Multilinear Relationships between Coordinates of Corresponding Image Points and Lines.

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Abstract

This paper describes how the fundamental matrix, F, the trifocal tensor T_i^{jk} and the quadrilinear relationship existing between corresponding points in four uncalibrated projective images may be derived in a common framework involving matrix determinants. Part of the paper contains a derivation of previous results, and is intended as a summary and reformulation. The derivations are based on the work of Faugeras and Mourrain [4] and Triggs [23, 22].

Some new results are included on the independence of the equations used to compute the trifocal and quadrilinear relationships, and methods of choosing those equations in a robust manner.

1 Introduction

The Fundamental Matrix, introduced by Longuet-Higgins ([15]) has proven to be a basic tool in the analysis of pairs of images. In particular, it has been been particularly fruitful in the analysis of uncalibrated image pairs since being applied to this problem in [3, 7, 6]. In a similar way, the trifocal (or trinocular) tensor has its roots ([20, 19, 24]) in the analysis of calibrated images, but was later rediscovered [8, 11, 17]) and applied to uncalibrated images. The discovery of a similar quadrilinear relationship followed, and was investigated in ([23, 22, 4, 25, 14, 13]). All these multilinear relationships were put in a common framework by the work of Triggs ([23]) and Faugeras and Mourrain [4].

This paper provides a summary of the derivation of the multilinear relationships, and their expression in terms of the multidimensional tensors, namely the fundamental matrix F_{ij}, the trifocal tensor T_i^{qr} and the quadrifocal tensor Q^{pqrs}. Specific formulae are given for each of these tensors in terms of the camera matrices. These tensors relate the coordinates of lines and points in the separate images, as summarized in Tables 1 and 2 of this paper. This gives a more complete set of relationships than has been written down previously.

Finally, the independence of the set of equations derived from point and line correspondences is considered. This analysis leads to a recommended method for formulating the equations derived from a correspondence in order to avoid singular cases and achieve greatest numerical robustness. In the four-view case, each point correspondence gives 16 independent equations in the 81 entries of the quadrifocal tensor. This suggests that the quadrifocal tensor ought to be computable from just five point correspondences, but other considerations mean thas this should be impossible. This conundrum is resolved by observing that whereas the set of equations derived from one correspondence has the expected full rank (16 independent equations), the equations derived from different point correspondences are not independent.

2 Bilinear Relations

We consider first the relationship that holds between the coordinates of a point seen in two separate views. Thus, let $\mathbf{u} \leftrightarrow \mathbf{u}'$ be a pair of corresponding points as seen in two separate images. It will be convenient, for clarity of notation, to represent the two camera matrices by A and B, instead of the usual notation, P and P'. Both the points $\mathbf{u}$ and $\mathbf{u}'$ are images of the same point $\mathbf{x}$ in space. For convenience, we write

$$\mathbf{u} = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \quad ; \quad \mathbf{u}' = \begin{pmatrix} u'^1 \\ u'^2 \\ u'^3 \end{pmatrix} \quad ; \quad \mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \tag{1}$$

The projection from space to image can now be expressed as follows.

$$\begin{aligned} k \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} &= A\mathbf{x} \\ k' \begin{pmatrix} u'^1 \\ u'^2 \\ u'^3 \end{pmatrix} &= B\mathbf{x} \end{aligned} \tag{2}$$

where k and k' are two undetermined constants.

These equations may be written down in one matrix equation. In order to do this, we denote the i-th row of the matrix A by $\mathbf{a}^i_\cdot$, and similarly the i-th row of matrix B by $\mathbf{b}^i_\cdot$. The projection due to the first camera may then be written as

$$\begin{bmatrix} \mathbf{a}^1_\cdot & u^1 \\ \mathbf{a}^2_\cdot & u^2 \\ \mathbf{a}^3_\cdot & u^3 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ -k \end{pmatrix} = 0 \ . \tag{3}$$

This expression may be compared with (2) which is is just another way of writing the same thing.

The projections of the point $\mathbf{x}$ into both views may be expressed as a single matrix equation by writing the equations derived from one view and derived from the other view in the same equation. This gives a set of equations

$$\left[\begin{array}{ccc} \mathbf{a}^1_\cdot & u^1 & \\ \mathbf{a}^2_\cdot & u^2 & \\ \mathbf{a}^3_\cdot & u^3 & \\ \hline \mathbf{b}^1_\cdot & & u'^1 \\ \mathbf{b}^2_\cdot & & u'^2 \\ \mathbf{b}^3_\cdot & & u'^3 \end{array}\right] \begin{pmatrix} \mathbf{x} \\ -k \\ -k' \end{pmatrix} = 0 \tag{4}$$

Now, this is a 6×6 set of equations which by hypothesis has a non-zero solution, the vector $(\mathbf{x}, -k, -k')^\top$. It follows that the matrix of coefficients in (4) must have zero determinant. This condition leads to a bilinear relationship between the entries of the vectors $\mathbf{u}$ and $\mathbf{u}'$ expressed by the fundamental matrix F. We will now look specifically at the form of this relationship.

Consider the matrix appearing in (4). Denote it by X. The determinant of X may be written as an expression in terms of the quantities u^i and u'^i. Notice that the entries u^i and u'^i appear in only two columns of X. This implies that the determinant of X may be expressed as a quadratic expression in terms of the u^i and u'^i. In fact, since all the entries u^i appear in the same column, there can be no terms of the form $u^i u^j$ or $u'^i u'^j$. Briefly, as an expression in terms of the u^i and u'^i, the determinant of X is a bilinear expression. The fact that the determinant is zero may be written as an equation

$$(u'^1, u'^2, u'^3) F (u^1, u^2, u^3)^\top = u^i u'^j F_{ji} = 0 \tag{5}$$

where F is a 3×3 matrix, the well-known fundamental matrix[1].

We may compute a specific formula for the entries of the matrix F as follows. The entry F_{ij} of F is the coefficient of the term $u'^i u^j$ in the expansion of the determinant of X. In order to find this coefficient, we must eliminate the rows and columns of the matrix containing u'^i and u^j, take the determinant of the resulting matrix and multiply by ± 1 as appropriate. For instance, the coefficient of $u'^1 u^1$ is obtained by eliminating two rows and the last two columns of the matrix X as shown in (4). The remaining matrix is

$$\begin{pmatrix} \mathbf{a}^2_{\cdot} \\ \mathbf{a}^3_{\cdot} \\ \mathbf{b}^2_{\cdot} \\ \mathbf{b}^3_{\cdot} \end{pmatrix}$$

and the coefficient of $u'^1 u^1$ is equal to the determinant of this 4×4 matrix. In general, we may write

$$F_{ji} = (-1)^{i+j} \det \begin{bmatrix} \sim \mathbf{a}^i_{\cdot} \\ \sim \mathbf{b}^j_{\cdot} \end{bmatrix} \ . \tag{6}$$

In this expression, the notation $\sim \mathbf{a}^i_{\cdot}$ has been used to denote the matrix obtained from A by **omitting** the row $\mathbf{a}^i_{\cdot}$. Thus the symbol $\sim$ may be read as *omit*, and $\sim \mathbf{a}^i_{\cdot}$ represents two rows of A. The determinant appearing on the right side of (6) is therefore a 4×4 determinant. This expression for the fundamental matrix was pointed out to me by Rajiv Gupta, and is also noted by Carlsson ([1]).

