psi) \tag{22}$$

which indicates that the filter output is the weighted sum of Wigner-D functions. The Wigner-D function $D_p^{q,q'}$ when expanded in terms of spherical harmonics basis indicates that it does not depend on spherical harmonics of degree greater than p. Thus, the convolution of a signal with the bandlimited filter function results in a bandlimited output function.

In the next section, we find a family of orthogonal bandlimited spatially concentrated eigenfunctions in a region $\mathcal{R}$ which can be used to perform filtering as defined in (20).

IV. Concentration Problem for Azimuthally Non-Symmetric Region

It is well appreciated that signals cannot be concentrated simultaneously in both finite time domain and finite frequency domain [9, 10]. If we make an analog of this for signals on the sphere, signals cannot have finite support at the same time both in spatial and spectral domains. The concentration problem to find the signals which are optimally concentrated in both the time and frequency domains has been investigated by Slepian, Pollak and Landau in 1960s [10].

Simons and his coauthors provided in-depth analysis of concentration problem on the sphere where the problem of finding bandlimited spatially concentrated functions and spacelimited spectrally concentrated functions has been formulated [1]. The problem to find the bandlimited spatially concentrated functions with maximum spherical harmonics degree L is formulated in (14) but has been analyzed only for azimuthally symmetric regions in the spatial domain. For azimuthally symmetric region on the sphere, due to the orthogonality of different orders spherical harmonics along longitude, the solution of concentration problem is equivalent to the solution of different subproblems, each for a fixed order $0 \le m \le L$. For the subproblem of fixed order m, the resulting eigenfunctions contain the contribution of spherical harmonics of degree $m \le \ell \le L$. These fixed order regions are shown in the spherical harmonics domain as shaded regions in Fig. 2(a) for $m = 1$ and $m = 2$.

Here, we revisit the concentration problem on the sphere to find the orthonormal family of bandlimited functions which have optimal spatial concentration in the azimuthally *non-symmetric* region $\mathcal{R} \in \mathbb{S}^2$ defined in (18). For the azimuthally non-symmetric region, the concenration problem does not reduce to subproblems of fixed order, therefore, we take into account the whole spectral domain up to maximum spherical harmonics degree and maximize the contribution of spherical harmonics of all orders and degrees within the spectral bandwidth. The resulting eigenfunctions are orthonormal over the span of whole spectral domain shown as the shaded region in Fig. 2(b).

Definition 1: If $f(\hat{\boldsymbol{x}})$ is the bandlimited function with maximum spherical harmonics degree L, we define $\mathbf{f}$ to be the spectral response of $f(\hat{\boldsymbol{x}})$ as

$$\mathbf{f} = [f_0,\ f_1,\ \cdots,\ f_C] \tag{23}$$

where $f_c = f_\ell^m = \langle f, Y_\ell^m \rangle$ and $C = L^2 + L$ and the mapping $(\ell, m) \leftrightarrow c$ as defined in (10) is used.

In order to maximize the concentration of $f(\hat{\boldsymbol{x}})$ in the strip region $\mathcal{R}$, we pose the concentration problem equivalent to (14) as

$$\lambda = \frac{\sum_{c=0}^{C} \sum_{d=0}^{C} f_c \overline{f_d} \mathcal{K}_{cd}}{\sum_{c=0}^{C} \|f_c\|^2} \tag{24}$$

and in matrix form as

$$\lambda = \frac{\mathbf{f}^H \boldsymbol{\mathcal{K}} \mathbf{f}}{\mathbf{f}^H \mathbf{f}} \tag{25}$$

where $\boldsymbol{\mathcal{K}}$ is a two dimensional matrix of size $(C+1) \times (C+1)$ of the form

$$\boldsymbol{\mathcal{K}} = \begin{pmatrix} \mathcal{K}_{00} & \mathcal{K}_{01} & \cdots & \mathcal{K}_{0C} \\ \mathcal{K}_{10} & \mathcal{K}_{11} & \cdots & \mathcal{K}_{1C} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{K}_{C0} & \mathcal{K}_{C1} & \cdots & \mathcal{K}_{CC} \end{pmatrix} \tag{26}$$

with each entry $\mathcal{K}_{cd}$ is given by

$$\mathcal{K}_{cd} \triangleq \int_{\mathcal{R}} Y_c(\hat{\boldsymbol{x}}) \overline{Y_d(\hat{\boldsymbol{x}})} ds(\hat{\boldsymbol{x}}) \tag{27}$$

where $Y_c = Y_\ell^m$ and $Y_d = Y_r^s$ with the mappings $(\ell, m) \leftrightarrow c$ and $(r, s) \leftrightarrow d$. Instead of solving different concentration

