Sensitivity of calibration to principal point position

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Abstract

A common practice when carrying out self-calibration of a camera from one or more views is to start with a guess at the principal point. The general belief is that inaccuracies in the estimation of the principal point do not have a significant effect on the other calibration parameters. It is the purpose of this paper to refute that belief. Indeed, it is demonstrated that the determination of the focal length of the camera is tied up very closely with the estimate of the principal point. Small changes in the estimated (sometimes merely guessed) principal point can cause very large changes in the estimated focal length. In fact, the relative uncertainty in the focal length is inversely proportional to the distance of the principal point to the epipolar line. This analysis is geometric and exact, rather than experimental.

1 Introduction

We are concerned throughout with natural pinhole cameras, that is cameras with zero skew and unit (or known) aspect ratio. First, we consider calibration of a camera from a single view, in which a horizon line and a vertical direction, or vertical vanishing point may be identified. Given knowledge of the principal point, the focal length may be determined by an easy geometric construction, from which the dependence of the focal length on the principal point estimate is easily seen.

This single-view case is next extended to the two-view case, in which the focal lengths of each camera may be determined from the fundamental matrix alone. It is shown that this problem may be reduced to the previous case of single-view calibration, by considering the horizon line and perpendicular vanishing direction of the plane formed by the base line and one of the camera's principal rays. The sensitivity of this process to variations in the principal point then becomes evident.

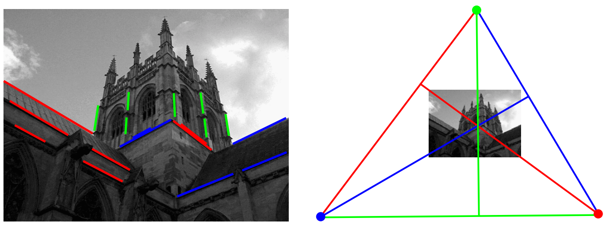

Figure 1: *Three orthogonal vanishing points are found from the image. The principal point lies at the orthocentre of the triangle formed by the vanishing points. Consequently, it lies on the perpendicular line from the apex to the horizon.*

2 Calibration from a single image

Throughout this paper, we assume that the skew of the camera is zero, and the aspect ratio is equal to one. Thus, the only remaining parameters of the camera are the focal length and the principal point.

We consider calibration of the camera (determination of the focal length and principal point) from a single view. Clearly this can not be done without some scene information, and many ways have been proposed to do this. It is not within the scope of this paper to investigate the sensitivity of all methods, and in fact, we will concentrate on one calibration approach. The method investigated here is related to the work of [LZ98], also considered in [HZ00]. Vanishing points of parallel lines are used to determine the calibration. In particular, if the vanishing points in three orthogonal directions are known, then the principal point may be identified as the orthocentre of the triangle that has the three orthogonal vanishing points as its vertices ([LZ98]). See Fig 1, which illustrates this point. As will be seen below, the focal length of the camera may be uniquely determined once the principal point is known.

If the vanishing points of all three orthogonal directions are known, then the camera may be completely calibrated.

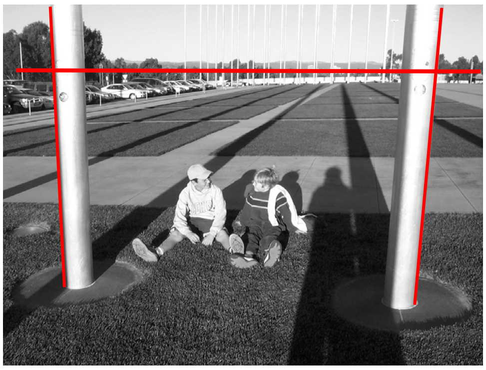

Figure 2: *An example of an image for which it is possible to compute the focal length of the camera, assuming knowledge of the principal point. To do this, it is sufficient to identify the horizon and a vertical vanishing direction. In this image, the exact position of the horizon can be established with the aid of the vanishing points of the ground markings and shadows. The image of the apex may be computed as the intersection of the edges of the two flag-poles. Focal length may then be computed using (1).*

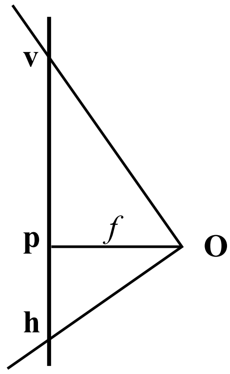

Figure 3: *The diagram shows the vertical plane containing the principal ray of the camera. The two rays to the horizon and the apex are perpendicular to each other, meeting the focal plane at points* $\mathbf{h}$ *and* $\mathbf{v}$ *respectively. The focal length is the distance from the camera centre to the focal plane. By similar triangles,* $f^2 = -d(\mathbf{p},\mathbf{h})d(\mathbf{p},\mathbf{v})$.

