Kruppa's Equations Derived from the Fundamental Matrix

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Abstract

The purpose of this note is to give a specific form for Kruppa's equations in terms of the Fundamental matrix. Kruppa's equations can be written explicitly in terms of the singular value decomposition (SVD) of the fundamental matrix. In the final section, the known relationship between the absolute conic and camera calibration is reviewed, for convenience' sake.

1 Derivation of Kruppa's Equations

Consider two camera matrices P and P' with the same calibration. Let C be the dual of the image of the absolute conic (abbreviated DIAC) as imaged by these two cameras. As is well known the image of the absolute conic (IAC) is independent of the pose of the camera, and so it is the same for the two cameras in question.

Let F be the fundamental matrix for the pair of cameras. We wish to apply projective transformations represented by 3×3 transformation matrices A and A' to the two images. After the transformation, the effective camera matrices will be AP and $A'P'$, corresponding to the camera projection followed by projective transformation of the image. This will of course change the DIAC to some new conic envelopes, which we will call D and D'. Since A and A' may be different, we can no longer assume that $D = D'$.

Suppose that A and A' are chosen so that the fundamental matrix for the two new camera matrices AP and $A'P'$ has the special form

$$E = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

This is a very special fundamental matrix having the property that the two epipoles are at the origin and that corresponding epipolar lines are identical in the two images.

Now, consider a plane passing through the two camera centres, tangent to the absolute conic. Such a plane will project to a pair of corresponding epipolar lines in the two images, and these two lines will be tangent to the IAC. Since there are two such tangential planes, there are two pairs of corresponding epipolar tangents.

We recall that corresponding epipolar lines in the two images are identical. Let $(\lambda, \mu, 0)^\top$ be a tangent to the IAC. Since D is the DIAC in the first image, this tangential relationship may be written as

$$(\lambda, \mu, 0) D (\lambda, \mu, 0)^\top = 0$$

and similarly, $(\lambda, \mu, 0)D'(\lambda, \mu, 0)^\top = 0$. Writing these two equations out explicitly gives

$$\lambda^2 d_{11} + 2\lambda\mu d_{12} + \mu^2 d_{22} = 0$$

and

$$\lambda^2 d'_{11} + 2\lambda\mu d'_{12} + \mu^2 d'_{22} = 0$$

where $D = (d_{ij}$ and $D' = (d'_{ij})$, both of which are symmetric.

Since the two tangent lines to the IAC must be the same two lines in the two images, these two equations must have the same pair of solutions for λ and μ. This means that they must be identical equations (up to scale), and so

$$\frac{d_{11}}{d'_{11}} = \frac{d_{12}}{d'_{12}} = \frac{d_{22}}{d'_{22}} \quad . \tag{1}$$

These are the Kruppa equations.

We now repeat this argument, this time being more precise about specific values. The purpose is to find explicit expressions for the matrices D and D' in terms of the fundamental matrix F.

Let the Singular Value Decomposition of the fundamental matrix be $F = UDV^\top$, where U and V are orthogonal, and $D = \mathrm{diag}(r, s, 0)$ is a diagonal matrix. We may write this as follows:

$$F = U \begin{pmatrix} r & & \\ & s & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^\top \ .$$

We write

$$A'^\top = U \begin{pmatrix} r & & \\ & s & \\ & & 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^\top \ .$$

Then, we see that $F = A'^\top E A$ with A and A' non-singular. For a pair of matching image points $\mathbf{u}' \leftrightarrow \mathbf{u}$ we have $\mathbf{u}'^\top F\mathbf{u} = 0$. Thus, $\mathbf{u}'^\top A'^\top E A\mathbf{u} = 0$. Setting $\hat{\mathbf{u}} = A\mathbf{u}$ and $\hat{\mathbf{u}}' = A'\mathbf{u}'$, we see that $\hat{\mathbf{u}}'^\top E\hat{\mathbf{u}} = 0$. Thus, A and A' are the two transforms that we require.

Next, we investigate the effect of this transformation of the DIAC. Consider a transformation A. How does this transformation transform lines ? Well, a point $\mathbf{u}$ lies on a line $\boldsymbol{\lambda}$ if and only if $\boldsymbol{\lambda}^\top \mathbf{u} = 0$. This can be written as $\boldsymbol{\lambda}^\top A^{-1} A\mathbf{u} = 0$. Thus, $\mathbf{u}$ lies on $\boldsymbol{\lambda}$ if and only if $A\mathbf{u}$ lies on $A^{-\top}\boldsymbol{\lambda}$. Thus, $A^{-\top}\boldsymbol{\lambda}$ is the transformed line. Now, a line $\boldsymbol{\lambda}$ belongs to a conic envelope C if and only if $\boldsymbol{\lambda}^\top C\boldsymbol{\lambda} = 0$. This can be written as $(\boldsymbol{\lambda}A^{-1})(ACA^\top)(A^{-\top}\boldsymbol{\lambda}) = 0$. Thus, the transformation A maps the conic envelope C to a $D = ACA^\top$, and similarly A' maps C to $D' = A'CA'^\top$.

