

A Dynamic Model of Contact between a Robot and an Environment with Unknown Dynamics

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Abstract

This paper presents an analysis of frictionless contact between a rigid body belonging to a robot mechanism and one belonging to its environment. According to this analysis, it is possible to design a hybrid motion/force controller such that the motion and force subsystems are instantaneously independent of each other, and both are instantaneously independent of the environmental dynamics. Such a control system should be able to operate in an environment with unknown dynamics.

1. Introduction

This paper presents a mathematical analysis of the dynamics of a robot mechanism in which a single body is making frictionless contact with a body in its environment. The latter is assumed to be part of an arbitrary rigid-body system with unknown dynamics. The contact is modelled as a general holonomic constraint between the two bodies; and it is assumed that the degree of constraint is constant at the current instant.

It is also assumed that the contact constraints are known, and that the positions and velocities of the two participating bodies are known and are consistent with the contact constraints.

In addition to the above assumptions, the method used in this paper requires that the participating robot body have six degrees of motion freedom (DMF). Apart from this, the method is completely general and is applicable to any kind of robot, including serial and parallel robots arms, mobile robots and robots with non-rigid mechanisms.

The analysis employs articulated-body equations to model the dynamics of the two participating bodies, and the change-of-basis technique [1] to resolve the equations into two independent subsystems. The analysis is conducted in a dual system of vector spaces, which assures the invariance and dimensional consistency of the resulting equations.

The outcome is a pair of equations, one describing the behaviour of the contacting robot body in the sub-

space of motion freedoms, and one in the subspace of contact forces. At the current instant, these equations are decoupled from each other, and the motion equation is independent of environmental dynamics. Furthermore, although the actual contact force does depend on the environment, the sum of this force with that required to maintain contact with the moving environment does not.

It is therefore possible to design a hybrid motion/force controller in which the motion and force subsystems are instantaneously decoupled, and the environment has no instantaneous effect on the controlled behaviour of either subsystem. There will be non-instantaneous effects, of course, but these are all felt at the position and velocity levels, not the acceleration level, so it may be possible to treat them as slowly-varying disturbances for the control system to reject.

Such a controller should be able to operate in contact with an environment having unknown dynamics.

The remainder of this paper is organized as follows. First, a little background material is presented in order to put this paper into context. This is followed by an introduction to the mathematics of systems of dual vector spaces, which forms the mathematical basis for all that follows. The remaining sections describe the contact model, the analysis and a possible control strategy.

2. Background

The main contribution of this paper concerns the degree to which the motion and force control subsystems of a hybrid motion/force controller can be decoupled when the robot is in contact with an unknown dynamic environment. The problems of dynamics compensation and decoupling have already received much attention; for example, see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In particular, it is already known that a hybrid motion/force controller with full compensation for the robot mechanism's dynamics will exhibit decoupled behaviour when the robot is in contact with a rigid environment [12]; and it is known that decoupling can

also be achieved when the robot is in contact with a general dynamic environment, provided the controller has at least partial knowledge of the environmental dynamics [13, 14]. This paper shows that the decoupling property extends to the case of contact with an unknown dynamic environment.

The subject matter of this paper is probably closest to that of [13], since both papers tackle essentially the same problem. The differences lie in the analytical method, the results obtained, and the proposed control scheme. This paper uses articulated-body models for both the robot and the environment. The former is closely related to the operational-space approach [6, 15]; and the latter is functionally equivalent to the second-order model in [13] for instantaneous dynamics.

The analysis is carried out using the change-of-basis technique described in [1]. The nearest similar idea is the modal decoupling of [3], which involves an eigenvalue-based decomposition of the product of an inverse mass matrix and a stiffness matrix. The change-of-basis technique uses a different kind of decomposition that reduces mass matrices and their inverses to block-diagonal form. Mathematically, the distinction is that modal decoupling works on a matrix that transforms vectors within a single vector space, whereas the change-of-basis decomposition applies to matrices that map vectors from one space to another.

Finally, this paper uses the concept of duality (or reciprocity) between motion and force vectors, in order to obtain an invariant formulation. It follows the formal approach advocated in [16], in which these vectors are assigned to two distinct vector spaces, each being the dual of the other. Many papers already use notions of duality and reciprocity, but often in a less explicit or less formal manner than here. (For example, generalized velocities and forces form a dual system.) More information on this topic can be found in [17, 18, 19, 20, 9].

3. Dual Vector Spaces

A system of dual vector spaces, or ‘dual system’ for short, is a mathematical structure comprising two vector spaces of equal dimension and a scalar product (a nondegenerate bilinear form) that takes one argument from each space. Each vector space is considered to be the dual of the other.