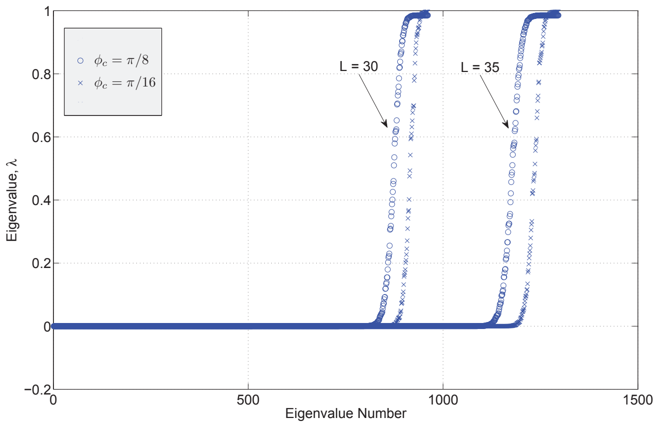


Fig. 3. Eigenvalues corresponding to $L^2 + 2L + 1$ eigenvectors are plotted which are obtained as a solution of the concentration problem in (29) for $L = 30$ and $L = 35$. The parameters of the region $\mathcal{R}$ are $\theta_c = \pi/4$ and ϕ_c as indicated in the plot.

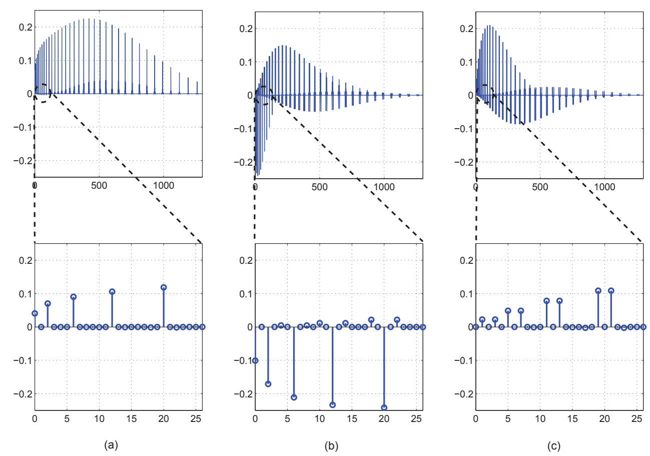


Fig. 4. Eigenvectors (a) $\mathbf{f}_1$, (b) $\mathbf{f}_2$ and (c) $\mathbf{f}_3$ which are obtained as a solution of the concentration problem in (29) for $L = 35$, $\phi_c = \pi/16$ and $\theta_c = \pi/4$.

problems for different orders [13], we want to maximize the contribution of spherical harmonics of all degrees and orders within the bandwidth within our desired spatial region. Consequently, and the resulting eigenfunctions are orthonormal to each other over the whole spectral domain. Our region $\mathcal{R}$ of interest in (18) is not azimuthally symmetric but symmetric around y-plane or around $\phi = 0$ and $\phi = \pi$ longitude. As the complex part of non-zero order spherical harmonics is anti-symmetric and real part is symmetric around y-plane, the each entry $\mathcal{K}_{cd}$ of matrix $\boldsymbol{\mathcal{K}}$ would be real and matrix itself would be positive definite ($\mathbf{f}\boldsymbol{\mathcal{K}}\mathbf{f}^H > 0$). Also, using the identity in (8), we can write

$$Y_r^s(\hat{\boldsymbol{x}})\overline{Y_\ell^m}(\hat{\boldsymbol{x}}) = (-1)^{s+m}\overline{Y_r^{-s}(\hat{\boldsymbol{x}})}Y_\ell^{-m}(\hat{\boldsymbol{x}}) \qquad (28)$$

and infer that $\boldsymbol{\mathcal{K}}$ is also Hermitian symmetric, but due to the symmetric properties of spherical harmonics, the Condon-Shortley phase as defined in the definition of Legendre polynomial in (4) would be missing if $m + s$ is even.

