Computation of the essential matrix from 6 points

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1 Computation of Essential Matrix

It is the present purpose to indicate how the essential matrix, Q, may be computed from a six point matches, provided that it is known that four of the points lie in a plane.

Thus, consider a set of matched points $\mathbf{u}'_i \leftrightarrow \mathbf{u}_i$ for $i = 1, \ldots, 6$ and suppose that the points $\mathbf{x}_1, \ldots, \mathbf{x}_4$ corresponding to the first four matched points lie in a plane in space. Let this plane be denoted by π. Suppose also that no three of the points $\mathbf{x}_1, \ldots, \mathbf{x}_4$ are collinear. Suppose further that the points $\mathbf{x}_5$ and $\mathbf{x}_6$ do **not** lie in that plane. Various other assumptions will be necessary in order to rule out degenerate cases. These will be noted as they occur.

The essential matrix, Q, satisfies the condition

$$\mathbf{u}'_i Q \mathbf{u}_i = 0 \tag{1}$$

for all i. It will be shown that Q is uniquely determined by the set of six point matches. Further, a method will be given for computing Q. The method is linear and non-iterative. This result is remarkable, since previously known methods have required 8 points for a linear solution ([2]) or 7 points for a solution involving finding the roots of a cubic equation ([1]). In addition, the solution using 7 points leads to three possible solutions, corresponding to the three roots of the cubic. Since Q has 7 degrees of freedom ([1]) it is not possible to compute Q from less than 7 arbitrary points. Therefore it is somewhat surprising that the condition that four of the points are co-planar should mean that a solution from six points is possible and unique.

First it will be shown how the problem of determining the matrix Q may be reduced to the case in which $\mathbf{u}'_i = \mathbf{u}_i$ for $i = 1, \ldots, 4$. From the assumption that points $\mathbf{x}_1, \ldots, \mathbf{x}_4$ lie in a plane and that no three of them are collinear, it may be deduced that no three of the points $\mathbf{u}_1, \ldots, \mathbf{u}_4$ are collinear in the first image and that no three of $\mathbf{u}'_1, \ldots, \mathbf{u}'_4$ are collinear in the second image. Given this, it is possible in a straight-forward manner to find a projective transformation, denoted P, such that $\mathbf{u}'_i = P\mathbf{u}_i$ for $i = 1, \ldots, 4$.

Denoting $P\mathbf{u}_i$ by the new symbol $\mathbf{u}''_i$, we see that $\mathbf{u}_i = P^{-1}\mathbf{u}''_i$ and so from (1)

$$0 = \mathbf{u}'_i Q \mathbf{u}_i = \mathbf{u}'_i Q P^{-1} \mathbf{u}''_i \ . \tag{2}$$

So, denoting $Q_1 = QP^{-1}$, the task now becomes that of determining Q_1 such that

$$\mathbf{u}'_i Q_1 \mathbf{u}''_i = 0 \tag{3}$$

for all i. In addition, $\mathbf{u}'_i = \mathbf{u}''_i$ for $i = 1, \ldots, 4$. Once Q_1 has been determined, the original matrix Q may be retrieved using the relationship

$$Q = Q_1 P \ . \tag{4}$$

Therefore, we will assume for now that $\mathbf{u}'_i = \mathbf{u}_i$ for $i = 1, \ldots, 4$. This being so, it is possible to characterize the points that lie in the plane π defined by $\mathbf{x}_1, \ldots \mathbf{x}_4$. A point $\mathbf{y}$ lies in the plane π if and only if it is mapped to the same point in both images.

Now consider any point $\mathbf{y}$ in space, and consider the plane defined by $\mathbf{y}$ and the two camera centres. This plane will meet the plane π in a straight line $\ell(\mathbf{y}) \subset \pi$. The line $\ell(\mathbf{y})$ must pass through the point $\mathbf{p}$ in which the line of the camera centres meets the plane π. This means that for all points $\mathbf{y}$ the lines $\ell(\mathbf{y})$ are concurrent, and meet at the point $\mathbf{p}$. Now we consider the images of the line $\ell(\mathbf{y})$ and the point $\mathbf{p}$ as seen from the two cameras. Since the line $\ell(\mathbf{y})$ lies in the plane π it must be the same as seen from both the cameras. Let the image of $\ell(\mathbf{y})$ as seen in either image be $L(\mathbf{y})$. If $\mathbf{u}_y$ and $\mathbf{u}'_y$ are the image points at which $\mathbf{y}$ is seen from the two cameras, then both points $\mathbf{u}_y$ and $\mathbf{u}'_y$ must lie on the line $L(\mathbf{y})$. Since the point $\mathbf{p}$ lies in the plane π, it must map to the same point in both images, so $\mathbf{u}_p = \mathbf{u}'_p$ and this point lies on the line $L(\mathbf{y})$. Therefore, $\mathbf{u}_y$, $\mathbf{u}'_y$ and $\mathbf{u}_p$ are collinear. The point $\mathbf{u}_p$ can be identified as the epipole in the first image, since points $\mathbf{p}$ and the two camera centres are collinear. Similarly, $\mathbf{u}'_p$ is the epipole in the second image.

