Computation of the essential matrix from 9 lines

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1 Computation from 9 lines

Consider the situation in which we have three images of four coplanar points and five lines that do not lie in the plane. Thus, let $\mathbf{x}_1, \ldots \mathbf{x}_4$ be four points lying in a plane π in $\mathcal{P}^3$. Let the images of these points as seen in three images be $\mathbf{u}_i$, $\mathbf{u}'_i$ and $\mathbf{u}''_i$. As before, we suppose that the images have been subjected to appropriate projective transforms so that $\mathbf{u}_i = \mathbf{u}'_i = \mathbf{u}''_i$ for all i. Then, a necessary and sufficient condition for any further point x to lie in the plane π is that x projects to the same point in all three images.

For convenience, it may be assumed that the image planes of the three images are all identical with the plane π itself. A point $\mathbf{x}$ in space is mapped to the image point $\mathbf{u}$ in which the line through $\mathbf{x}$ and the camera centre pierces the image plane. Coordinates for $\mathcal{P}^3$may be chosen so that the plane π is the plane at infinity and the first camera is placed at the point $(0, 0, 0, 1)^\top$. Let the other two cameras be placed at the points $\begin{pmatrix} \mathbf{a} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix}$. The three camera transformation matrices are then $P = (I \mid 0)$, $P' = (I \mid -\mathbf{a})$ and $P'' = (I \mid -\mathbf{b})$.

As with the 6 point case, if we know the positions in the epipoles, it will be possible to deduce the essential matrices corresponding to each pair of cameras. With three views, there are three epipoles in the plane π, namely the points at which lines through pairs of camera centres meet the image plane. It may easily be seen that the three epipoles are collinear, since they must all lie on the line in which the plane defined by the three camera centres meets the image plane. This may be verified by computing the epipoles explicitly. With the choice of camera locations above one computes that the three epipoles are at locations $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{b} - \mathbf{a}$, which are indeed collinear points in the image plane. It follows that determining $\mathbf{a}$ and $\mathbf{b}$ is sufficient to determine the three essential matrices.

Now consider a line λ in $\mathcal{P}^3$which does not lie in the image plane. Let the projections of λ with respect to the three cameras be ℓ, ℓ' and ℓ''. Since λ does not lie in the image plane, its three images will be distinct lines. However, lines ℓ, ℓ' and ℓ'' must all meet at a common point, namely the point at which λ meets the image plane. Let us pause to consider this interesting fact. Given three (or more) images of a set of lines in space, it is possible to apply projective transformations to two of the images so that corresponding triples of lines in the three images are coincident in a common point in the transformed images.

Given ℓ, ℓ' and ℓ'' the line λ may be retrieved as the intersection of the three planes defined by each line and its correponding camera centre. Each such plane may be computed explicitly. In particular (see [1]) the plane through a line ℓ as projected by a camera with matrix P is given by $P^\top \ell$. Therefore, the three planes are equal to $P^\top \ell = \begin{pmatrix} \ell \\ 0 \end{pmatrix}$, $P'^\top \ell' = \begin{pmatrix} \ell' \\ \mathbf{a}^\top \ell' \end{pmatrix}$, and $P''^\top \ell'' = \begin{pmatrix} \ell' \\ \mathbf{b}^\top \ell'' \end{pmatrix}$. The fact that these three planes meet in a common line implies that the 4×3 matrix

$$A = \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & \ell'^\top \mathbf{a} & \ell''^\top \mathbf{b} \end{pmatrix} .$$

has rank 2. Hence, there must be a linear dependency between the columns of A.

As remarked above, the lines ℓ, ℓ' and ℓ'' are coincident, so there is a relationship $\alpha\ell + \beta\ell' + \gamma\ell'' = 0$. This gives a linear dependency between the first three rows of A. Since ℓ, ℓ' and ℓ'' are known, the weights α, β and γ may be computed explicitly. Since A has rank 2, this dependency must also apply to the last row as well which means that

$$\beta\ell'^\top \mathbf{a} + \gamma\ell''^\top \mathbf{b} = 0 \ .$$

This is a single linear equation in the coordinates of the two vectors $\mathbf{a}$ and $\mathbf{b}$. Given five such equations, arising from five lines not lying in the plane π, it is possible to solve for $\mathbf{a}$ and $\mathbf{b}$ up to an unknown (but insignificant) scale factor.

Summary of the algorithm The algorithm for determining the essential matrices from four coplanar points and five lines in three images is as follows. We start with coordinates $\mathbf{u}_i$, $\mathbf{u}'_i$ and $\mathbf{u}''_i$, the images of the points in the three images and also ℓ, ℓ' and ℓ'', the images of the lines. The steps of the algorithm are as follows.

1. Determine two-dimensional projective transformations represented by non-singular 3×3 matrices K' and K'' such that for each $i = 1, \ldots 4$ we have $u_i = K'u'_i = K''u''_i$.

2. Replace each line ℓ' by the transformed line $K'^*\ell'$, and each ℓ'' by $K''^*\ell''$. The notation K'^* represents the inverse transpose or cofactor matrix of K'.

3. For each $i = 1, \ldots, 5$ find coefficients α_i, β_i and γ_i such that $\alpha_i\ell + \beta_i\ell' + \gamma_i\ell'' = 0$.

4. Solve the set of five linear equations $\beta_i\ell_i'^\top \mathbf{a} + \gamma_i\ell_i''^\top \mathbf{b} = 0$ to find the vectors $\mathbf{a}$ and $\mathbf{b}$, up to an indeterminate scale.

5. The three essential matrices are $K'^\top[\mathbf{a}]_\times$, $K''^\top[\mathbf{b}]_\times$ and $K''^\top[\mathbf{b} - \mathbf{a}]_\times K'$.

The above discussion was concerned with the case in which the plane π was defined by four points. Any other planar object which uniquely defines a projective basis for the plane may be used just as well, for example four coplanar lines. This shows that four coplanar lines plus five lines not in the plane are sufficient (in 3 views) to determine the essential matrices. This can be compared with the method for determining the essential matrices from 13 unrestricted lines in three views ([1]).

References

[1] R. Hartley, *"Invariants of Points Seen in Multiple Images"*, Submitted for publication, available from author.