Critical Configurations for N-view Projective Reconstruction

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in some order to be determined

Abstract

In this paper we give a complete characterization of critical configurations for projective reconstruction with any number of points and views. A set of cameras and points is said to be critical if the projected image points are insufficient to determine the placement of the points and the cameras uniquely, up to a projective transformation. For two views, the critical configurations are well-known. In this paper it is shown that a configuration of $n \geq 3$ cameras and m points is critical if all points and cameras lie on the intersection of two distinct ruled quadrics. Contrary to the two-view case, which in general allows two ambiguous solutions, there is a family of ambiguous reconstructions for the n-view case. Conversely, it is shown that (except for minimal cases) for any critical configuration, all the points and cameras lie on the intersection of two ruled quadrics.

1 Introduction

A key problem in computer vision is to recover the shape of an object from a number of images. This inverse problem has a number of inherent ambiguities. It is well-known that from image measurements alone, the cameras and the 3D points can only be determined up to an unknown projective transformation. For two views, additional ambiguities occur if and only if all points and cameras lie on a ruled quadric. This critical surface or "gefährlicher Ort" was studied by Krames [8] in 1941. See [9, 4] for a more recent treatment.

In this paper, we consider the problem of ambiguity of projective reconstruction from three or more views, and give a complete description of the critical configurations. A configuration of cameras $\mathtt{P}^i$ and points $\mathbf{P}_j$ is called *critical* if there exists a second configuration of cameras $\mathtt{Q}^i$ and points $\mathbf{Q}_j$ such that

$$\mathtt{P}^i\mathbf{P}_j = \mathtt{Q}^i\mathbf{Q}_j$$

for all i, j [1].

Understanding of the two-view case will be helpful for reading this paper, though the important prerequisite results will be quoted here for convenience. We follow the approach and terminology used in [2, 4]. Other work on critical curves and surfaces can be found in [6, 1, 7]. Partial results concerning ambiguous configurations with more than two views have been reported previously in the literature. Maybank and Shashua ([10]) considered the case of many views of 6 points, showing that a configuration is critical if and only if the points and camera all lie on a quadric. This was shown to be dual to the two-view ambiguity in [3].

The first non-trivial examples of critical configurations for the 3-view case were given in [2] in which it was shown that three cameras always belong to some critical configuration of points in which the cameras and points all lie on the intersection of two ruled quadrics, known as an elliptic quartic curve[2]. Furthermore in *any* critical configuration, all points and cameras must lie on the intersection of three (and hence two) ruled quadrics. However, the example given in [2] was somewhat special, and the full range of critical curves was not given. In the present paper, we show that *any* curve formed as the intersection of two ruled quadrics is critical – a set of cameras and points all lying on this intersection curve allow an ambiguous reconstruction. This is a much more general result than the single example of a critical curve given in [2].

The result just quoted gives a complete description of the critical configurations for three views of a set of points. This paper extends that result by showing that the same quartic curve is a critical set for n-view reconstruction. Thus, one may add any number of further cameras with centres located on the critical curve without removing the reconstruction ambiguity – the curve remains critical for all the views. Thus, the critical configurations for n views of ≥ 7 points are completely described.

The results presented here are of both practical and theoretical interest. On a theoretical level, a classification of all critical configurations is important and it has been an open problem for a long time. The classification is also useful in practical situations when designing measurement paths for structure and motion estimation. Naturally, one wants to avoid the critical configurations.

[1] As part of this definition we exclude the trivial cases of ambiguity for (i) projective coordinate system (ii) camera resection and (iii) point intersection.

[2] In the classification of space curves [11], there are two types of irreducible quartics, elliptic quartics (the intersection of two quadrics) and rational quartics (the intersection of a cubic and a quadric minus two lines).

2 Statement of the problem

This section is largely to define the notation used in subsequent sections. We consider the case of $n \geq 3$ cameras, and denote the camera matrices by $\mathtt{P}^i$, for $i = 0, \ldots, n-1$. Consider also a set of points $\mathbf{P}_j$. The question considered is under what circumstances there exists an alternative set of camera matrices $\mathtt{Q}^i$ and points $\mathbf{Q}_j$ such that $\mathtt{P}^i\mathbf{P}_j = \mathtt{Q}^i\mathbf{Q}_j$ for all i, j, but $\{\mathbf{P}_j\}$ and $\{\mathbf{Q}_j\}$ are projectively inequivalent point sets. If such an alternative set of points and cameras exist, then we say that the configuration $\{\mathtt{P}^i, \mathbf{P}_j\}$ is a *critical configuration*. In this case, the alternative configuration $\{\mathtt{Q}^i, \mathbf{Q}_j\}$ is its *conjugate configuration*.

It is expedient immediately to dispose of a trivial form of ambiguity in which the point sets $\mathbf{P}_i$ and $\mathbf{Q}_i$ are projectively equivalent, and the only ambiguity is in the cameras. This form of ambiguity is just a result of the ambiguity of camera resectioning for a single camera. Such an ambiguity arises (see [2]) when the points all lie on a common twisted cubic with one of the camera centres ([1]). Other ambiguities for camera resectioning are given in [4]. Another trivial form of ambiguity arises in ray intersectioning, i.e. when computing a 3D point from known camera positions. Ambiguity occurs if and only if all cameras and (at least) one point lie on a line. Since these forms of reconstruction ambiguity are of limited interest, they are excluded from further discussion. Consequently the definition of critical configuration will be strengthened to require that (i) the point sets *and* (ii) the camera sets in the two alternative reconstructions are not projectively equivalent.