Thus, the focal length and the principal point are completely determined, and in particular, the focal length may be determined without any extra assumptions about the principal point. Therefore, we consider a calibration problem in which there is a little less information available. Accordingly, we assume that the horizon line and the vertical vanishing point (henceforth known as the apex) are identified in an image. An example of such an image is given in Fig 2. The principal point in the image must lie on the perpendicular from the apex to the horizon. Thus, the principal point has a single remaining degree of freedom. Suppose that the principal point is the point $\mathbf{p}$ in the image, $\mathbf{v}$ is the vertex, and $\mathbf{h}$ is the foot of the perpendicular from the apex to the horizon. In this case, the focal length may be computed by a simple formula:

$$f^2 = -d(\mathbf{p},\mathbf{h})d(\mathbf{p},\mathbf{v}) \qquad (1)$$

where f is the focal length, and $d(\cdot,\cdot)$ represents the (signed) Euclidean distance. The negative sign in this expression expresses the fact that $d(\cdot,\cdot)$ is a signed quantity, and that the direction vectors from the principal point to the horizon and the apex must be in opposite directions in order for f^2 to be positive, as given by this formula. In other words, the principal point must lie between the horizon and the apex. Proof of this formula is given in Fig 3.

The focal length may be computed by a simple geometric construction, as follows. Let C be the circle with the line from apex $\mathbf{v}$ to horizon $\mathbf{h}$ as diameter. The line in the image through the principal point $\mathbf{p}$ meets the circle C in two points $\mathbf{a}$ and $\mathbf{b}$. By elementary geometry of a circle, $d(\mathbf{p},\mathbf{a})d(\mathbf{p},\mathbf{b}) = d(\mathbf{p},\mathbf{v})d(\mathbf{p},\mathbf{h})$, and since $d(\mathbf{p},\mathbf{a}) = -d(\mathbf{p},\mathbf{b})$ it follows that $f = d(\mathbf{p},\mathbf{a})$. The circle constructed with centre $\mathbf{p}$ passing through $\mathbf{a}$ and $\mathbf{b}$ is the $45°$ circle in the image. This is illustrated in Fig 4, and on a real image in Fig 5. Since f is the distance from $\mathbf{p}$ to the circle C along a line drawn perpendicular to the diameter $\mathbf{vh}$, the way f varies as a function of the presumed principal point location is easily visualized. In particular, as the principal point moves towards the horizon, or the apex in the image, the corresponding value of the focal length f diminishes towards zero.

If an image is taken with a camera aimed directly at the horizon, the principal point corresponds with the point $\mathbf{h}$ on the horizon, and so $d(\mathbf{p},\mathbf{h}) = 0$. In this case, however, the distance to the apex will be infinite, and hence $f^2 = 0 \times \infty$, and the computed value of f will be indeterminate. From this, we see that it is impossible to compute f from this configuration if the camera is pointed directly at the horizon, or the apex.

2.1 A different formula.

It is sometimes difficult to measure the distance of the principal point to the apex, particularly if the apex is far away from the visible area of the image. A simple modification of the image takes care of that problem. Suppose that $\mathbf{l}$ is a line from the apex to the horizon, not passing through the principal point. This can be the image of a vertical object in the scene. Let $s_1 = d(\mathbf{p},\mathbf{h})$ be the orthogonal distance from the principal point to the horizon, and let $s_2 = d(\mathbf{p},\mathbf{l})$ be the orthogonal distance from the principal point to the line $\mathbf{l}$. Let θ be the angle between line $\mathbf{l}$ and the horizon. Then

$$f^2 = s_1 s_2 / cos\theta \qquad (2)$$

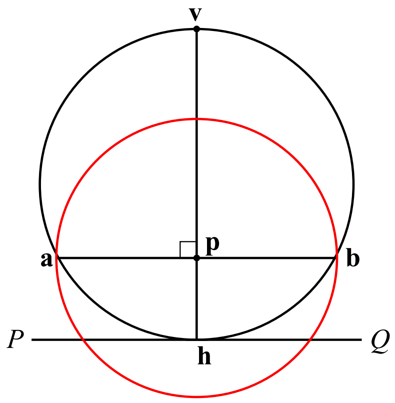

Figure 4: *A circle (black) is drawn with diameter the perpendicular line between the apex* $\mathbf{v}$ *and the horizon* PQ*. The principal point must lie on this line. A line is drawn through* $\mathbf{p}$ *perpendicular to* $\mathbf{vh}$*. This meets the circle in two points* $\mathbf{a}$ *and* $\mathbf{b}$*. The focal length equals the distance* $d(\mathbf{p}, \mathbf{a})$*, and the circle with diameter* $\mathbf{ab}$ *is the* $45°$ *circle in the image (shown in red).*

The proof of this is immediate from Fig 6.

Sensitivity of f**.** In terms of (2) it is easy to write down an expression for the uncertainty of the focal length in terms of small errors in the three parameters s_1, s_2 and θ. Since $df = \partial f/\partial s_1 ds_1 + \partial f/\partial s_2 ds_2 + \partial f/\partial\theta d\theta$, simple calculus leads to the following formula for the relative change in f.

$$df/f = 1/2\,(ds_1/s_1 + ds_2/s_2 + tan\theta d\theta) \qquad (3)$$

From this we see that f is sensitive to small changes in the measurements of s_1, s_2 and θ when s_1 or s_2 are small, or θ is close to $90°$. In particular, it is impossible to estimate f accurately when the principal point is close to the horizon line, since in this case s_1 is small, and the angle θ will be close to $90°$.

