# Kruppa's Equations Derived from the Fundamental Matrix

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*Abstract*— **The purpose of this note is to give a specific form for Kruppa's equations in terms of the Fundamental matrix. Kruppa's equations can be written explicitly in terms of the singular value decomposition (SVD) of the fundamental matrix.**

*Keywords*— **Singular Value Decomposition, Fundamental Matrix, Kruppa's equations, camera calibration**

## I. Introduction

KRUPPA'S equations ([1]) received prominence in the computer vision field through the work of Maybank, Faugeras and Luong ([2], [3], [4]) where they were used for autocalibration of a camera. They represent the basic constraint on camera calibration induced by image correspondences in two or more views. Though these equations were derived in [2], the lack of a simple explicit form has been an impediment to research in this area. This note provides a simple derivation and form for Kruppa's equations, based on the Singular Value Decomposition of the fundamental matrix.

## II. The DIAC and calibration

Kruppa's equations are intimately connected with the absolute conic. The properties of this conic and its connection with calibration are reviewed next, for the readers' convenience. It is seen that knowledge of the image of the absolute conic is equivalent to calibration of the camera that took the image. A camera mapping is represented ([5]) by a $3 \times 4$ matrix $P = K[R \mid -R\mathbf{t}]$, where $K$ is an upper triangular matrix of internal parameters of the camera, and $R$ and $\mathbf{t}$ represent the orientation and position of the camera with respect to the world coordinate system. The camera maps a point $\mathbf{x}$ in homogeneous coordinates to the point $\mathbf{u} = P\mathbf{x}$ in an image. Calibration of the camera refers to the determination of the matrix $K$.

The absolute conic is a conic lying on the plane at infinity, having equation $x^2 + y^2 + z^2 = 0; \quad t = 0$, where $(x, y, z, t)^\top$ are coordinates of a point in 3-dimensional space. The absolute conic does not contain any points with real coordinates – it is composed entirely of complex points. The image of the absolute conic in an image is however representable by a real symmetric $3 \times 3$ matrix, as will be seen next.

Define a vector $\mathbf{x} = (x, y, z)^\top$. A point $(x, y, z, 0)^\top$ is on the absolute conic if and only if $\mathbf{x}^\top\mathbf{x} = 0$. Consider a camera matrix $P = K[R \mid -R\mathbf{t}]$. A point $(x, y, z, 0)^\top$ on the absolute conic maps to $\mathbf{u} = P(x, y, z, 0)^\top = KR\mathbf{x}$. Thus, $\mathbf{x} = R^\top K^{-1}\mathbf{u}$, and the condition $\mathbf{x}^\top\mathbf{x} = 0$ becomes $\mathbf{u}^\top K^{-\top} R R^\top K^{-1}\mathbf{u} = \mathbf{u}^\top K^{-\top} K^{-1}\mathbf{u} = 0$. Thus, a point $\mathbf{u}$ is on the image of the absolute conic if and only if it lies on the conic represented by the matrix $K^{-\top}K^{-1}$. In other words, $K^{-\top}K^{-1}$ is the matrix representing the image of the absolute conic. Taking inverses (dual conics, [6]) reveals that $KK^\top$ is the dual of the image of the absolute conic. We will denote $KK^\top$ by $C$. If $C$ is known then the calibration matrix $K$ may be retrieved by Choleski factorization. Specifically, any symmetric positive-definite matrix (such as $C$) may be uniquely factored as a product $KK^\top$ such that $K$ is an upper triangular matrix with positive diagonal entries ([7]). For an algorithm for Choleski factorization, see [8].

Since we will be considering the absolute conic in subsequent pages, we adopt the following abbreviations :

AC means the absolute conic.
IAC means the image of the absolute conic.
DIAC means the dual of the image of the absolute conic.

We have shown how the calibration matrix $K$ may be retrieved if the matrix $C$ representing the DIAC is known. Conversely, if $K$ is known, then $C = KK^\top$ is determined. This shows the intimate link between the IAC and calibration. An important point to note is that the formula $C = KK^\top$ for the DIAC depends only on the calibration matrix, and not on the orientation $R$ or the position $\mathbf{t}$ of the camera. The IAC is fixed under Euclidean motions of the camera.