Standard camera configurations. It is useful to put the three camera matrices in a canonical form. The first remark is that the precise form of the camera matrices $\mathtt{P}^i$ is not important for criticality, just their centres. This is expressed in the following result that is proved in [2] (Proposition 1).

Theorem 2.1. *If $\{\mathtt{P}^i, \mathbf{P}_j\}$ is a critical configuration, and $\widehat{\mathtt{P}}^i$ are cameras with the same centres as $\mathtt{P}^i$, then $\{\widehat{\mathtt{P}}^i, \mathbf{P}_j\}$ is a critical configuration as well.*

For convenience, it may therefore be assumed that the three camera matrices are of the form $\mathtt{P}^i = [\mathtt{I} \mid \mathbf{v}^i]$, where $\mathbf{v}^i$ is the camera centre.

Critical surfaces for pairs of cameras. If $\{\mathtt{P}^i, \mathbf{P}_j\}; i = 0, \ldots, 2$ is a critical configuration for three views, then it is critical for any of the three pairs of cameras. Therefore, it follows from the well-known results on 2-view critical surfaces that the set of points $\mathbf{P}_k$ and each of the camera centres of $\mathtt{P}^i$ and $\mathtt{P}^j$ lie on a ruled quadric surface, denoted $S^{ij}_{\mathtt{P}}$. According to [2], the quadric is given by the formula

$$S^{ij}_{\mathtt{P}} = \mathtt{P}^{i\top}\mathtt{F}^{ij}_{\mathtt{Q}}\mathtt{P}^j \tag{1}$$

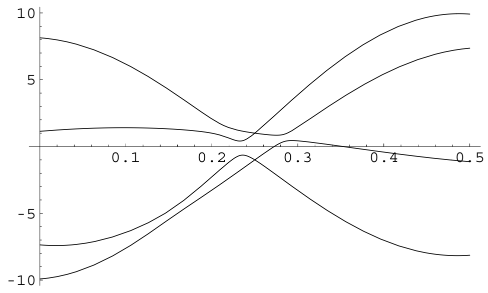

Figure 1: *Example of the variation of the four eigenvalues of the quadrics in a pencil, as a function of a parameter (range 0 to 1) parametrizing the pencil. For a given value of the parameter (horizontal axis) the quadric is ruled if there are two positive and two negative eigenvalues, and otherwise unruled. The quadric is degenerate at the points where one of the eigenvalues becomes zero. See Figure 2 for more details.*

where $\mathtt{F}^{ij}_{\mathtt{Q}}$ is the fundamental matrix for the pair of cameras $(\mathtt{Q}^i, \mathtt{Q}^j)$ in the conjugate configuration.

3 Pencils of quadrics

Given two quadrics represented by symmetric 4×4 matrices A and B, the pencil generated by them consists of all quadrics of the form $\alpha A + \beta B$.

Proposition 3.2. *If C is the intersection of the quadrics A and B, then C lies on each of the quadrics $\alpha A + \beta B$ in the pencil. If*

$$\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \neq 0$$

then the intersection of quadrics $\alpha A + \beta B$ and $\gamma A + \delta B$ is the same as the intersection of the quadrics A and B.

A pencil of quadrics may be defined by either ruled or unruled quadrics. In non-degenerate cases these correspond to hyperboloids of one sheet, or ellipsoids. It is important to note that a pencil of quadrics defined by two ellipsoids may contain hyperboloids, and even imaginary quadrics (those with no real points). Similarly, a pencil defined by hyperboloids may contain ellipsoids and imaginary conics. This is easily seen by example. Figures 1 and 2 illustrate this point. If the equation $\det(A + \lambda B)$ has distinct real roots, then it is possible to simultaneously diagonalize a pair of real symmetric matrices by changing coordinate systems, to get a canonical form. However, this is otherwise not possible. See [12] for more details.

The intersection of two quadrics is a fourth-degree curve called an *elliptic quartic* [11]. It has 16 degrees of freedom ($2 \times 9 - 2 = 16$ – two quadrics minus the choice of base quadrics defining the pencil). Thus, 8 points in general position define an elliptic quartic uniquely. Generically, a pencil

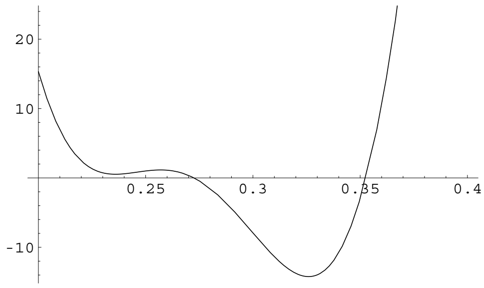

Figure 2: *The transition from ruled to unruled quadric takes place at points where the determinant of the quadric in the pencil vanishes. This is when $\det(A - \lambda B) = 0$, namely at the generalized eigenvalues. This equation may have zero, two or four real roots. This plot shows the value of $\det(A - \lambda B)$ for the example in Figure 1 showing that there are only two real generalized eigenvalues for this pencil.*