This discussion may now be applied to the points $\mathbf{x}_5$ and $\mathbf{x}_6$. Since $\mathbf{x}_5$ and $\mathbf{x}_6$ do not lie in the plane π it follows that $\mathbf{u}'_5 \neq \mathbf{u}_5$ and $\mathbf{u}'_6 \neq \mathbf{u}_6$. Then the point $\mathbf{u}_p$ may easily be found as the point of intersection of the lines $< \mathbf{u}'_5, \mathbf{u}_5 >$ and $< \mathbf{u}'_6, \mathbf{u}_6 >$.

As an aside, the point of intersection of the lines $< \mathbf{u}_5, \mathbf{u}_6 >$ and $< \mathbf{u}'_5, \mathbf{u}'_6 >$ is of interest as being the image of the point where the line through $< \mathbf{x}_5, \mathbf{x}_6 >$ meets the plane π.

The previous discussion indicates how the epipole may be found. This construction will succeed unless the two lines $< \mathbf{u}'_5, \mathbf{u}_5 >$ and $< \mathbf{u}'_6, \mathbf{u}_6 >$ are the same. The two lines will be distinct unless the two points $\mathbf{x}_5$ and $\mathbf{x}_6$ lie in a common plane with the two camera centres.

Now, if Q is the essential matrix corresponding to the set of matched points, then since $\mathbf{u}_p$ is the epipole in the first image, we have an equation

$$Q\mathbf{u}_p = 0$$

and since $\mathbf{u}'_p = \mathbf{u}_p$ is the epipole in the second image, it follows also that

$$\mathbf{u}_p{}^\top Q = 0$$

Furthermore, for $i = 1, \ldots, 4$, we have $\mathbf{u}_i = \mathbf{u}'_i$, and so, $\mathbf{u}_i{}^\top Q \mathbf{u}_i = 0$. For $i = 5, 6$, we have $\mathbf{u}'_i = \mathbf{u}_i + \alpha_i \mathbf{u}_p$. Therefore, $0 = \mathbf{u}'_i{}^\top Q \mathbf{u}_i = (\mathbf{u}_i + \alpha_i \mathbf{u}_p)^\top Q \mathbf{u}_i = \mathbf{u}_i{}^\top Q \mathbf{u}_i$. So for all $i = 1, \ldots, 6$,

$$\mathbf{u}_i{}^\top Q \mathbf{u}_i = 0 \ .$$

This should give more than enough equations in general to solve for Q, however, the existence and uniqueness of the solution need to be proven

Now, a new piece of notation will be introduced. For any vector $\mathbf{t} = (t_x, t_y, t_y)^\top$ we define a skew-symmetric matrix, $S(\mathbf{t})$ according to

$$S(\mathbf{t}) = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \ . \tag{5}$$

Any 3×3 skew-symmetric matrix can be represented in this way for some vector $\mathbf{t}$. Matrix $S(\mathbf{t})$ is a singular matrix of rank 2, unless $\mathbf{t} = 0$. Furthermore, the null-space of

$S(\mathbf{t})$ is generated by the vector $\mathbf{t}$. This means that $\mathbf{t}^\top S(\mathbf{t}) = S(\mathbf{t})\,\mathbf{t} = 0$ and that any other vector annihilated by $S(\mathbf{t})$ is a scalar multiple of $\mathbf{t}$.

We now prove the existence and uniqueness of the solution for the essential matrix.

Lemma 1.1. *Let $\mathbf{u}_p$ be a point in projective 2-space and let $\{\mathbf{u}_i\}$ be a further set of points. If there are at least three distinct lines among the lines $< \mathbf{u}_p, \mathbf{u}_i >$ then there exists a unique matrix Q such that*

$$\mathbf{u}_p{}^\top Q = Q\mathbf{u}_p = 0$$

and for all i

$$\mathbf{u}'_i Q \mathbf{u}_i = 0$$

Furthermore, Q is skew-symmetric, and hence $Q \approx S(\mathbf{u}_p)$.

