Is Epipolar Geometry Necessary?

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Abstract

We examine and contrast the projective properties of two simple 3D configurations. The first consists of six points, four of which are coplanar. We prove that epipolar geometry and the essential matrix can be recovered uniquely for this structure and give a constructive algorithm for this. The second configuration has four coplanar points and a single non-coplanar line. In this case it is not possible to determine the epipolar geometry. However, both structures have two projective invariants, and these are recoverable from two (uncalibrated) perspective images. We include examples of the invariants for real objects.

1 Introduction

A number of recent papers have demonstrated that vision tasks such as recognition and structure recovery can be accomplished using only projective properties [4, 7, 14]. This contrasts with "conventional" approaches where full Euclidean reconstruction of 3-space is sought. One of the immediate advantages of the projective approach is that no camera calibration is required. The intrinsic parameters need not be known since only projective properties of the rays (not angles) are used.

Given two perspective images of particular 3D configurations and assuming only image feature correspondences, we consider the following questions:

1. Can the epipolar geometry of the two cameras be uniquely recovered?
2. Can projective invariants of the 3D structure be computed?
   (So called "multiple view invariants").

The invariants sought will be to transformation by the projective group (i.e. multiplication of the homogeneous representation of 3-space by an arbitrary non-singular 4×4 matrix).

If the epipolar geometry is known, then 3D structure can be recovered up to 3D collineation i.e. up to an arbitrary projective transformation [4, 7]. Consequently, invariants to this transformation can be computed from the recovered structure (since they are unaffected by the projective transformation relating the recovered and "true" Euclidean configurations). Here we examine cases where multiple view invariants can be obtained in the absence of epipolar geometry. In particular we contrast two structures:

1. Four coplanar points, and two non-coplanar points. The non-coplanar points must be in "general position". This is made more precise below.
2. Four coplanar points, and a non-coplanar line.

The essential benefit of the four coplanar points is that they define a projective basis for the plane which can be used to *transfer* [1, 12] coordinates between the world plane and images. Any other planar configuration which uniquely defines a projective basis for the plane could equally well be used. For example, four coplanar lines.

That the epipolar geometry can be recovered for the six point structure has been established by [2, 11]. The derivation is repeated here, see figure 1. We extend this analysis to the determination of the essential matrix[1], Q. It is shown that Q is uniquely determined by the set of six point matches. Further, a method will be given for computing Q. The method is linear and non-iterative. This result is remarkable, since previously known methods have required 8 points for a linear solution [9] or 7 points for a solution involving finding the roots of a cubic equation [6]. In addition, the solution using 7 points leads to three possible solutions, corresponding to the three roots of the cubic. Since Q has 7 degrees of freedom [6] it is not possible to compute Q from less than 7 arbitrary points. Therefore it is somewhat surprising that the condition that four of the points are co-planar should mean that a solution from six points is possible and unique.

The second structure (four coplanar points and a line) is interesting because the simple replacement of two points by a line generates two significant changes: First, it is not possible to recover the epipolar geometry from this alone; second, the structure has an isotropy under the projective group. However, both structures have two projective invariants which can be recovered from two views.

1.1 Number of Invariants and Isotropies

As described in [13] the number of (functionally independent scalar) invariants to the action of a group G is given by:

$$\#\text{invar} = \dim S - \dim G + \dim G_{\mathbf{x}}$$

where $\dim S$ is the "dimension" of the structure, $\dim G$ the dimension of G, in this case 15, and $\dim G_{\mathbf{x}}$ the dimension of the isotropy sub-group (if any) which leaves the structure unaffected under the action of G. Examples are given in table 1.

The key point about an isotropy is that a structure with fewer degrees of freedom than the group dimension can still have invariants. In section 3.3 we discuss the isotropy of the line and four coplanar point configuration.

2 Six points, four coplanar

Consider a set of matched points $\mathbf{x}'_i \leftrightarrow \mathbf{x}_i$ for $i = 1, \ldots, 6$ and suppose that the points $\mathbf{X}_1, \ldots, \mathbf{X}_4$[2] corresponding to the first four matched points lie in a plane in space. Let this plane be denoted by Π. Suppose also that no three of the points $\mathbf{X}_1, \ldots, \mathbf{X}_4$ are collinear. Suppose further that the points $\mathbf{X}_5$ and $\mathbf{X}_6$ do **not** lie in that plane. Various

[1] This matrix was introduced by Longuet-Higgins [9] assuming the two cameras were calibrated, and has since been extensively investigated e.g. [10]. Most of the results also apply to uncalibrated cameras of the type considered in this paper [6].

[2] We adopt the notation that corresponding points in the world and image are distinguished by large and small letters. Vectors are written in bold font, e.g. $\mathbf{x}$ and $\mathbf{X}$. Homogeneous representations are used e.g. $\mathbf{X}_i = (X_i, Y_i, Z_i, 1)^t$. $\mathbf{x}$ and $\mathbf{x}'$ are corresponding image points in two views.

