A Conjecture of Erdős the Ramsey Number $r(W_6)$

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Abstract

It was conjectured by Paul Erdős that if G is a graph with chromatic number at least k, then the diagonal Ramsey number $r(G) \geq r(K_k)$. That is, the complete graph K_k has the smallest diagonal Ramsey number among the graphs of chromatic number k. This conjecture is shown to be false for $k = 4$ by verifying that $r(W_6) = 17$, where W_6 is the wheel with 6 vertices, since it is well known that $r(K_4) = 18$. Computational techniques are used to determine $r(W_6)$ as well as the Ramsey numbers for other pairs of small order wheels.

1 Introduction

The following well known conjecture is due to Paul Erdős.

CONJECTURE 1 *If G is a graph with chromatic number $\chi(G) \geq k$, then the Ramsey number*

$$r(G) \geq r(K_k).$$

The strong form of the Erdős conjecture is that if $\chi(G) \geq k$, and G does not contain a copy of K_k, then $r(G) > r(K_k)$.

For $k = 3$ it is trivial to verify this stronger conjecture. If $G \not\supseteq K_3$ and $\chi(G) \geq 3$, then G has at least 4 vertices. Thus $r(G) > 6 = r(K_3)$, since neither the graph $K_3 \cup K_3$ or its complement will contain a copy of G.

For $k = 4$ some similar observations can be made. If $\chi(G) \geq 4$ and G has at component with at least 7 vertices, then $K_6 \cup K_6 \cup K_6$ does not contain G and the complement contains no 4-chromatic graph. Thus, $r(G) > 18 = r(K_4)$. The only 4-chromatic graph with 4, 5, or 6 vertices that does not contain a K_4 is the wheel $W_6 = K_1 + C_5$ with 6 vertices. Thus, the Erdős conjecture in the case $k = 4$ is equivalent to $r(W_6) \geq 18$ (strict inequality in the strong form of the conjecture). However, the following result can be verified with computational techniques described in section 3, so the Erdős conjecture is false for $k = 4$.

THEOREM 1 *$r(W_6) = 17$.*

There is an off-diagonal version of the Erdős conjecture: If $\chi(G)$, $\chi(H) \geq k$, then $r(G, H) \geq r(K_k, K_k)$. For $k = 3$ this off-diagonal version of the conjecture is easily verified. Of course, it is not true for $k = 4$, since it is not even true in the diagonal case. However, the next result, which can also be verified by a computer search, verifies that the only exception to the off-diagonal form of the conjecture for $k = 4$ comes from the pair (W_6, W_6).

THEOREM 2 *$r(K_4, W_6) = 19$.*

It is rather surprising that $r(K_4, K_4) = 18$, $r(W_6, W_6) = 17$, but $r(K_4, W_6) = 19$, since one would normally expect that the off-diagonal Ramsey number would not exceed the maximum of the corresponding diagonal Ramsey numbers.

The wheel $W_5 = K_1 + C_4$ with 4 spokes has chromatic number 3, thus it would not be surprising for $r(W_5) < r(K_4)$. In fact, the next result verifies that this is true.

THEOREM 3 *$r(W_5) = 15$.*

2 Ramsey Numbers for Wheels

For any integer $k \geq 4$, W_k will denote the wheel $K_1 + C_{k-1}$ with k vertices and $k - 1$ spokes. Thus, W_4 is isomorphic to K_4. Also, W_3 will denote the complete graph K_3. For graphs G and H the Ramsey number $r(G, H)$ is the smallest positive integer n such that for every graph F with at least n vertices, either $F \supseteq G$ or $\overline{F} \supseteq H$. If $H = G$, then $r(G, H)$ will simply be denoted by $r(G)$. Specific notation and terminology will be introduced as needed, but we will generally follow the notation of [2].

Using computational techniques described in section 3 the Ramsey numbers $r(W_i, W_j)$ for $3 \leq i, j \leq 6$ can be determined. They are listed in Table 1.

