Classification of regular two-graphs on 36 and 38 vertices

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Abstract

In a previous paper an incomplete investigation into regular two-graphs on 36 vertices established the existence of at least 227. Using a more efficient algorithm, the two authors have independently verified that in fact these 227 comprise the complete set. An immediate consequence of this is that all strongly regular graphs with parameters $(35, 16, 6, 8)$, $(36, 14, 4, 6)$, $(36, 20, 10, 12)$ and their complements are now known.

Similar techniques were attempted in the case of regular two-graphs on 38 vertices, but without success on account of the vast amount of computer time required. Instead a different approach was used which managed to increase the known number of such regular two-graphs from 11 to 191.

Key words and phrases: regular two-graph, classification, strongly regular graph

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1 Introduction

In their paper [3] Bussemaker, Mathon and Seidel made the first attempt to classify regular two-graphs on at most 50 vertices. The results they obtained are summarised in the following table, where $N(n)$ denotes the number of regular two-graphs on n vertices and a bar indicates that the number is complete.

n	6	10	14	16	18	26	28	30	36	38	42	46	50
$N(n)$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{4}$	$\overline{1}$	6	91	11	18	80	18

Some of their results were extended in [8], where a complete classification of the regular two-graphs on 30 vertices was announced together with an improvement in the estimate of the number of (non-isomorphic) regular two-graphs on 36 vertices, from 91 to 227. However, it proved impossible to complete the computer search for these regular two-graphs, mainly on account of the inefficiency of the algorithm used. By using a new and vastly improved approach the present authors have independently established that the 227 regular two-graphs on 36 vertices found in [8] constitute the complete set. Moreover, some aspects of this new method were applied to regular two-graphs on 38 vertices and this gave rise to a new lower bound for their number, namely 191. The method is fully explained in Section 3.

2 Preliminaries

We begin by defining some of our terminology.

Definition 1 *An* (n, k) graph *is a regular graph of order* n *and degree* k. *An* (n, k, λ, μ) strongly regular graph*, sometimes abbreviated to* $\mathrm{SRG}(n, k, \lambda, \mu)$*, is an* (n, k) *graph such that every pair of adjacent vertices have exactly* λ *common neighbours and every pair of non-adjacent vertices have exactly* μ *common neighbours.*

The trivial cases $k = 0$ and $k = n-1$ are commonly excluded from the class of strongly regular graphs. It is easily seen that the complement of an $\mathrm{SRG}(n, k, \lambda, \mu)$ is an $\mathrm{SRG}(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$.

We also give a brief informal description of a regular two-graph; for a more formal definition the reader is referred to [9]. For our purposes, it is sufficient to consider a regular two-graph as a *switching class* of graphs.

Definition 2 *Let* A *and* B *be the* ∓ 1 *adjacency matrices of two graphs* Γ_1 *and* Γ_2 *on the same vertex set* Ω *(*-1 *for adjacency and* $+1$ *for non-adjacency). Then* Γ_1 *and* Γ_2 *are said to be switching equivalent if there exists a diagonal matrix* D *with entries* ± 1*, such that* $DAD = B$.

Clearly, switching equivalence is an equivalence relation on the set of all graphs on n vertices. A *two-graph* is an equivalence class of graphs under the relation of switching (a *switching class*). It is also clear that all graphs in the switching class of a two-graph have the same Seidel spectrum (the eigenvalues of the ∓ 1 adjacency matrix), so we can define the spectrum of a two-graph to be the Seidel spectrum of any one of the graphs in its switching class. A two-graph is said to be *regular* if its spectrum contains just two distinct eigenvalues.

From this point on we only consider regular two-graphs on n vertices with eigenvalues ρ_1 and ρ_2, where $\rho_1 > \rho_2$. The only possibilities for ρ_1 and ρ_2 are that either they are both odd integers, or $\rho_1 = -\rho_2 = \sqrt{n-1}$ and $n-1$ is the sum of squares of integers.