Dependence of f **on the principal point.** It was seen above that if the horizon and apex are identifiable in an image, then the principal point position may vary with one degree of freedom, along the line through the apex perpendicular to the horizon. If there is only one vertical line feature visible in the image, then the apex may not be identified, but instead may lie anywhere along the image of the vertical line. The principal point may now vary with two degrees of freedom anywhere in the image. Figure 7 gives an example of such an image. For a choice of the principal point, the focal length may be computed using (2), or by direct geometric construction, as shown previously. However, because of the added degree of freedom, the computed focal length depends more dramatically on the assumed position of the principal point. It may vary from near zero, when the principal point is assumed to be close to the intersection of the horizon and vertical line, to large values when the principal point is far from both these lines.

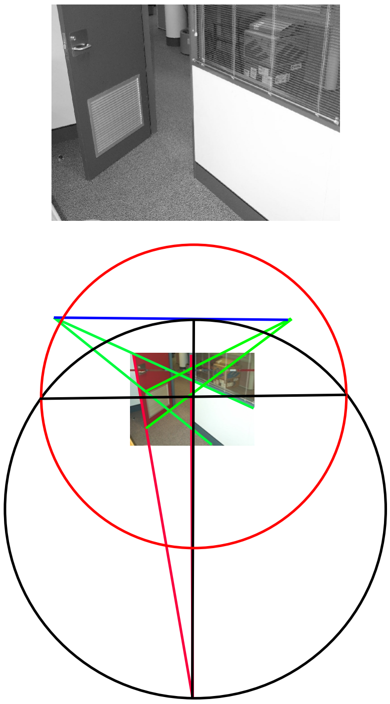

Figure 5: *This image shows the computation of the focal length of the camera assuming a principal point at the centre of the image. The original image is shown above, and the construction below. Since the horizon line (blue) is not directly visible in this image, it is computed as the line joining two horizontal vanishing points, computed from the right-hand wall and the grid on the door (green lines). Note that these are not orthogonal vanishing points. The vertical vanishing point is given by the sides of the door and the door frame (red lines). The red circle is the* $45°$ *circle, with radius* f*, assuming the principal point is placed as shown (at the centre of the red circle).*

3 Calibration from two views

Next, the calibration methods for a single view discussed above will be extended to consideration of calibration from two views. This problem was first addressed in [Har92] where an algorithm for computing the two focal lengths was given. This algorithm was quite complicated, and the problem was later considered by several authors. The most compact solution was found by Bougnoux in [Bou98], where a simple formula was given for the two focal lengths in terms of the fundamental matrix and the two principal points (see

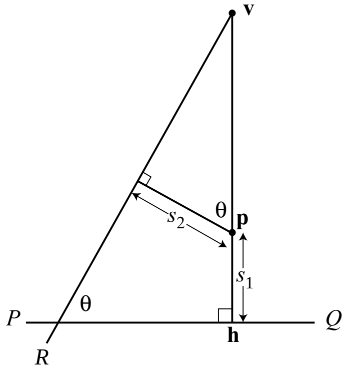

Figure 6: *Since $d(\mathbf{p}, \mathbf{v}) = s_2/cos\theta$, it follows from (1) that $f^2 = -s_1 s_2/cos\theta$.*

equation (5) below). This formula will be rederived here in a way that sheds light on its geometric meaning, and its sensitivity to variations in the assumed positions of the principal point. It will be shown that Bougnoux's formula is closely related to (2), and has a simple geometric interpretation.

First, we note that in order to compute the focal lengths, the positions of the two principal points need to be known. The geometry of two views may be summarized in Fig 8. The two camera centres O_1 and O_2 are joined by a baseline O_1O_2 and the view direction of each camera is defined by the principal rays, R_1 and R_2 of the cameras. Of most interest is the case where these three lines are not coplanar; in other words, the principal rays of the two cameras do not meet. The principal ray R_1 and the baseline O_1O_2 define a plane, which will be denoted by $R_1O_1O_2$. In addition, let A_1 be the ray through the camera centre O_1 perpendicular to the plane $R_1O_1O_2$. This plane, and the ray A_1 form an orthogonal pair in space. If their projections into the second image can be identified, then we may use the methods of section 2 to compute the focal length of the second camera.

First of all, consider the first (left) image taken from camera centre O_1. We want to identify the projection in this image of the plane $R_1O_1O_2$ and the ray A_1. The projection of the plane is easily identified as the epipolar line through the principal point, namely $[\mathbf{e}_1]_\times \mathbf{p}_1 = \mathbf{e}_1 \times \mathbf{p}_1$. Since the ray A_1 is perpendicular to the plane containing the principal ray, it projects to a point at infinity in the image, in the direction perpendicular to the line $[\mathbf{e}_1]_\times \mathbf{p}_1$. This infinite point may be written as $\mathtt{I}(\mathbf{e}_1 \times \mathbf{p}_1)$, or $\mathtt{I}[\mathbf{e}_1]_\times \mathbf{p}_1$, where $\mathtt{I}$ is the 3×3 matrix $\text{diag}(1, 1, 0)$.