Those entries in Table 1 not marked with an asterisk are already in the literature. The classical Ramsey numbers $r(W_3) = 6$ and $r(W_3, W_4) = 9$ are well known. In [9] Greenwood

TABLE 1
$r(W_i, W_j)$

$_i{}^j$	3	4	5	6
3	6	9	11	11
4	9	18	17	19*
5	11	17	15	17*
6	11	19*	17*	17*

and Gleason proved $r(W_4) = 18$. More generally, Chvátal and Harary in [3] and [4] determined all Ramsey numbers of pairs of graphs with at most 4 vertices. Clancy in [6] calculated most of the Ramsey numbers for pairs of graphs with at most 4 and 5 vertices respectively. In particular, $r(W_3, W_5) = 11$ can be found there. The Ramsey table of Clancy was expanded by Hendry in [10] to include most pairs of graphs with at most 5 vertices, and this table includes $r(W_4, W_5) = 17$ and $r(W_5) = 15$. All Ramsey numbers for a triangle versus any graph with at most 6 vertices were determined in [8]; in particular, $r(W_3, W_6) = 11$.

Chvátal and Schwenk in [5] proved that $17 \leq r(W_6) \leq 20$, and the upper bound was lowered to 19 in [7]. The following graph H_1 on 16 vertices described in [5] gives the lower bound for $r(W_6)$. (We will, when possible, denote (i, j) by just ij.)

$$V(H_1) = \{0, 1\} \times \{0, 1, \cdots, 6, 7\}$$

$$E(H_1) = \{(ij, k\ell) : |j - \ell| = 0, 1, 4, 7\}.$$

The graph H_1 is transitive, so the neighborhoods of all of the vertices of H_1 are isomorphic. It is straightforward to verify that the neighborhoods of 00 in H_1 and $\overline{H}_1$ are the graphs in Figure 1. Neither of these neighborhoods of 00 contain a C_5, so clearly $r(W_6) > 16$. Also, the neighborhood in H_1 contains no C_4, so $r(W_5, W_6) > 16$.

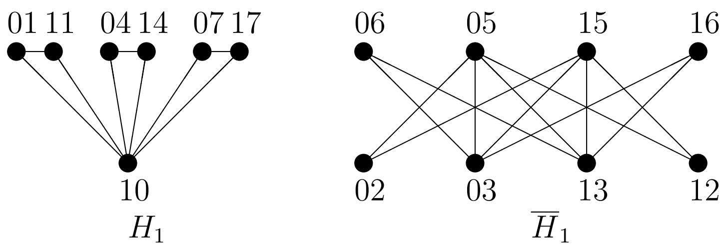

Figure 1: **Neighborhood of the vertex** 00

For a lower bound for $r(W_4, W_6)$, first consider the graph H_2' of order 16 defined as follows:

$$V(H_2') = \{0,1\} \times \{0,1,\cdots,6,7\}$$
$$E(H_2') = \{(ij, k\ell) : |j-\ell| = 1,2,6,7\}.$$

(This graph is sometimes expressed as $\overline{K}_2 \otimes C_8^2$.) Let H_2 be the graph of order 18 obtained from H_2' by adding adjacent vertices α and β with α adjacent to those pairs in H_2' that end in an even integer and β adjacent to those pairs that end in an odd integer. The graph H_2, which is pictured in Figure 2, is a 9-regular graph of order 18.