Corresponding to each vertex in the vertex set of a regular two-graph there is a graph in the switching class that has that vertex as an isolated vertex. Deleting the chosen (isolated) vertex gives rise to a graph on $n-1$ vertices whose ∓ 1 adjacency matrix, B say, satisfies

$$(B - \rho_1 I)(B - \rho_2 I) = -J \text{ and } B\boldsymbol{j} = (\rho_1 + \rho_2)\boldsymbol{j}.$$

Here J and $\boldsymbol{j}$ denote, as usual, the all-ones matrix and the all-ones vector, respectively. Thus the graph is regular and its $(0,1)$ adjacency matrix has three distinct eigenvalues, and hence the graph is in fact strongly regular. In terms of ρ_1 and ρ_2 the three eigenvalues are

$$(n-2-\rho_1-\rho_2)/2 \text{ (the degree)}, \quad (-1-\rho_1)/2, \quad (-1-\rho_2)/2.$$

This strongly regular graph is called a *descendant* of the regular two-graph with respect to the chosen vertex. Conversely, a strongly regular graph of degree k and other eigenvalues r and s that take the above form for suitable ρ_1 and ρ_2 gives rise to a regular two-graph with eigenvalues ρ_1 and ρ_2 by adjoining an isolated vertex. Thus a regular two-graph is determined by any one of its descendants. For fuller details the reader is referred to [1].

It is possible, when ρ_1 and ρ_2 are odd integers, that the switching class of a regular two-graph contains *regular* graphs. In that case, it is not difficult to see that these regular graphs must in fact be strongly regular.

One result that is particularly useful in the computer search for the regular two-graphs is the following. It will be used in the next section.

Theorem 1 [4, Theorem 1.2.2] *Suppose that*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$$

is a hermitian matrix. Then the eigenvalues of A_{11} interlace the eigenvalues of A.

What this means is that if A has size n, A_{11} has size m, A has eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and A_{11} has eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$, then for $i = 1, 2, \ldots, m$

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}.$$

The two cases of interest in this article are when $n = 36$ and $n = 38$. We deal with each of these cases in turn.

3 Regular Two-graphs on 36 vertices

Here, by complementation if necessary, we may assume that $\rho_1 = 7$ and $\rho_2 = -5$. Then the parameters of any descendant are $(35, 16, 6, 8)$, which gives eigenvalues $16^1, 2^{20}, -4^{14}$ (exponents denote multiplicities). Also, in this case, if the switching class of the regular two-graph contains a regular graph it must be either an SRG$(36, 14, 4, 6)$ or an SRG$(36, 20, 10, 12)$. All such SRG's arise in this fashion and it is a relatively simple matter to determine all them from the regular two-graphs. Of course, the converse is not true. There may be (indeed there are) regular two-graphs that have none of these regular graphs in their switching classes.

As mentioned earlier, a regular two-graph is determined by any one of its descendants. Thus we concentrate on finding the $(35, 16, 6, 8)$ strongly regular graphs. Of course, it is not necessary to obtain all of these to determine the two-graphs, but it turns out to be just as easy to search for them all.

Let A be the adjacency matrix of a $(35, 16, 6, 8)$ strongly regular graph. Then $2J - 5(A - 2I)$ is positive semi-definite and has eigenvalues 0^{21} and 30^{14}, so that its rank is 14. Now we may assume that A takes the form

$$\begin{bmatrix} 0 & \boldsymbol{j}^t & \mathbf{0} \\ \boldsymbol{j} & C & N \\ \mathbf{0} & N^t & D \end{bmatrix}, \tag{1}$$

where $\boldsymbol{j}$ is the all-ones vector of size 16 and C is the adjacency matrix of the neighbour graph of a vertex. This neighbour graph has 16 vertices and is regular of degree 6. Moreover, by Theorem 1 the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{16}$ of $2J - 5(C - 2I)$ interlace those of $2J - 5(A - 2I)$. Thus

$$30 \geq \lambda_1 \geq \ldots \geq \lambda_{15} = \lambda_{16} = 0. \tag{2}$$

It is also true that the eigenvalues of C interlace those of A so that C has second eigenvalue 2 and all its eigenvalues bounded below by -4.