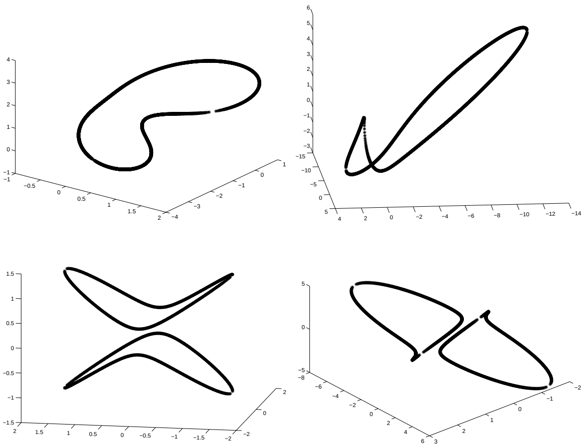

Figure 3: *Examples of randomly generated elliptic quartics. In all examples, the pencils contain ruled quadrics.*

of quadrics, with a non-empty set of points lying on the intersection, contains distinct ruled quadrics. Some examples of randomly generated elliptic quartics are given in Figure 3.

4 Critical surfaces for 3 views

Now, we prove the main theorem concerning critical configurations in three views.

Theorem 4.3. *A configuration of three cameras $\mathtt{P}^i$ and points $\mathbf{P}_j$, is critical if all points and cameras lie on the intersection of two distinct ruled quadrics.*

The proof consists of exhibiting an explicit formula for the alternative configurations.

Proof. Without loss of generality, we may assume that $S^{10}_{\mathtt{P}}$ is the quadric $z = xy$, represented by the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad (2)$$

since all hyperboloids of one sheet are projectively equivalent. Further, we may assume that the camera centres are the points $(0,0,0)^\top$ and $(1,1,1)^\top$ and $(-1,-1,1)$.[3] According to Theorem 2.1 the camera matrices may then be taken as $\mathtt{P}^0 = [\,\mathtt{I}\,|\,0\,]$ and $\mathtt{P}^1 = [\,\mathtt{I} \mid (-1,-1,-1)^\top\,]$ and $\mathtt{P}^2 = [\,\mathtt{I} \mid (1,1,-1)^\top\,]$

Next, we want another quadric B that passes through the camera centres. As B varies over all possible quadrics passing through the three camera centres, the intersection of A and B encompasses all elliptic quartics passing through the three camera centres. In homogeneous coordinates, the three camera centres are $(0,0,0,1)$, $(1,1,1,1)$ and $(-1,-1,1,1)$. Since $(0,0,0,1)$ lies on the quadric B, the bottom right-hand entry of B is zero. The condition that the other two points lie on the quadric implies that the sum of entries in the top right 2×2 block of B, is zero, and the sum of the remaining entries of B is also zero. This means that quadric is of the special form $B = B' + B'^\top$, where

$$B' = \begin{bmatrix} p & q & s-t & -s-u \\ 0 & r & s+t & -s+u \\ 0 & 0 & -p-q-r-v & v \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (3)$$

Furthermore, we will be considering the pencil defined by A and B, so we may assume that $v = 0$, since this can otherwise be achieved by adding vA, without changing the intersection of the two quadrics A and B. Thus, the matrix B is defined by 6 parameters, $\{p,q,r,s,t,u\}$.

Two alternative reconstructions involving cameras $\mathtt{Q}^i$ and points $\mathbf{Q}$ are given in Table 1 and Table 2. It may be verified directly that $\mathtt{P}^i\mathbf{P} = \mathtt{Q}^i\mathbf{Q}$ for all points $\mathbf{P} = (x,y,xy,1)^\top$ and corresponding points $\mathbf{Q}$, provided that $\mathbf{P}$ lies on the quadric B. (It always lies on quadric A). The easiest way to see this is to verify that $(\mathtt{P}^i\mathbf{P}) \times (\mathtt{Q}^i\mathbf{Q}) = 0$ for all such points. In fact for $i = 0,1$, the cross-product is always zero, whereas for $i = 2$ it may be verified by direct computation that

$$(\mathtt{P}^2\mathbf{P}) \times (\mathtt{Q}^2\mathbf{Q}) = \left(\mathbf{P}^\top B\mathbf{P}\right)(4,-4x,4)^\top$$

for the first solution, and

$$(\mathtt{P}^2\mathbf{P}) \times (\mathtt{Q}^2\mathbf{Q}) = \left(\mathbf{P}^\top B\mathbf{P}\right)(-4y,4,4)^\top$$

for the second solution. Thus $\mathtt{P}^2\mathbf{P} = \mathtt{Q}^2\mathbf{Q}$ if and only if $\mathbf{P}$ lies on B. ■

[3] Assuming that the three camera centres do not lie on the same generator of the quadric. In this case the three centres are collinear. An analagous proof can be derived for this case.

