Generation of simple quadrangulations of the sphere

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Abstract

A *simple quadrangulation of the sphere* is a finite simple graph embedded on the sphere such that every face is bounded by a walk of 4 edges. We consider the following classes of simple quadrangulations: arbitrary, minimum degree 3, 3-connected, and 3-connected without non-facial 4-cycles. In each case we show how the class can be generated by starting with some basic graphs in the class and applying a sequence of local modifications. The duals of our algorithms generate classes of quartic (4-regular) planar graphs.

In the case of minimum degree 3, our result is a strengthening of a theorem of Nakamoto and almost implicit in Nakamoto's proof. In the case of 3-connectivity, a corollary of our theorem matches a theorem of Batagelj. However, Batagelj's proof contained a serious error which cannot easily be corrected.

We also give a theoretical enumeration of rooted planar quadrangulations of minimum degree 3, and some counts obtained by a program of Brinkmann and McKay that implements our algorithm.

Keywords: quadrangulation, planar, graph, map, quartic

1 Introduction

A *quadrangulation of the sphere* is a finite graph embedded on the sphere such that every face is bounded by a walk of 4 edges. We will be primarily concerned with *simple quadrangulations*, which are those which do not have multiple edges. We will also follow some (but not all) of the literature by not counting the path of three vertices as a simple quadrangulation. With this proviso, the boundary of each face of a simple quadrangulation corresponds to a 4-cycle of the graph.

Two quadrangulations are regarded as the same if there is an embedding-preserving isomorphism (possibly reflectional) between them. That is, we are not concerned with abstract graph isomorphisms.

It is a standard fact that a simple plane graph is 2-connected if and only if every face is bounded by a cycle. The connectivity of a simple quadrangulation cannot be more than 3, since Euler's formula implies that the average degree of the vertices is less than 4. Therefore a simple quadrangulation of the sphere has connectivity either 2 or 3.

We are concerned with generating all the simple quadrangulations in some given class by beginning with a "starting set" of basic quadrangulations in the class and recursively applying some "expansions" to them.

The starting sets we will use consist of the square or the pseudo-double wheels. A *square* is just a 4-cycle. A *pseudo-double wheel* is a quadrangulation with $n \geq 8$ vertices, n even, consisting of a cycle $(v_0, v_1, \ldots, v_{n-3})$, a vertex adjacent to $v_0, v_2, \ldots, v_{n-4}$, and a vertex adjacent to $v_1, v_3, \ldots, v_{n-3}$. All of these have a unique embedding on the sphere. The smallest pseudo-double wheel is also called the *cube*. Figure 1 shows the square and the two smallest pseudo-double wheels.

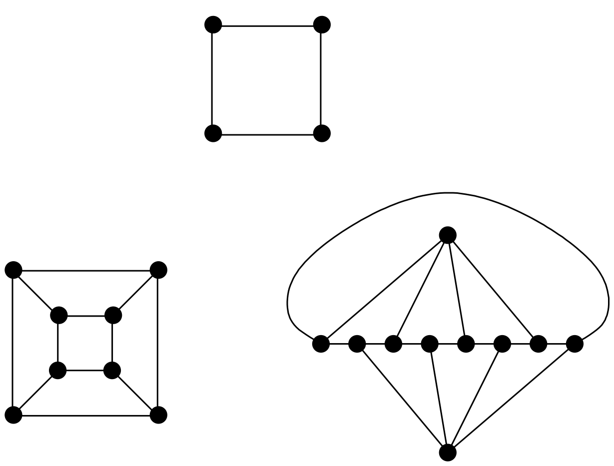

Figure 1: The square and the two smallest pseudo-double wheels

The pictures in this paper should be interpreted using the following rules. The

pictures show actual embeddings, so the cyclic ordering of edges around each vertex is important. Moreover:
(i) each displayed vertex is distinct from the others;
(ii) edges that are completely drawn must occur in the cyclic order given in the picture;
(iii) half-edges indicate that at an edge *must* occur at this position in the cyclic order around the vertex;
(iv) a triangle indicates that one or more edges *may* occur at this position in the cyclic order around the vertex (but they need not);
(v) if neither a half-edge nor a triangle is present in the angle between two edges in the picture, then these two edges must follow each other directly in the cyclic ordering of edges around that vertex.

By convention, throughout this paper, we write the vertices bounding a face in clockwise order.