Structure (S)	$\dim S$	$\dim G_{\mathbf{x}}$	#invar
6 points general position	18	0	3
7 points general position (*)	21	0	6
5 points, 4 coplanar	14	1	0
6 points, 4 coplanar (*)	17	0	2
line and 4 coplanar points	15	2	2
line and 5 points, 4 coplanar	18	0?	3?
2 lines and 4 coplanar points	19	0?	4?

Table 1: The number of functionally independent scalar invariants for 3D configurations under the action of the projective group. In all cases general position is assumed e.g. the line is not coplanar with any other two points. (*) indicates that the epipolar geometry can be determined from two views of the structure (though not uniquely in the case of 7 points).

other assumptions will be necessary in order to rule out degenerate cases. These will be noted as they occur.

In the following sections we first explain how the epipolar geometry is determined. We then prove that this configuration is sufficient to uniquely define the Q matrix and hence five of the points may be used as a projective basis for $\mathcal{P}^3$.

2.1 Epipolar Geometry

First it will be shown that the problem may be reduced to the case in which $\mathbf{x}'_i = \mathbf{x}_i$ for $i = 1, \ldots, 4$. From the assumption that points $\mathbf{X}_1, \ldots, \mathbf{X}_4$ lie in a plane and that no three of them are collinear, it may be deduced that no three of the points $\mathbf{x}_1, \ldots, \mathbf{x}_4$ are collinear in the first image and that no three of $\mathbf{x}'_1, \ldots, \mathbf{x}'_4$ are collinear in the second image. Given this, it is possible in a straight-forward manner to find a 3×3 projective transformation matrix $\mathtt{T}$, such that $\mathbf{x}'_i = \mathtt{T}\mathbf{x}_i, i \in \{1, .., 4\}$. Denoting $\mathtt{T}\mathbf{x}_i$ by the new symbol $\mathbf{x}''_i$, we see that $\mathbf{x}'_i = \mathbf{x}''_i$ for $i = 1, \ldots, 4$.

Therefore, we will assume for now that $\mathbf{x}'_i = \mathbf{x}_i$ for $i = 1, \ldots, 4$. This being so, it is possible to characterize the points that lie in the plane Π defined by $\mathbf{X}_1, \ldots \mathbf{X}_4$. A point $\mathbf{Y}$ lies in the plane Π if and only if it is mapped to the same point in both images.

Now consider any point $\mathbf{Y}$ in space, not on Π, and consider the epipolar plane defined by $\mathbf{Y}$ and the two camera centres (see figure 1). This plane will meet the plane Π in a straight line $L(\mathbf{Y}) \subset \Pi$. The line $L(\mathbf{Y})$ must pass through the point $\mathbf{P}$ in which the line of the camera centres meets the plane Π. This means that for all points $\mathbf{Y}$ the lines $L(\mathbf{Y})$ are concurrent, and meet at the point $\mathbf{P}$. Now we consider the images of the line $L(\mathbf{Y})$ and the point $\mathbf{P}$ as seen from the two cameras. Since the line $L(\mathbf{Y})$ lies in the plane Π it must be the same as seen from both the cameras. Let the image of $L(\mathbf{Y})$ as seen in either image be $\ell(\mathbf{Y})$. If $\mathbf{y}$ and $\mathbf{y}'$ are the image points at which $\mathbf{Y}$ is seen from the two cameras, then both points $\mathbf{y}$ and $\mathbf{y}'$ must lie on the line $\ell(\mathbf{Y})$. Since the point $\mathbf{P}$ lies in the plane Π, it must map to the same point in both images, so $\mathbf{p} = \mathbf{p}'$ and this point lies

Figure 1: Epipolar geometry. The points $\mathbf{X}_1, \ldots, \mathbf{X}_4$ are coplanar, with images $\mathbf{x}_i$ and $\mathbf{x}'_i$ in the first and second images respectively. The epipolar plane defined by the point $\mathbf{Y}$ and optical centers $\mathbf{O}$ and $\mathbf{O}'$ intersects the plane Π in the line $L(\mathbf{Y}) =< \mathbf{Y}_1, \mathbf{Y}_2 >$, where $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are the intersections of Π with the lines $< \mathbf{Y}, \mathbf{O} >$ and $< \mathbf{Y}, \mathbf{O}' >$ respectively.
The epipolar line may be constructed in the second image as follows: Determine the plane projective transformation such that $\mathbf{x}'_i = \mathtt{T}\mathbf{x}, i \in \{1, .., 4\}$. Use this transformation to transfer the point $\mathbf{y}$ to $\mathbf{y}'_1 = \mathtt{T}\mathbf{y}$. This determines two points in the second image, $\mathbf{y}'$ and $\mathtt{T}\mathbf{y}$, which are projections of points ($\mathbf{Y}$ and $\mathbf{Y}_1$) on the line $< \mathbf{O}, \mathbf{Y} >$. This defines the epipolar line of $\mathbf{Y}$ in the second image. A second point, not on Π, will define its corresponding epipolar lines. The epipole lies on both lines, so is determined by their intersection. A similar construction gives epipolar lines and hence the epipole in the first image.