The camera matrices are

$$\mathtt{Q}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad , \quad \mathtt{Q}^1 = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

and

$$\mathtt{Q}^2 = \begin{bmatrix} -4\,(2p+q-t+u) & 8r & 4\,(p+q+2r+s+t) & -2\,(p+q-s-t) \\ 0 & 8\,(r+s-u) & -2\,(q-t+u) & -q+t-u \\ 8p & -8r & -2\,(2p+q-2s+3t+3u) & 2p+q-2s-t-u \end{bmatrix}$$

The conjugate point to $\mathbf{P} = (x, y, xy, 1)^\top$ is $\mathbf{Q} = ((x-1)x, (x-1)y, (x-1)xy, -2x\,(-2+y+xy))^\top$.

Table 1: *First conjugate solution to reconstruction problem for cameras $\mathtt{P}^i$ and points on the intersection of quadrics A and B given by (2) and (3) respectively.*

The camera matrices are

$$\mathtt{Q}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad , \quad \mathtt{Q}^1 = \begin{bmatrix} 0 & 0 & -2 & 1 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

and

$$\mathtt{Q}^2 = \begin{bmatrix} -8\,(p+s+u) & 0 & 2\,(q+t-u) & -q-t+u \\ -8p & 4\,(q+2r+t-u) & -4\,(2p+q+r+s-t) & -2\,(q+r-s+t) \\ 8p & -8r & 2\,(q+2r-2s-3t-3u) & q+2r-2s+t+u \end{bmatrix}$$

The conjugate point to $\mathbf{P} = (x, y, xy, 1)^\top$ is $\mathbf{Q} = ((y-1)x, (y-1)y, (y-1)xy, 2y\,(-2+x+xy))^\top$.

Table 2: *Second conjugate solution to reconstruction problem.*

5 A family of solutions

Given a quartic curve defined as the intersection of a pencil of quadrics spanned by A and B, Table 1 and Table 2 give examples of conjugate solutions for which the critical quadric $S^{10}_{\mathtt{P}}$ is equal to A. However, A is just one of a family of quadrics that may be used to span the pencil. For all values of a parameter γ, the quadric $\gamma A + B$ may be chosen as one of such a pair of spanning quadrics. Provided $\gamma A + B$ is a ruled quadric, the argument and examples of section 4 show that there is a conjugate configuration for which $S^{10}_{\mathtt{P}} = \gamma A + B$.

It is easy to see that the conjugate solutions arising from two different quadrics A and A' are projectively inequivalent. For let $\mathtt{F}^{10}_{\mathtt{Q}}$ and $\mathtt{F}'^{10}_{\mathtt{Q}}$ be the fundamental matrices for the two different configurations. Since $A = \mathtt{P}^{1\top}\mathtt{F}^{10}_{\mathtt{Q}}\mathtt{P}^0$ and $A' = \mathtt{P}^{1\top}\mathtt{F}'^{10}_{\mathtt{Q}}\mathtt{P}^0$ are different, it follows that $\mathtt{F}^{10}_{\mathtt{Q}} \neq \mathtt{F}'^{10}_{\mathtt{Q}}$. Since the fundamental matrices are different, the two configurations are not projectively equivalent. This shows

Theorem 5.4. *Suppose three cameras, and a set of points $\{\mathtt{P}^i, \mathbf{P}_j\}$ lie on a curve defined as the intersection of a pencil of quadrics. Then for each ruled quadric A in the pencil, there exists a pair of conjugate configurations, each of the form $\{\mathtt{Q}^i, \mathbf{Q}_j\}$, such that the critical quadric $S^{10}_{\mathtt{P}} = \mathtt{P}^{1\top}\mathtt{F}^{10}_{\mathtt{Q}}\mathtt{P}^0 = A$.*

Thus, unlike critical configurations in the two-view case, which allow two conjugate solutions, critical configurations for three views give rise to two one-parameter families of conjugate configurations.

6 Linear mapping

Since the conjugate solutions were seemingly pulled out of a hat, we will give a more theoretical treatment of the problem now, which will partly elucidate the method used to arrive at this solution.

We start by giving some properties of critical quadrics for two views, as given by (1). For the proofs, see [2].

Theorem 6.5. *Given camera matrices $\mathtt{P}$ and $\mathtt{P}'$ and a 3×3 matrix $\mathtt{F}$, define $S = \mathtt{P}'^\top\mathtt{F}\mathtt{P}$. Let $S_{\rm sym} = S + S^\top$ represent a quadric surface. Then*

1. *$S_{\rm sym}$ is zero if and only if $\mathtt{F}$ is the fundamental matrix corresponding to the pair $(\mathtt{P}, \mathtt{P}')$.*

2. *If non-zero, then S_{sym} represents a ruled quadric, if* $\det \mathtt{F} = 0$.

3. *The camera centres of both* P *and* P *lie on* S_{sym}.

Note the important information that this construction defines a *ruled* quadric, provided $\det \mathtt{F} = 0$.

Let P and P′ be fixed, and consider the mapping $f : \mathtt{F} \mapsto S_{\text{sym}}$ defined in Theorem 6.5. This mapping is defined over the set of all 3×3 matrices, and its range is in the space of all symmetric 4×4 matrices representing quadrics passing through the centres of P and P′. Let us count the dimensions of the range and domain of this mapping.