Note that $\mathtt{I}\mathbf{l}$ represents the point at infinity in the direction perpendicular to $\mathbf{l}$ for any line $\mathbf{l}$. For example, (a, b) is the vector perpendicular to the line $ax + by + c = 0$, represented by coordinates $(a, b, c)^\top$, and $(a, b, 0)$ is the vanishing point in this direction. This relation will be used often in this paper, so we emphasize it:

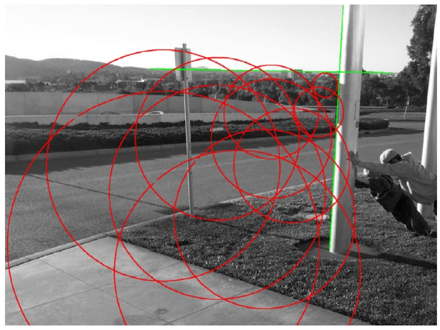

Figure 7: *In this image, the apex is unknown, but is constrained to lie somewhere along the vertical direction shown in the image (green). The principal point has two degrees of freedom, but once a position of the principal point is assumed, the focal length of the camera may be computed. The red circles represent various estimates of the focal length for different assumed principal points. Each circle is the locus of all directions 10° from the principal point, which is at the centre of the circle. Thus, the radius of each circle is equal to $0.176f = \tan(10^\circ)f$. The focal length has been computed in each case using (2).*

3.1. *If $\mathbf{p}$ is a point in an image, and $\mathbf{e}$ is the epipole, then $\mathtt{I}[\mathbf{e}]_\times \mathbf{p}$ represents the point at infinity in the direction normal to the epipolar line $[\mathbf{e}]_\times \mathbf{p} = \mathbf{e} \times \mathbf{p}$.*

This formula may be applied repeatedly. For instance $\mathtt{I}[\mathbf{e}]_\times \mathtt{I}[\mathbf{e}]_\times \mathbf{p}$ is the point at infinity on the line $\mathbf{e} \times \mathbf{p}$.

It is now easy to compute the epipolar lines in the second image corresponding to the plane $R_1O_1O_2$ and the ray A_1. This is done simply by transferring the epipolar lines from the first image to the second, using the fundamental matrix. First, consider the vanishing line of the plane $R_1O_1O_2$. Since this plane passes through the camera centre O_2, it is viewed edge-on in the second image. Consequently, its vanishing line is nothing more than the image of any line lying in the plane $R_1O_1O_2$. Since the principal ray of the first camera is just such a line we see:

3.2. *The vanishing line in the second image of the plane $R_1O_1O_2$ is the epipolar line $\mathtt{F}\mathbf{p}_1$ corresponding to the principal point of the first camera. Here $\mathtt{F}$ is the fundamental matrix.*

Secondly, the image of the ray A_1 is easily computed by transferring its vanishing point using the fundamental matrix:

3.3. *The image of the ray A_1 in the second image is the epipolar line $\mathtt{F}\mathtt{I}[\mathbf{e}_1]_\times \mathbf{p}_1$.*

Thus, we may conclude:

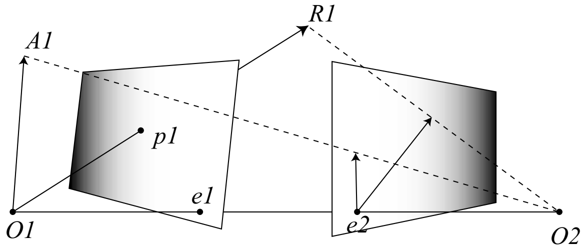

Figure 8: *The plane defined by the principal ray R_1 and the base-line O_1O_2 is imaged as an epipolar line in the second (right-hand) image. The ray A_1 is perpendicular to this plane, and also projects to an epipolar line in the second image. The two epipolar lines are the images of an orthogonal plane/line pair. Assuming a position for the principal point $\mathbf{p}_2$, the focal length f_2 of the second camera may be computed using the method of section 2.*

3.4. *Given the fundamental matrix $\mathtt{F}$ for two images, and assumed positions of the principal points $\mathbf{p}_1$ and $\mathbf{p}_2$ in the two images, the focal length f_2 may be computed as follows.*

1. *Compute the two epipolar lines $\mathtt{F}\mathbf{p}_1$ and $\mathtt{F}\mathtt{I}[\mathbf{e}_1]_\times\mathbf{p}_1$ in the second image.*
2. *Apply equation (2) where s_1 and s_2 are the distances from the principal point $\mathbf{p}_2$ to these two lines, and θ is the angle between them.*

An explicit formula is possible. Let $\mathbf{l}$ and $\mathbf{l}'$ be the two epipolar lines computed in (3.4). The distance from a point $\mathbf{p} = (x, y, 1)^\top$ to a line $\mathbf{l}$ is $\mathbf{l}^\top\mathbf{p}/\alpha$ where $\alpha = (l_1^2 + l_2^2)^{1/2}$, and l_i here represents the i-th component of the vector $\mathbf{l}$. Furthermore, the angle between two lines $\mathbf{l}$ and $\mathbf{l}'$ satisfies the relation $\cos\theta = \mathbf{l}'^\top\mathtt{I}\mathbf{l})/\alpha\alpha'$. Now, writing (2) in terms of the homogeneous coordinates for the two lines $\mathbf{l}_2$ and $\mathbf{l}_2'$ involved, yields

$$f_2^2 = -\frac{(\mathbf{l}_2'^\top\mathbf{p}_2)(\mathbf{l}_2{}^\top\mathbf{p}_2)}{\mathbf{l}_2{}^\top\mathtt{I}\mathbf{l}_2'} \qquad (4)$$

since the factors α and α' cancel top and bottom.