A different way of writing the expression for F_{ji} makes use of the tensor ϵ_{rst} which is defined to be 0 unless all of r, s and t are different, and is ± 1 depending on whether the indices (r, s, t) consitute an even or odd permutation of $(1, 2, 3)$. The tensor ϵ_{ijk} (or its contravariant counterpart, ϵ^{ijk}) is connected with the cross product of two vectors. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, and $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ is their cross product, then the following formula may easily be verified.

$$c_i = (\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a^j b^k \ .$$

[1]Here and elsewhere we use the tensor summation convention that an index repeated in upper (contravariant) and lower (covariant) positions implies summation over the range of indices. It may be more sensible to define F_{ij} by the formula $u^i u'^j F_{ij} = 0$, but the formula (5) is conventional.

Using this notation, one may derive the following formula.

$$F_{ji} = \left(\frac{1}{4}\right) \epsilon_{ipq}\epsilon_{jrs} \det \begin{bmatrix} \mathbf{a}^p_{\cdot} \\ \mathbf{a}^q_{\cdot} \\ \mathbf{b}^r_{\cdot} \\ \mathbf{b}^s_{\cdot} \end{bmatrix} . \tag{7}$$

To see this, note that F_{ji} is defined in (7) in terms of a sum of determinants over all values of p, q, r and s. However for a given value of i, the tensor ϵ_{ipq} is zero unless p and q are different from i and from each other. This leaves only two remaining choices of p and q (for example if $i = 1$, then we may choose $p = 2, q = 3$ or $p = 3, q = 2$). Similarly, there are only two different choices of r and s giving rise to non-zero terms. Thus the sum consists of 4 non-zero terms only. Furthermore, the determinants appearing in these four terms consists of the same four rows of the matrices A and B and hence have equal values, except for sign. However, the value of $\epsilon_{ipq}\epsilon_{jrs}$ is such that the four terms all have the same sign and are equal. Thus, the sum (7) is equal to the single term appearing in (6).

A similar formula involving the fundamental matrix is

$$F_{ji}\epsilon^{ipq}\epsilon^{jrs} = \det \begin{bmatrix} \mathbf{a}^p_{\cdot} \\ \mathbf{a}^q_{\cdot} \\ \mathbf{b}^r_{\cdot} \\ \mathbf{b}^s_{\cdot} \end{bmatrix} . \tag{8}$$

This formula may be derived in a straight-forward manner from (7).

2.1 Invariants of Lines

In this brief section it will be shown how the fundamental matrix may be used to define invariants of spatial objects (in this particular case, lines) in terms of the images of those objects in a pair of images. This method was discovered by Carlsson ([1]). Given two lines λ and μ in one image, and the corresponding lines λ' and μ' in the other image. From (8) we may see that

$$\begin{aligned}
(\lambda' \times \mu')^\top F(\lambda \times \mu) &= F_{ji}(\lambda_p \mu_q \epsilon^{ipq})(\lambda'_r \mu'_s \epsilon^{jrs}) \\
&= \lambda_p \mu_q \lambda'_r \mu'_s \det \begin{bmatrix} \mathbf{a}^p_{\cdot} \\ \mathbf{a}^q_{\cdot} \\ \mathbf{b}^r_{\cdot} \\ \mathbf{b}^s_{\cdot} \end{bmatrix} \\
&= \det \begin{bmatrix} \lambda_p \mathbf{a}^p_{\cdot} \\ \mu_q \mathbf{a}^q_{\cdot} \\ \lambda'_r \mathbf{b}^r_{\cdot} \\ \mu'_s \mathbf{b}^s_{\cdot} \end{bmatrix} \\
&= \det \left[A^\top \lambda, A^\top \mu, B^\top \lambda', B^\top \mu'\right] \qquad (9)
\end{aligned}$$

The cross-products on the left side of this sequence of equations represent the point of intersection of the lines in the two images. A term such as $A^\top \lambda$ on the right represents a plane in space that projects via camera matrix A onto the line λ. We now consider four line correspondences in two views. For $i = 1, \ldots, 4$ let $\lambda^{(i)} \leftrightarrow \lambda'^{(i)}$ be the i-th line correspondence, where the upper index indicating the line number is put in parentheses to emphasize that it is not a tensorial index. We denote $\det\left[A^\top\lambda^{(i)}, A^\top\lambda^{(j)}, B^\top\lambda'^{(i)}, B^\top\lambda'^{(j)}\right]$ by I_{ij}. It was shown in [9] that the homogeneous vector

$$I = (I_{12}I_{34}, I_{13}I_{24}, I_{14}I_{23}) \tag{10}$$

is a complete projective invariant of the four lines in space corresponding to the matched lines in the images. According to (9), we may write $I_{ij} = (\lambda'^{(i)} \times \lambda'^{(j)})^\top F(\lambda^{(i)} \times \lambda^{(j)})$ Substituting this formula into (10) yields a neat formula due to Carlsson ([1]) for the projective invariants of four lines in space, in terms of their projections in two views.

3 Trilinear relations

The basic idea behind the derivation of the fundamental matrix can be used to derive relationships between the coordinates of points seen in three views. This analysis results in the definition of a triply-indexed tensor, known as the trifocal tensor, with one covariant and two contravariant indices. Unlike the Fundamental Matrix, this trifocal tensor relates both lines and points in the three images. We begin by describing the way matching points are related by the trifocal tensor.

3.1 Point relations

Consider a point correspondence across three views : $\mathbf{u} \leftrightarrow \mathbf{u}' \leftrightarrow \mathbf{u}''$. Let the third camera matrix be C and let $\mathbf{c}^i_.$ be its i-th row. Analogous to (4) we can write an equation describing the projection of a point $\mathbf{x}$ into the three images.

$$\left[\begin{array}{cccc}
\mathbf{a}^1_. & u^1 & & \\
\mathbf{a}^2_. & u^2 & & \\
\mathbf{a}^3_. & u^3 & & \\
\hline
\mathbf{b}^1_. & & u'^1 & \\
\mathbf{b}^2_. & & u'^2 & \\
\mathbf{b}^3_. & & u'^3 & \\
\hline
\mathbf{c}^1_. & & & u''^1 \\
\mathbf{c}^2_. & & & u''^2 \\
\mathbf{c}^3_. & & & u''^3
\end{array}\right]
\begin{pmatrix} \mathbf{x} \\ -k \\ -k' \\ -k'' \end{pmatrix} = 0 \ . \tag{11}$$

This matrix, which as before we will call X, has 9 rows and 7 columns. From the existence of a solution to this set of equations, we deduce that its rank must be at most 6. Hence any 7×7 minor has zero determinant. This fact gives rise to the trilinear relationships that hold between the coordinates of the points $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$.