Expansions will be defined using an operation called *face contraction.* Let Q be a simple quadrangulation with a face $F = (x, u, v, w)$. The *contraction* of F at the vertices $\{x, v\}$ produces a quadrangulation Q' obtained from Q by identifying the vertices x and v to form a new vertex x', identifying the edges $\{x, u\}$ and $\{v, u\}$ to form a new edge $\{x', u\}$, and identifying the edges $\{x, w\}$ and $\{v, w\}$ to form a new edge $\{x', w\}$. The faces of Q' are the faces of Q other than F, with the vertex and edge identifications as specified. A face contraction is illustrated in Figure 2. In the case that x and v lie on only one common face, we might omit mention of the face.

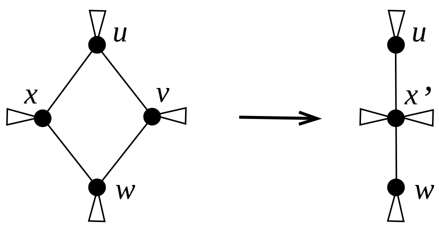

Figure 2: A face contraction

For $0 \le i \le 4$, a P_i*-expansion* is an operation that takes a quadrangulation to a larger quadrangulation. We will define each expansion as the inverse of the corresponding reduction, which consist of one or more restricted face contractions, as follows. In all cases, the named vertices must be distinct.

- Suppose there are distinct faces (u, v, w, x) and (u, y, w, v), so that v has degree 2. A P_0*-reduction* consists of a face contraction at $\{x, v\}$.
- A P_1*-reduction* consists of a contraction of a face (x, u, v, w) at $\{x, v\}$, where x has degree 3 and u, v and w each have degree at least 3.

- A P_2*-reduction* is defined as a sequence of two face contractions. We have faces labelled clockwise as follows: (p, t, y, u), (t, z, x, y), (u, y, x, w), (z, v, w, x), (t, q, v, z). Here all labelled vertices except t and possibly w have degree exactly 3, while t has degree at least 4 and w has degree at least 3. A P_2-reduction consists of a face contraction at $\{x, v\}$ followed by a face contraction at $\{z, q\}$.

- A P_3*-reduction* is defined as a sequence of four contractions. We have faces (u, v, w, x), (a, b, v, u), (b, c, w, v), (c, d, x, w) and (d, a, u, x). Here u, v, w and x all have degree 3, and we assume that a, b, c, d all have degree at least 4. A P_3-reduction consists of a face contraction at $\{a, v\}$, followed by one at $\{b, w\}$, followed by one at $\{c, x\}$, followed by one at $\{d, u\}$. (Thus, a P_3-reduction is essentially just the removal of the vertices u, v, w, x and their incident edges.)

- A P_4*-reduction* consists of a contraction of the face (x, u, v, w) at $\{x, v\}$, where each of x, u, v, w has degree at least 2.

Figure 3 shows the five expansions in pictorial form.

We can now define the generation concept formally. Given a class $\mathcal{Q}$ of simple quadrangulations, and a set of expansions $\mathcal{P} \subseteq \{P_0, P_1, P_2, P_3, P_4\}$, the relation $R(\mathcal{Q}, \mathcal{P})$ is defined by

$$R(\mathcal{Q}, \mathcal{P}) = \{(G, G') \in \mathcal{Q} \times \mathcal{Q} \mid G' \text{ can be obtained from } G \text{ by applying some } P \in \mathcal{P}\}.$$

We say that $\mathcal{Q}$ *is generated from* $S \subseteq \mathcal{Q}$ *by* $\mathcal{P}$ if the closure of S under $R(\mathcal{Q}, \mathcal{P})$ is $\mathcal{Q}$. An equivalent definition is that, for any $G \in \mathcal{Q}$, there is a sequence $G_0, G_1, \ldots, G_k = G$ of quadrangulations in $\mathcal{Q}$ such that $G_0 \in S$ and, for each i, G_{i+1} can be obtained from G_i by applying some $P \in \mathcal{P}$. It is important to note that each G_i must lie in $\mathcal{Q}$; we do not allow the type of generation that can step outside the class and then step back in again.