## III. Kruppa's Equations

Given two views of a scene taken with two different cameras, one may compute a projective reconstruction of the scene. If one has additional information that the cameras used to image the two scenes have the same internal calibration, then this implies a certain restriction on the class of possible reconstruction. This in turn implies a restriction on the internal calibration of the camera. This restriction may be expressed by Kruppa's equations, which will be derived in this section. Kruppa's equations will be formulated here in terms of simple conditions on the DIAC, which we have just seen is equivalent to internal calibration. The purpose of this section is to give a specific form for Kruppa's equations in terms of the Fundamental matrix. Kruppa's equations can be written explicitly in terms of the Singular Value Decomposition [7] of the fundamental matrix.

Consider two camera matrices $P$ and $P'$ with the same calibration. Let $C$ be the DIAC as imaged by these two cameras. As was shown in section II the image of the abso-

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lute conic (IAC) is independent of the pose of the camera, and so it is the same for the two cameras in question.

Let $F$ be the fundamental matrix for the pair of cameras. For reasons to become apparent later, we wish to apply projective transformations represented by $3 \times 3$ transformation matrices $A$ and $A'$ to the two images. After the transformations, the effective camera matrices will be $AP$ and $A'P'$, corresponding to the camera projection followed by projective transformation of the image. This will of course change the DIAC to some new conic envelopes, which we will call $D$ and $D'$. Since $A$ and $A'$ may be different, we can no longer assume that $D = D'$.

Suppose that $A$ and $A'$ are chosen so that the fundamental matrix for the two new camera matrices $AP$ and $A'P'$ has the special form

$$E = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{1}$$

This is a very special fundamental matrix having the property that the two epipoles are at the origin and that corresponding epipolar lines are identical in the two images.

Now, consider a plane passing through the two camera centres, tangent to the absolute conic. Such a plane will project to a pair of corresponding epipolar lines in the two images, and these two lines will be tangent to the IAC. Since there are two such tangential planes, there are two pairs of corresponding epipolar tangents.

We recall that corresponding epipolar lines in the two images are identical. Let $(\lambda, \mu, 0)^\top$ be a tangent to the IAC. Since $D$ is the DIAC in the first image, this tangential relationship may be written as

$$(\lambda, \mu, 0) D (\lambda, \mu, 0)^\top = 0$$

and similarly, $(\lambda, \mu, 0) D' (\lambda, \mu, 0)^\top = 0$. Writing these two equations out explicitly gives

$$\lambda^2 d_{11} + 2\lambda\mu d_{12} + \mu^2 d_{22} = 0$$

and

$$\lambda^2 d'_{11} + 2\lambda\mu d'_{12} + \mu^2 d'_{22} = 0$$

where $D = [d_{ij}]$ and $D' = [d'_{ij}]$, both of which are symmetric.

Since the two tangent lines to the IAC must be the same two lines in the two images, these two equations must have the same pair of solutions for $\lambda$ and $\mu$. This means that they must be identical equations (up to scale), and so

$$\frac{d_{11}}{d'_{11}} = \frac{d_{12}}{d'_{12}} = \frac{d_{22}}{d'_{22}} . \tag{2}$$

Next, we determine the form of the coefficients $d_{ij}$ and $d'_{ij}$ in terms of the DIAC, $C$ and the two transformations $A$ and $A'$. Consider a transformation $A$. How does this transformation transform lines ? Well, a point $\mathbf{u}$ lies on a line $\boldsymbol{\lambda}$ if and only if $\boldsymbol{\lambda}^\top \mathbf{u} = 0$. This can be written as $\boldsymbol{\lambda}^\top A^{-1} A \mathbf{u} = 0$. Thus, $\mathbf{u}$ lies on $\boldsymbol{\lambda}$ if and only if $A\mathbf{u}$ lies on $A^{-\top}\boldsymbol{\lambda}$. Thus, $A^{-\top}\boldsymbol{\lambda}$ is the transformed line. Now, a line $\boldsymbol{\lambda}$ belongs to a conic envelope $C$ if and only if $\boldsymbol{\lambda}^\top C \boldsymbol{\lambda} = 0$. This can be written as $(\boldsymbol{\lambda}^\top A^{-1})(ACA^\top)(A^{-\top}\boldsymbol{\lambda}) = 0$. Thus, the transformation $A$ maps the conic envelope $C$ to $D = ACA^\top$, and similarly $A'$ maps $C$ to $D' = A'CA'^\top$.