Substituting the formulas (3.2) and (3.3) for the lines $\mathbf{l}_2$ and $\mathbf{l}_2'$ finally gives the formula

$$f_2^2 = -\frac{(\mathbf{p}_1{}^\top[\mathbf{e}_1]_\times\mathtt{I}\mathtt{F}^\top\mathbf{p}_2)(\mathbf{p}_1{}^\top\mathtt{F}^\top\mathbf{p}_2)}{\mathbf{p}_1{}^\top([\mathbf{e}_1]_\times\mathtt{I}\mathtt{F}^\top\mathtt{I}\mathtt{F})\mathbf{p}_1} \quad . \qquad (5)$$

In this formula, $\mathbf{e}_2$ is the epipole in the second image, and $\mathbf{p}_1$ and $\mathbf{p}_2$ are homogeneous 3-vectors representing the principal points in the two images. It is required however, that $\mathbf{p}_1 = (x, y, 1)^\top$, with last coordinate equal to 1.

The formula for the second focal length may be obtained by interchanging the role of the first and second cameras in this formula.

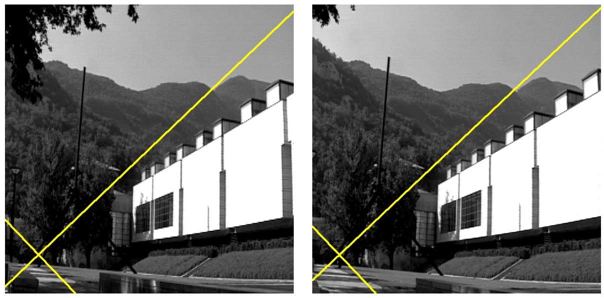

Figure 9: *A pair of images from which one wishes to compute the focal lengths. It is assumed that the principal point $\mathbf{p}_1$ in the first image is the centre of the image. Shown overlaid on the first (left-hand) image are the epipolar line $\mathbf{e}_1 \times \mathbf{p}_1$ corresponding to the principal ray, R_1, and the perpendicular epipolar line $\mathbf{e}_1 \times \mathtt{I}[\mathbf{e}_1]_\times\mathbf{p}_1$. Overlaid on the second (right-hand) image are the corresponding epipolar lines $\mathtt{F}\mathbf{p}_1$ and $\mathtt{F}\mathtt{I}[\mathbf{e}_1]_\times\mathbf{p}_1$. From these latter two lines one may compute the focal length f_2. Computation of the focal length f_2 will be very imprecise in this case, since the angle θ between these two lines is close to 90°, and also because one of the lines (actually $\mathtt{F}\mathbf{p}_1$) passes very close to the centre of the image, and hence (presumably) close to the principal point $\mathbf{p}_2$. Relative accuracy of the focal length $\mathbf{f}_2$ is given by (3), and is proportional to the distance of $\mathbf{p}_2$ from each of these two epipolar lines.*

4 Effect of varying the principal points

We consider now the effect of varying the principal point in either or both the images. Without loss of generality, let us restrict attention to (5), and computation of f_2^2. Since the left hand side of (5) is the square of the focal length, it is necessary that the right hand side be positive, otherwise, no solution exists. To analyze this further, we note that the expression for f_2^2 may be split into parts as follows:

$$A(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1{}^\top[\mathbf{e}_1]_\times\mathtt{I}\mathtt{F}^\top\mathbf{p}_2$$
$$B(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1{}^\top\mathtt{F}^\top\mathbf{p}_2$$
$$D(\mathbf{p}_1) = \mathbf{p}_1{}^\top([\mathbf{e}_1]_\times\mathtt{I}\mathtt{F}^\top\mathtt{I}\mathtt{F})\mathbf{p}_1$$

Thus, $f_2^2 = -A(\mathbf{p}_1, \mathbf{p}_2)B(\mathbf{p}_1, \mathbf{p}_2)/D_2(\mathbf{p}1)$. Note that the estimated value of the focal length becomes zero where the numerator (that is, either A or B) vanishes, and goes to infinity where the denominator D vanishes. The value of f_2^2 will change sign on the union of the vanishing sets of A, B and D. We look at each of these terms independently.

4.1 Varying the principal point $\mathbf{p}_2$

This situation has been explored in section 2. The denominator $D(\mathbf{p}_1)$ does not depend on $\mathbf{p}_2$, and so is constant. We look at the vanishing sets of the two terms $A(\mathbf{p}_1, \mathbf{p}_2)$ and $B(\mathbf{p}_1, \mathbf{p}_2)$. As discussed above, $\mathbf{p}_1{}^\top \mathtt{F}^\top$ and $\mathbf{p}_1{}^\top [\mathbf{e}_1]_\times \mathtt{IF}^\top$ represent the epipolar lines in the second image, corresponding to the principal ray in the first image, and a ray perpendicular to it. When $\mathbf{p}_2$ lies on one of these epipolar lines, the numerator of (5) vanishes. Consequently, the zero set of $A(\mathbf{p}_1, \mathbf{p}_2)B(\mathbf{p}_2, \mathbf{p}_2)$ for fixed $\mathbf{p}_1$ and varying $\mathbf{p}_2$ consists of a pair of epipolar lines in the second image. The image plane is divided into four sections by the images of the principal ray R_1 and the line A_1 orthogonal to the plane $R_1 O_1 O_2$. The principal point must lie in the acute-angled region of the plane. This is illustrated in Fig 10.