A dual system can be defined by listing its constituent parts, so we shall use the expression $\langle U, V, \cdot \rangle$ to denote the dual system consisting of the vector spaces U and V and the scalar product ‘ \cdot ’. If $\mathbf{u} \in U$ and

$\mathbf{v} \in V$ then the scalar product may be written either $\mathbf{u} \cdot \mathbf{v}$ or $\mathbf{v} \cdot \mathbf{u}$, both meaning the same, but the expressions $\mathbf{u} \cdot \mathbf{u}$ and $\mathbf{v} \cdot \mathbf{v}$ are not defined.

Dual systems arise naturally in the mathematical modelling of physical systems in which a scalar physical quantity equates with the scalar product of two different types of vector. In the case of rigid-body systems, the scalar is work (or power, virtual work, etc.) and the two types of vector are motions and forces. We therefore define a dual system $\langle M^n, F^n, \cdot \rangle$ in which M^n is a space of n -dimensional motion vectors, F^n is a space of n -dimensional force vectors, and the scalar product is the work done by a force vector acting on a motion vector. Examples of motion vectors include (generalized) velocities, accelerations, infinitesimal displacements and directions of motion freedom; and examples of force vectors include (generalized) forces, momenta and contact normals.

The same basic structure appears in bond graphs, where effort and flow vectors combine to form energy scalars, and also in tensor calculus, where covariant and contravariant vectors combine to form invariant scalars. In hybrid motion/force control, dual systems support invariant formulations [16, 9].

3.1. Reciprocal Coordinates

A basis on a dual system $\langle U, V, \cdot \rangle$ is a set of vectors, half of which form a basis on U , while the other half form a basis on V .

Let P be a basis on $\langle M^n, F^n, \cdot \rangle$, and let $P = \{P_M, P_F\}$ where $P_M = \{\mathbf{d}_1, \dots, \mathbf{d}_n\} \subset M^n$ is a basis on M^n and $P_F = \{\mathbf{e}_1, \dots, \mathbf{e}_n\} \subset F^n$ is a basis on F^n . In P coordinates, motion vectors are expressed using P_M and force vectors using P_F .

If the elements of P_M and P_F satisfy

$$\mathbf{d}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

then P_M is said to be reciprocal to P_F , and P defines a reciprocal coordinate system on $\langle M^n, F^n, \cdot \rangle$. (The term ‘dual coordinates’ is also used.) The special property of reciprocal coordinates is that the scalar product takes the form $\mathbf{m} \cdot \mathbf{f} = \mathbf{m}^T \mathbf{f}$ and is invariant with respect to any change of basis from one reciprocal coordinate system to another.

Reciprocal coordinates are the dual-system equivalent of orthonormal (or Cartesian) coordinates in a Euclidean vector space, but there is one important difference: there are n^2 freedoms to choose a reciprocal basis on a dual system, compared with only $n(n-1)/2$ freedoms to choose an orthonormal basis on a Eu-

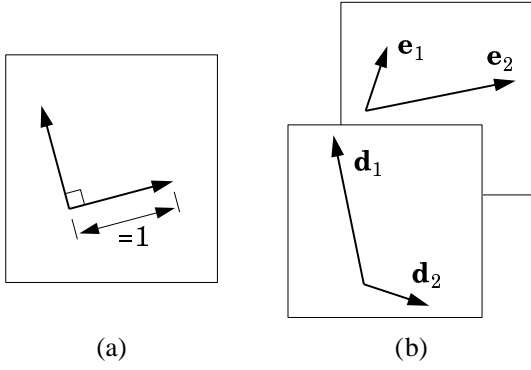


Figure 1: The difference between an orthonormal basis on a 2D Euclidean space (a) and a reciprocal basis on a 2D system of dual vector spaces (b).

clidean n -space. These extra freedoms encompass generalized coordinates, and they provide the change-of-basis technique with enough freedoms to work.

The difference is illustrated in Figure 1. The orthonormal basis in Figure 1(a) consists of two unit vectors at right angles. Since neither the lengths of the vectors nor the angle between them can be altered, the only parameter that can be varied freely is the overall orientation of the basis. In contrast, the reciprocal basis shown in Figure 1(b) consists of four vectors in total, two in each space. In general, one may choose any two of these vectors freely, leaving the other two to be determined by the reciprocity conditions; so there are a total of four freedoms to choose a reciprocal basis on this dual system.