on the line $\ell(\mathbf{Y})$. Therefore, $\mathbf{y}$, $\mathbf{y}'$ and $\mathbf{p}$ are collinear. The point $\mathbf{p}$ can be identified as the epipole in the first image, since points $\mathbf{p}$ and the two camera centres are collinear. Similarly, $\mathbf{p}'$ is the epipole in the second image. Thus one point not on Π is sufficient to determine a line in each image on which the epipole must lie[3].

This discussion may now be applied to the points $\mathbf{X}_5$ and $\mathbf{X}_6$. Since $\mathbf{X}_5$ and $\mathbf{X}_6$ do not lie in the plane Π it follows that $\mathbf{x}'_5 \neq \mathbf{x}_5$ and $\mathbf{x}'_6 \neq \mathbf{x}_6$. Then the point $\mathbf{p}$ may easily be found as the point of intersection of the lines $< \mathbf{x}'_5, \mathbf{x}_5 >$ and $< \mathbf{x}'_6, \mathbf{x}_6 >$. As an aside, the point of intersection of the lines $< \mathbf{x}_5, \mathbf{x}_6 >$ and $< \mathbf{x}'_5, \mathbf{x}'_6 >$ is of interest as being the image of the point where the line through $< \mathbf{X}_5, \mathbf{X}_6 >$ meets the plane Π, see section 2.3.

The previous discussion indicates how the epipole may be found. This construction will succeed unless the two lines $< \mathbf{x}'_5, \mathbf{x}_5 >$ and $< \mathbf{x}'_6, \mathbf{x}_6 >$ are the same. The two lines will be distinct unless the two points $\mathbf{X}_5$ and $\mathbf{X}_6$ lie in a common plane with the two camera centres.

To summarise:

1. Calculate the plane projective transformation matrix $\mathtt{T}$, such that $\mathbf{x}'_i = \mathtt{T}\mathbf{x}_i, i \in \{1, .., 4\}$.
2. Determine the epipole, $\mathbf{p}'$, in the second image as the intersection of the lines $< \mathtt{T}\mathbf{x}_5, \mathbf{x}'_5 >$ and $< \mathtt{T}\mathbf{x}_6, \mathbf{x}'_6 >$. Note, these lines are given by $\mathtt{T}\mathbf{x}_i \wedge \mathbf{x}'_i,\ i \in \{5, 6\}$ [16]. Similarly, the epipole in the first image is the intersection of the lines $\mathtt{T}^{-1}\mathbf{x}'_i \wedge \mathbf{x}_i,\ i \in \{5, 6\}$.
3. The epipolar line in the second image corresponding to a point $\mathbf{x}$ in the first is given by $\mathtt{T}\mathbf{x} \wedge \mathbf{p}'$.

2.2 Computation of Essential Matrix

The essential matrix, Q, satisfies the condition

$$\mathbf{x}'_i Q \mathbf{x}_i = 0 \tag{1}$$

for all i. As in the previous section the problem of determining the matrix Q is reduced to the case in which $\mathbf{x}'_i = \mathbf{x}_i$ for $i = 1, \ldots, 4$. If $\mathbf{x}''_i = \mathtt{T}\mathbf{x}'_i$ for $i = 1, \ldots, 4$, then

$$0 = \mathbf{x}'_i Q \mathbf{x}_i = \mathbf{x}'_i Q \mathtt{T}^{-1} \mathbf{x}''_i \ . \tag{2}$$

So, denoting $Q_1 = Q\mathtt{T}^{-1}$, the task now becomes that of determining Q_1 such that

$$\mathbf{x}'_i Q_1 \mathbf{x}''_i = 0 \tag{3}$$

for all i. In addition, $\mathbf{x}'_i = \mathbf{x}''_i$ for $i = 1, \ldots, 4$. Once Q_1 has been determined, the original matrix Q may be retrieved using the relationship

$$Q = Q_1 \mathtt{T} \ . \tag{4}$$

[3] Another way to see this is that $\mathbf{Y}$ and $\mathbf{Y}_1$ (a virtual point), see figure 1, are collinear in the first image. This is the condition for motion parallax. As described in [8], their positions in the second image ($\mathbf{y}'$ and $\mathtt{T}\mathbf{y}$) are coincident with the focus of expansion (the epipole). We are grateful to Andrew Blake for this observation.