If $\mathcal{Q}$ is a class of simple quadrangulations and $0 \le i \le 4$, we say that $G' \in \mathcal{Q}$ is P_i-*reducible in* $\mathcal{Q}$ if there is $G \in \mathcal{Q}$ such that G' can be obtained by applying a P_i-reduction to G. When we wish to emphasise that both G and G' lie in $\mathcal{Q}$, we will call the P_i-reduction in question a $P_i(\mathcal{Q})$-reduction, and the reverse operation a $P_i(\mathcal{Q})$-expansion. However, remember that in fact no other expansions are permitted in generating $\mathcal{Q}$ according to our definitions.

A *separating 4-cycle* is a 4-cycle which is not the boundary of a face. We can now state our main theorems, which will be proved in Sections 3 and 4. (Note that a 3-connected quadrangulation is necessarily simple.)

Theorem 1 *The class* $\mathcal{Q}_1$ *of all simple quadrangulations of the sphere is generated from the square by the* $P_0(\mathcal{Q}_1)$*-expansions and* $P_1(\mathcal{Q}_1)$*-expansions.*

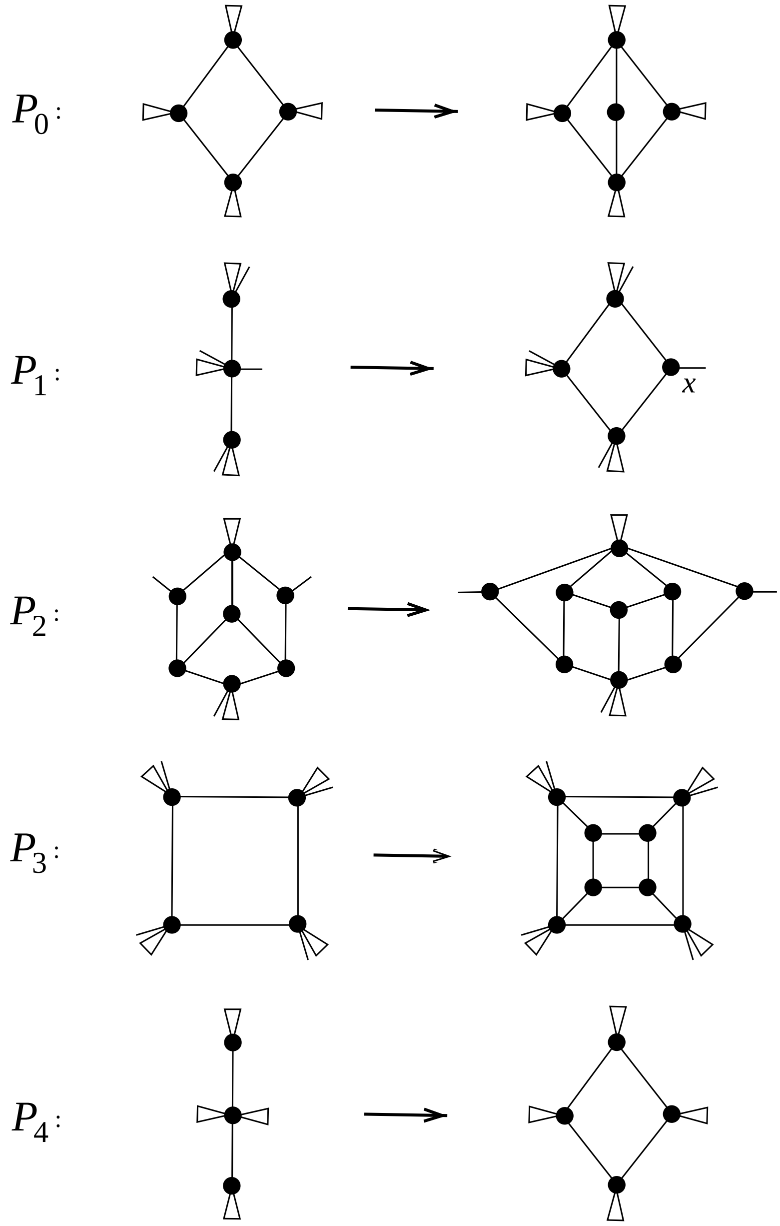

Figure 3: Expansion operations

Theorem 2 *The class $\mathcal{Q}_2$ of all simple quadrangulations of the sphere with minimum degree 3 is generated from the pseudo-double wheels by the $P_1(\mathcal{Q}_2)$-expansions and $P_3(\mathcal{Q}_2)$-expansions.*