Although Figure 1(b) uses the visual cues of angle and magnitude to suggest the reciprocity conditions (for example, \mathbf{d}_1 is shown at right angles to \mathbf{e}_2 to suggest $\mathbf{d}_1 \cdot \mathbf{e}_2 = 0$), it should be realized that the concepts of angle and magnitude are usually undefined in a dual system.

3.2. Coordinate Transforms

In general, motion and force vectors obey different coordinate transformation rules. If P and Q are two reciprocal bases on $\langle M^n, F^n, \cdot \rangle$, and \mathbf{m}_P , \mathbf{m}_Q , \mathbf{f}_P and \mathbf{f}_Q are coordinate vectors representing $\mathbf{m} \in M^n$ and $\mathbf{f} \in F^n$ in P and Q coordinates, respectively, then

$$\mathbf{m}_Q = \mathbf{X}_m \mathbf{m}_P, \quad \mathbf{f}_Q = \mathbf{X}_f \mathbf{f}_P,$$

$$\mathbf{m}_P = \mathbf{X}_m^{-1} \mathbf{m}_Q, \quad \mathbf{f}_P = \mathbf{X}_f^{-1} \mathbf{f}_Q,$$

where \mathbf{X}_m and \mathbf{X}_f are the coordinate transformation matrices that perform the transformations from P_M to Q_M coordinates, and P_F to Q_F coordinates, respec-

mapping	transformation rule	transform type
$M^n \mapsto M^n$	$\mathbf{A}_Q = \mathbf{X}_m \mathbf{A}_P \mathbf{X}_m^{-1}$	similarity
$M^n \mapsto F^n$	$\mathbf{B}_Q = \mathbf{X}_f \mathbf{B}_P \mathbf{X}_m^{-1}$	congruence
$F^n \mapsto M^n$	$\mathbf{C}_Q = \mathbf{X}_m \mathbf{C}_P \mathbf{X}_f^{-1}$	congruence
$F^n \mapsto F^n$	$\mathbf{D}_Q = \mathbf{X}_f \mathbf{D}_P \mathbf{X}_f^{-1}$	similarity

Table 1: Transformation rules for linear mappings defined on a dual system.

tively. The reciprocity property ensures that

$$\mathbf{X}_f = (\mathbf{X}_m^{-1})^T,$$

which is sufficient to guarantee the invariance of the scalar product.

The two vector spaces of a dual system give rise to four types of linear mapping: $M^n \mapsto M^n$, $M^n \mapsto F^n$, $F^n \mapsto M^n$ and $F^n \mapsto F^n$. They can all be represented using $n \times n$ matrices, but each has its own transformation rule, as shown in Table 1. (They are essentially the same as the rules for transforming the four types of dyadic tensor.) Observe that two mappings obey similarity transforms, which preserve eigenvalues, while the other two obey congruence transforms, which preserve symmetry and definiteness.

3.3. Subspaces

Any subspace of a vector space can be expressed as the range of a suitable matrix. If S is an m -dimensional subspace of an n -dimensional vector space then it can be expressed as the range of an $n \times m$ matrix \mathbf{S} whose columns are the coordinates of any m linearly-independent vectors that span S . These vectors form a basis on S , so \mathbf{S} serves to define both a subspace and a basis. The matrix transforms like its column vectors; and any element of S can be expressed in the form $\mathbf{S} \boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is an $m \times 1$ vector of coordinates.

Subspaces can be used to define linear decompositions of the parent space. If a vector space V is the direct sum of two subspaces S_1 and S_2 (written $V = S_1 \oplus S_2$) then any vector $\mathbf{v} \in V$ can be decomposed uniquely into $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in S_1$ and $\mathbf{v}_2 \in S_2$. Any two subspaces will direct-sum to V provided they have no nonzero element in common and their dimensions sum to the dimensions of V .

The decomposition can also be written in the form

$$\mathbf{v} = \mathbf{S}_1 \boldsymbol{\alpha}_1 + \mathbf{S}_2 \boldsymbol{\alpha}_2 = [\mathbf{S}_1 \ \mathbf{S}_2] \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix}.$$

In this equation, $[\mathbf{S}_1 \ \mathbf{S}_2]$ is a square matrix that can be interpreted as a coordinate transform, and $[\boldsymbol{\alpha}_1^T \ \boldsymbol{\alpha}_2^T]^T$ is

a representation of \mathbf{v} in the coordinate system defined by the columns of $[\mathbf{S}_1 \mathbf{S}_2]$.