The domain of f is the 9-dimensional space consisting of all 3×3 matrices, effectively $\mathcal{R}^9$. The dimension of the space of all 4×4 symmetric matrices is 10. However, the constraint that the quadric S_{sym} should pass through a given point (a camera centre) gives a single linear constraint on the entries of S_{sym}. Hence, the set of symmetric matrices representing quadrics passing through two given points has dimension 8, and is in fact an 8-dimensional subspace of $\mathcal{R}^{10}$. Thus, f is a linear mapping from $\mathcal{R}^9$ to $\mathcal{R}^8$. According to Theorem 6.5, the mapping f has a 1-dimensional kernel. It follows that f is an epimorphism (onto-mapping). We have shown the following result.

Theorem 6.6. *For two given fixed camera matrices* P *and* P′*, the mapping $f : \mathtt{F} \mapsto S_{\text{sym}}$ is a linear transformation, mapping the vector space of all 3×3 matrices* F *onto the vector space of all symmetric 4×4 matrices representing quadrics containing the two camera centres for* P *and* P′*. The kernel of f is the linear subspace generated by the fundamental matrix of the pair* $(\mathtt{P}, \mathtt{P}')$.

This result does not take into account the condition $\det F = 0$ for a matrix to be a fundamental matrix. The restriction of this map to the set of all zero-determinant fundamental matrices defines a 2-to-1 mapping from the set of all fundamental matrices to a set of *ruled* quadrics, as enunciated in the next theorem.

Theorem 6.7. *Let* P *and* P′ *be fixed and define* $f(\mathtt{F}) = (\mathtt{P}'^{\top}\mathtt{F}\mathtt{P})_{\text{sym}}$. *If* F *has zero determinant (hence is a fundamental matrix), then $f(\mathtt{F})$ represents a ruled quadric passing through the camera centres of* P *and* P′*. Conversely, let S be any* ruled *non-degenerate quadric passing through the centres of* P *and* P′*. Then there exist exactly two fundamental matrices* F *(i.e. with zero determinant) such that $f(\mathtt{F}) = S$.*

Proof. We prove just the case that S is a non-degenerate quadric (i.e. a hyperboloid of one sheet) and that the two camera centres do not lie on the same generator.

Without loss of generality, we may assume that S is the quadric $z = xy$, since all hyperboloids of one sheet are projectively equivalent. Further, we may assume that the camera centres are the points $(0, 0, 0)^{\top}$ and $(1, 1, 1)^{\top}$. The two camera matrices must then be of the form $\mathtt{P} = \mathtt{K}[\mathtt{I}|0]$ and $\mathtt{P}' = \mathtt{K}'[\mathtt{I} \mid (1, 1, 1)^{\top}]$, where K and K′ are non-singular matrices.

Now, it may be verified easily (see the example on page 929 of [2]) that the matrix $\mathtt{F} = \mathtt{K}'^{-\top}\mathtt{G}\mathtt{K}^{-1}$ gives the required result, where

$$\mathtt{G} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad or \quad \mathtt{G} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Thus, there are two separate choices of fundamental matrix F that map onto the ruled quadric S. ■

7 Compatible fundamental matrices

We return to the consideration of three camera matrices $\mathtt{P}^i; i = 0, \ldots, 2$. For each pair of camera matrices, $\mathtt{P}^i$ and $\mathtt{P}^j$, the mapping f is defined as in Theorem 6.6. Let the three mappings obtained in this way be denoted by f^{ij}, namely f^{10}, f^{20} and f^{21}. Given three fundamental matrices $\mathtt{F}_{\mathtt{Q}}^{ij}$, we can form the three quadrics $f^{ij}(\mathtt{F}_{\mathtt{Q}}^{ij})$.

In this context, the three fundamental matrices $\mathtt{F}_{\mathtt{Q}}^{ij}$ are the fundamental matrices corresponding to pairs of views in a conjugate configuration involving camera matrices $\mathtt{Q}^i; i = 0, \ldots, 2$. An essential issue here is the *compatibility* of the three fundamental matrices.

Definition 7.8. Three fundamental matrices $\mathtt{F}^{01}$, $\mathtt{F}^{02}$ and $\mathtt{F}^{12}$ are *compatible* if they satisfy the conditions

$$\mathbf{e}^{12\top}\mathtt{F}^{10}\mathbf{e}^{02} = 0, \quad \mathbf{e}^{21\top}\mathtt{F}^{20}\mathbf{e}^{01} = 0, \quad \mathbf{e}^{20\top}\mathtt{F}^{21}\mathbf{e}^{10} = 0.$$

The importance of this condition is given by the following theorem [5].

Theorem 7.9. *Given a compatible set of three fundamental matrices $\mathtt{F}^{10}$, $\mathtt{F}^{20}$ and $\mathtt{F}^{21}$, there exist three camera matrices $\mathtt{Q}^i, \mathtt{Q}^1, \mathtt{Q}_2$ such that $\mathtt{F}^{ij}$ is the fundamental matrix corresponding to the pair $(\mathtt{Q}^i, \mathtt{Q}^j)$.*

8 Discovering the conjugate solutions

The conjugate reconstructions that are given in Table 1 and Table 2 may be verified directly, which is convincing proof of their correctness, and hence constitutes a sufficiency proof of Theorem 4.3. This may however be unsatisfying, since the formulas have been pulled out of a hat by some sort of sleight of hand. In this section, some indication is given of how they were discovered.