Now, if Q is the essential matrix corresponding to the set of matched points, then since $\mathbf{p}$ is the epipole in the first image, we have an equation

$$Q\mathbf{p} = 0$$

and since $\mathbf{p}' = \mathbf{p}$ is the epipole in the second image, it follows also that

$$\mathbf{p}^\top Q = 0$$

Furthermore, for $i = 1, \ldots, 4$, we have $\mathbf{x}_i = \mathbf{x}'_i$, and so, ${\mathbf{x}_i}^\top Q\mathbf{x}_i = 0$. For $i = 5, 6$, we have $\mathbf{x}'_i = \mathbf{x}_i + \alpha_i\mathbf{p}$. Therefore, $0 = {\mathbf{x}'_i}^\top Q\mathbf{x}_i = (\mathbf{x}_i + \alpha_i\mathbf{p})^\top Q\mathbf{x}_i = {\mathbf{x}_i}^\top Q\mathbf{x}_i$. So for all $i = 1, \ldots, 6$,

$${\mathbf{x}_i}^\top Q\mathbf{x}_i = 0 \ .$$

This should give more than enough equations in general to solve for Q, however, the existence and uniqueness of the solution need to be proven

Now, a new piece of notation will be introduced. For any vector $\mathbf{t} = (t_x, t_y, t_y)^\top$ we define a skew-symmetric matrix, $S(\mathbf{t})$ according to

$$S(\mathbf{t}) = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \ . \tag{5}$$

Any 3×3 skew-symmetric matrix can be represented in this way for some vector $\mathbf{t}$. Matrix $S(\mathbf{t})$ is a singular matrix of rank 2, unless $\mathbf{t} = 0$. Furthermore, the null-space of $S(\mathbf{t})$ is generated by the vector $\mathbf{t}$. This means that $\mathbf{t}^\top S(\mathbf{t}) = S(\mathbf{t})\,\mathbf{t} = 0$ and that any other vector annihilated by $S(\mathbf{t})$ is a scalar multiple of $\mathbf{t}$.

We now prove the existence and uniqueness of the solution for the essential matrix.

Lemma 2.1. *Let $\mathbf{p}$ be a point in projective 2-space and let $\{\mathbf{x}_i\}$ be a further set of points. If there are at least three distinct lines among the lines $< \mathbf{p}, \mathbf{x}_i >$ then there exists a unique matrix Q such that*

$$\mathbf{p}^\top Q = Q\mathbf{p} = 0$$

and for all i

$$\mathbf{x}'_i Q\mathbf{x}_i = 0$$

Furthermore, Q is skew-symmetric, and hence $Q \approx S(\mathbf{p})$.

Proof : Let us assume without loss of generality that the lines $< \mathbf{p}, \mathbf{x}_i >$ for $i = 1, \ldots, 3$ are distinct.

Let $\mathtt{T}_2$ be a non-singular matrix such that

$$\begin{aligned} \mathtt{T}_2\mathbf{p} &= (0, 0, 1)^\top \\ \mathtt{T}_2\mathbf{x}_1 &= (1, 0, 0)^\top \\ \mathtt{T}_2\mathbf{x}_2 &= (0, 1, 0)^\top \end{aligned}$$

Suppose that $\mathtt{T}_2\mathbf{x}_3 = (r, s, t)^\top$. Since the lines $< \mathbf{p}, \mathbf{x}_i >$ are distinct, so must be the lines $< \mathtt{T}_2\mathbf{p}, \mathtt{T}_2\mathbf{x}_i >$. From this it follows that both r and s are non-zero, for otherwise, the line $< \mathtt{T}_2\mathbf{p}, \mathtt{T}_2\mathbf{x}_3 >$ must be the same as $< \mathtt{T}_2\mathbf{p}, \mathtt{T}_2\mathbf{x}_i >$ for $i = 1$ or 2. Now, define the matrix $Q_2 = {\mathtt{T}_2}^\top Q\mathtt{T}_2$. Then

$${\mathtt{T}_2}^\top Q_2(0, 0, 1)^\top = {\mathtt{T}_2}^\top Q_2\mathtt{T}_2\mathbf{p} = Q\mathbf{p} = 0$$

and so

$$Q_2(0,0,1)^\top = 0 \tag{6}$$

Similarly,

$$(0,0,1)Q_2 = 0 \tag{7}$$

Next,

$$(1,0,0)Q_2(1,0,0)^\top = \mathbf{x}_1{}^\top \mathtt{T}_2{}^\top Q_2 \mathtt{T}_2 \mathbf{x}_1 = \mathbf{x}_1{}^\top Q \mathbf{x}_1 = 0 \tag{8}$$

and similarly,

$$(0,1,0)Q_2(0,1,0)^\top = 0 \ . \tag{9}$$

and

$$(r,s,t)Q_2(r,s,t)^\top = 0 \ . \tag{10}$$

Now, writing

$$Q_2 = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

equation (6) implies $c = f = j = 0$. Equation (7) implies $g = h = j = 0$. Equation (8) implies $a = 0$ and equation (9) implies $e = 0$. Finally, equation (10) implies $rs(b+d) = 0$ and since $rs \neq 0$ this yields $b + d = 0$. So,

$$Q_2 = \begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is skew-symmetric. Therefore, $Q = \mathtt{T}_2^{-1\top} Q_2 \mathtt{T}_2^{-1}$ is also skew-symmetric.