If two subspaces $S \subset M^n$ and $T \subset F^n$ have the property that $\mathbf{s} \cdot \mathbf{t} = 0$ for every $\mathbf{s} \in S$ and $\mathbf{t} \in T$ then we say that they are reciprocal, and write $S \perp T$. If, in addition, they satisfy $\dim(S) + \dim(T) = n$ (i.e., their dimensions sum to n) then we say that T is the *reciprocal complement* of S , and write $T = S^\perp$. Reciprocal complements are unique, and satisfy $(S^\perp)^\perp = S$.

Notice that the word ‘reciprocal’ has a different meaning in ‘reciprocal complement’ to that in ‘reciprocal basis’. In the former, it refers to the screw-theoretic definition of reciprocity between twists and wrenches; while in the latter, it refers to the fact that the product of the two sets of basis vectors is the identity matrix. In multilinear algebra, S^\perp would be called the annihilator of S .

3.4. M^6 and F^6

The dual system $\langle M^6, F^6, \cdot \rangle$ is ideally suited to describing rigid-body dynamics; and most existing 6D vector notations can easily be translated into a dual-system format, so that existing formulas and equations can be re-used. In general, the translation process involves the following steps:

1. Formally assign each vector quantity to the appropriate vector space (M^6 or F^6), and classify mappings and dyadics as per Table 1.
2. Adopt a reciprocal coordinate system (see below).
3. Convert conventional accelerations to spatial accelerations [21], so that they behave like vectors.
4. Abandon certain concepts that are incompatible with duality (e.g. inner products, orthogonal complements and the common-screw relation between twists and wrenches).

Most 6-D vector notations use Plücker ray and/or axis coordinates (see [20]). If ray coordinates are used for M^6 and axis coordinates for F^6 , or vice versa, then the coordinates are reciprocal.

4. Dynamic Model of Contact

This section presents a dynamic model of a general state of contact between a robot body B^r and an environment body B^e . The model consists of an equation of motion for each participating body (in the absence

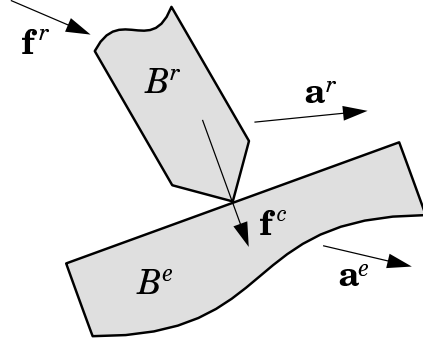


Figure 2: Contact between robot body B^r and environment body B^e .

of contact) and a description of the contact constraint. The equations are:

$$\mathbf{a}^r = \Phi^r (\mathbf{f}^r - \mathbf{f}^c) + \mathbf{b}^r, \quad (1)$$

$$\mathbf{a}^e = \Phi^e \mathbf{f}^c + \mathbf{b}^e, \quad (2)$$

$$\mathbf{a}^r - \mathbf{a}^e = \mathbf{S}^c \dot{\alpha} + \dot{\mathbf{S}}^c \alpha \quad (3)$$

and

$$(\mathbf{S}^c)^T \mathbf{f}^c = \mathbf{0}. \quad (4)$$

(See Figure 2.)

Eqs. 1 and 2 express the dynamic behaviour of B^r and B^e in the form of articulated-body equations of motion [21]. This type of equation allows each body to be a member of an arbitrary rigid-body system of unlimited size and complexity. Neither equation describes the dynamics of an entire rigid-body system, but each describes the totality of dynamic effects that are felt at the relevant body. They are therefore fully general for the task at hand.

\mathbf{a}^r and \mathbf{a}^e are the spatial accelerations of B^r and B^e , respectively; Φ^r and Φ^e are their articulated-body inverse inertias; and \mathbf{b}^r and \mathbf{b}^e are their bias accelerations. \mathbf{b}^r is defined as the acceleration that B^r would have if $\mathbf{f}^r - \mathbf{f}^c = \mathbf{0}$, and therefore accounts for the sum of all forces acting on B^r other than $\mathbf{f}^r - \mathbf{f}^c$. \mathbf{b}^e is defined similarly. Φ^r and Φ^e are SPSD matrices with ranks equal to the DMF of their respective bodies. At one extreme, if B^e had no freedom to move then the rank of Φ^e would be zero (which implies that $\Phi^e = \mathbf{0}$). At the other extreme, if B^r had a full 6 DMF then Φ^r would have full rank and hence be an SPD matrix.