The concentration problem in (27) gives rise to following eigenvalue problem

$$\boldsymbol{\mathcal{K}}\mathbf{f} = \lambda\mathbf{f} \qquad (29)$$

whose solution gives family of $C + 1 = L^2 + 2L + 1$ orthonormal real eigenvectors because $\boldsymbol{\mathcal{K}}$ is real, symmetric and positive definite. The eigenvalue $0 < \lambda < 1$ associated with each eigenvector is a measure of concentration of the corresponding eigenfunction in the region $\mathcal{R}$.

A. Analysis of Bandlimited Spatially Concentrated Eigenfunctions

Each eigenvector obtained as a solution of eigenvalue problem in (29) denotes the spectral response of corresponding eigenfunctions. Similar to the case of azimuthally symmetric region, we find that most of the eigenvalues are near to zero which indicate that they are not concentrated in the desired region. We are only interested in the eigenvectors whose concentration measure λ is near unity. If N' denotes the number of such concentrated eigenvectors, we order them in the decreasing order of their eigenvalues as $\mathbf{f}_1, \mathbf{f}_1, \cdots, \mathbf{f}_{N'}$ such that the corresponding eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N'}$. Each eigenvector $\mathbf{f}_a$ denotes the spectral response of corresponding eigenfunction $f_a(\hat{\boldsymbol{x}})$ and can be written in the form of (23) as

$$\mathbf{f}_a = [f_{a,0},\ f_{a,1},\ \cdots,\ f_{a,C}] \qquad (30)$$

for $a = 1, 2, 3 \cdots N'$ and $f_{a,c} = f_{a,\ell}^m$ with mapping $(\ell, m) \leftrightarrow c$. Since we have used the orthonormalized spherical harmonics, the resulting eigenvectors are also orthonormal over the whole spectral domain, *i.e.*,

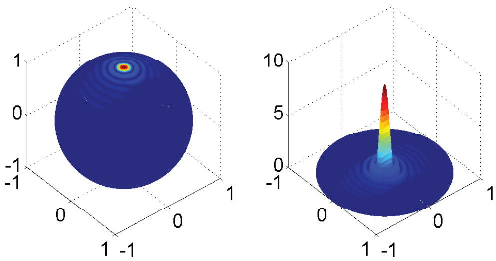
$$\mathbf{f_a}\mathbf{f_b}^H = \delta_{ab} \qquad (31)$$

where δ_{ab} denotes the kronecker delta and is equal to 1 for $a = b$ and zero otherwise.

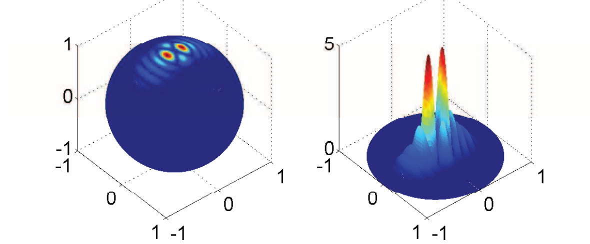
Fig. 3 shows the eigenvalues corresponding to eigenvectors which are obtained as a solution of the concentration problem posed in (29) for different bandwidths and strips of different widths. The eigenvalues are plotted for maximum spherical harmonic degree $L = 30$ and $L = 35$, and considering two different regions $\mathcal{R}$ parameterized by $\phi_c = \pi/8, \theta_c = \pi/4$ and $\phi_c = \pi/16, \theta_c = \pi/4$ for each L. We see that most of the eigenvalues either lie near zero or unity and there is a sharp transition between the two extremes. This observation is in accordance with the findings in the existing literature for azimuthally symmetric case [1]. In addition, we observe that the spectral parameter L and the number of eigenvalues which have more than $80\%(\lambda > 0.8)$ concentration in the spatial region $\mathcal{R} \in \mathbb{S}^2$ are approximately related by (16), where A denotes the area of the region $\mathcal{R}$. The relation in (16) holds if there is direct transition from zero to one in the eigenvalues as N denotes the sum of all of the eigenvalues and serve

TABLE I
NUMBER OF SIGNIFICANT EIGENVALUES - CALCULATED SUM (N) IN (16), NUMBER OF EIGENVALUES ≥ 0.8 (N') AND SUM OF ALL EIGENVALUES (N''). (ϕ_c, θ_c) DEFINES THE REGION $\mathcal{R}$.