The first part of the lemma has been proven. Now, since Q is skew-symmetric and $Q\mathbf{p} = 0$, it follows that $Q = S(\mathbf{p})$, as required. This shows uniqueness of the essential matrix Q. To show the existence of a matrix Q satisfying all the conditions of the lemma, it is sufficient to observe that a skew-symmetric matrix Q has the property that $\mathbf{x}_i{}^\top Q \mathbf{x}_i = 0$ for any vector $\mathbf{x}_i$. □

This lemma allows us to give an explicit form for the matrix Q expressed in terms of the original matched points.

Theorem 2.2. *Let $\{\mathbf{x}'_i\} \leftrightarrow \{\mathbf{x}_i\}$ be a set of 6 image correspondences derived from 6 points $\mathbf{X}_i$ in space, and suppose it is known that the points $\mathbf{X}_1, \ldots, \mathbf{X}_4$ lie in a plane. Let $\mathtt{T}$ be a 3×3 matrix such that $\mathbf{x}'_i = \mathtt{T}\mathbf{x}_i$ for $i = 1, \ldots, 4$. Suppose that the lines $< \mathbf{x}'_5, \mathtt{T}\mathbf{x}_5 >$ and $< \mathbf{x}'_6, \mathtt{T}\mathbf{x}_6 >$ are distinct and let $\mathbf{p}$ be their intersection. Suppose further that among the lines $< \mathbf{x}'_i, \mathbf{p} >$ there are at least three distinct lines. Then there exists a unique essential matrix Q satisfying the point correspondences and the condition of coplanarity of the points $\mathbf{X}_1, \ldots, \mathbf{X}_4$ and Q is given by the formula*

$$Q = S(\mathbf{p})\mathtt{T}$$

The conditions under which a unique solution exists may be expressed in geometrical terms. Namely :

1. Points $\mathbf{X}_1, \ldots, \mathbf{X}_4$ lie in a plane Π, but no three of them are collinear.

Figure 2: The projective invariant of 6 points, 4 coplanar (points 1-4), can be computed by intersecting the line, Λ, through the non-planar points (5 and 6) with the common plane. There are then 5 coplanar points, for which two invariants to the plane projective group can be calculated.

2. Points $\mathbf{X}_5$ and $\mathbf{X}_6$ do not lie in the plane Π, and do not lie in a common plane passing through the two camera centres.

3. The points $\mathbf{X}_1, \ldots, \mathbf{X}_6$ do not all lie in two planes passing through the camera centres.

Under the above conditions, the essential matrix Q is determined uniquely by the set of image correspondences. Note that according to [4, 7], this in turn determines the locations of the points themselves and the cameras up to a projective transformation of 3-space.

2.3 Projective invariants

The meaning of the 3D projective invariants can most readily be appreciated from figure 2. The line, Λ, formed from the two non-coplanar points intersects the plane Π in a unique point $\mathbf{X}_I$. This construction is unaffected by projective transformations of $\mathcal{P}^3$. There are then 5 coplanar points and consequently two plane projective invariants - which are also invariants of the 3D transformation.

As described in section 2.1, the image of $\mathbf{X}_I$ can be determined from two views (see [15] for an alternative derivation). We then have the image of five coplanar points, for which plane projective invariants may be calculated. These invariants have the same value calculated on Π or from any projection of Π. This construction does not require epipolar calibration to be known.

To summarise:

1. Determine the imaged intersection $\mathbf{x}'_I$ of the plane Π and the line Λ in the second image as the intersection of the lines $\mathtt{T}\mathbf{x}_5 \wedge \mathtt{T}\mathbf{x}_6$ and $\mathbf{x}'_5 \wedge \mathbf{x}'_6$.

2. Calculate the two plane projective invariants of five points, $\mathbf{x}'_i,\ i \in \{1,..,4\}$, and $\mathbf{x}'_5 = \mathbf{x}'_I$. These are given by

$$I_1 = \frac{|m_{125}||m_{134}|}{|m_{124}||m_{135}|} \qquad I_2 = \frac{|m_{124}||m_{235}|}{|m_{234}||m_{125}|} \tag{11}$$

where m_{jkl} is the matrix $[\mathbf{x}'_j \mathbf{x}'_k \mathbf{x}'_l]$ and $||$ is a determinant.