A first step in the computer search for SRG's $(35, 16, 6, 8)$ was to determine all matrices C satisfying the above conditions. This was done independently by the two authors using different methods. In one a combination of the conditions that $2J-5(C-2I)$ has rank at most 14 and eigenvalues that satisfy condition (2) was used, while in the other the condition that was used was the interlacing of the eigenvalues of any subgraph of C with those of A. There was also a difference in the way that the subgraphs of C were generated. We give a description of one of these methods.

The paper [7] gives a general algorithm for generating graphs without isomorphs. It is easily adjusted to generate only $(16, 6)$ graphs, but there are far too many of them to allow generation and testing of the full set. Instead, we took advantage of the fact that the generation procedure operates on induced subgraphs and adds one point at a time. For each intermediate graph an eigenvalue test was applied to determine if it was feasible as an induced subgraph of a $(35, 16, 6, 8)$ neighbourhood. If not, backtracking took place. To speed up the testing for each subgraph, eigenvectors as well as eigenvalues were obtained and these were used in the one-vertex extensions to find bounds on the smallest and largest non-Perron eigenvalues (which must lie between 2 and -4). These estimates were improved by an iteration process. In all, approximately 5×10^9 subgraphs were tested and of these all but 3.6% were rejected. The precise number of feasible neighbour graphs was 137,668. However, it was possible to reduce this number still further as we shall see.

Suppose that C is the adjacency matrix of a feasible neighbour graph; then for C to be embedded in the matrix (1) it is necessary that, for each column $\boldsymbol{x}$ of N, the matrix

$$C' = \begin{bmatrix} 0 & \boldsymbol{j}^t & 0 \\ \boldsymbol{j} & C & \boldsymbol{x} \\ 0 & \boldsymbol{x}^t & 0 \end{bmatrix}$$

is also feasible. By this we mean that $\boldsymbol{x}$ should have weight 8 and the eigenvalues of C' interlace those of A. Call such a vector $\boldsymbol{x}$ *permissible*.

The set of permissible vectors $\boldsymbol{x}$ must satisfy some simple conditions. For example, for each vertex v of C, there must be at least 9 vectors $\boldsymbol{x}$ which include v. In fact, there must be a set of 9 vectors such that the required number of common neighbours of v and each other vertex in C is achieved. A similar requirement holds for each pair of vertices in C. Often these constraints can only be met if certain permissible vectors are avoided, or certain vectors are used, which strengthens the requirements related to other vertices those vectors contain. This gives a converging iterative process of increasingly stronger constraints on C.

As it turned out, all but 12,699 of the graphs in our collection of 137,668 failed to meet these requirements. There remained the problem of attempting to embed these in a $(35, 16, 6, 8)$ graph. This was done most efficiently using the following considerations.

For each feasible C, the corresponding permissible columns $\boldsymbol{x}$ were generated in lexicographical order. The question is: how can these be used in an efficient way? Applying a back-tracking procedure on them turned out to be extremely slow. The way forward was to generate the rows $1, 2, \ldots, 16$ of N in (1), entry by entry, in such a way that

(a) at entry (r, s), column s thus far completed was a sub-column of one of the permissible $\boldsymbol{x}'$s,

(b) the columns were in lexicographical order,

(c) the inner product of row i with row j was 6 or 8 according as $A_{i+1,j+1} = 1$ or 0.

This turned out to be very easy and efficient to implement, since the partial columns of N could be regarded as binary integers and tests could be performed with few machine instructions.