There are essentially two different types of 7×7 minors of X. In choosing 7 rows of X, we may choose either

1. Three rows from each of two camera matrices and one row from the third, or

2. Three rows from one camera matrix and two rows from each of the two others.

Let us consider the first type. A typical such 7×7 minor of X is of the form

$$\left[\begin{array}{cccc}
\mathbf{a}^{1}_{\cdot} & u^{1} & & \\
\mathbf{a}^{2}_{\cdot} & u^{2} & & \\
\mathbf{a}^{3}_{\cdot} & u^{3} & & \\
\hline
\mathbf{b}^{1}_{\cdot} & & u'^{1} & \\
\mathbf{b}^{2}_{\cdot} & & u'^{2} & \\
\mathbf{b}^{3}_{\cdot} & & u'^{3} & \\
\hline
\mathbf{c}^{i}_{\cdot} & & & u''^{i}
\end{array}\right] . \tag{12}$$

Note that this matrix contains only one entry in the last column, namely u''^{i}. Expanding the determinant by cofactors down this last column reveals that the determinant is equal to

$$u''^{i} \det \left[\begin{array}{ccc}
\mathbf{a}^{1}_{\cdot} & u^{1} & \\
\mathbf{a}^{2}_{\cdot} & u^{2} & \\
\mathbf{a}^{3}_{\cdot} & u^{3} & \\
\hline
\mathbf{b}^{1}_{\cdot} & & u'^{1} \\
\mathbf{b}^{2}_{\cdot} & & u'^{2} \\
\mathbf{b}^{3}_{\cdot} & & u'^{3}
\end{array}\right] .$$

Apart from the factor u''^{i}, this just leads to the bilinear relationship expressed by the fundamental matrix, as discussed in section 2.

The other sort of 7×7 minor is of more interest. An example of such a determinant is of the form

$$\det \left[\begin{array}{cccc}
\mathbf{a}^{1}_{\cdot} & u^{1} & & \\
\mathbf{a}^{2}_{\cdot} & u^{2} & & \\
\mathbf{a}^{3}_{\cdot} & u^{3} & & \\
\hline
\mathbf{b}^{j}_{\cdot} & & u'^{j} & \\
\mathbf{b}^{l}_{\cdot} & & u'^{l} & \\
\hline
\mathbf{c}^{k}_{\cdot} & & & u''^{k} \\
\mathbf{c}^{m}_{\cdot} & & & u''^{m}
\end{array}\right] . \tag{13}$$

By the same sort of argument as with the bilinear relations one sees that this leads to a trilinear relation of the form $\det X = f(\mathbf{u}, \mathbf{u}', \mathbf{u}'') = 0$. By expanding this determinant down the column containing u^{i}, one can find a specific formula for $\det X$, namely

$$\det X = \pm \frac{1}{2} u^{i} u'^{j} u''^{k} \epsilon_{ilm} \epsilon_{jqx} \epsilon_{kry} \det \begin{bmatrix} \mathbf{a}^{l}_{\cdot} \\ \mathbf{a}^{m}_{\cdot} \\ \mathbf{b}^{q}_{\cdot} \\ \mathbf{c}^{r}_{\cdot} \end{bmatrix} = 0_{xy} \tag{14}$$

where x and y are free indices corresponding to the rows omitted from the matrices B and C to produce (13). We introduce the tensor

$$T_{i}^{qr} = \frac{1}{2} \epsilon_{ilm} \det \begin{bmatrix} \mathbf{a}^{l}_{\cdot} \\ \mathbf{a}^{m}_{\cdot} \\ \mathbf{b}^{q}_{\cdot} \\ \mathbf{c}^{r}_{\cdot} \end{bmatrix} \tag{15}$$

The trilinear relationship (14) may then be written

$$u^i u'^j u''^k \epsilon_{jqx} \epsilon_{kry} T_i^{qr} = 0_{xy} \ . \tag{16}$$

The tensor T_i^{qr} is the trifocal tensor, and (16) is Shashua's trilinear relation. The indices x and y are free indices, and each choice of x and y leads to a different trilinear relation.

Just as in the case of the fundamental matrix, one may write the formula for the tensor T_i^{qr} in a slighly different way :

$$T_i^{qr} = (-1)^{i+1} \det \begin{bmatrix} \sim \mathbf{a}^i_\cdot \\ \mathbf{b}^q_\cdot \\ \mathbf{c}^r_\cdot \end{bmatrix} \ . \tag{17}$$

As in section 2, the expression $\sim \mathbf{a}^i$ means the matrix A with row i omitted. Note that we omit row i from the first camera matrix, but *include* rows q and r from the other two camera matrices.

In the often-considered case where the first camera matrix A has the canonical form $[I \mid 0]$, the expression (17) for the trifocal tensor may be written simply ([10, 18]) :

$$T_i^{qr} = b_i^q c_4^r - b_4^q c_i^r \ . \tag{18}$$

Note that there are in fact 27 possible trilinear relations that may be formed in this way (refer to (13)). Specifically, note that each relation arises from taking all three rows from one camera matrix along with two rows from each of the other two matrices. This gives the following computation.

- 3 ways to choose the first camera matrix from which to take all three rows.
- 3 ways to choose the row to omit from the second camera matrix.
- 3 ways to choose the row to omit from the third camera matrix.

This gives a total of 27 trilinear relations. However, among the 9 ways of choosing two rows from the second and third camera matrices, only 4 are linearly independent. This means that there are a total of 12 linearly independent trilinear relations.

It is important to distinguish between the number of trilinear relations, however, and the number of different trifocal tensors. As is shown by (16), several different trilinear relations may be expressed in terms of just one trifocal tensor. In (16) each distinct choice of the free indices x and y gives rise to a different trilinear relation, all of which are expressible in terms of the same trifocal tensor T_i^{qr}. On the other hand, in the definition of the trifocal tensor given in (15), the camera matrix A is treated differently from the other two, in that A contributes two rows (after omitting row i) to the determinant defining any given entry of T_i^{qr}, whereas the other two camera matrices contribute just one row. This means that there are in fact three different trifocal tensors corresponding to the choice of which of the three camera matrices contributes two rows.

3.2 Line relations

A line in an image is represented by a covariant vector λ_i, and the condition for a point $\mathbf{u}$ to lie on the line is that $\lambda_i u^i = 0$. Let x^j represent a point in space, and a_j^i represent a camera matrix. The 3D point x^j is mapped to the image point $u^i = a^i_j x^j$. It follows that the condition for the point x^j to project to a point on the line λ_i is that $\lambda_i a^i_j x^j = 0$. Another way of looking at this is that $\lambda_i a^i_j$ represents a plane consisting of all points that project onto the line λ_i. Once more, the condition for the point x^j to line on this plane is that $\lambda_i a^i_j x^j = 0$.

Consider the situation where a point x^j maps to a point u^i in one image and to some point on lines λ'_q and λ''_r in two other images. This may be expressed by equations

$$\begin{aligned} u^i &= k a^i_j x^j \\ \lambda'_q b^q_j x^j &= 0 \\ \lambda''_r c^r_j x^j &= 0 \end{aligned}$$

This may be written as a single matrix equation of the form

$$\left[\begin{array}{cc} \mathbf{a}^1_{\cdot} & u^1 \\ \mathbf{a}^2_{\cdot} & u^2 \\ \mathbf{a}^3_{\cdot} & u^3 \\ \hline \lambda'_q \mathbf{b}^q_{\cdot} & 0 \\ \hline \lambda''_r \mathbf{c}^r_{\cdot} & 0 \end{array}\right] \begin{pmatrix} \mathbf{x} \\ -k \end{pmatrix} = 0 \ . \tag{19}$$