Proof : Let us assume without loss of generality that the lines $< \mathbf{u}_p, \mathbf{u}_i >$ for $i = 1, \ldots, 3$ are distinct.

Let P_2 be a non-singular matrix such that

$$\begin{aligned} P_2\mathbf{u}_p &= (0,0,1)^\top \\ P_2\mathbf{u}_1 &= (1,0,0)^\top \\ P_2\mathbf{u}_2 &= (0,1,0)^\top \end{aligned}$$

Suppose that $P_2\mathbf{u}_3 = (r,s,t)^\top$. Since the lines $< \mathbf{u}_p, \mathbf{u}_i >$ are distinct, so must be the lines $< P_2\mathbf{u}_p, P_2\mathbf{u}_i >$. From this it follows that both r and s are non-zero, for otherwise, the line $< P_2\mathbf{u}_p, P_2\mathbf{u}_3 >$ must be the same as $< P_2\mathbf{u}_p, P_2\mathbf{u}_i >$ for $i = 1$ or 2. Now, define the matrix $Q_2 = P_2{}^\top Q P_2$. Then

$$P_2{}^\top Q_2 (0,0,1)^\top = P_2{}^\top Q_2 P_2 \mathbf{u}_p = Q\mathbf{u}_p = 0$$

and so

$$Q_2(0,0,1)^\top = 0 \tag{6}$$

Similarly,

$$(0,0,1)Q_2 = 0 \tag{7}$$

Next,

$$(1,0,0)Q_2(1,0,0)^\top = \mathbf{u}_1{}^\top P_2{}^\top Q_2 P_2 \mathbf{u}_1 = \mathbf{u}_1{}^\top Q \mathbf{u}_1 = 0 \tag{8}$$

and similarly,

$$(0,1,0)Q_2(0,1,0)^\top = 0 \ . \tag{9}$$

and

$$(r,s,t)Q_2(r,s,t)^\top = 0 \ . \tag{10}$$

Now, writing

$$Q_2 = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

equation (6) implies $c = f = j = 0$. Equation (7) implies $g = h = j = 0$. Equation (8) implies $a = 0$ and equation (9) implies $e = 0$. Finally, equation (10) implies $rs(b+d) = 0$ and since $rs \neq 0$ this yields $b + d = 0$. So,

$$Q_2 = \begin{pmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which is skew-symmetric. Therefore, $Q = P_2^{-1\top} Q_2 P_2^{-1}$ is also skew-symmetric.

The first part of the lemma has been proven. Now, since Q is skew-symmetric and $Q\mathbf{u}_p = 0$, it follows that $Q = S(\mathbf{u}_p)$, as required. This shows uniqueness of the essential matrix Q. To show the existence of a matrix Q satisfying all the conditions of the lemma, it is sufficient to observe that a skew-symmetric matrix Q has the property that $\mathbf{u}_i{}^\top Q \mathbf{u}_i = 0$ for any vector $\mathbf{u}_i$. $\square$

This lemma allows us to give an explicit form for the matrix Q expressed in terms of the original matched points.

Theorem 1.2. *Let $\{\mathbf{u}'_i\} \leftrightarrow \{\mathbf{u}_i\}$ be a set of 6 image correspondences derived from 6 points $\mathbf{x}_i$ in space, and suppose it is known that the points $\mathbf{x}_1, \ldots, \mathbf{x}_4$ lie in a plane. Let P be a 3×3 matrix such that $\mathbf{u}'_i = P\mathbf{u}_i$ for $i = 1, \ldots, 4$. Suppose that the lines $< \mathbf{u}'_5, P\mathbf{u}_5 >$ and $< \mathbf{u}'_6, P\mathbf{u}_6 >$ are distinct and let $\mathbf{u}_p$ be their intersection. Suppose further that among the lines $< \mathbf{u}'_i, \mathbf{u}_p >$ there are at least three distinct lines. Then there exists a unique essential matrix Q satisfying the point correspondences and the condition of coplanarity of the points $\mathbf{x}_1, \ldots, \mathbf{x}_4$ and Q is given by the formula*

$$Q = S(\mathbf{u}_p)P$$

The conditions under which a unique solution exists may be expressed in geometrical terms. Namely :

1. Points $\mathbf{x}_1, \ldots, \mathbf{x}_4$ lie in a plane π, but no three of them are collinear.
2. Points $\mathbf{x}_5$ and $\mathbf{x}_6$ do not lie in the plane π, and do not lie in a common plane passing through the two camera centres.
3. The points $\mathbf{x}_1, \ldots, \mathbf{x}_6$ do not all lie in two planes passing through the camera centres.