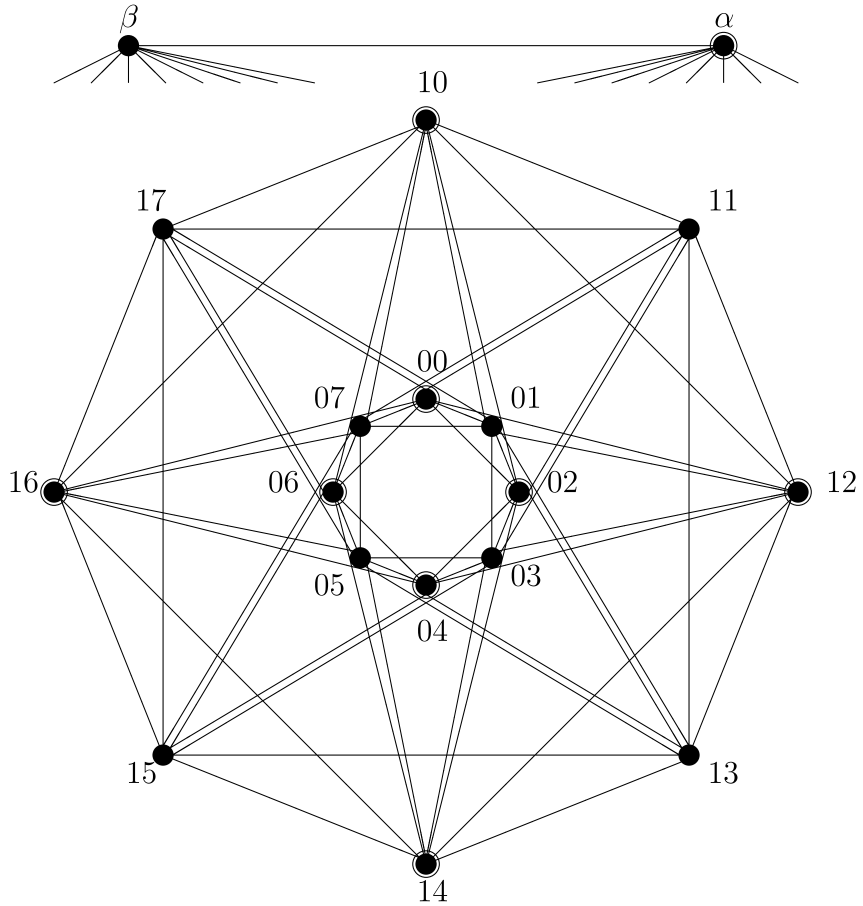

Figure 2: H_2

There are clearly two orbits of the automorphism group of H_2, with one class containing α and β, and with the other class containing the remaining 16 vertices. It is straightforward to check that the neighborhood of α (and likewise β) in H_2 is isomorphic to $K_{4,4} \cup K_1$. Also, the neighborhood of α in $\overline{H}_2$ is isomorphic to $K_4 \cup K_4$. The neighborhoods of 00 in H_2 and $\overline{H}_2$ are pictured in Figure 3. There is no C_3 in the neighborhood of any vertex in H_2, and no C_5 in the neighborhood of any vertex in $\overline{H}_2$. It follows that $r(W_4, W_6) > 18$.

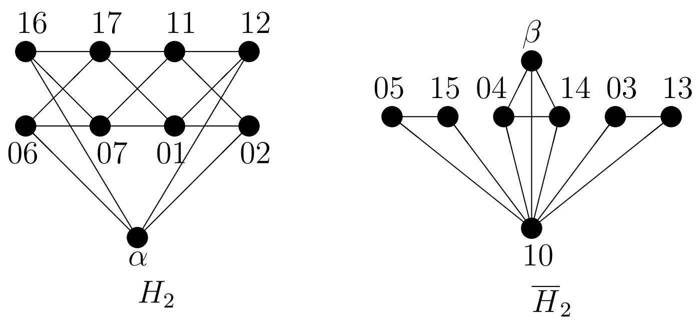

Figure 3: **Neighborhood of the vertex** 00

Given graphs G and H, a graph F is said to be (G, H)- free if $F \not\supseteq G$ and $\overline{F} \not\supseteq H$. Of course, the Ramsey number $r(G, H)$ is the smallest number such that there is no (G, H)-free graph with that number of vertices. The verification that $r(W_6, W_6) = 17$, $r(W_4, W_6) = 18$ and $r(W_5, W_6) = 17$ involves an exhaustive computer search to determine all nonisomorphic graphs that are (G, H)-free for the appropriate graphs G and H. The procedure is described in detail in the next section. These calculations yielded the numbers in the Tables 2, 3 and 4 for the pairs (W_6, W_6), (W_5, W_6), and (W_4, W_6) respectively.