2.3.1 Relation between 2D invariants and algebraic invariants of 3D points

Six 3D points in $\mathcal{P}^3$ in general position have three projective invariants. The coplanarity reduces by one the number of invariants (one of the invariants will be zero). We may arbitrarily choose coordinates for 5 of the points of the six point configuration (any other coordinates of the five points can be transformed to these by a collineation of $\mathcal{P}^3$):

$$\begin{aligned}\mathbf{X}_1 &= (1,0,0,0)^\top \\ \mathbf{X}_2 &= (0,1,0,0)^\top \\ \mathbf{X}_3 &= (0,0,1,0)^\top \\ \mathbf{X}_5 &= (0,0,0,1)^\top \\ \mathbf{X}_6 &= (1,1,1,1)^\top\end{aligned}$$

The fourth coplanar point then has coordinates:

$$\mathbf{X}_4 = (\alpha, \beta, \gamma, 1)^\top$$

The coordinates of this point give rise to the two independent projective invariants of the six points:

$$I_1^3 = \alpha/\gamma \qquad I_2^3 = \beta/\gamma \tag{12}$$

For the five point planar invariants we use the coordinates of the first four points restricted to the plane (the subordinate geometry):

$$\begin{aligned}\bar{\mathbf{X}}_1 &= (1,0,0)^\top \\ \bar{\mathbf{X}}_2 &= (0,1,0)^\top \\ \bar{\mathbf{X}}_3 &= (0,0,1)^\top \\ \bar{\mathbf{X}}_4 &= (\alpha,\beta,\gamma)^\top\end{aligned}$$

The intersection of the line $< \mathbf{X}_5, \mathbf{X}_6 >$ with Π is given by

$$\bar{\mathbf{X}}_I = (1,1,1)^\top$$

Then from (11) the five point planar invariants are:

$$I_1 = \beta/\gamma \qquad I_2 = \gamma/\alpha \tag{13}$$

i.e. simply functions of the two 3D invariants in (12) above as expected.

3 A line and four coplanar points

Here the two non-coplanar points of the previous section are "replaced" by a line, Λ. As with the previous configuration this structure has two projective invariants (which are determined from the two five point invariants of the four coplanar points and the intersection of the line with the plane Π) which can be determined from two views. However, it is no longer possible to recover the epipolar geometry, it is not even possible to restrict the epipole to a line in each image.

3.1 Constraints on epipolar geometry

Surprisingly the image of Λ in each view *adds no constraints at all* towards solving for the epipolar geometry or essential matrix. To see this geometrically consider the back projection of a point imaged in two views. Each point back projects to a ray. In general two lines are skew in $\mathcal{P}^3$, so the condition that they intersect (since they arise from a common point) constrains the imaging geometry. In contrast the back projection of a line is a plane, and in general two planes intersect in a line in $\mathcal{P}^3$. Consequently, no constraint is given. Adding a third view does constrain the geometry since three planes intersect in a point in general, not a line, in $\mathcal{P}^3$.

3.2 Projective invariants

As in the six point case the invariant under projective transformations of $\mathcal{P}^3$ can be obtained from the five point planar invariants of the four coplanar points and the intersection of Λ with Π. Again this can be calculated from two views, where here the plane projective transformation is used to transfer a line, the image of Λ. This construction does not require epipolar calibration to be known.

As described in table 1, the line and four coplanar point configuration has only 15 degrees of freedom. That two invariants can be constructed indicates the presence of an isotropy sub-group. The action of this group is described below.

To summarise:

Given a set of matched points $\mathbf{x}'_i \leftrightarrow \mathbf{x}_i$ for $i = 1, \ldots, 4$ which are the images of coplanar points, together with the images $\mathbf{l}$ and $\mathbf{l}'$ of a line in $\mathcal{P}^3$ not on Π[4]

1. Determine the imaged intersection $\mathbf{x}'_I$ of the plane Π and the line Λ in the second image as the intersection of the lines $\mathbf{l}'$ and $\mathtt{T}^{-T}\mathbf{l}$. This is given by $\mathbf{x}'_I = \mathbf{l}' \wedge \mathtt{T}^{-T}\mathbf{l}$ [16].
2. Calculate the two plane projective invariants of five points, $\mathbf{x}'_i,\ i \in \{1, .., 4\}$, and $\mathbf{x}'_5 = \mathbf{x}'_I$ using equation (11).

[4] If Λ does lie on Π, the transferred line will be coincident with $\mathbf{l}'$, i.e. $\mathbf{l}' = \mathtt{T}^{-T}\mathbf{l}$.

3.3 Existence of an Isotropy

As explained in section 1.1 in order for the line and four coplanar points to have two invariants under collineation of $\mathcal{P}^3$there must by an isotropy sup-group acting. In this section we give a simple derivation of this sub-group which leaves the structure unchanged under the action of the projective group, and determine its action on $\mathcal{P}^3$.

The construction of the sub-group is in two stages:

1. Construct the sub-group for which Π is a plane of fixed points. This is necessary since four points remain fixed under the action of the sub-group, and consequently every point on the plane is unchanged (as four points define a basis for the plane).
2. Construct the sub-group of (1) for which the line Λ is a fixed line. Note this does not have to be a line of fixed points since only one point on the line (its intersection with Π must be unchanged).