\mathbf{f}^r is a force transmitted to B^r from other parts of the robot mechanism, and \mathbf{f}^c is the force transmitted from B^r to B^e through the contact. \mathbf{f}^r is assumed to contain all controllable forces acting on B^r , but its exact definition can be chosen to suit individual circumstances. Any force that is not included in \mathbf{f}^r will have its effects incorporated into \mathbf{b}^r instead.

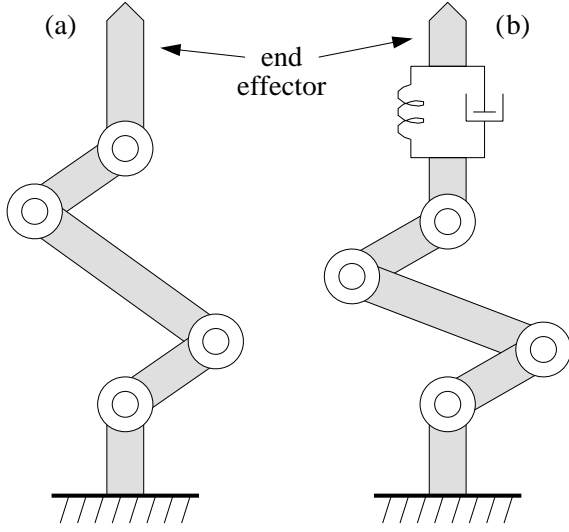


Figure 3: A rigid robot mechanism (a) and a robot with a compliance between the end effector and the arm (b).

$S^c \subset M^6$ is the subspace of instantaneous motion freedoms permitted by the contact constraints at the current configuration, and S^c is a matrix representing S^c . If the contact imposes d constraints on the relative motion of B^r and B^e then S^c has dimension $6 - d$ and S^c is a $6 \times (6 - d)$ matrix.

Eq. 3 expresses the acceleration constraint imposed by the contact, which is simply the time-derivative of the velocity constraint equation: $\mathbf{v}^r - \mathbf{v}^e = \mathbf{S} \alpha$. At the acceleration level, all velocities are assumed to be known; so $\dot{\alpha}$ is treated as a vector of unknown acceleration variables, while α is treated as known. \dot{S}^c is also assumed to be known.

Finally, Eq. 4 expresses the fact that the constraint force does no work in any direction of motion that is compatible with the motion constraints.

Eq. 1 is capable of modelling the dynamics of any body in a general rigid-body system. It is therefore applicable to practically any robot, including mobile robots, parallel robots and so on. A couple of examples are shown in Figure 3. Both are serial robot arms, and in both cases B^r is the end effector.

Figure 3(a) shows a robot with a rigid mechanism—one composed entirely of rigid bodies and ideal joints. In this example, Eq. 1 can be equated with the operational-space formulation of end-effector dynamics, such as Eqs. 14 or 50 in [6] or Eqs. 3, 9 or 23 in [15]. If we take

$$\Lambda \ddot{\mathbf{x}} + \boldsymbol{\mu} + \mathbf{p} = \mathbf{F}$$

to be a representative operational-space formulation, where Λ is the operational-space inertia of the end-effector and $\boldsymbol{\mu}$ and \mathbf{p} contain velocity-product and

gravitational terms, respectively, then $\Phi^r = \Lambda^{-1}$ and $\mathbf{b}^r = -\Lambda^{-1}(\boldsymbol{\mu} + \mathbf{p})$.

The robot in Figure 3(b) consists of an end-effector body connected to the end of a rigid robot arm via a generalized spring and damper. A system like this could be used to model a robot with a wrist-mounted 6-axis force sensor. The end effector in this example is kinematically independent of the arm, so Eq. 1 should be the equation of motion of just this one body. Φ^r should be the inverse of the end-effector's rigid-body inertia; \mathbf{f}^r should be the force transmitted to the end-effector through the spring and damper; and \mathbf{b}^r should account for gravity and velocity-product terms. Bear in mind that \mathbf{a}^r refers to the acceleration of the end-effector, not the end of the arm.

5. Analysis

This section derives an equation of motion for B^r , including the effect of the contact, using the change-of-basis technique described in [1]. This method requires Φ^r to be an SPD matrix, so it is applicable only to systems in which B^r has 6 DMF in the absence of the contact constraint. If B^r has fewer DMF then a different analytical procedure must be used, which is outside the scope of this paper.