4.2 Varying the principal point $\mathbf{p}_1$

Next we consider the effect of varying the principal point $\mathbf{p}_1$ on the estimate of the focal length f_2. In this section we will assume that the principal point $\mathbf{p}_2$ of the second camera is known and fixed at a given value. As $\mathbf{p}_1$ varies, formula (5) give an estimate for the focal length f_2.

Vanishing set of $A(\mathbf{p}_1, \mathbf{p}_2)B(\mathbf{p}_1, \mathbf{p}_2)$. Since $\mathtt{F}$ is the fundamental matrix, $\mathtt{F}\mathbf{p}_1$ is the epipolar line corresponding to the point $\mathbf{p}_1$. Thus the set of points $\mathbf{p}_1$ at which $B(\mathbf{p}_1, \mathbf{p}_2)$ vanishes is precisely the set of points on the epipolar line $\mathtt{F}^\top \mathbf{p}_2$.

On the other hand, $\mathbf{p}_1{}^\top [\mathbf{e}_1]_\times \mathtt{I}$ is a point at infinity lying on the epipolar line perpendicular to the one through $\mathbf{p}_1$. Hence $A(\mathbf{p}_1, \mathbf{p}_2)$ vanishes when $\mathbf{p}_1$ lies on an epipolar line perpendicular to $\mathtt{F}^\top \mathbf{p}_2$.

4.5. *For a given value of $\mathbf{p}_2$, the set of points $\mathbf{p}_1$ such that the numerator $A(\mathbf{p}_1, \mathbf{p}_2)B(\mathbf{p}_1, \mathbf{p}_2)$ of (5) vanishes consists of the epipolar line $\mathtt{F}^\top \mathbf{p}_2$ and the epipolar line perpendicular to it.*

Vanishing set of D. We now turn to the denominator of (5), namely $D(\mathbf{p}_1) = \mathbf{p}_1{}^\top ([\mathbf{e}_1]_\times \mathtt{IF}^\top \mathtt{IF})\mathbf{p}_1$. Note that this is independent of $\mathbf{p}_2$. From the form of this expression, it is easily seen that the vanishing set is a conic. It will be shown that in fact it is a degenerate conic, consisting of two perpendicular lines through the epipole $\mathbf{e}_1$.

First of all one observes (by simple substitution and using the fact that $\mathtt{F}\mathbf{e}_1 = 0$) that the epipole $\mathbf{e}_1$ belongs to this vanishing set. If $\mathbf{p}_1$ is any point such that $D(\mathbf{p}_1) = 0$, then $D(\mathbf{p}_1 + \alpha \mathbf{e}_1) = 0$ for any α, and hence the vanishing set is made up of lines passing through $\mathbf{e}_1$, epipolar lines.

Now let $\mathbf{p}_1$ be a point at infinity, and let $\mathbf{p}'_1$ be the point at infinity on the epipolar line normal to $[\mathbf{e}_1]_\times \mathbf{p}_1$. By symmetry, $\mathbf{p}''_1 = (\mathbf{p}'_1)' = \mathbf{p}_1$. By (3.1), $\mathbf{p}'_1 = \mathtt{I}[\mathbf{e}_1]_\times \mathbf{p}_1$, and so $D(\mathbf{p}_1) = \mathbf{p}'_1{}^\top (\mathtt{F}^\top \mathtt{IF})\mathbf{p}_1$. Substituting $\mathbf{p}'_1$ for $\mathbf{p}_1$ in this expression gives

$$
\begin{aligned}
D(\mathbf{p}'_1) &= \mathbf{p}''^{\top}_1 (\mathtt{F}^\top \mathtt{IF})\mathbf{p}'_1 \\
&= \mathbf{p}_1{}^\top (\mathtt{F}^\top \mathtt{IF})\mathbf{p}'_1 \\
&= \mathbf{p}'^{\top}_1 (\mathtt{F}^\top \mathtt{IF})\mathbf{p}_1 \text{ (by transposing)} \\
&= D(\mathbf{p}_1) \ .
\end{aligned}
$$

Thus, $D(\mathbf{p}'_1)$ vanishes if and only if $D(\mathbf{p}_1)$ does. Consequently, the zero set of $D(\mathbf{p}_1)$ consists of a pair of orthogonal epipolar lines.

Summarizing this complete discussion.

4.6. *For a given value of $\mathbf{p}_2$, the set of points $\mathbf{p}_1$ for which the estimated value of f_2^2 changes sign consists of two pairs of orthogonal epipolar lines in image 1. The value of f_2^2 tends to zero at one pair of lines, and to infinity at the other pair of lines.*

This is illustrated in Fig 11.