Theorem 3 *The class $\mathcal{Q}_3$ of all 3-connected quadrangulations of the sphere is generated from the pseudo-double wheels by the $P_1(\mathcal{Q}_3)$-expansions and $P_3(\mathcal{Q}_3)$-expansions.*

Theorem 4 *The class $\mathcal{Q}_4$ of all 3-connected quadrangulations of the sphere without separating 4-cycles is generated from the pseudo-double wheels by the $P_1(\mathcal{Q}_4)$-expansions.*

Recall that the definition of $R(\mathcal{Q}, \mathcal{P})$ insists that an expansion in $\mathcal{P}$ may only be applied to a quadrangulation in $\mathcal{Q}$ if the result is also in $\mathcal{Q}$. It is not difficult to check that this is true in Theorems 1, 2 and 4; that is, applying expansion P_0 or P_1 to a simple quadrangulation produces a simple quadrangulation, and so on. For Theorem 3, any application of P_3 to a 3-connected quadrangulation produces a 3-connected quadrangulation. However, the expansion P_1 may only be applied to a 3-connected quadrangulation when the 3 vertices involved in the expansion (that is, those depicted in the picture on the left of the second line of Figure 3) do *not* form a 3-cut.

By taking the planar dual, Theorems 1–4 also imply generation methods for various classes of 4-regular planar graphs. The duals of the pseudo-double wheels are just the antiprisms. For convenience, we list here the exact definition of the dual class for each of Theorems 1–4.
(1) The dual of $\mathcal{Q}_1$ is the class of 4-regular, 4-edge-connected planar graphs (which are not necessarily simple).
(2) The dual of $\mathcal{Q}_2$ is the class of simple, 4-regular, 4-edge-connected planar graphs.
(3) The dual of $\mathcal{Q}_3$ is the class of simple, 4-regular, 3-connected planar graphs.
(4) The dual of $\mathcal{Q}_4$ is the class of 4-regular, 3-connected, 6-cyclically-edge-connected planar graphs (which are necessarily simple).

In establishing some of the above, it is useful to realise that planar 4-regular graphs cannot have cuts consisting of an odd number of edges. This corresponds to the fact that quadrangulations are bipartite.

Weaker versions of each of Theorems 1–4 have appeared in the literature. However most of them use P_4 instead of the more restricted operation P_1. The use of P_1 represents a substantial improvement, since for any given quadrilateral there are only linearly many places to apply a P_1-expansion, but possibly a quadratic number of places to apply a P_4-expansion. This has considerable benefit for the efficiency of generation algorithms. It might also help in the construction of inductive proofs, though we don't have an example.

Nakamoto [12] proved Theorem 2 using P_4 instead of P_1. Nakamoto also considered surfaces other than the sphere.

Negami and Nakamoto [13] showed that all simple quadrangulations are generated from the square using P_4; Batagelj [1] also gave this result.

For 3-connected quadrangulations, Broersma [4] showed that the cube, together with P_3, P_4 and a less restricted version of P_2, are enough. Batagelj [1] had in fact earlier claimed a stronger result, namely that the class of 3-connected quadrangulations was generated from the cube by the operations P_1, P_2 and P_3. (Batagelj defines P_1 in a slightly more restricted form than we do, but the same restriction is implied by our requirement that operations are only applied if the resulting graph remains in the desired class—in this case the class of 3-connected quadrangulations.) Note that Batagelj's claim is true: it follows from Theorem 3 because the set of pseudo-double wheels is generated from the cube by P_2. However, Batagelj's proof is incorrect (as he acknowledges). This is shown by Figure 4, which (contrary to one of Batagelj's key steps) is not P_1-reducible in $\mathcal{Q}_3$ even though it has a vertex of degree 3 not adjacent to any other.

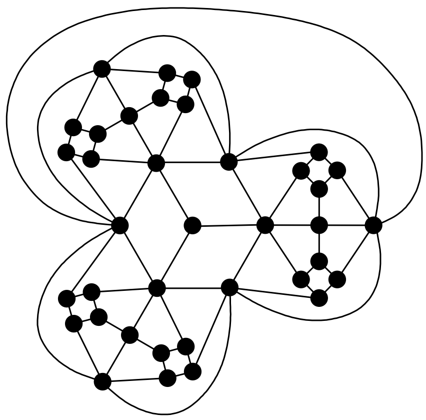

Figure 4: Example showing that Batagelj's proof is incomplete

The 3-connected quadrangulations without separating 4-cycles are closely related to the 3-connected planar graphs, as we shall show in Section 5. Via this relationship, Tutte's method [15] for generating all 3-connected planar graphs is equivalent to generating the 3-connected quadrangulations without separating 4-cycles from the pseudo-double wheels using P_4. Theorem 3 thus implies a strengthening of Tutte's theorem, which we state as Corollary 1.