We adopt the notation of section 2.3.1 for the six points. The line Λ is given in its homogeneous parametric representation by

$$\Lambda = \zeta(1,1,1,1)^\top + \eta(0,0,0,1)^\top \tag{14}$$

First, in order for Π to be a plane of fixed points it is necessary and sufficient that the 4×4 transformation matrix $\mathtt{T}$ satisfies

$$\mathbf{X}_i = \mathtt{T}\mathbf{X}_i, \qquad i \in \{1,..,4\}$$

It is a simple matter to show that $\mathtt{T}$ must have the form

$$\mathtt{T} = \begin{pmatrix} \mu_1 & 0 & 0 & \mu_2 \\ 0 & \mu_1 & 0 & \mu_3 \\ 0 & 0 & \mu_1 & \mu_4 \\ 0 & 0 & 0 & \mu_5 \end{pmatrix} \tag{15}$$

where $\mu_i, i \in \{1,..,5\}$ parameterise the sub-group which has four dof (only their ratio is significant).

Second, we determine the sub-group which leaves Λ fixed. This can be carried out using Pluckerian line coordinates [16], but here we use the parametric representation (14) above. Under the action of the isotropy group the points on the line need not be fixed, but the transformed points must still lie on Λ. The transformation of two points is sufficient to determine the transformed lines (three are required to determine the transformation of all the points on the line). By inspection $\mathtt{T}\mathbf{X}_5$ and $\mathtt{T}\mathbf{X}_5$ satisfy (14) iff $\mu_2 = \mu_3 = \mu_4$. Hence we arrive at

$$\mathtt{T} = \begin{pmatrix} \mu_1 & 0 & 0 & \mu_2 \\ 0 & \mu_1 & 0 & \mu_2 \\ 0 & 0 & \mu_1 & \mu_2 \\ 0 & 0 & 0 & \mu_5 \end{pmatrix} \tag{16}$$

which is a two dimensional sub-group of the collineations of $\mathcal{P}^3$.

It is interesting to examine the transformation of $\mathcal{P}^3$under the action of $\mathtt{T}$. The clearest way to see this is to determine the eigen-vectors of $\mathtt{T}$. These are the fixed points of the

Figure 3: Images of a hole punch captured with different lenses and viewpoints.

Figure 4: Line drawing of the hole punch extracted from image A in figure 3. Points 1 and 5 are occluded in this view.

collineation. We find

$$\begin{aligned}\mathbf{E}_1 &= (1,0,0,0)^\top \\ \mathbf{E}_2 &= (0,1,0,0)^\top \\ \mathbf{E}_3 &= (0,0,1,0)^\top \\ \mathbf{E}_4 &= (\mu_2,\mu_2,\mu_2,\mu_5-\mu_1)^\top\end{aligned}$$

The first three are degenerate with eigen-value μ_1, the fourth has eigen-value μ_5. As expected any point on the plane $\mathbf{X} = \nu_1\mathbf{X}_1 + \nu_2\mathbf{X}_2 + \nu_3\mathbf{X}_3$ is unchanged by $\mathtt{T}$ (since after the transformation all the basis vectors are multiplied by μ_1). The fourth eigen-vector is a fixed point on Λ. To see the effect of the isotropy group on points not on Π, consider any line L containing $\mathbf{E}_4$. This will intersect Π at some point, $\mathbf{X}_\Pi$ say, and any point on the line $\mathbf{X}$ is given by $\mathbf{X} = \zeta\mathbf{E} + \eta\mathbf{X}_\Pi$. After the transformation the point is $\mathtt{T}\mathbf{X} = \mu_5\zeta\mathbf{E} + \mu_1\eta\mathbf{X}_\Pi$ which still lies on L i.e. any line through $\mathbf{E}$ is a fixed line under the isotropy. Consequently, since every point in $\mathcal{P}^3$ lies on a line through $\mathbf{E}$, the action of $\mathtt{T}$ on $\mathcal{P}^3$is to move points towards (or away from) $\mathbf{E}$, with only $\mathbf{E}$ and points on Π remaining unchanged.

4 Experimental Results

The images used for acquisition and assessment are shown in figure 3.

A local implementation of Canny's edge detector [3] is used to find edges to sub-pixel accuracy. These edge chains are linked, extrapolating over any small gaps. A piecewise linear graph is obtained by incremental straight line fitting. Edgels in the vicinity of tangent discontinuities ("corners") are excised before fitting as the edge operator localisation degrades with curvature. Vertices are obtained by extrapolating and intersecting the fitted lines. Figure 4 shows a typical line drawing.

Although invariants obtained from two views are fairly stable, improvements in stability

Images	I_1	I_2
D,A	0.440	-0.968
D,B	0.378	-1.117
B,A	0.371	-1.170
C,E	0.370	-1.150
F,A	0.333	-1.314
D,A,B	0.372	-1.151
C,E,D	0.369	-1.148
F,A,C	0.370	-1.196
C,A,B,D,E	0.375	-1.140
F,A,B,C,D,E	0.369	-1.170

Table 2: This table shows the line and four coplanar point invariants extracted from several combinations of views using points 2,4,14,17 and the line between points 6 and 13.

are achieved by augmenting with measurements from other views. At present this is carried out in a primitive fashion by determining the intersection point in a least squares manner. See table 2.