Since this set of equations has a solution, one deduces that $\det X = 0$, where X is the matrix on the left of the equation. Expanding this determinant down the last column gives

$$\begin{aligned} 0 = \det X &= \frac{1}{2} u^i \epsilon_{ilm} \det \begin{bmatrix} \mathbf{a}^l_{\cdot} \\ \mathbf{a}^m_{\cdot} \\ \lambda'_q \mathbf{b}^q_{\cdot} \\ \lambda''_r \mathbf{c}^r_{\cdot} \end{bmatrix} \\ &= \frac{1}{2} u^i \lambda'_q \lambda''_r \epsilon_{ilm} \det \begin{bmatrix} \mathbf{a}^l_{\cdot} \\ \mathbf{a}^m_{\cdot} \\ \mathbf{b}^q_{\cdot} \\ \mathbf{c}^r_{\cdot} \end{bmatrix} \\ &= u^i \lambda'_q \lambda''_r T_i^{qr} \end{aligned} \tag{20}$$

This shows the connection of the trifocal tensor with sets of lines. The two lines λ'_q and λ''_r back project to planes meeting in a line in space. The image of this line in the first image is a line, which may be represented by λ_i. For any point u^i on that line the relation (20) holds. It follows that $\lambda'_q \lambda''_r T_i^{qr}$ is the representation of the line λ_i. Thus, we see that for three corresponding lines in the three images:

$$\lambda_p \approx \lambda'_q \lambda''_r T_p^{qr} \tag{21}$$

The symbol $\approx$ means that the two sides are equal up to a scale factor. Since the two sides of the relation (21) are vectors, this may be interpreted as meaning that the vector product of the two sides vanishes. Expressing this vector product using the tensor ϵ^{ijk}, we arrive at an equation

$$\lambda_p \lambda'_q \lambda''_r \epsilon^{ipw} T_i^{qr} = 0^w \ . \tag{22}$$

In an analogous manner to the derivation of (16) and (20) one may derive a relationship between corresponding points in two images and a line in a third image. In particular, if a point x^j in space maps to points u^i and u'^i in the first two images, and to some point on a line λ''_r in the third image, then one may derive a relation

$$u^i u'^j \lambda''_r \epsilon_{jqx} T_i^{qr} = 0_x \quad . \tag{23}$$

In this relation, the index x is free, and there is one such relation for each choice of $x = 1, \ldots, 3$, of which two are linearly independent.

We can summarize the results of this section in the following table, in which the final column denotes the number of linearly independent equations.

Correspondence	Relation	number of equations
three points	$u^i u'^j u''^k \epsilon_{jqx} \epsilon_{kry} T_i^{qr} = 0_{xy}$	4
two points, one line	$u^i u'^j \lambda''_r \epsilon_{jqx} T_i^{qr} = 0_x$	2
one point, two lines	$u^i \lambda'_q \lambda''_r T_i^{qr} = 0$	1
three lines	$\lambda_p \lambda'_q \lambda''_r \epsilon^{piw} T_i^{qr} = 0^w$	2

Table 1. Trilinear Relations

Note how the different equation sets are related to each other. For instance, the second line of the table is derived from the first by replacing $u''^k \epsilon_{kry}$ by the line λ''_r and deleting the free index y.

3.3 Point relation as a special case of Line relations

It will now be shown that the trilinear relation (16) is in fact nothing but a special case of the trilinear relation (20) for lines. In the trilinear relation $u^i u'^j u''^k \epsilon_{jqx} \epsilon_{kry} T_i^{qr} = 0_{xy}$ for points, we may write $\lambda'_{qx} = u'^j \epsilon_{jqx}$ and $\lambda''_{ry} = u''^k \epsilon_{kry}$. The trilinear relation then becomes

$$u^i \lambda'_{qx} \lambda''_{ry} T_i^{qr} = 0_{xy} \tag{24}$$

which is beginning to looking much like the trilinear relation (20) for lines. Observe as before that the indices x and y are free variables in this expression. We will now show that for any choice of the free variables x and y, the terms λ'_{qx} and λ''_{ry} do represent geometrically meaningful lines. We concentrate on λ'_{qx}, since the analysis for λ''_{ry} is identical.

Consider the case $x = 1$. Then $\lambda'_{q1} = u'^j \epsilon_{jq1}$, and expanding this out, we see that $(\lambda'_{11}, \lambda'_{21}, \lambda'_{31}) = (0, -u'^3, u'^2)$. We see that $\lambda'_{\cdot 1}$ is a line passing through the point $u'^j = (u'^1, u'^2, u'^3)$ and parallel with the first coordinate axis. A similar thing occurs for the other choices $x = 2, 3$. Specifically, $\lambda'_{\cdot 2}$ is the line through u'^j parallel with the second coordinate axis, and $\lambda'_{\cdot 3}$ is the line through u'^j passing also through the coordinate origin, $(0, 0, 1)$.

What is all this saying ? Let $\mathbf{u} \leftrightarrow \mathbf{u}' \leftrightarrow \mathbf{u}''$ be corresponding points in three image. If we choose any lines λ' and λ'' that pass respectively through the two points $\mathbf{u}'$ and $\mathbf{u}''$ then from (20) we obtain a relation $u^i \lambda'_j \lambda''_k T_i^{jk} = 0$. For any arbitrary choice of the lines λ'

and λ'' we can write down such a relation. In the particular case where the lines λ' and λ'' are chosen to be lines through $\mathbf{u}'$ and $\mathbf{u}''$ either horizontal, vertical or passing through the origin, then one obtains Shashua's trilinear relation for points (16). One may choose two lines through each of the points $\mathbf{u}'$ and $\mathbf{u}''$, resulting in four independent trilinear relations.

This interpretation of the point relationships has been previously observed in [16] and [4].

4 Quadrilinear Relations

Similar arguments work in the case of four views. Once more, consider a point correspondence across 4 views : $\mathbf{u} \leftrightarrow \mathbf{u}' \leftrightarrow \mathbf{u}'' \leftrightarrow \mathbf{u}'''$. With camera matrices A, B, C and D, the projection equations may be written as

$$\left[\begin{array}{ccccc}
\mathbf{a}^1_{\cdot} & u^1 & & & \\
\mathbf{a}^2_{\cdot} & u^2 & & & \\
\mathbf{a}^3_{\cdot} & u^3 & & & \\
\hline
\mathbf{b}^1_{\cdot} & & u'^1 & & \\
\mathbf{b}^2_{\cdot} & & u'^2 & & \\
\mathbf{b}^3_{\cdot} & & u'^3 & & \\
\hline
\mathbf{c}^1_{\cdot} & & & u''^1 & \\
\mathbf{c}^2_{\cdot} & & & u''^2 & \\
\mathbf{c}^3_{\cdot} & & & u''^3 & \\
\hline
\mathbf{d}^1_{\cdot} & & & & u'''^1 \\
\mathbf{d}^2_{\cdot} & & & & u'''^2 \\
\mathbf{d}^3_{\cdot} & & & & u'''^3
\end{array}\right]
\begin{pmatrix} \mathbf{x} \\ -k \\ -k' \\ -k'' \\ k''' \end{pmatrix} = 0 \ . \tag{25}$$

Since this equation has a solution, the matrix X on the left has rank at most 7, and so all 8×8 determinants are zero. As in the trilinear case, any determinants containing only one row from one of the camera matrices gives rise to a trilinear or bilinear relation between the remaining cameras. A different case occurs when we consider 8×8 determinants containing two rows from each of the camera matrices. Such a determinant leads to a new quadrilinear relationship of the form

$$u^i u'^j u''^k u'''^l \epsilon_{ipw} \epsilon_{jqx} \epsilon_{kry} \epsilon_{lsz} Q^{pqrs} = 0_{wxyz} \tag{26}$$

where each choice of the free variables w, x, y and z gives a different equation, and the 4-dimensional tensor Q^{pqrs} is defined by

$$Q^{pqrs} = \det \begin{bmatrix} \mathbf{a}^p_{\cdot} \\ \mathbf{b}^q_{\cdot} \\ \mathbf{c}^r_{\cdot} \\ \mathbf{d}^s_{\cdot} \end{bmatrix} \tag{27}$$

Note that the four indices of the four-view tensor are contravariant, and there is no distinguished view as there is in the case of the trifocal tensor. There is only one four-view tensor corresponding to four given views, and this one tensor gives rise to 81 different quadrilinear relationships, of which 16 are linearly independent.