A method for generating quadrangulations which are not necessarily simple, but whose planar dual is simple, was claimed by Manca [9]. However, Lehel [8] showed a counterexample to Manca's theorem and corrected it by adding an additional expansion (making 5 expansions altogether).

2 Preliminary observations

Here we will list some elementary properties of simple quadrangulations of the sphere.

Lemma 1 *The following are true of all simple quadrangulations.*
(i) The minimum degree is 2 or 3. In particular, if the minimum degree is 3, then there are at least 8 vertices of degree 3.
(ii) The graph is bipartite.

Proof. The first claim in part (i) follows from Euler's formula, while the second claim in part (i) can be found in [1]. Part (ii) is well-known. □

Lemma 2 *Let G be a simple quadrangulation with minimum degree 3 and let H be a component of the subgraph induced by the vertices of degree 3. Then H is one of the following graphs:*
(i) a cycle of even length at least 8, in which case G is a pseudo-double wheel;
(ii) a path (possibly of a single vertex);
(iii) a cube, in which case $G = H$;
(iv) one of the four graphs of Figure 5.

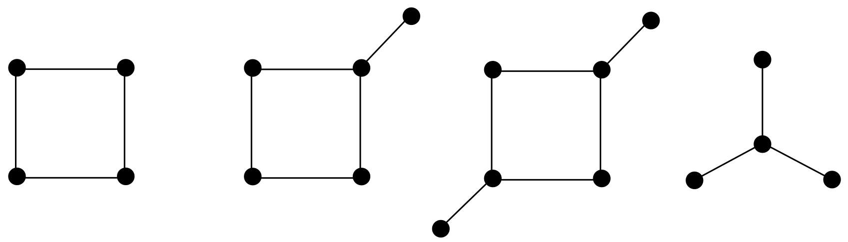

Figure 5: Graphs for Lemma 2 (iv)

Proof. See Nakamoto [12, Lemma 2]. □

If G is a simple quadrangulation embedded on the sphere and C is a 4-cycle in G, then excising C from the sphere leaves two open disks D_1 and D_2. If both D_1 and D_2 contain vertices of G, then C is a *separating 4-cycle*. (Equivalently, C is a separating 4-cycle if it is not the boundary of any face.) The subgraphs of G induced by the closures of D_1 and D_2 (which include C itself in both cases) are $\bar{C}_1$ and $\bar{C}_2$. Both of these subgraphs are themselves simple quadrangulations. If either $\bar{C}_1$ or $\bar{C}_2$ has no separating 4-cycles, denote that quadrangulation by $\bar{C}$ (chosen arbitrarily if there are two choices) and call C a *minimal separating 4-cycle*.

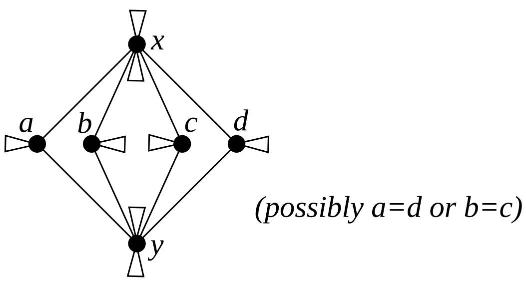

Figure 6: Configuration for Lemma 3

Lemma 3 *If G is a simple quadrangulation with at least 6 vertices and no separating 4-cycle, then G is 3-connected.*

Proof. If $\{x, y\}$ is a 2-cut of G, there are two distinct faces (x, b, y, a) and (x, d, y, c) which have x and y on their boundaries. (A 2-cut of adjacent vertices is impossible in a simple quadrangulation.) However, it might be that $a = d$ or $b = c$ (see Figure 6). If neither $a = d$ nor $b = c$, then (x, c, y, a) is a separating 4-cycle, while if both equalities hold then G has only 4 vertices. So suppose that $b = c$ but $a \neq d$. Then (x, d, y, a) is not a face since G has at least 6 vertices, so (x, d, y, a) is a separating 4-cycle. Similarly we also can find a separating 4-cycle if $b \neq c$ but $a = d$. □

Lemma 4 *Let G be a 3-connected quadrangulation which is not a pseudo-double wheel and has no separating 4-cycles. Let F be a face of G. Then G has a vertex x of degree 3 not on F such that x is adjacent to at most one other vertex of degree 3.*

Proof. Let H be a component of the subgraph of G induced by the vertices of degree 3. By Lemma 1(i), we can choose H to include at least one vertex not on F.