Table 2 indicates that there are precisely two (W_6, W_6)-free graphs of order 16. These two graphs are the graphs H_1 and $\overline{H}_1$ described by Chvátal and Schwenk in [5]. Likewise, the graph H_1 is the unique (W_5, W_6)-free graph of order 16 indicated in Table 3. The two (W_4, W_6)-free graphs of order 18 indicated in Table 4 are the graph H_2 pictured in Figure 2, and the graph obtained from H_2 by deleting the edge between the vertices α and β of H_2.

It is surprising that the two maximum (W_6, W_6)-free graphs are so different in the diagonal Ramsey case $r(W_6)$. Usually, the extremal graphs in such cases are self-complementary or nearly so. Also, Table 2 shows the extremely large decrease in the number of (W_6, W_6)-free graphs as the order changes from 15 to 16, which is unusual. The more typical pattern is given in Table 3 and Table 4.

The known values in Table 1 were also verified using the same search techniques. As a consequence of these calculations, the number of (W_i, W_j)-free graphs were determined for

TABLE 2
Nonisomorphic
(W_6, W_6)-free graphs

# of vertices	# of graphs
1	1
2	2
3	4
4	11
5	34
6	140
7	762
8	5541
9	46148
10	371620
11	2155354
12	8472354
13	18466346
14	38024924
15	62287938
16	2
total	129831181

TABLE 3
Nonisomorphic
(W_5, W_6)-free graphs

# of vertices	# of graphs
1	1
2	2
3	4
4	11
5	31
6	122
7	581
8	3427
9	21014
10	105463
11	306169
12	448371
13	34242
14	3299
15	5
16	1
total	922743

TABLE 4
Nonisomorphic
(W_4, W_6)-free graphs

# of vertices	# of graphs
1	1
2	2
3	4
4	10
5	29
6	112
7	543
8	3546
9	28233
10	232337
11	1651381
12	8437954
13	27039916
14	43625194
15	34035296
16	43072
17	148
18	2
total	115097780

$i, j = 3$ or 4, and these values are given in Tables 5, 6 and 7. (Table 7 can be found in [12].)

The same computational techniques used to verify the entries in Table 1 can also be used to determine $r(W_3, W_k)$ for larger values of k. In particular, it can be shown that $r(W_3, W_7) = 13$, $r(W_3, W_8) = 15$, $r(W_3, W_9) = 17$, $r(W_3, W_{10}) = 19$, and $r(W_3, W_{11}) = 21$. ¿From this information it is reasonable to conjecture that $r(W_3, W_k) = 2k - 1$ for $k \geq 6$. This is consistent with the result in [1] that $r(K_3, G_k) = 2k - 1$ for any sparse graph G_k of order k (sparse, in this case, means at most $17k/15$ edges). Information on the number of (W_3, W_j)-free graphs for $3 \leq j \leq 11$ is given in Table 8.

3 Computational Procedures

This section describes the computational process by which we generated the set of all (W_i, W_j)-free graphs for various (i, j). Similar methods have been used before in the construction of cubic graphs [13] and several Ramsey number computations [12], [14].

TABLE 5
Nonisomorphic
(W_5, W_5)-free graphs

# of vertices	# of graphs
1	1
2	2
3	4
4	11
5	28
6	104
7	402
8	1876
9	7246
10	18162
11	18792
12	6028
13	533
14	62
total	53247

TABLE 6
Nonisomorphic
(W_4, W_5)-free graphs

# of vertices	# of graphs
1	1
2	2
3	4
4	10
5	26
6	94
7	401
8	2307
9	15452
10	104314
11	531892
12	1437877
13	865055
14	111153
15	2891
16	82
total	3071561

TABLE 7
Nonisomorphic
(W_4, W_4)-free graphs

# of vertices	# of graphs
1	1
2	2
3	4
4	9
5	24
6	84
7	362
8	2079
9	14701
10	103706
11	546356
12	1449166
13	1184231
14	130816
15	640
16	2
17	1
total	3432184

Let us denote by $WF(i, j; n)$ the set of all (W_i, W_j)-free graphs with n vertices. The result of applying the permutation α to the labels of any labelled object X will be denoted by X^α, and also $\mathrm{Aut}(G)$ is the automorphism group of the graph G, represented as a group of permutations of $V(G)$.