The first step is to define four subspaces, S_1, S_2, T_1 and T_2 , that satisfy the following equations:

$$\begin{aligned} S_1 \oplus S_2 &= M^6, & T_1 \oplus T_2 &= F^6, \\ S_1 &= S^c, & T_1 &= (\Phi^r)^{-1} S_1, \\ S_2 &= \Phi^r T_2, & T_2 &= (S^c)^\perp. \end{aligned}$$

These spaces are defined uniquely by S^c and Φ^r , and they have the effect of decomposing $\langle M^6, F^6, \cdot \rangle$ into a pair of dual subsystems, $\langle S_1, T_1, \cdot \rangle$ and $\langle S_2, T_2, \cdot \rangle$, which are aligned with the directions of freedom and constraint as specified by S^c and $(S^c)^\perp$.

The second step is to define matrices S_1, S_2, T_1 and T_2 to represent the above four subspaces and simultaneously satisfy

$$[S_1 \ S_2]^T [T_1 \ T_2] = \mathbf{1}_{6 \times 6}.$$

This condition ensures that the column vectors of the four matrices form a reciprocal basis on $\langle M^6, F^6, \cdot \rangle$. In this special basis, the coordinates fall naturally into two groups: one associated with $\langle S_1, T_1, \cdot \rangle$ and one with $\langle S_2, T_2, \cdot \rangle$. Furthermore, Φ^r takes block-diagonal form, with one block in $\langle S_1, T_1, \cdot \rangle$ and one in $\langle S_2, T_2, \cdot \rangle$. If S_1 and T_2 are any two matrix representations of S_1 and T_2 then the reciprocity condition can be satisfied by choosing

$$T_1 = (\Phi^r)^{-1} S_1 (S_1^T (\Phi^r)^{-1} S_1)^{-1}$$

and

$$\mathbf{S}_2 = \Phi^r \mathbf{T}_2 (\mathbf{T}_2^T \Phi^r \mathbf{T}_2)^{-1}.$$

The next step is to transform Eqs. 1–4 to the special basis. If we define \mathbf{X}_m and \mathbf{X}_f to be the coordinate transformation matrices for motion and force vectors, respectively, from the given basis to the special basis, then

$$\mathbf{X}_m = [\mathbf{T}_1 \ \mathbf{T}_2]^T, \quad \mathbf{X}_m^{-1} = [\mathbf{S}_1 \ \mathbf{S}_2],$$

$$\mathbf{X}_f = [\mathbf{S}_1 \ \mathbf{S}_2]^T, \quad \mathbf{X}_f^{-1} = [\mathbf{T}_1 \ \mathbf{T}_2].$$

If $[(\mathbf{a}_1^r)^T (\mathbf{a}_2^r)^T]^T$ is the coordinate vector representing \mathbf{a}^r in the special basis then

$$\begin{bmatrix} \mathbf{a}_1^r \\ \mathbf{a}_2^r \end{bmatrix} = \mathbf{X}_m \mathbf{a}^r = \begin{bmatrix} \mathbf{T}_1^T \mathbf{a}^r \\ \mathbf{T}_2^T \mathbf{a}^r \end{bmatrix}$$

and

$$\mathbf{a}^r = \mathbf{X}_m^{-1} \begin{bmatrix} \mathbf{a}_1^r \\ \mathbf{a}_2^r \end{bmatrix} = \mathbf{S}_1 \mathbf{a}_1^r + \mathbf{S}_2 \mathbf{a}_2^r.$$

All other motion vectors transform similarly; and force vectors obey similar equations with \mathbf{X}_f in place of \mathbf{X}_m .

Transforming Eqs. 1 and 2 to the special basis produces

$$\begin{bmatrix} \mathbf{a}_1^r \\ \mathbf{a}_2^r \end{bmatrix} = \begin{bmatrix} \Phi_{11}^r & \mathbf{0} \\ \mathbf{0} & \Phi_{22}^r \end{bmatrix} \begin{bmatrix} \mathbf{f}_1^r - \mathbf{f}_1^c \\ \mathbf{f}_2^r - \mathbf{f}_2^c \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1^r \\ \mathbf{b}_2^r \end{bmatrix} \quad (5)$$

and

$$\begin{bmatrix} \mathbf{a}_1^e \\ \mathbf{a}_2^e \end{bmatrix} = \begin{bmatrix} \Phi_{11}^e & \Phi_{12}^e \\ \Phi_{21}^e & \Phi_{22}^e \end{bmatrix} \begin{bmatrix} \mathbf{f}_1^c \\ \mathbf{f}_2^c \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1^e \\ \mathbf{b}_2^e \end{bmatrix}, \quad (6)$$

where $\Phi_{ii}^r = \mathbf{T}_i^T \Phi^r \mathbf{T}_i$ and $\Phi_{ij}^e = \mathbf{T}_i^T \Phi^e \mathbf{T}_j$, $i, j \in \{1, 2\}$, from the appropriate formula in Table 1.