As in the case of the trifocal tensor, there are also relations between fixed lines and points in the case of the four-view tensor. Equations relating points are really just special cases of the relationship for lines. In the case of a 4-line correspondence, however, something different happens, as will now be explained. The relationship between a set of four lines and the quadrifocal tensor is given by the formula

$$\lambda_p \lambda'_q \lambda''_r \lambda'''_s Q^{pqrs} = 0 \tag{28}$$

for any set of corresponding lines λ_p, λ'_q, λ''_r and λ'''_s. However, the derivation shows that this condition will hold as long as there is a single point in space that projects onto the four image lines. It is not necessary that the four image lines correspond (in the sense that they are the image of a common line in space). Now, consider the case where three of the lines (for instance λ_p, λ'_q and λ''_r) correspond by deriving from a single 3D line. Now let λ'''_s be any *arbitrary* line in the fourth image. The back projection of this line is a plane, which will meet the 3D line in a single point, $\mathbf{X}$ and the conditions are present for (28) to hold. Since this is true for any arbitrary line λ'''_s, it must follow that $\lambda_p \lambda'_q \lambda''_r Q^{pqrs} = 0^s$. This gives three equations involving λ_p, λ'_q and λ''_r of which two are linearly independent. However given a set of corresponding lines in four images, as above, one may choose a subset of three lines, and for each line-triplet obtain a pair of equations in this way. Thus for a set of four corresponding lines, one obtains a total of 8 equations, which may be verified (empirically) to be independent.

The 4-view relations may be summarized in the following table.

Correspondence	Relation	number of equations
4 points	$u^i u'^j u''^k u'''^l \epsilon_{ipw} \epsilon_{jqx} \epsilon_{kry} \epsilon_{lsz} Q^{pqrs} = 0_{wxyz}$	16
3 points, 1 line	$u^i u'^j u''^k \lambda'''_s \epsilon_{ipw} \epsilon_{jqx} \epsilon_{kry} Q^{pqrs} = 0_{wxy}$	8
2 points, 2 lines	$u^i u'^j \lambda''_r \lambda'''_s \epsilon_{ipw} \epsilon_{jqx} Q^{pqrs} = 0_{wx}$	4
3 lines	$\lambda_p \lambda'_q \lambda''_r Q^{pqrs} = 0^s$	2
4 lines	$\lambda_p \lambda'_q \lambda''_r Q^{pqrs} = 0^s, \lambda_p \lambda'_q \lambda'''_s Q^{pqrs} = 0^r, \ldots$	8

Table 2. Quadrilinear Relations.

No equation is given here for the case of three lines and one point, since this gives no more restrictions on the tensor than just the 3-line correspondence.

5 Number of Independent Equations

It was asserted in considering the definition of the quadrifocal tensor Q^{pqrsl} that each point correspondence gives rise to 16 linearly independent equations. Similarly each point correspondence across three views gives rise to four linearly independent equations in the entries of the trifocal tensor T_i^{qr}. We now examine this point more closely. We begin with the four view case.

5.1 Four View Case

Given sufficiently many point matches across four views, one may solve for the tensor Q^{pqrs}. Subsequently, one may retrieve the camera matrices and carry out projective reconstruction. Details of how the step of retrieving the camera matrices is done are omitted here, but are given by Heyden ([14, 13]). A curious phenomenon occurs however, when one counts the number of point matches necessary to do this. As indicated above, it appears to be the case that each point match gives 16 linearly independent equations in the entries of the tensor Q^{pqrs}. On the other hand, it seems unlikely that the equations derived from two totally unrelated sets of point correspondences could have any dependencies. It would therefore appear that from 5 point correspondences one obtains 80 equations, which is enough to solve for the entries of Q^{pqrs} up to scale. From this argument it would appear that it is possible to solve for the tensor from only 5 point matches across 4 views, and thence one may solve for the camera matrices, up to the usual projective ambiguity. This conclusion however is contradicted by the following remark.

Proposition 5.1. *It is not possible to determine the positions of 4 (or any number of) cameras from the images of 5 points.*

Proof. Since any two sets of five points in $\mathcal{P}^3$are projectively equivalent (barring the case where 3 points are in a plane), we may assume that the five points form a projective basis for $\mathcal{P}^3$. Consider the first camera. The situation is that each point $\mathbf{x}_i$ is known for $i = 1, \ldots, 5$, and the images $\mathbf{u}_i$ of the points are also known. However, each such 3D to 2D correspondence gives two linear equations in the entries of the camera matrix M, a total of 10 equations in all from the 5 points. Since M has 11 degrees of freedom, it can not be determined uniquely from 10 equations. This is in agreement with the observation of Sutherland ([21]) that $5\frac{1}{2}$ such 3D to 2D correspondences are required to determine M. This means that the camera matrix M is not determined uniquely with respect to a fixed projective basis. The same applies to the other cameras, and thus the proposition is demonstrated. □

Obviously there is some error in our counting of equations. In fact, Heyden states ([14]) that six point correspondences are necessary to compute Q^{pqrs}. The truth is that our counting argument is false, as is shown by the following two propositions.

Proposition 5.2. *Consider a single point correspondence* $\mathbf{u} \leftrightarrow \mathbf{u}' \leftrightarrow \mathbf{u}'' \leftrightarrow \mathbf{u}'''$ *across four views. Letting the four free indices w, x, y and z in (26) vary from 1 to 3 one obtains from this correspondence a set of 81 equations in the entries of Q^{pqrs}. The rank of this set of equations is 16. Furthermore, let the equations be written as* $A\mathbf{q} = 0$ *where A is an* 81×81 *matrix and* $\mathbf{q}$ *is a vector containing the entries of Q^{pqrs}. Then the 16 non-zero singular values of A are all equal.*

What this result is saying is that indeed as expected one obtains 16 linearly independent equations from one point correspondence, and in fact it is possible to reduce this set of equations by an orthogonal transform (multiplication of the equation matrix A on the left by an orthogonal matrix U) to a set of orthogonal equations. The rank of the set of equations is a “very solid” 16. This is a very favourable result as far as the conditioning of the problem is concerned.

The key point in the proof of Proposition 5.2 concerns the singular values of a skew-symmetric matrix.

Lemma 5.3. *A 3×3 skew-symmetric matrix has two equal non-zero singular values.*

Although this is well known, a brief proof is given.

Proof. Defining matrices

$$E = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Z = ED = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the Schurr normal form ([5]) of a 3×3 skew-symmetric matrix S can be written $S = kVZV^\top$ where k is a scalar. Thus the SVD of S is given by $S = kUDV^\top$ where $U = VE$. □

The rest of the proof of Proposition 5.2 is quite straight-forward as long as one does not get lost in notation.