5 Varying both principal points

The previous sections have assumed that one of the principal points was fixed, and the other was allowed to vary. In some instances it may be more realistic to assume that the principal point is the same in both images, albeit not known exactly. Therefore, in this section, we assume that $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$, and examine how the focal length estimate depends on the position of this common principal point. In this case, the formula for f_2 is as follows.

$$
f_2^2 = -\frac{(\mathbf{p}^\top [\mathbf{e}_1]_\times \mathtt{IF}^\top \mathbf{p})(\mathbf{p}^\top \mathtt{F}^\top \mathbf{p})}{\mathbf{p}^\top ([\mathbf{e}_1]_\times \mathtt{IF}^\top \mathtt{IF})\mathbf{p}} = \frac{A(\mathbf{p}, \mathbf{p})B(\mathbf{p}, \mathbf{p})}{D(\mathbf{p})} \ . \tag{6}
$$

We have seen already that the zero-set of $D(\mathbf{p})$ consists of a pair of orthogonal epipolar lines through the epipole $\mathbf{e}_1$. The zero sets of the expressions $A(\mathbf{p}, \mathbf{p}) = \mathbf{p}^\top \mathtt{F}^\top \mathbf{p}$ and $B(\mathbf{p}, \mathbf{p}) = \mathbf{p}^\top [\mathbf{e}_1]_\times \mathtt{IF}^\top \mathbf{p}$ are conics. It is easy to see that each of these expressions vanishes at both epipoles $\mathbf{e}_1$ and $\mathbf{e}_2$, hence the conics pass through the two epipoles. In addition, it may be verified that the tangents to these two conics at the epipole $\mathbf{e}_1$ are orthogonal. To see that, recall that the tangent at a point $\mathbf{x}$ on a conic defined by matrix $\mathtt{C}$ is the line $(\mathtt{C} + \mathtt{C}^\top)\mathbf{x}$. Applying this, one finds that the tangents to these two conics at the point $\mathbf{e}_1$ are $\mathtt{F}^\top \mathbf{e}_1$ and $[\mathbf{e}_1]_\times \mathtt{IF}^\top \mathbf{e}_1$, which are easily seen to be orthogonal lines.

Summarizing this gives the following result.

5.7. *As the position of the principal point $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$ varies, the singular set where the value of the focal length f_2^2 changes sign has the following components.*

1. *Two orthogonal epipolar lines passing through the epipole $\mathbf{e}_1$. As the assumed position of $\mathbf{p}$ approaches these lines, the value of f_2^2 tends to infinity.*

2. *Two conics passing through the two epipoles $\mathbf{e}_1$ and $\mathbf{e}_2$. The conics are orthogonal at $\mathbf{e}_1$. As $\mathbf{p}$ approaches these conics, the estimated value of f_1^2 tends to zero.*

These lines divide the plane into regions. The value of f_2^2 is negative on alternate regions, which therefore represent impossible positions for the principal point.

For an illustration of this, see Fig 12.

6 Conclusion

We have argued that the assumed position of the principal point may have a large effect on the estimated focal length of the cameras for certain single-view and two-view calibration scenarios. The two-view calibration method analyzed here has been shown previously ([NHBP96]) to have degenerate configurations when the principal rays of the cameras meet. We have given here a simple analysis of how the quality of the focal length estimate degenerates when the principal ray of one camera lies close to the epipolar line corresponding to the other principal point. It is shown that in fact, the relative uncertainty in the the focal length estimated by this method is inversely proportional to the distance of the principal point to this epipolar line. Because of this sensitivity, the practicality of estimating focal lengths from two views is doubtful. Figure 13 shows a selection of images that have been used to compute the fundamental matrix, in all of which computation of the focal lengths will be quite inaccurate.

The method and analysis of this paper apply to other imaging scenarios, such as planar motion ([AZH96] in which it is also possible to compute the image positions of the horizon and apex. A new interpretation of Bougnoux's focal length formula, is also given here, relating it to the geometry of the principal rays.

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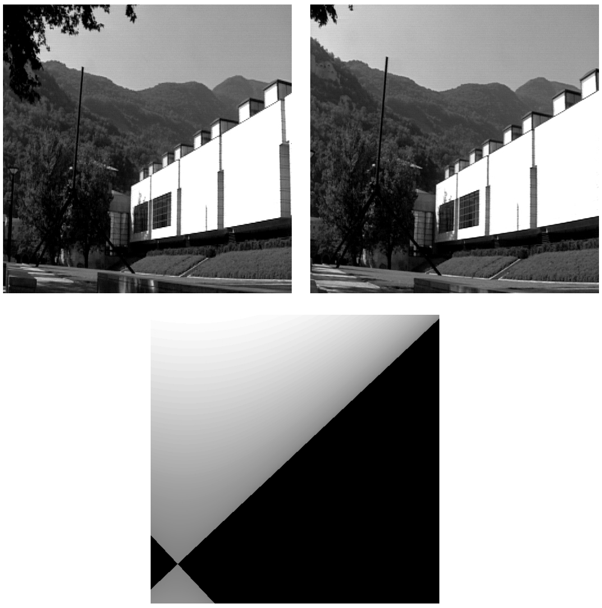