Suppose that θ is a function defined for any $G \in \bigcup_{n\geq 2} WF(i, j; n)$ which satisfies these properties:

(i) $\theta(G)$ is an orbit of $\mathrm{Aut}(G)$,

(ii) the vertices in $\theta(G)$ have maximum degree in G, and

(iii) for any G, and any permutation α of $V(G)$, $\theta(G^\alpha) = \theta(G)^\alpha$.

It is easy to implement a function satisfying the requirements for θ by using the program *nauty* [11]. Given θ, and $G \in WF(i, j; n)$ for some $n \geq 2$, the *parent* of G is the graph $\mathrm{par}(G)$ formed from G by removing the first vertex in $\theta(G)$ and its incident edges. The properties of θ imply that isomorphic graphs have isomorphic parents. It is also easily seen that $\mathrm{par}(G) \in WF(n{-}1)$. Since $WF(i, j; 1) = \{K_1\}$, we find that the relationship "par" defines a

TABLE 8
Nonisomorphic (W_3, W_j)-free graphs

j	$r(W_3, W_j)$	Total # $r(W_3, W_j)$-free graphs	# $r(W_3, W_j)$-free graphs of order $r(W_3, W_j)$ - 1
3	6	9	1
4	9	48	3
5	11	106	1
6	11	269	37
7	13	808	61
8	15	2862	92
9	17	13268	141
10	19	107355	201
11	21	1742876	288

rooted directed tree T whose vertices are the isomorphism classes of $\bigcup_{n\geq 1} WF(i,j;n)$, with the isomorphism class $\{K_1\}$ as the root. If ν is a node of T, then the *children* of ν are those nodes ν' of T such that for any $G \in \nu'$ we have $\mathrm{par}(G) \in \nu$. The set of children of ν can be found by the following algorithm, whose correctness follows easily from the definitions:

(a) Let G be any representative of the isomorphism class ν. Suppose that G has n vertices and maximum degree Δ.

(b) Let $L = L(G)$ be a list of all subsets X of $V(G)$ such that

- **(b.1)** either $|X| > D$, or $|X| = D$ and X does not include any vertex of degree D,
- **(b.2)** if $G(X)$ is the graph of order $n+1$ formed from G by appending a new vertex x adjacent to X, then $G(X)$ is (W_i, W_j)-free and $x \in \theta(G(X))$.

(c) Remove isomorphs from amongst the set $\{G(X) \mid X \in L\}$. The remaining graphs form a set of distinct representatives for the children of ν.

It is not hard to show that any isomorphs at step (c) are due to the equivalence of two sets X under $\mathrm{Aut}(G)$. In the computational process, we have the choice of using generators for $\mathrm{Aut}(G)$ to remove such equivalent sets, or using canonical labelling to detect isomorphs amongst the graphs $G(X)$. Both methods are available using the facilities of *nauty*; we chose the second for computational convenience even though the first should be slightly faster.

This approach has a number of important advantages over other methods. In particular, isomorph rejection need be performed only within very small families of graphs. For example, even though $|WF(4,4,14)| = 43625194$, no isomorphism class of $WF(4,4,13)$ has more than 49 children. Secondly, the guaranteed non-isomorphism of the outputs permits them to be processed as they are generated, without the need to store them. Thirdly, it is very easy to split the generation task into completely independent pieces which can be run separately.

The execution time for these computations was typically about 50 milliseconds per output graph on a Sun workstation rated at 12 mips. The longest computation was for (W_4, W_6)-free graphs, which took about 2.5 mip-years in total.

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