Transforming Eq. 3 to the special basis produces

$$\begin{bmatrix} \mathbf{a}_1^r - \mathbf{a}_1^e \\ \mathbf{a}_2^r - \mathbf{a}_2^e \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \dot{\boldsymbol{\alpha}} + \begin{bmatrix} \mathbf{b}_1^c \\ \mathbf{b}_2^c \end{bmatrix}$$

where $\mathbf{b}^c = \dot{\mathbf{S}}^c \boldsymbol{\alpha}$. (Note the special form of \mathbf{S}^c in the special basis.) As $\dot{\boldsymbol{\alpha}}$ is a free variable, there is actually no constraint on the value of $\mathbf{a}_1^r - \mathbf{a}_2^e$; so the constraint equation can be simplified to

$$\mathbf{a}_2^r - \mathbf{a}_2^e = \mathbf{b}_2^c. \quad (7)$$

Finally, transforming Eq. 4 to the special basis, and simplifying the result, produces

$$\mathbf{f}_1^c = \mathbf{0}. \quad (8)$$

Now that the equations are all expressed in the special basis, all that remains is to solve them. Substituting Eq. 8 into Eqs. 5 and 6 produces

$$\mathbf{a}_1^r = \Phi_{11}^r \mathbf{f}_1^r + \mathbf{b}_1^r, \quad (9)$$

$$\mathbf{a}_2^r = \Phi_{22}^r (\mathbf{f}_2^r - \mathbf{f}_2^c) + \mathbf{b}_2^r \quad (10)$$

and

$$\mathbf{a}_2^e = \Phi_{22}^e \mathbf{f}_2^c + \mathbf{b}_2^e. \quad (11)$$

(We are not interested in \mathbf{a}_1^e .) Substituting Eqs. 10 and 11 into Eq. 7 produces

$$\Phi_{22}^r (\mathbf{f}_2^r - \mathbf{f}_2^c) + \mathbf{b}_2^r - \Phi_{22}^e \mathbf{f}_2^c - \mathbf{b}_2^e = \mathbf{b}_2^c,$$

from which we obtain

$$\mathbf{f}_2^c = (\Phi_{22}^r + \Phi_{22}^e)^{-1} (\Phi_{22}^r \mathbf{f}_2^r + \mathbf{b}_2^r - \mathbf{b}_2^e - \mathbf{b}_2^c). \quad (12)$$

Eqs. 9, 10 and 12 between them describe the dynamic behaviour of B^r , taking into account the effect of the contact. Eq. 9 describes the behaviour of B^r in $\langle S_1, T_1, \cdot \rangle$, while Eq. 10 (with \mathbf{f}_2^c given by Eq. 12) describes its behaviour in $\langle S_2, T_2, \cdot \rangle$.

The most striking feature of these equations is that Eq. 9 is independent of environmental dynamics. This means that the behaviour of B^r in $\langle S_1, T_1, \cdot \rangle$ is instantaneously independent of environmental dynamics, although the environment will, of course, have an integral effect that is evident over time. Another useful feature is that Eqs. 9 and 10 are decoupled from each other, which means that the behaviour of B^r in $\langle S_1, T_1, \cdot \rangle$ is instantaneously independent of its behaviour in $\langle S_2, T_2, \cdot \rangle$; and a third interesting feature is that of all the quantities appearing in Eq. 6, only Φ_{22}^e and \mathbf{b}_2^e have any instantaneous effect on B^r .

To summarize, the equation of motion of B^r can be expressed as the sum of two instantaneously independent subsystems: one in $\langle S_1, T_1, \cdot \rangle$, which is aligned with the motion freedoms of the contact constraint, and one in $\langle S_2, T_2, \cdot \rangle$, which is aligned with the directions of constraint. The former is instantaneously independent of environmental dynamics, while the latter depends on only a subset of the dynamics of B^e . The only assumption needed to achieve these results is that Φ^r is an SPD matrix.

6. Control

This section applies the results of the previous section to the analysis of a control law for a hybrid motion/force controller. It is shown that the motion and force subsystems are instantaneously decoupled, and that the former is decoupled from the environmental

dynamics. A modification to the force subsystem is suggested that decouples it also from the environmental dynamics.