Proof. (Proposition 5.2) The full set of 81 equations derived from a single point correspondence is of the form $u^i \epsilon_{ipw} u'^j \epsilon_{jqx} u''^k \epsilon_{kryv} u'''^l \epsilon_{lsz} Q^{pqrs} = 0_{wxyz}$. A total of 81 equations are generated by varying w, x, y, z over the range $1, \ldots, 3$. Thus, the equation matrix A may be written as

$$A_{(wxyz)(pqrs)} = u^i \epsilon_{ipw} u'^j \epsilon_{jqx} u''^k \epsilon_{kry} u'''^l \epsilon_{lsz} \tag{29}$$

where the indices $(wxyz)$ index the row and $(pqrs)$ index the column of A. We will have occasion frequently to consider a set of indices, such as $(wxyz)$ in this case, as a single index for the row or column of a matrix. This situation will be indicated by enclosing the indices in parentheses as here, and referring to them as a *combined index.*

We consider now the expression $u^i \epsilon_{ipw}$. This may be considered as a matrix indexed by the free indices p and w. Furthermore, since $u^i \epsilon_{ipw} = -u^i \epsilon_{iwp}$ we see that it is a skew-symmetric matrix, and hence has equal singular values. We denote this matrix by S_{wp}. Writing the result of Lemma 5.3, using tensor notation we have

$$U_a^w S_{wp} V_e^p = k D_{ae} \tag{30}$$

where the matrix D is as in Lemma 5.3. Now, the matrix A in (29) may be written as $A_{(wxyz)(pqrs)} = S_{wp} S'_{xq} S''_{yr} S'''_{zs}$. Consequently, applying (30) we may write

$$U_a^w U_b'^x U_c''^y U_d'''^z A_{(wxyz)(pqrs)} V_e^p V_f'^q V_g''^r V_h'''^s = kk'k''k''' D_{ae} D_{bf} D_{cg} D_{dh} \ . \tag{31}$$

Now, writing

$$\hat{U}_{(abcd)}^{(wxyz)} = U_a^w U_b'^x U_c''^y U_d'''^z$$

$$\hat{V}_{(efgh)}^{(pqrs)} = V_e^p V_f'^q V_g''^r V_h'''^s$$

$$\hat{D}_{(abcd)(efgh)} = D_{ae} D_{bf} D_{cg} D_{dh}$$

and

$$\hat{k} = kk'k''k'''$$

we see that (31) may be written as

$$\hat{U}^{(wxyz)}_{(abcd)} A_{(wxyz)(pqrs)} \hat{V}^{(pqrs)}_{(efgh)} = \hat{k}\hat{D}_{(abcd)(efgh)} \quad . \tag{32}$$

As a matrix, $D_{(abcd)(efgh)}$ is diagonal with 16 non-zero diagonal entries, all equal to unity. To show that (32) is the SVD of the matrix $A_{(pqrs)(tuvw)}$, and hence to complete the proof, it remains only to show that $U^{(wxyz)}_{(abcd)}$ and $V^{(pqrs)}_{(efgh)}$ are orthogonal matrices. To this end, we show that $U^{(wxyz)}_{(abcd)}$ has unit norm orthogonal columns. Thus, for two columns with combined indices $(pqrs)$ and $(tuvw)$ respectively, we verify

$$\begin{aligned}
\sum_{i,j,k,l} \hat{U}^{(ijkl)}_{(pqrs)} \hat{U}^{(ijkl)}_{(tuvw)} &= \sum_{i,j,k,l} (U^i_p U'^j_q U''^k_r U'''^l_s)(U^i_t U'^j_u U''^k_v U'''^l_w) \\
&= \sum_i U^i_p U^i_t \sum_j U'^j_q U'^j_u \sum_k U''^k_r U''^k_v \sum_l U'''^l_s U'''^l_w \\
&= \delta_{pt}\delta_{qu}\delta_{rv}\delta_{sw} \\
&= \delta_{(pqrs)(tuvw)}
\end{aligned}$$

and so $\hat{U}$ is orthogonal, as required. A similar argument shows that $\hat{V}$ is orthogonal as well. This completes the proof that A has rank 16, and all non-zero singular values are equal. □

Thus, each point correspondence gives 16 equations. The surprising fact however is that the equation sets corresponding to two unrelated point correspondences have a dependency, as stated in the following proposition.

Proposition 5.4. *The set of equations (26) derived from a set of n general point correspondences across four views has rank $16n - \binom{n}{2}$, for $n \leq 5$.*

The notation $\binom{n}{2}$ means the number of choices of 2 among n, specifically, $\binom{n}{2} = n(n-1)/2$. Thus for 5 points there are only 70 independent equations, not enough to solve for Q^{pqrs}. For $n = 6$ points, $16n - \binom{n}{2} = 81$, and we have enough equations to solve for the 81 entries of Q^{pqrs}. These propositions will be proven below.

Proof. We consider two point correspondences across four views, namely $u^i \leftrightarrow u'^i \leftrightarrow u''^i \leftrightarrow u'''^i$ and $v^i \leftrightarrow v'^i \leftrightarrow v''^i \leftrightarrow v'''^i$. The first correspondence gives rise to a set of equations : $u^i u'^j u''^k u'''^l \epsilon_{ipw}\epsilon_{jqx}\epsilon_{kry}\epsilon_{lsz} Q^{pqrs} = 0_{wxyz}$ where there is a different equation for each choice of w, x, y and z. There are a total of 81 equations in the 81 entries of the tensor Q^{pqrs}. The coefficients of each equation may be considered as a vector in the Euclidean space R^{81}. According to Proposition 5.2, however, the 81 such vectors span a subspace $S_{\mathbf{u}}$ of dimension 16 in R^{81}.

A similar set of equations may be derived from the second correspondence, and these equations span a second 16-dimensional subspace $S_{\mathbf{v}}$ of R^{81}. If the two subspaces $S_{\mathbf{u}}$ and $S_{\mathbf{v}}$ intersect only in the zero vector, then together the two subspaces generate a subspace of dimension 32 of R^{81}. The proposition we are proving asserts that this is not so, however. Therefore, it is our goal to show that these two subspaces have non-trivial intersection.

The vectors generating $S_{\mathbf{u}}$ may be denoted by $\hat{\mathbf{u}}_{(wxyz)}$ where $(wxyz)$ is a combined index for the vector, and each $\hat{\mathbf{u}}_{(wxyz)}$ is a vector with components

$$\hat{u}_{(wxyz)(pqrs)} = u^i u'^j u''^k u'''^l \epsilon_{ipw}\epsilon_{jqx}\epsilon_{kry}\epsilon_{lsz}$$

and $(pqrs)$ is a combined index for the component of the vector. We consider a specific linear combination of the vectors $\hat{\mathbf{u}}_{(wxyz)}$ given by $\mathbf{s} = v^w v'^x v''^y v'''^z \hat{\mathbf{u}}_{(wxyz)}$. This is a vector with components

$$s_{(pqrs)} = v^w v'^x v''^y v'''^z u^i u'^j u''^k u'''^l \epsilon_{ipw}\epsilon_{jqx}\epsilon_{kry}\epsilon_{lsz} \quad . \tag{33}$$

In a similar fashion, the subspace $S_{\mathbf{v}}$ is generated by vectors $\hat{\mathbf{v}}_{(ijkl)}$, which have components

$$\hat{v}_{(ijkl)(pqrs)} = v^w v'^x v''^y v'''^z \epsilon_{wpi}\epsilon_{xqj}\epsilon_{yrk}\epsilon_{zsl} = v^w v'^x v''^y v'''^z \epsilon_{ipw}\epsilon_{jqx}\epsilon_{kry}\epsilon_{lsz} \quad .$$

The last expression is obtained from the previous one by swapping the first and last indices of each ϵ.