Under the above conditions, the essential matrix Q is determined uniquely by the set of image correspondences. Note that according to [1], this in turn determines the locations of the points themselves and the cameras up to a projective transformation of 3-space.

2 Why does this work ?

With 8 points of more it is possible to solve for the matrix Q by solving a set of linear equations. If there are fewer than 8 points, the set of linear equations will be under-determined, and hence there will be a family of solutions. It is instructive to consider how the extra condition that four of the points should be coplanar cuts this family down to a single solution. Let us consider a particular example.

Consider a set of 6 matched points $\mathbf{u}'_i \leftrightarrow \mathbf{u}_i$ as follows :

$$\begin{array}{c}
(1,0,0)^\top \leftrightarrow (1,0,0)^\top \\
(0,1,0)^\top \leftrightarrow (0,1,0)^\top \\
(0,0,1)^\top \leftrightarrow (0,0,1)^\top \\
(1,1,1)^\top \leftrightarrow (1,1,1)^\top \\
(1,0,0)^\top \leftrightarrow (-1,1,1)^\top \\
(0,1,0)^\top \leftrightarrow (-1,1,1)^\top
\end{array} \tag{11}$$

Assume that the first 4 points lie in a plane. From the previous discussion, it is obvious that the epipole is the point $(-1,1,1)^\top$, and hence that

$$Q = S((-1,1,1)^\top = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} .$$

However, we will compute Q directly. Each of the six point correspondences gives rise to an equation $\mathbf{u}_i' Q \mathbf{u}_i = 0$ which is linear in the entries of Q. Since there are six equations in nine unknowns, there will be a 3-parameter family of solutions. It is easily verified, therefore, that the general solution is given by

$$Q = \begin{bmatrix} 0 & A & -A \\ B & 0 & B \\ C & -C-2B & 0 \end{bmatrix} . \tag{12}$$

Now, the condition $\det(Q) = 0$ yields an equation $2AB(C+B) = 0$, and hence, either $C = -B$ or $A = 0$ or $B = 0$. Thus, Q has one of the forms

$$Q = \begin{bmatrix} 0 & A & -A \\ B & 0 & B \\ -B & -B & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \\ B & 0 & B \\ C & -C-2B & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & A & -A \\ 0 & 0 & 0 \\ C & -C & 0 \end{bmatrix} . \tag{13}$$

We consider the first one of these solutions Since Q is determined only up to scale, we may choose $B = 1$, and so

$$Q = \begin{bmatrix} 0 & A & -A \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} . \tag{14}$$

Next, we investigate the condition that the first four matched points lie in a plane. To do this, it is necessary to find a pair of camera matrices that realize (see [1]) the matrix Q. It does not matter which realization of Q is picked, since any other choice will be equivalent to a projective transformation of object space (see [1]), which will take planes to planes. Accordingly, since Q factors as

$$Q = \begin{bmatrix} -A & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

a realization of Q is given by the two camera matrices

$$M = (I \mid 0) \quad \text{and} \quad M' = \left(\begin{array}{ccc|c} 1 & & & 1 \\ & -A & & A \\ & & -A & A \end{array} \right)$$

Then it is easily verified that the points

$$\mathbf{x}_1 = (1,0,0,0)^\top \ , \ \mathbf{x}_2 = (0,1,0,0)^\top \ , \ \mathbf{x}_3 = (0,0,1,0)^\top \ , \ \mathbf{x}_4 = (1,1,1,k)^\top \ ,$$

where k is defined by $1 + k = -A + kA$, are mapped by the two cameras to the required image points as specified by (11). However, the requirement that these four points lie in a plane means that $k = 0$ and hence that $A = -1$. Substituting this value in (14) yields the expected matrix $Q = S((-1,1,1)^\top)$. It may be verified that the two other choices for Q given in (13) do not lead to any further solution.

The role of the coplanarity condition now becomes clear. Without this condition, there are a family of solutions for the essential matrix Q. Only one of the family of solutions is consistent with the condition that the four points lie in a plane.

References

[1] R. Hartley, *"Estimation of Relative Camera Positions for Uncalibrated Cameras,"*, Technical Report, GE Corporate R&D, 1 River Road, Schenectady, NY 12301, Oct., 1991.

[2] Longuet-Higgins, H. C., "A computer algorithm for reconstructing a scene from two projections," Nature, Vol. 293, **10,** Sept. 1981.