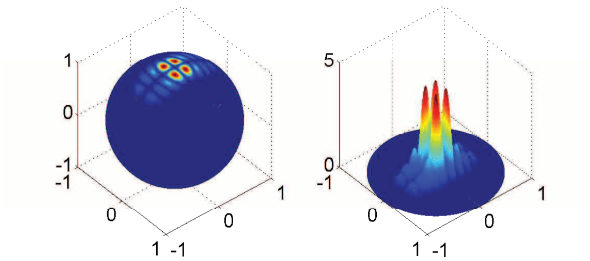
(θ_c, ϕ_c)	L	N	N'	N''
$\pi/8, \pi/16$	35	30.2	23	30.4
$\pi/4, \pi/16$	30	46.6	34	46.3
$\pi/4, \pi/16$	35	62.8	50	62.5
$\pi/4, \pi/8$	30	88.7	74	88.3
$\pi/4, \pi/8$	35	119.7	102	119.1



(a)



(b)

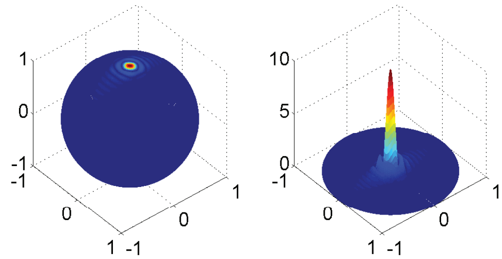


(c)

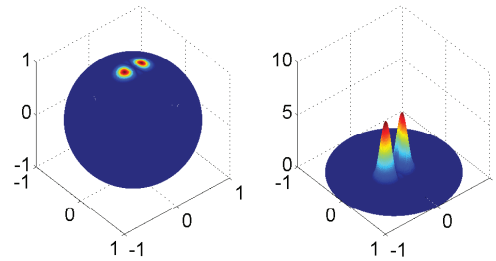
Fig. 5. Eigenvectors (a) $f_1(\theta, \phi)$, (b) $f_2(\theta, \phi)$ and (c) $f_2(\theta, \phi)$ which are obtained as a solution of concentration problem in (29) for $L = 30$, $\phi_c = \pi/8$ and $\theta_c = \pi/4$.

as an upper bound on the number of significant eigenvalues. This is illustrated in Table I which shows the calculated N in (16), the sum of all eigenvalues N'' and the number of significant eigenvalues N' with concentration measure $\lambda \geq 0.8$ for different bandwidths L and regions $\mathcal{R}$.

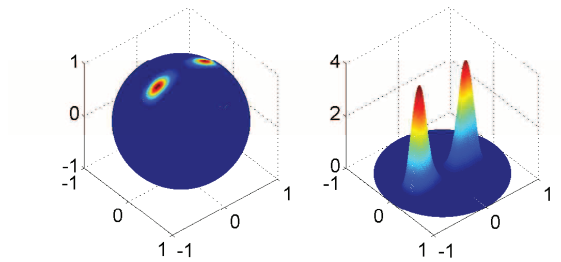
As we have ordered the eigenvectors in the order of decreasing eigenvalue, the Condon-Shortley phase may be missing in some of the eigenvectors. Fig. 4 shows the first three most spatially concentrated eigenvectors $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{f}_3$ for $L = 35$, $\phi_c = \pi/16$ and $\theta_c = \pi/4$. For each of the eigenfunction f_a, we see that $|f_{a,\ell}^m| = |f_{a,\ell}^{-m}|$ but the information about Condon-



(a)



(b)



(c)

Fig. 6. Eigenvectors (a) $f_1(\theta, \phi)$, (b) $f_2(\theta, \phi)$ and (c) $f_3(\theta, \phi)$ which are obtained as a solution of concentration problem in (29) for $L = 35$, $\phi_c = \pi/16$ and $\theta_c = \pi/4$.