5 Conclusions

We have demonstrated that multiple view invariants can indeed be recovered without epipolar calibration being necessary. The discussion applies as well to analogues of this configuration, for example: four coplanar lines and two non-coplanar points, and five lines (four coplanar).

We have also shown that for the structure with an isotropy it is not possible to determine the epipolar geometry. We conjecture that this is always the case i.e. if the 3D structure has an isotropy under the projective group then it is not possible to determine the epipolar geometry (it can only be constrained up to a family of solutions). Of course the converse is not true - four coplanar points and n non-coplanar lines is not sufficient to determine the epipolar geometry for any n.

Appendix: Why are six points (four coplanar) sufficient ?

With 8 points or more it is possible to solve for the matrix Q by solving a set of linear equations. If there are fewer than 8 points, the set of linear equations will be underdetermined, and hence there will be a family of solutions. It is instructive to consider how the extra condition that four of the points should be coplanar cuts this family down to a single solution. Let us consider a particular example.

Consider a set of 6 matched points $\mathbf{x}'_i \leftrightarrow \mathbf{x}_i$ as follows :

$$\begin{array}{l}
(1,0,0)^\top \leftrightarrow (1,0,0)^\top \\
(0,1,0)^\top \leftrightarrow (0,1,0)^\top \\
(0,0,1)^\top \leftrightarrow (0,0,1)^\top \\
(1,1,1)^\top \leftrightarrow (1,1,1)^\top \\
(1,0,0)^\top \leftrightarrow (-1,1,1)^\top \textbf{THESE NEED CORRECTION} \\
(0,1,0)^\top \leftrightarrow (-1,1,1)^\top \textbf{THESE NEED CORRECTION}
\end{array} \tag{17}$$

Assume that the first 4 points lie in a plane. From the previous discussion, it is obvious that the epipole is the point $(-1,1,1)^\top$, and hence that

$$Q = S((-1,1,1)^\top = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} .$$

However, we will compute Q directly. Each of the six point correspondences gives rise to an equation $\mathbf{x}'_i Q \mathbf{x}_i = 0$ which is linear in the entries of Q. Since there are six equations in nine unknowns, there will be a 3-parameter family of solutions. It is easily verified, therefore, that the general solution is given by

$$Q = \begin{bmatrix} 0 & A & -A \\ B & 0 & B \\ C & -C-2B & 0 \end{bmatrix} . \tag{18}$$

Now, the condition $\det(Q) = 0$ yields an equation $2AB(C+B) = 0$, and hence, either $C = -B$ or $A = 0$ or $B = 0$. Thus, Q has one of the forms

$$Q = \begin{bmatrix} 0 & A & -A \\ B & 0 & B \\ -B & -B & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \\ B & 0 & B \\ C & -C-2B & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & A & -A \\ 0 & 0 & 0 \\ C & -C & 0 \end{bmatrix} . \tag{19}$$

We consider the first one of these solutions Since Q is determined only up to scale, we may choose $B = 1$, and so

$$Q = \begin{bmatrix} 0 & A & -A \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} . \tag{20}$$

Next, we investigate the condition that the first four matched points lie in a plane. To do this, it is necessary to find a pair of camera matrices that realize (see citeHartley91) the matrix Q. It does not matter which realization of Q is picked, since any other choice will be equivalent to a projective transformation of object space (see [6]), which will take planes to planes. Accordingly, since Q factors as

$$Q = \begin{bmatrix} -A & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

a realization of Q is given by the two camera matrices

$$M = (I \mid 0) \quad \text{and} \quad M' = \left(\begin{array}{ccc|c} 1 & & & 1 \\ & -A & & A \\ & & -A & A \end{array} \right)$$

Then it is easily verified that the points

$$\mathbf{X}_1 = (1,0,0,0)^\top \ , \ \mathbf{X}_2 = (0,1,0,0)^\top \ , \ \mathbf{X}_3 = (0,0,1,0)^\top \ , \ \mathbf{X}_4 = (1,1,1,k)^\top \ ,$$

where k is defined by $1 + k = -A + kA$, are mapped by the two cameras to the required image points as specified by (17). However, the requirement that these four points lie in a plane means that $k = 0$ and hence that $A = -1$. Substituting this value in (20) yields the expected matrix $Q = S((-1,1,1)^\top)$. It may be verified that the two other choices for Q given in (19) do not lead to any further solution.

The role of the coplanarity condition now becomes clear. Without this condition, there are a family of solutions for the essential matrix Q. Only one of the family of solutions is consistent with the condition that the four points lie in a plane.

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