Now, we want to compute d_{ij}, where $D = (d_{ij})$. Let

$$A = \begin{pmatrix} \mathbf{a}_1{}^\top \\ \mathbf{a}_2{}^\top \\ \mathbf{a}_3{}^\top \end{pmatrix}$$

where $\mathbf{a}_i$ is the i-th row of A. Then, from $D = ACA^\top$ we compute $d_{ij} = \mathbf{a}_i{}^\top C \mathbf{a}_j$.

Then (1) leads to the following explicit form for the Kruppa equations :

$$\frac{\mathbf{a}_1{}^\top C \mathbf{a}_1}{\mathbf{a}_1'{}^\top C \mathbf{a}_1'} = \frac{\mathbf{a}_1{}^\top C \mathbf{a}_2}{\mathbf{a}_1'{}^\top C \mathbf{a}_2'} = \frac{\mathbf{a}_2{}^\top C \mathbf{a}_2}{\mathbf{a}_2'{}^\top C \mathbf{a}_2'} \tag{2}$$

We can write these equations directly in terms of the SVD of the fundamental matrix $F = U\mathrm{diag}(r, s, 0)V^\top$. Specifically, we have

$$A' = \begin{pmatrix} r & & \\ & s & \\ & & 1 \end{pmatrix} U^\top$$

from which we have

$$A' = \begin{pmatrix} \mathbf{a}_1'{}^\top \\ \mathbf{a}_2'{}^\top \\ \mathbf{a}_3'{}^\top \end{pmatrix} = \begin{pmatrix} r\mathbf{u}_1{}^\top \\ s\mathbf{u}_2{}^\top \\ \mathbf{u}_3{}^\top \end{pmatrix} \quad .$$

where $\mathbf{u}_i$ is the i-th column of U.

For A we have

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^\top$$

and so

$$A = \begin{pmatrix} \mathbf{a}_1{}^\top \\ \mathbf{a}_2{}^\top \\ \mathbf{a}_3{}^\top \end{pmatrix} = \begin{pmatrix} \mathbf{v}_2{}^\top \\ -\mathbf{v}_1{}^\top \\ \mathbf{v}_3{}^\top \end{pmatrix} \quad .$$

where $\mathbf{v}_i$ is the i-th column of V.

From (2) we obtain

$$\frac{\mathbf{v}_2{}^\top C \mathbf{v}_2}{r^2 \mathbf{u}_1{}^\top C \mathbf{u}_1} = \frac{-\mathbf{v}_2{}^\top C \mathbf{v}_1}{rs\mathbf{u}_1{}^\top C \mathbf{u}_2} = \frac{\mathbf{v}_1{}^\top C \mathbf{v}_1}{s^2 \mathbf{u}_2{}^\top C \mathbf{u}_2} \tag{3}$$

2 The DIAC and calibration

Finally, we investigate the exact relationship between the DIAC and the camera calibration.

The absolute conic has equation $x^2 + y^2 + z^2 = 0; \quad t = 0$. Define a vector $\mathbf{x} = (x, y, z)^\top$. Thus a point $(x, y, z, 0)^\top$ is on the absolute conic if and only if $\mathbf{x}^\top \mathbf{x} = 0$. Let a camera matrix $P = K(R \mid -R\mathbf{t})$. A point $(x, y, z, 0)^\top$ on the absolute conic maps to $\mathbf{u} = P(x, y, z, 0)^\top = KR\mathbf{x}$. Thus, $\mathbf{x} = R^\top K^{-1}\mathbf{u}$, and the condition $\mathbf{x}^\top \mathbf{x}$ becomes $\mathbf{u}^\top K^{-\top} R R^\top K^{-1} \mathbf{u} = \mathbf{u}^\top K^{-\top} K^{-1} \mathbf{u} = 0$. Thus, a point $\mathbf{u}$ is on the IAC if and only if it lies on the conic represented by the matrix $K^{-\top}K^{-1}$. In other words, $K^{-\top}K^{-1}$ is the matrix representing the IAC. Taking inverses (dual conics) reveals that $KK^\top$ is the DIAC, denoted C in equations (3). Thus, we may find C by solving the Kruppa equations (3) and then find K by Choleski factorization.