We suppose given a pair of ruled quadrics A and B which intersect in a space curve. Contained on this curve are the centres of three cameras $\mathtt{P}^0$, $\mathtt{P}^1$ and $\mathtt{P}^2$, as well as any number of points $\mathbf{P}_j$. If this is to be a critical configuration, then there will exist three conjugate cameras $\mathtt{Q}^0$, $\mathtt{Q}^1$ and $\mathtt{Q}^2$ and also points $\mathbf{Q}_j$. The three-view critical configuration will also be critical as a two-view configuration for each of the pairs of cameras, and hence there will exist three critical quadrics $S_{\mathtt{P}}^{10}$, $S_{\mathtt{P}}^{20}$ and $S_{\mathtt{P}}^{21}$ defined by (1), all meeting along the curve defined by A and B. We attempt to construct three conjugate camera matrices $\mathtt{Q}^i$ by finding a set of compatible fundamental matrices. The task then, is to find three fundamental matrices $\mathtt{F}_{\mathtt{Q}}^{ij}$, which must satisfy the following conditions.

1. The three quadrics $S_{\mathtt{P}}^{ij}$ defined by (1) in terms of the given $\mathtt{F}_{\mathtt{Q}}^{ij}$ must belong to the pencil of quadrics defined by A and B.

2. The three fundamental matrices $\mathtt{F}_{\mathtt{Q}}^{ij}$ must be compatible in the sense of Definition 7.8.

If these conditions hold, then it is possible to extract three camera matrices $\mathtt{Q}^i$ that give rise to the three fundamental matrices $\mathtt{F}_{\mathtt{Q}}^{ij}$. Furthermore, the three camera matrices $\mathtt{P}^i$ and the set of points on the intersection of the three critical quadrics $S_{\mathtt{P}}^{ij}$ form a critical configuration. Actually for this last claim to be true, it is necessary that the three cameras $\mathtt{Q}^i$ not be collinear, as shown in [2][4].

Consider the condition that $S_{\mathtt{P}}^{ij}$ must belong to the pencil defined by A and B. We assume that

$$S_{\mathtt{P}}^{10} = A, \quad S_{\mathtt{P}}^{20} = \alpha_0 A + B \quad \text{and} \quad S_{\mathtt{P}}^{21} = \alpha_1 A + B.$$

Now, according to Theorem 6.7 there exist two choices of fundamental matrix $\mathtt{F}_{\mathtt{Q}}^{10}$ such that $f^{10}(\mathtt{F}_{\mathtt{Q}}^{10}) = A$. We select one of them – in fact the two different conjugate configurations given in Table 1 and Table 2 correspond to the choice of the two possible $\mathtt{F}_{\mathtt{Q}}^{10}$ at this stage. Now, for $(i,j) = (2,0)$ or $(2,1)$, find matrices $\mathtt{G}^{ij}$ and $\mathtt{H}^{ij}$ (not necessarily singular) such that

$$f^{ij}(\mathtt{G}^{ij}) = A, \quad f^{ij}(\mathtt{H}^{ij}) = B.$$

Now, set

$$\begin{aligned} \mathtt{F}_{\mathtt{Q}}^{20} &= \alpha_0 \mathtt{G}^{20} + \mathtt{H}^{20} + \beta_0 \mathtt{F}_{\mathtt{P}}^{20}, \\ \mathtt{F}_{\mathtt{Q}}^{21} &= \alpha_1 \mathtt{G}^{21} + \mathtt{H}^{21} + \beta_1 \mathtt{F}_{\mathtt{P}}^{21}. \end{aligned}$$

Since the mapping f^{ij} is linear, and maps $\mathtt{F}_{\mathtt{P}}^{ij}$ to zero, it follows that

$$f^{ij}(\mathtt{F}_{\mathtt{Q}}^{ij}) = \alpha_j f^{ij}(\mathtt{G}^{ij}) + f^{ij}(\mathtt{H}^{ij}) = \alpha_j A + B = S_{\mathtt{P}}^{ij}.$$

Unfortunately, the $\mathtt{F}_{\mathtt{Q}}^{ij}$ arbitrarily chosen in this way are neither true fundamental matrices (having determinant zero) nor are compatible. We need to choose α_j and β_j (four parameters) to satisfy the constraints $\det(\mathtt{F}_{\mathtt{Q}}^{ij}) = 0$, and the three compatibility constraints of Definition 7.8. Note that $\mathtt{F}_{\mathtt{Q}}^{10}$ was chosen such that $\det(\mathtt{F}_{\mathtt{Q}}^{10}) = 0$, so there are only five remaining constraints.

We are faced with the problem of satisfying five non-linear constraints with only four parameters. In addition, the constraints are non-linear – the determinant constraints are cubic, whereas the compatibility constraints involve extracting epipoles (null-spaces of the matrices). No symbolic algebra package (Mathematica or Maple) is capable of solving this system without help even for numerical examples, and let alone the general case where B is a function of symbolic parameters, as in (3). Clearly we are going to need luck!