Now, one forms the linear combination $\mathbf{s} = u^i u'^j u''^k u'''^l \hat{\mathbf{v}}_{(ijkl)}$, which when expanded turns out to have the same components as the vector $\mathbf{s}_{(pqrs)}$ in (33). In brief, we have

$$\mathbf{s} = v^w v'^x v''^y v'''^z \hat{\mathbf{u}}_{(wxyz)} = u^i u'^j u''^k u'''^l \hat{\mathbf{v}}_{(ijkl)} \quad . \tag{34}$$

This shows that the subspaces $S_{\mathbf{u}}$ and $S_{\mathbf{v}}$ or R^{81} intersect in a subspace generated by the vector $\mathbf{s}$. Thus, the dimension of the subspace generated by vectors $\hat{\mathbf{u}}_{(wxyz)}$ and $\hat{\mathbf{v}}_{(ijkl)}$ is at most 31, provided that $\mathbf{s}$ is non-zero. To consider the possibility that $\mathbf{s} = 0$, we rearrange (33) to get

$$\begin{aligned} s_{(pqrs)} &= (u^i v^w \epsilon_{ipw})(u'^j v'^x \epsilon_{jqx})(u''^k v''^y \epsilon_{kry})(u'''^l v'''^z \epsilon_{lsz}) \\ &= (\mathbf{u} \times \mathbf{v})_p (\mathbf{u}' \times \mathbf{v}')_q (\mathbf{u}'' \times \mathbf{v}'')_r (\mathbf{u}''' \times \mathbf{v}''')_s \end{aligned}$$

where $(\mathbf{u} \times \mathbf{v})$ represents the vector product. Such a vector product is nonzero unless the $\mathbf{u}$ and $\mathbf{v}$ are equal, up to scale, and hence represent the same point. Thus, $\mathbf{s}$ is non-zero unless $\mathbf{u}$ and $\mathbf{v}$ (or $\mathbf{u}'$ and $\mathbf{v}'$, etc) represent the same point. To take care of the case where for instance $\mathbf{u} = \mathbf{v}$, we note that then

$$v'^x v''^y v'''^z \hat{\mathbf{u}}_{(wxyz)} = -u'^j u''^k u'''^l \hat{\mathbf{v}}_{(wjkl)} \tag{35}$$

for each value of w. To verify this, note that the components of the vectors on each side of (35) are

$$v'^x v''^y v'''^z u^i u'^j u''^k u'''^l \epsilon_{ipw}\epsilon_{jqx}\epsilon_{kry}\epsilon_{lsz} = -u'^j u''^k u'''^l v^i v'^x v''^y v'''^z \epsilon_{ipw}\epsilon_{xqj}\epsilon_{yrk}\epsilon_{zsl} \quad .$$

This means that $S_{\mathbf{u}}$ and $S_{\mathbf{v}}$ intersect in at least a 3-dimensional subspace. Thus, in all cases, we have shown that the subspace generated by $S_{\mathbf{u}}$ and $S_{\mathbf{v}}$ has dimension at most 31.

We now consider the possibility that the dimension of the subspace is less than 31. In such a case, all 31×31 sub-determinants of the matrix having as rows the vectors $\hat{\mathbf{u}}_{(wxyz)}$ and $\hat{\mathbf{v}}_{(ijkl)}$ must vanish. These subdeterminants may be expressed as polynomial expressions in the coefficients of the points $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{u}''$, $\mathbf{u}'''$, $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{v}''$ and $\mathbf{v}'''$. These coefficients together make up a 24-dimensional space. Thus, there is a function $f : R^{24} \rightarrow R^N$ for some N (equal to the number of such 31×31 subdeterminants), such that the equation matrix has rank less than 31 only on the set of zeros of the function f. Any arbitrarily chosen example may be used to show that the function f is not identically zero. It follows, that the set of point correspondences for which the set of equations has rank less than 31 is a variety in R^{24}, and hence is nowhere dense. Thus, for a general pair of point correspondences, the set of equations generated by a pair of point correspondences across 4 views has rank 31.

We now turn to the general case of n point correspondences across all 4 views. Note that the linear relationship (34) that holds for two point correspondences is non-generic, but depends on the pair of correspondences. In general, therefore, given n point correspondences, there will be $\binom{n}{2}$ such relationships. This reduces the dimension of the space spanned by the set of equations to $16n - \binom{n}{2}$ as required. $\square$

5.2 Three View Case

In this section, we consider the set of equations relating the entries of the trifocal tensor T_i^{jk} generated by a single point correspondence across three views. We find the following favourable situation holds.

Proposition 5.5. *Consider a single point correspondence $\mathbf{u} \leftrightarrow \mathbf{u}' \leftrightarrow \mathbf{u}''$ across three views. Letting the two free indices x and y in (16) vary from 1 to 3 one obtains from this correspondence a set of 9 equations in the entries of T_i^{qr}. The rank of this set of equations is 4. Furthermore, let the equations be written as $A\mathbf{t} = 0$ where A is a 9×27 matrix and $\mathbf{t}$ is a vector containing the entries of T_i^{qr}. Then the 4 non-zero singular values of A are equal.*

Proof. This proposition is similar to Proposition 5.2, and is proven in much the same way. The full set of 9 equations derived from a single point correspondence is of the form $u^i u'^j \epsilon_{jqx} u''^k \epsilon_{kry} T_i^{qr} = 0_{xy}$. A total of 9 equations are generated by varying x and y over the range $1, \ldots, 3$. Thus, the equation matrix A may be written as

$$A_{(xy)\binom{i}{qr}} = u^i u'^j u''^k \epsilon_{jqx} \epsilon_{kry} \tag{36}$$

where the indices (xy) index the row and $\binom{i}{qr}$ index the column of A. As in the proof of Proposition 5.2, we may write $u'^j \epsilon_{jqx} = S'_{xq}$ and $u''^k \epsilon_{kry} = S''_{yr}$. Then the matrix A in (36) may be written as $A_{(xy)\binom{i}{qr}} = u^i S'_{xq} S''_{yr}$. Consequently, applying (30) we may write

$$U'^x_a U''^y_b A_{(xy)\binom{i}{qr}} V'^q_e V''^r_f = k' k'' u^i D_{ae} D_{bf} \ . \tag{37}$$

Next, introducing a vector u_i with covariant (lower) index, defined such that $u_i = u^i$ for all i, we have

$$U'^x_a U''^y_b A_{(xy)\binom{i}{qr}} u_i V'^q_e V''^r_f = k' k'' u_i u^i D_{ae} D_{bf} = k' k'' ||\mathbf{u}||^2 D_{ae} D_{bf} \ .$$

Now, writing

$$\hat{U}^{(xy)}_{(ab)} = U'^x_a U''^y_b$$

$$\hat{V}^{\binom{qr}{i}}_{(ef)} = u_i V'^q_e V''^r_f$$

$$\hat{D}_{(ab)(ef)} = D_{ae} D_{bf} \quad .$$

and

$$\hat{k} = k' k'' ||\mathbf{u}||^2$$

we see that (37) may be written as

$$\hat{U}^{(xy)}_{(ab)} A_{(xy)\binom{i}{qr}} \hat{V}^{\binom{qr}{i}}_{(ef)} = \hat{k} \hat{D}_{(ab)(ef)} \tag{38}$$