The next stage, when a possible N had been found, proceeded very quickly. Generally speaking, it was possible to determine whether or not the partially completed matrix could be extended to the full adjacency matrix in a matter of a second or two. It turned out that of the 12,699 feasible neighbour graphs, 7,440 could actually be embedded in an SRG $(35, 16, 6, 8)$ and these gave rise to 3,854 non-isomorphic SRG's. This was precisely the number obtained in the previous incomplete search of [8], so the number of non-isomorphic regular two-graphs on 36 vertices is also as obtained there, namely 227. Since, as mentioned earlier, classification of the two-graphs leads to the classification of the strongly regular graphs in the switching classes, the figures quoted in [8] as lower bounds are in fact upper bounds also. Thus there are precisely 32,548 non-isomorphic $(36, 20, 10, 12)$ graphs and 180 non-isomorphic $(36, 14, 4, 6)$ graphs (in the complement: $(36, 15, 6, 6)$ and $(36, 21, 12, 12)$, respectively). As an additional check, the $(36, 14, 4, 6)$ graphs were also generated directly.

Remark In their paper [2] Bussemaker et al. proved the non-existence of an SRG $(49, 16, 3, 6)$ by considering possible neighbour graphs on 16 vertices and of degree 3. In this case there are precisely 4207 such graphs, all but 13 of which could be ruled out using properties of the putative strongly regular graph. However, to eliminate these 13 graphs as candidates for a possible neighbour graph, the authors used the condition that a certain matrix associated with each of them was in fact non-singular. This was based on the fact that none of the 13 graphs had an eigenvalue -2. Had this not been the case, their method would have failed. The procedure that that we have outlined is more general and could have been used had the 13 graphs not had the required property.

4 Regular two-graphs on 38 vertices

In this case the eigenvalues are $\rho_1 = -\rho_2 = \sqrt{37}$ and the parameters of the descendants are $(37, 18, 8, 9)$. Whereas in the previous section we had to determine a subset of the regular graphs on 16 vertices of degree 6, here, if the same method were adopted, we would have to consider regular graphs on 18 vertices of degree 8. To get some idea of the scale of the problem, the following procedure was used. From the 11 regular two-graphs found in [3] generate the descendants and the complements of these descendants (also SRG's$(37, 18, 8, 9)$) to get a set G_0 of SRG's. From G_0

generate all non-isomorphic neighbour graphs N_0. Now at level $k \geq 0$ pursue this scheme:

1. Embed the neighbour graphs of N_k in the full SRG using the methods outlined in the previous section; delete those that lie in $\bigcup_{i=0}^{k-1} G_i$;
2. enlarge the resulting set of SRG's by first determining the regular two-graphs from which they come and then extract the descendants and their complements; again delete those SRG's that lie in $\bigcup_{i=0}^{k-1} G_i$ and call the resulting set G_k;
3. extract from G_k all non-isomorphic neighbour graphs that do not lie in N_k to get N_{k+1};

until N_{k+1} is empty.

When this was done it was discovered that $k = 9$ and that $\bigcup_{i=0}^{9} G_i$ contained 6,760 non-isomorphic $(37, 18, 8, 9)$ graphs that arose as descendants of 191 regular two-graphs of which 63 are self-dual and the remaining 128 comprise 64 dual pairs. In the table below, the numbers of graphs, neighbour graphs and two-graphs (TG_k) are given at each level $k = 0, 1, \ldots, 9$.

k	0	1	2	3	4	5	6	7	8	9
$\lvert G_k \rvert$	82	152	380	684	1648	2118	850	446	324	76
$\lvert N_k \rvert$	1229	2636	6941	12409	29494	36964	15487	8139	5920	1402
$\lvert TG_k \rvert$	11	4	10	18	44	57	23	13	9	2

Are there any more to be found? The following information may persuade the reader that this is so, but if there are it may require a different approach to locate them. In total these 6,760 SRG's produced 120,621 non-isomorphic neighbour graphs and when an attempt was made to determine feasible $(18, 8)$ graphs as possible neighbour graphs, an incomplete search found over 1,000,000, none of which was among the 120,621.

Mathon [6] investigated self-complementary strongly regular graphs, among which were precisely two with parameters $(37, 18, 8, 9)$. Since these were already included in the initial list G_0 referred to above, they are, a fortiori, contained in our list of $6,760$.

Most of the strongly regular graphs reported in this paper are available either from `http://gauss.maths.gla.ac.uk/~ted` or `http://cs.anu.edu.au/~bdm`.

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