Shortley phase of $(-1)^m$ for $m < 0$ is missing for $\mathbf{f}_3$ and needs to be corrected so that the corresponding eigenfunction will be real. If $\hat{\mathbf{f}}_a$ be the eigenvector after the Condon-Shortley phase correction such that the spherical harmonic coefficients satisfy $\hat{f}_{a,\ell}^m = (-1)^m \hat{f}_{a,\ell}^{-m}$, we can get the corresponding real eigenfunction $f_a(\hat{\boldsymbol{x}})$ using spherical harmonics synthesis as

$$f_a(\hat{\boldsymbol{x}}) = \sum_{c=0}^{C+1} \hat{f}_{a,c} Y_\ell^m(\hat{\boldsymbol{x}}) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \hat{f}_{a,\ell}^m Y_\ell^m(\hat{\boldsymbol{x}}) \quad (32)$$

where the mapping $(\ell, m) \leftrightarrow c$ has been used.

B. Illustrations

We illustrate that the bandlimited eigenfunctions, corresponding to eigenvectors which are obtained as a solution of concentration problem in (29), are spatially concentrated in the spatial strip region at the north pole. We demonstrate with the first three most concentrated eigenfunctions $f_1(\hat{\boldsymbol{x}})$, $f_2(\hat{\boldsymbol{x}})$ and $f_3(\hat{\boldsymbol{x}})$ for different bandwidths and different spatial regions $\mathcal{R}$. Fig. 5 and Fig. 6 show the magnitude of the eigenfunctions for $L = 35$,$\phi_c = \pi/16$, $\theta_c = \pi/4$ and $L = 30$, $\phi_c = \pi/8$,

$\theta_c = \pi/4$ respectively. As the functions are localized around north pole, we have shown these functions on the unit sphere domain and on the displaced unit sphere for an illustration.

C. Filter Realization

We see that these bandlimited eigenfunctions are spatially concentrated in the desired region $\mathcal{R}$ and can be used as building blocks to construct a desired filter for directional smoothing as defined in (20). One simpler realization to construct a desired filter $\mathbf{h}$ would be the weighted sum of eigenvectors. If $\mathbf{w} = [w_1, w_2, \cdots, w_{N'}]$ denotes the weights, the filter $\hat{\mathbf{h}}$ be the weighted sum of N' eigenvectors whose corresponding eigenfunctions are spatially concentrated and can be expressed as

$$\hat{\mathbf{h}} = \mathbf{w}\,[\mathbf{f}_1, \mathbf{f}_2, \cdots \mathbf{f}_{N'}]^T \tag{33}$$

where the weights are calculated such that error $\|\mathbf{h} - \hat{\mathbf{h}}\|$ is minimized. Note that since there are N' degrees of freedom to design the $(L+1)^2$ spectral coefficients of the filter, we have an overdetermined linear system which can be solved using standard ℓ_2-minimization technique [17].

V. Conclusions and Future Work

In this work, we defined the filtering operation on the unit sphere using the azimuthally non-symmetric functions. We posed and solved the concentration problem to find a family of bandlimited functions which are spatially concentrated in the azimuthally non-symmetric region and orthonormal over the whole spectral domain within maximum spherical harmonic degree. We also presented an analysis of eigenfunctions obtained as a solution of concentration problem and found that the number of eigenfunctions with significant concentration in the desired spatial region are upper bounded by the result in derived in [1]. We illustrated with the help of examples and then finally presented the simple realization of bandlimited filter as a weighted sum of computed eigenfunctions. This is the first step towards concentration problem for azimuthally non-symmetric region with the whole spectral domain under consideration. Following directions of future work can also be pursued:

1) We have considered only one type of azimuthally non-symmetric strip region at the north pole to solve the concentration problem as this type of region can be used to perform directional smoothing on the sphere. The north pole at $\theta = 0$ is only a convention and any point on the sphere can be considered as north pole. One can solve the concentration problem for different types of non-symmetric regions.
2) Filtering as convolution using azimuthally symmetric filter functions, when evaluated in spectral domain, results in the multiplication of spherical harmonics components of the signal and the filter function, but the component of all orders for a particular degree are multiplied by same component *i.e.*, the zero order component of the filter [14]. There is a need to define a filter of non-zero order components such that the convolution in spatial domain corresponds to multiplication of spherical harmonics components. Such a filter can be constructed using the proposed eigenfunctions.
3) The problem of spatially localized cross spectral analysis and spectral estimation on the sphere has been considered using azimuthally symmetric eigenfunctions as spatial windows, but needs to be revisited for the non-symmetric regions.

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