If H includes a 4-cycle (not necessarily the whole of H), either that 4-cycle or the surrounding 4-cycle is a separating 4-cycle. (If both 4-cycles are faces then G is a cube, which is a pseudo-double wheel.) Consequently, by Lemma 2, either the required vertex x exists or H is a path whose endpoints lie on F.

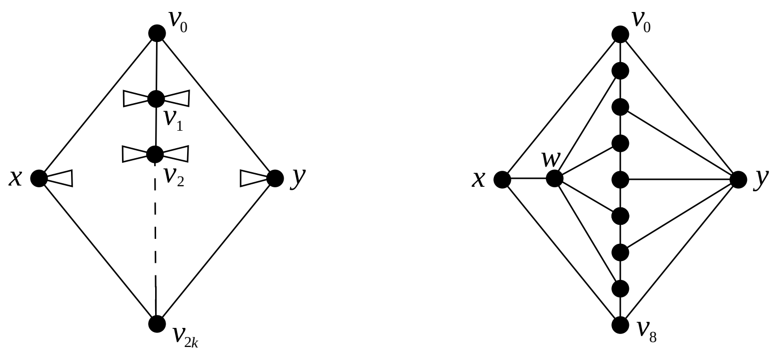

Figure 7: Configurations for Lemma 4

Since H is a path and G is bipartite, H has even length and connects two opposite vertices of F. Let x and y be the other vertices of F. Drawing G with F on the outside, the situation is represented on the left of Figure 7, where $V(H) = \{v_0, v_1, \ldots, v_{2k}\}$. Since G is a quadrangulation and H is a path, the extra neighbour of v_1 must be a vertex w, which we can assume to be in the interior of the cycle formed by H and x. We can now complete the graph: (v_0, y, v_2, v_1) must be a face, (v_1, v_2, v_3, w) must be a face, (v_2, y, v_4, v_3) must be a face, and so on. We find that w is adjacent to $v_1, v_3, \ldots, v_{2k-1}$

and y is adjacent to $v_0, v_2, v_4, \ldots, v_{2k}$. Finally, we must have that (v_0, v_1, w, x) and (v_{2k-1}, v_{2k}, x, w) are faces, so w is adjacent to x. However, this means that x has degree 3 and therefore H is a cycle, a contradiction. The situation for $k = 4$ is shown on the right of Figure 7. □

3 Proofs of Theorems 1 and 2

In each case, the proof requires us to show that each quadrangulation within a given class (other than the starting graphs) is reducible within that class by one of the defining operations.

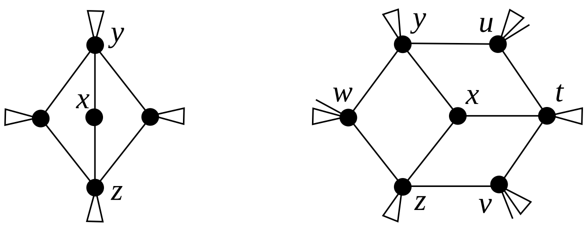

Figure 8: Cases in the proof of Theorem 1

Proof of Theorem 1. Let G be a simple quadrangulation larger than the square. By Lemma 1(i), G has minimum degree either 2 or 3.

Let x be a vertex of minimum degree. If x has degree 2, its neighbours y and z have degree at least 3, as otherwise G would be a square. Hence the situation is as depicted on the left of Figure 8, showing that G is P_0-reducible in $\mathcal{Q}_1$.

So suppose that G has minimum degree 3, and let x be a vertex of degree 3. Consider the faces (x, y, u, t), (x, t, v, z) and (x, z, w, y) incident with x, as at the right of Figure 8. A $P_1(\mathcal{Q}_1)$-reduction can be applied to (x, z, w, y) unless w is adjacent to t (in which case the reduction would create a parallel edge). However, if w is adjacent to t then v cannot be adjacent to y (by planarity), so a $P_1(\mathcal{Q}_1)$-reduction applies to (x, t, v, z).