Since A may be assumed to be of a simple form (2), it is easy to compute $\mathtt{F}_{\mathtt{Q}}^{10}$, and from it extract the epipoles $\mathbf{e}^{10}$ and $\mathbf{e}^{01}$.[5] Now, one of the consistency conditions is $\mathbf{e}^{21\top}(\mathtt{F}_{\mathtt{Q}}^{20}\mathbf{e}^{01}) = 0$, which may be thought of as an equation involving the epipole $\mathbf{e}^{21}$. The equation $\mathbf{e}^{21\top}\mathtt{F}_{\mathtt{Q}}^{21}$ gives three further equations involving the epipole $\mathbf{e}^{21}$, which also ensure that $\mathtt{F}_{\mathtt{Q}}^{21}$ has zero determinant. Numerical examples suggested that there were solutions to this set of equations independent of the particular form of $\mathtt{F}_{\mathtt{Q}}^{20}$, that is, valid for all values of α_0 and β_0. This would imply that for some values of (α_1, β_1) there is a solution for $\mathbf{e}^{21}$ to the set of equations

$$\begin{aligned} \mathbf{e}^{21\top}(\mathtt{G}^{20}\mathbf{e}^{01}) &= 0 \\ \mathbf{e}^{21\top}(\mathtt{H}^{20}\mathbf{e}^{01}) &= 0 \\ \mathbf{e}^{21\top}(\mathtt{F}_{\mathtt{P}}^{20}\mathbf{e}^{01}) &= 0 \\ \mathbf{e}^{21\top}\mathtt{F}_{\mathtt{Q}}^{21} &= \mathbf{0}. \end{aligned}$$

Note that the first three of these equations do not involve any of the variables α_j or β_j, since $\mathtt{G}^{20}$, $\mathtt{H}^{20}$, $\mathtt{F}_{\mathtt{P}}^{20}$ are known quantities (dependent only on the parameters of A and B). The fourth equation (actually three equations) is linear in α_1 and β_1. If such a solution for $\mathbf{e}^{21}$ exists, then a solution for (α_1, β_1) can be obtained as follows:

1. Find the value of $\mathbf{e}^{21\top}$ as the left null-space of the matrix $[\mathtt{G}^{20}\mathbf{e}^{01}, \mathtt{H}^{20}\mathbf{e}^{01}, \mathtt{F}_{\mathtt{P}}^{20}\mathbf{e}^{01}]$.

2. Linearly solve the equation $\mathbf{e}^{21\top}\mathtt{F}_{\mathtt{Q}}^{21} = \mathbf{0}$ for (α_1, β_1).

Thus, the solution for (α_1, β_1) is obtained by solving only linear equations, which explains the absence of radicals in the solution given in Table 1 and Table 2. A similar procedure may be used to solve for the remaining variables (α_0, β_0) defining fundamental matrix $\mathtt{F}_{\mathtt{Q}}^{20}$. The epipoles were next extracted from the computed $\mathtt{F}_{\mathtt{Q}}^{20}$ and $\mathtt{F}_{\mathtt{Q}}^{21}$ and the third consistency condition $\mathbf{e}^{12\top}\mathtt{F}_{\mathtt{Q}}^{10}\mathbf{e}^{02} = 0$ verified. Finally, the camera matrices $\mathtt{Q}^i$ were extracted from the three consistent fundamental matrices. Necessary simplifications of the obtained matrices were done interactively.

[4] As also shown in [2] points in the plane of the centres of the $\mathtt{Q}$ cameras may not be critical. However this need not concern us, since the theorem is being used only to guide our search for the conjugate configuration. The result is justified by the displayed conjugate configurations.

[5] Epipoles are labelled according to the convention that $\mathtt{F}_{\mathtt{Q}}^{ij}\mathbf{e}^{ji} = 0$.

This whole computation was carried out for a general symbolic value of B given by (3), resulting in the general solution given in Table 1 and Table 2, and thereby justifying the procedure.

9 n-view critical configurations

We now wish to extend this result to n views. By adding a fourth camera to a critical three-view configuration, one might suspect that the criticality would disappear. However, this is not the case.

The goal of this section is the following theorem.

Theorem 9.10. *Given $n \geq 3$ cameras $\mathtt{P}^i; i = 0, \ldots, n-1$ and $\mathbf{P}_j$, then the configuration is critical if all points and cameras lie on the intersection of two distinct ruled quadrics A and B.*

Proof. Assume that configuration is on the intersection of A and B. For simplicity of notation, we consider the 4-view case. The result for n views follows by induction. The three cameras $\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^2$ along with the points form a critical configuration, and hence a conjugate configuration exists. Similarly a conjugate configuration exists for the points along with the three cameras $\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^3$. The goal is to show that these two conjugate configurations are compatible. This will be done by showing that the cameras $\mathtt{Q}^0$ and $\mathtt{Q}^1$ may be chosen to be the same in each of the two configurations derived from the two camera triples.