The matrix $D_{(ab)(ef)}$ is diagonal with 4 unit diagonal entries. As before, to complete the proof, we need only show that $\hat{U}$ and $\hat{V}$ are orthogonal, The matrix $\hat{U}$ is orthogonal as is shown using the same argument as before. Since the matrix $\hat{V}^{\binom{qr}{i}}_{(ef)}$ is not square (it has dimension 27×9), we need to show that is has orthogonal columns. Details are as follows.

$$\begin{aligned}
\sum_{q,r,i} \hat{V}^{\binom{qr}{i}}_{(ef)} \hat{V}^{\binom{qr}{i}}_{(e'f')} &= \sum_{i,p,q} (V'^q_e V''^r_f u_i)(V'^q_{e'} V''^r_{f'} u_i) \\
&= \sum_q V'^q_e V'^q_{e'} \sum_r V''^r_f V''^r_{f'} \sum_i (u_i^2) \\
&= \delta_{ee'} \delta_{ff'} ||\mathbf{u}||^2 \\
&= \delta_{(ef)(e'f')} ||\mathbf{u}||^2 \quad .
\end{aligned}$$

Thus, in fact, the rows of $\hat{V}$ are orthogonal, and each one has the same norm equal to $||\mathbf{u}||$. This completes the proof. □

5.3 Choosing equations

In the previous two sections, proofs have been given that the singular values of the full set of equations derived from three or four point equations are all equal. The key point in the argument is that the two non-zero singular values of a 3×3 skew-symmetric matrix are equal. This proof may clearly be extended to apply to any of the other sets of equations derived from line or point correspondences given in sections 3 and 4.

Consider once more the case of three point correspondences in three views. The results on singular values show that it is in general advisable to include all 9 equations derived from this correspondence, rather than selecting just four independent equations. This will avoid difficulties with near singular situations. This conclusion is supported by experimental observation. Indeed, numerical examples show that the condition of a set of equations derived from a set several point correspondences is substantially better when all 9 equations are included for each point correspondence. In this context, the condition of the equation set is given by the ratio of the first (largest) to the n-th singular value, where n is the number of linearly independent equations.

Including all 9 equations rather than just 4 means that the set of equations is larger, leading to increased complexity of solution. However, whether the equations are solved using the Singular Value Decomposition, or the method of normal equations, the increase in complexity needs only to be linear. This point is explained in [12]. For formulae about the complexity of the SVD, see [5].

An alternative to including all 9 equations (or all 81 in the 4-view case) is to include the minimum number (4 or 16 respectively) of correctly chosen equations. This notion will be illustrated for the three-view case. As we saw in section 3.3, the equations $u^i u'^j u''^k \epsilon_{jqx} \epsilon_{kry} T_i^{qr} = 0_{xy}$ derived from a point correspondence across three views may be considered as a set of line equations $u^i \lambda'_{qx} \lambda''_{ry} T_i^{qr} = 0_{xy}$ by writing $\lambda'_{qx} = u'^j \epsilon_{jqx}$ and $\lambda''_{ry} = u''^k \epsilon_{kry}$. Each choice of x or y gives a different line through the points u'^j and u''^k, a total of 9 choices.

Now, as a matrix, λ'_{qx} is skew-symmetric, and hence has two equal singular values. As remarked, this is the basis for the set of all 9 equations having rank 4, and 4 equal singular values. On the other hand, if we select just two lines passing through u'^j by making a choice of two values for the index x, then the two columns of the matrix λ'_{qx} are not orthogonal, and the resulting 3×2 matrix does not have equal singular values, and in fact may be nearly rank-deficient. As previously seen, one remedy is to include the equations derived from all 3 choices of the index x. The corresponding three lines λ'_{qx} are the lines parallel with the coordinate axes and through the origin and passing through the point u'^j. An alternative is to select two other lines $\hat{\lambda}'_{qx}$, for $x = 1, 2$, passing through u'^j and represented by an orthonormal pair of vectors. In this case, the matrix $\hat{\lambda}'_{qx}$ of dimension 3×2 will have two equal singular values. If this is done also for the point u''^k, then arguments of section 5.2 apply, and the resulting set of 4 equations $u^i \hat{\lambda}'_{qx} \hat{\lambda}''_{ry} T_i^{qr} = 0_{xy}$ for $x, y = 1, 2$ will be independent and orthonormal. Note that the condition that the vectors $\hat{\lambda}'_{q1}$ and $\hat{\lambda}'_{q2}$ are orthonormal has nothing to do with geometric orthogonality of the lines. A simple way of finding a pair of orthonormal vectors $\hat{\lambda}'_{qx}$ such that $u'^q \hat{\lambda}'_{qx} = 0_x$ is using Householder transforms ([5]). A Householder matrix h_{qx} is an orthogonal matrix such that $u'^q h_{qx} = \delta_{3x} = (0, 0, 1)$. Setting $\hat{\lambda}_{qx} = h_{qx}$ for $x = 1, 2$ gives the required pair of lines passing through the point u'^q.

We summarize this discussion as follows.

Recommended method for formulating point equations.

Given a point correspondence $\mathbf{u} \leftrightarrow \mathbf{u}' \leftrightarrow \mathbf{u}''$ across three views :

1. Find Householder matrix h'_{qx} and h''_{ry} such that $u'^q h'_{qx} = \delta_{3x}$ and $u''^r h'_{ry} = \delta_{3y}$.
2. For $x, y = 1, 2$ set $\hat{\lambda}'_{qx} = h'_{qx}$ and $\hat{\lambda}''_{ry} = h''_{ry}$.
3. The formula $u^i \hat{\lambda}'_{qx} \hat{\lambda}''_{ry} T_i^{qr} = 0_{xy}$ for $x, y = 1, 2$ gives a set of four orthonormal equations in the entries of T_i^{qr}.

Once more, it is evident that essentially this method will work for all the types of equations summarized in Tables 1 and 2.

Acknowledgement This discussion was prompted by remarks of an anonymous reviewer to my paper [12].

6 Summary and Other Work

Although using a slightly different approach, this paper summarized previous results of Triggs ([22]) and Faugeras and Mourrain ([4]) on the derivation of multilinear relationships between corresponding image coordinates. The formulae for relations between mixed point and line correspondences are extensions of the result of [10, 12]. This paper suggests that the most basic relations are the point-line-line correspondence equation $u^i\lambda'_q\lambda''_r T_i^{qr} = 0$ in the three-view case, and the line-correspondence equation $\lambda_p\lambda'_q\lambda''_r\lambda'''_s Q^{pqrs} = 0$ for four views. Indeed numerical robustness may be enhanced by reducing other correspondences to this type of correspondence, for carefully selected lines.

There are several aspects of multilinear relations that have not been addressed in this paper. Most notable is the inverse problem of reconstruction, in other words, retrieval of the camera matrices (up to a common projective transformation) from the multilinear tensor. For the two-view case this is the now somewhat classic problem of two-view projective reconstruction considered by many authors, including [6, 2]. In the three-view case a solution was given in [10, 12, 18]. The four-view case has been considered by Heyden ([14, 13]) in the context of his algorithm for reconstruction from six points in four views. Both for the three and four view reconstruction problem further work remains to be done before a complete understanding is achieved.

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