Now, we want to compute $d_{ij}$, where $D = [d_{ij}]$. Let

$$A = \begin{bmatrix} \mathbf{a}_1{}^\top \\ \mathbf{a}_2{}^\top \\ \mathbf{a}_3{}^\top \end{bmatrix}$$

where $\mathbf{a}_i$ is the $i$-th row of $A$. Then, from $D = ACA^\top$ we compute $d_{ij} = \mathbf{a}_i{}^\top C \mathbf{a}_j$. A similar formula holds for $d'_{ij}$. Then (2) leads to the following explicit formula :

$$\frac{\mathbf{a}_1{}^\top C \mathbf{a}_1}{\mathbf{a}'_1{}^\top C \mathbf{a}'_1} = \frac{\mathbf{a}_1{}^\top C \mathbf{a}_2}{\mathbf{a}'_1{}^\top C \mathbf{a}'_2} = \frac{\mathbf{a}_2{}^\top C \mathbf{a}_2}{\mathbf{a}'_2{}^\top C \mathbf{a}'_2} \tag{3}$$

These are the Kruppa equations.

We conclude the derivation by finding an explicit form for the matrices $A$ and $A'$ in terms of the Singular Value Decomposition of the fundamental matrix $F$. Let $F$ be written as $F = UWV^\top$, where $U$ and $V$ are orthogonal, and $W = \mathrm{diag}(r, s, 0)$ is a diagonal matrix. We may write this as follows:

$$F = U \begin{bmatrix} r & & \\ & s & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^\top .$$

We write

$$A' = \begin{bmatrix} r & & \\ & s & \\ & & 1 \end{bmatrix} U^\top$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^\top .$$

Then, we see that $F = A'^\top E A$ with $A$ and $A'$ non-singular. For a pair of matching image points $\mathbf{u}' \leftrightarrow \mathbf{u}$ we have $\mathbf{u}'^\top F \mathbf{u} = 0$. Thus, $\mathbf{u}'^\top A'^\top E A \mathbf{u} = 0$. Setting $\hat{\mathbf{u}} = A\mathbf{u}$ and $\hat{\mathbf{u}}' = A'\mathbf{u}'$, we see that $\hat{\mathbf{u}}'^\top E \hat{\mathbf{u}} = 0$. Thus, $A$ and $A'$ are the two transforms that we require.

Matrix $A'$ can be written as

$$A' = \begin{bmatrix} \mathbf{a}'_1{}^\top \\ \mathbf{a}'_2{}^\top \\ \mathbf{a}'_3{}^\top \end{bmatrix} = \begin{bmatrix} r\mathbf{u}_1{}^\top \\ s\mathbf{u}_2{}^\top \\ \mathbf{u}_3{}^\top \end{bmatrix} .$$

where $\mathbf{u}_i$ is the $i$-th column of $U$.

For $A$ we have

$$A = \begin{bmatrix} \mathbf{a}_1{}^\top \\ \mathbf{a}_2{}^\top \\ \mathbf{a}_3{}^\top \end{bmatrix} = \begin{bmatrix} \mathbf{v}_2{}^\top \\ -\mathbf{v}_1{}^\top \\ \mathbf{v}_3{}^\top \end{bmatrix} .$$

where $\mathbf{v}_i$ is the i-th column of $V$.