Consider the way that the cameras $\mathtt{Q}^0$ and $\mathtt{Q}^1$ are constructed. Let $\mathtt{P}^i; i = 0, \ldots, 2$ be the first triple of cameras. They and the points $\mathbf{P}_j$ lie (without loss of generality) on the quadric A. The fundamental matrix $\mathtt{F}^{10}_{\mathtt{Q}}$ is chosen so that $A = f^{10}(\mathtt{F}^{10}_{\mathtt{Q}})$. From this value of $\mathtt{F}^{10}_{\mathtt{Q}}$ two camera matrices $\mathtt{Q}^0$ and $\mathtt{Q}^1$ may be computed. The choice of the pair $(\mathtt{Q}^0, \mathtt{Q}^1)$ is unique up to a projectivity.

We proceed in the same way with the second triple of cameras $\mathtt{P}^0, \mathtt{P}^1, \mathtt{P}^3$. Since the camera $\mathtt{P}^2$ or $\mathtt{P}^3$ does not take part in this construction, the resulting fundamental matrix $\mathtt{F}^{10}_{\mathtt{Q}}$ is the same, and from it one may extract two camera matrices, which will be denoted $(\mathtt{Q}'^0, \mathtt{Q}'^1)$, which must be projectively equivalent to the pair $(\mathtt{Q}^0, \mathtt{Q}^1)$. However (and this is the main point) a conjugate configuration is defined only up to projectivity. Consequently, it is possible to choose the camera pair $(\mathtt{Q}'^0, \mathtt{Q}'^1)$ to be identical to $(\mathtt{Q}^0, \mathtt{Q}^1)$. In this way, from the two conjugate configurations corresponding to the two triples of cameras with indices $(0, 1, 2)$ and $(0, 1, 3)$, we get four camera matrices $\mathtt{Q}^i; i = 0, \ldots, 3$.

So far we have not mentioned the points. Now let $\mathbf{Q}_j$ be points derived from the $(0, 1, 2)$-configuration, and let $\mathbf{Q}'_j$ be the points derived from the $(0, 1, 3)$-configuration. By the definition of criticality,

$$\mathtt{P}^i\mathbf{P}_j = \mathtt{Q}^i\mathbf{Q}_j \text{ for } i = 0, 1, 2$$

and

$$\mathtt{P}^i\mathbf{P}_j = \mathtt{Q}^i\mathbf{Q}'_j \text{ for } i = 0, 1, 3.$$

Considering only the first two cameras here, we see that

$$\mathtt{Q}^i\mathbf{Q}_j = \mathtt{Q}^i\mathbf{Q}'_j \text{ for } i = 0, 1.$$

Since the image of a point in two views is sufficient to determine its position (by triangulation), it follows that $\mathbf{Q}_j = \mathbf{Q}'_j$. Consequently $\mathtt{P}^i\mathbf{P}_j = \mathtt{Q}^i\mathbf{Q}_j$ for $i = 0, \ldots, 3$, and so $\{\mathtt{Q}^i, \mathbf{Q}_j\}$ forms a conjugate configuration to $\{\mathtt{P}^i, \mathbf{P}_j\}$, which therefore is a critical configuration.

∎

10 Converse

It is natural to ask whether the critical configurations considered here account for all possible critical configurations involving $n \geq 3$ cameras. The answer is that this is true, except for certain minimal configurations involving small numbers of points.

According to Theorem 6.5 and [2], the centre of the camera $\mathtt{P}^i$ belonging to a critical configuration lies on each of the quadrics $S^{ij}_{\mathtt{P}}$, as do all the points $\mathbf{P}_j$ belonging to the critical configuration. By intersecting these quadrics for two different values of j, we see that the centre of $\mathtt{P}^i$ must lie on the intersection of two quadrics also passing through the points. It follows that if the set of all points $\mathbf{P}_j$ lie on a unique quadric, then all the camera centres must lie on this same quadric. Therefore, we can state:

- *If $\{\mathtt{P}^i, \mathbf{P}_j\}$ is a critical configuration with at least 3 cameras, then all the points $\mathbf{P}_j$ must lie on the intersection of two ruled quadrics. If there is a unique such quadric intersection containing all the points, then it must contain all the camera centres as well.*

One may wonder whether there are critical configurations in which the quartic curve passing through the points is not unique, and in which all the points and cameras do not lie on a single elliptic quartic. The answer to this question is yes, according to ([10]). In that paper, it is shown that a configuration of six points and any number of cameras lying on a ruled quadric surface is critical. Note that six points are insufficient to define a unique quadric. In this case, the points and camera centres do not all lie on the same quartic curve.

Note that in general 8 points are necessary to define a unique quartic curve, but not sufficient. For instance three quadrics will in general intersect in 8 points, which therefore do not lie on a unique quartic. Furthermore, the set of all quadrics intersecting in a given twisted cubic do not belong to a single pencil, forming in fact a two-parameter family. However, three independent quadrics can not intersect in more than 8 isolated points ([11].

11 Conclusion

We have given a complete categorization of all ambiguous configurations for projective reconstruction from $n \geq 2$ views, except for the case where the set of points is insufficient to define a unique elliptic quartic. The main ambiguity is when all object points (regardless of how many) and all camera centres (again, regardless of the number of cameras) lie on an elliptic quartic curve. Furthermore, unlike the two-view case, there is a one-parameter family of conjugate reconstructions which give the same image projections.

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