Figure 10: *Pair of images of a museum, and the estimated value of the focal length* f_2^2*. The principal point* $\mathbf{p}_1$ *is assumed to be in the centre of the first image. The fundamental matrix is computed from point correspondences between the images. For each assumed position of the principal point* $\mathbf{p}_2 = (x, y)$ *in the second image, formula (5) gives an estimate of the focal length* f_2^2*. The image at the right shows the focal length* f_2^2 *as a function of* (x, y)*. The focal length is represented by the intensity of the image. For points* (x, y) *in the black region of the image, the estimated value of* f_2^2 *is negative, meaning that the assumptions on position of the principal point are not viable (impossible situation).*

The image is divided into four regions by the two epipolar lines corresponding to the principal ray R_1 *and its perpendicular* A_1 *in the first image. For* $\mathbf{p}_2$ *in the light region, the value of the focal length may be computed by (2), and its uncertainty by (3). In this example, one of the epipolar lines passes close to the centre of the image, and hence probably close to the principal point* $\mathbf{p}_2$*. Hence, according to (3) the estimate of* f_2 *will be very sensitive to variation in the position of* $\mathbf{p}_2$.

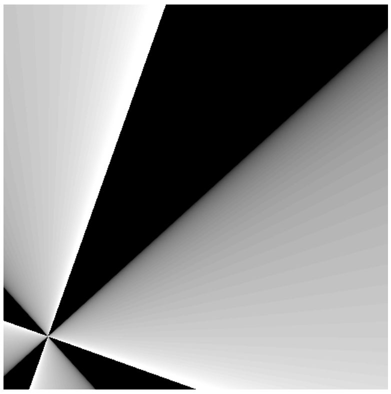

Figure 11: *Assuming* $\mathbf{p}_2$ *is fixed, and* $\mathbf{p}_1$ *is varying, the singular set for estimation of* f_2 *consists of points* $\mathbf{p}_1$ *lying on two pairs of orthogonal epipolar lines. The estimated value of* f_2*, according to (5) tends to* 0 *and* ∞ *as* $\mathbf{p}_1$ *approaches the alternating epipolar lines. The value of* f_2 *is only defined for* $\mathbf{p}_1$ *in alternating sectors of the image as shown, since* f_2^2 *is negative for* $\mathbf{p}_1$ *in the other (dark) sectors.*

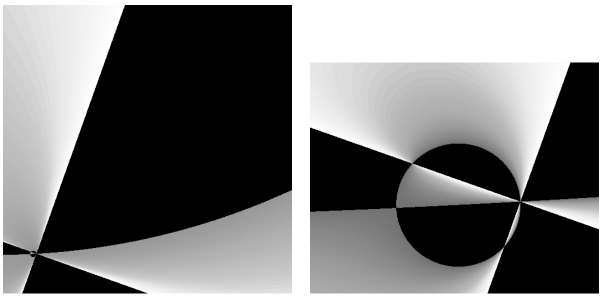

Figure 12: *Museum images : the geometry of the allowable positions of the principal point, assuming that it is the same in both images (shown at left). The plane is divided into regions by three conics (one of them degenerate). Black regions represent impossible locations for the principal point. Inside the allowable (white) regions, the focal length approaches infinity at the pair of orthogonal lines (degenerate conic), and zero at the non-degenerate conics. At right is an enlargement of the bottom right-hand corner, showing the conic corresponding to the denominator of (6).*

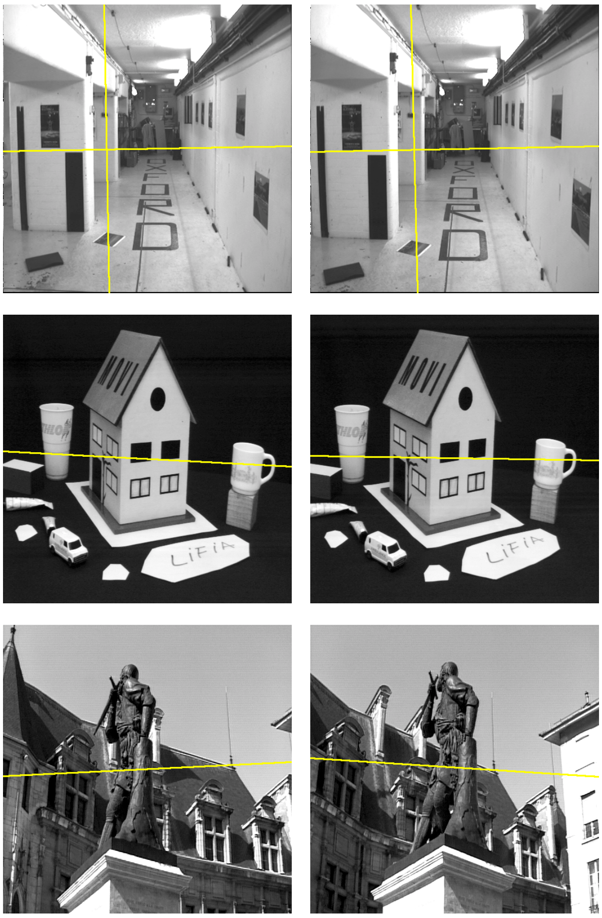

Figure 13: *This is the same as Fig 9 for other image pairs. In some cases, the epipoles are outside of the image area, and so only one of the epipolar lines is visible. Nevertheless, in all cases, the epipolar line lies close to the centre of the image, and it may be seen that computation of* f_2 *will be very dependent on the precise position of the principal point.*