From (3) we obtain

$$\frac{\mathbf{v}_2{}^\top C \mathbf{v}_2}{r^2 \mathbf{u}_1{}^\top C \mathbf{u}_1} = \frac{-\mathbf{v}_2{}^\top C \mathbf{v}_1}{rs \mathbf{u}_1{}^\top C \mathbf{u}_2} = \frac{\mathbf{v}_1{}^\top C \mathbf{v}_1}{s^2 \mathbf{u}_2{}^\top C \mathbf{u}_2} \tag{4}$$

This is the promised explicit form for the Kruppa equations.

### *A. Consequences of Kruppa's Equations*

Since the numerator and denominator of (3) or (4) are linear expressions in the entries of the matrix $C$, the resulting equations are quadratic. From (4) one obtains two independent quadratic equations in the entries of $C$. Since $C$ is symmetric, and defined only up to a scale, it has 5 degrees of freedom. A single pair of views is clearly not sufficient to determine $C$. However from three views, one obtains three fundamental matrices, one for each of the pairs of views. Now we have six quadratic equations in five unknowns. It is not clear, of course that the six equations are independent. However, as demonstrated empirically by Luong ([4]) they are sufficient (at least with noise-free data) to determine $C$. Direct determination of the calibration by the solution of these simultaneous quadratics is possible, as Luong showed. Solution is difficult because of the existence of multiple solutions to non-linear equations, and the difficulties involved with solving redundant systems. Luong's method was to solve subsets of five equations, giving up to $2^5 = 32$ solutions, and then to select solutions common to all subsets of equations.

Special cases of the Kruppa equations are of interest, as considered below.

**Calibrated Cameras.** If we assume that the calibration matrix of the cameras is the identity matrix $K = I$ in both cases, then the matrix $C = KK^\top$ representing the DIAC is also equal to the identity. In this case, Kruppa's equations (4) reduce to the form

$$1/r^2 = 0/0 = 1/s^2$$

where $r$ and $s$ are the singular values of the fundamental (or essential) matrix. This implies $r = s$. In other words if the calibration matrix is the identity, then the two non-zero singular values of the fundamental matrix are equal. This is an easy proof of a result of Huang and Faugeras ([9]).

**Camera with known principal point.** If the principal point of the camera is known, then by a suitable change of image coordinates one may assume that the principal point is at the origin. If it is further assumed that there is no *skew* parameter, then the calibration matrix is a diagonal matrix of the form $\mathrm{diag}(k_u, k_v, 1)$. In this case, $C = \mathrm{diag}(k_u^2, k_v^2, 1)$ and the Kruppa equations (4) are quadratic in the variables $k_u^2$ and $k_v^2$. From two images we have a pair of quadratic equations in two variables – sufficient to solve for $k_u^2$ and $k_v^2$ in a relatively straight-forward manner. Thus under assumptions of known principal point, and no skew one may solve for $k_u^2$ and $k_v^2$, and hence for $k_u$ and $k_v$. There may be up to four solutions, but only positive solutions for $k_u^2$ and $k_v^2$ need be considered. This result may be derived also from the work of Luong, and is generally known.

**Translatory motion.** In the case of pure translatory motion of the camera, the fundamental matrix $F$ is skew-symmetric. This may be seen in a variety of ways. For instance, if $K[I \mid 0]$ and $K[I \mid \mathbf{t}]$ are the two cameras, then (as follows immediately from [10], Lemma 2) the fundamental matrix is given by $F = K^{-\top}[\mathbf{t}]_\times K^{-1}$, where

$$[\mathbf{t}]_\times = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}$$

is a skew-symetric matrix. Thus, $F$ is skew-symmetric and its Schur decomposition ([7]) is $F = UEU^\top$ where $U$ is orthogonal and $E$ is as given in (1). The two matrices $A$ and $A'$ required to transform $F$ to the simple form (1) are both equal. Thus both numerator and denominator are equal in the form (3) of the Kruppa equations. Then equations (3) takes the simple form $1 = 1 = 1$. Although this is a significant fact[1], it is not a useful constraint on camera calibration. Thus, a translatory motion of the cameras does not impose any constraint on camera calibration. It is interesting that affine scene reconstruction is nevertheless possible from translatory motions ([11]).

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