Consider the following control law:

$$\mathbf{f}^r = (\Phi^r)^{-1} (\mathbf{S}_1 \mathbf{u}_m - \mathbf{b}^r) + \mathbf{T}_2 \mathbf{u}_f, \quad (13)$$

where \mathbf{u}_m and \mathbf{u}_f are vectors computed by the motion- and force-control subsystems, respectively. Observe that it does not require any knowledge of the environment's dynamics. Transforming this equation to the special basis produces

$$\begin{bmatrix} \mathbf{f}_1^r \\ \mathbf{f}_2^r \end{bmatrix} = \begin{bmatrix} (\Phi_{11}^r)^{-1} (\mathbf{u}_m - \mathbf{b}_1^r) \\ \mathbf{u}_f - (\Phi_{22}^r)^{-1} \mathbf{b}_2^r \end{bmatrix};$$

and substituting the expressions for \mathbf{f}_1^r and \mathbf{f}_2^r from this equation into Eqs. 9 and 10 produces

$$\mathbf{a}_1^r = \mathbf{u}_m \quad (14)$$

and

$$\mathbf{f}_2^c + (\Phi_{22}^r)^{-1} \mathbf{a}_2^r = \mathbf{u}_f. \quad (15)$$

These are the equations of motion for B^r under closed-loop control via Eq. 13. Eq. 14 describes the behaviour of B^r in $\langle S_1, T_1, \cdot \rangle$ as a function of the motion control signal \mathbf{u}_m ; and Eq. 15 describes the behaviour of B^r in $\langle S_2, T_2, \cdot \rangle$ as a function of the force control signal \mathbf{u}_f .

These equations describe a pair of decoupled subsystems, in the sense that \mathbf{u}_m has no instantaneous effect in $\langle S_2, T_2, \cdot \rangle$ and \mathbf{u}_f has no instantaneous effect in $\langle S_1, T_1, \cdot \rangle$. Therefore a hybrid motion/force controller that uses Eq. 13 to combine the outputs of the force and motion control laws exhibits no instantaneous crosstalk between the motion and force control channels. There will, of course, still be some degree of cross-coupling between the two subsystems; but these effects are carried via position- and velocity-dependent terms, so it may be feasible to treat them as slowly-varying disturbances for the control system to reject.

Observe that neither Eq. 14 nor Eq. 15 contains any instantaneous dependency on the environment. We have already established this property for Eq. 9, so it is not surprising if it is inherited by Eq. 14; but Eq. 15 demonstrates that the environmental dependency of \mathbf{f}_2^c exactly cancels that of $(\Phi_{22}^r)^{-1} \mathbf{a}_2^r$, so that the expression $\mathbf{f}_2^c + (\Phi_{22}^r)^{-1} \mathbf{a}_2^r$ is independent of environmental dynamics. This quantity can be thought of as the sum of the contact force and the force required to accelerate the robot so as to maintain contact with the environment.

If the force control subsystem is designed so that the objective is to control the value of $\mathbf{f}_2^c + (\Phi_{22}^r)^{-1} \mathbf{a}_2^r$

rather than \mathbf{f}_2^c , then the controlled behaviour in both the motion and force subsystems will be independent of environmental dynamics. The environment will, of course, still have an effect on the robot's behaviour, but it does so via position- and velocity-dependent terms which the control system could treat as slowly-varying disturbances. A control system that seeks to control $\mathbf{f}_2^c + (\Phi_{22}^r)^{-1} \mathbf{a}_2^r$ in the first instance, still has the option of controlling \mathbf{f}_2^c via an outer loop operating at a lower frequency.

Finally, note that Eq. 13 is not a new control law; so the results in this section apply also to any existing control scheme that happens to use the same control law, or an equivalent one.

7. Conclusion

This paper has presented an analysis of frictionless rigid-body contact between a general robot and a general dynamic environment, which assumes only that the participating robot body has six degrees of motion freedom. The analysis uses articulated-body equations to describe the dynamics of the participating bodies, and the change-of-basis technique to resolve the equations into independent subsystems.

It was shown that the equation of motion of the participating robot body can be resolved into two subsystems, one in the space of motion freedoms and one in the space of contact constraints, and that the former is instantaneously independent of environmental dynamics.

A hybrid motion/force controller based on these results is free of instantaneous cross-coupling between the motion and force control channels, and the controlled motion behaviour is instantaneously independent of environmental dynamics. If the force-control subsystem is designed to control the sum of the actual contact force and the force required to accelerate the robot in pursuit of maintaining contact, then the controlled force behaviour is also instantaneously independent of environmental dynamics.

An experimental investigation of this theory is being planned.

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