In any case, G is either P_0-reducible in $\mathcal{Q}_1$ or P_1-reducible in $\mathcal{Q}_1$, which proves the theorem. □

Proof of Theorem 2. Let G be a simple quadrangulation with minimum degree 3, other than a pseudo-double wheel. If H is a component of the subgraph induced by the vertices of degree 3, Lemma 2 tells us that H is either a 4-cycle or has a vertex of degree at most 1. In the first case, G is P_3-reducible in $\mathcal{Q}_2$, since the vertices neighbouring H have degree at least 4 in G (otherwise H would not be a component).

Therefore, we may assume that there is some vertex x of degree 3 with at least two neighbours y and z of degree at least 4. As in the previous proof, consider the incident faces (see Figure 9). A $P_1(\mathcal{Q}_2)$-reduction can be applied to (x, z, w, y) unless

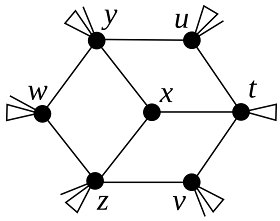

Figure 9: Configuration in the proof of Theorem 2

w is adjacent to t. However, in that case v cannot be adjacent to y (by planarity) and also t has degree at least 4. Therefore, a $P_1(\mathcal{Q}_2)$-reduction can be applied to (x, t, v, z). (It is important to check that the two neighbours of x on the face where the reduction is applied have degree at least 4, otherwise the result of the reduction has minimum degree 2 and is therefore outside $\mathcal{Q}_2$.)

In any case, G is either P_1-reducible in $\mathcal{Q}_2$ or P_3-reducible in $\mathcal{Q}_2$, which proves the theorem. □

4 Proofs of Theorems 3 and 4

Throughout this section, quadrangulations are 3-connected and hence simple. We will say that G is *P_1-reducible at x* if there is a P_1-reduction where the vertex x plays the part of the vertex of that label in Figure 3. Recall that a reduction can only be applied if the resulting graph is still in the class of interest.

Lemma 5 *Let G be a 3-connected quadrangulation which is not a pseudo-double wheel and has no separating 4-cycles. Let x be any vertex of degree 3 with at least two neighours of degree at least 4. Then G is P_1-reducible in $\mathcal{Q}_3$ at x.*

Proof. Suppose that x has degree 3 and is adjacent to at least two vertices, y and z of degree at least 4. We have faces (x, z, w, y), (x, y, u, t), (x, t, v, z), just as in Figure 9. Suppose that a $P_1(\mathcal{Q}_3)$-reduction cannot be applied to (x, z, w, y) because the resulting graph is not 3-connected. Then at least one of $\{w, x, u\}$, $\{w, x, t\}$, $\{w, x, v\}$ forms a 3-cut in G. Hence, without loss of generality, w lies on a face with either u or t. If w is on a face with t then, by parity, w must be adjacent to t. Hence (x, t, w, y) forms a separating 4-cycle. On the other hand, if w lies on a face with u then w and u must be opposite vertices of that face. Thus they have a common neighbour $a \neq y$. But then (w, y, u, a) forms a separating 4-cycle, since $\deg(y) \geq 4$. In either case, we find that the absence of separating 4-cycles implies that G is P_1-reducible in $\mathcal{Q}_3$ at x. □

Lemma 6 *If there exists a minimal separating 4-cycle C such that $\bar{C}$ is not a pseudo-double wheel, then G is P_1-reducible in $\mathcal{Q}_3$.*

Proof. Using Lemma 3, it is clear that $\bar{C}$ is a 3-connected quadrangulation without a separating 4-cycle. Applying Lemma 4 to $\bar{C}$ shows that there is a vertex $x \in \bar{C} \setminus C$, which has degree 3 in $\bar{C}$ and is adjacent to at least two vertices of degree at least 4 in $\bar{C}$. Therefore, by Lemma 5, the quadrangulation $\bar{C}$ is P_1-reducible at x in $\mathcal{Q}_3$. Let (x, z, w, y) be the face where a $P_1(\mathcal{Q}_3)$-reduction applies in $\bar{C}$.

Clearly $\deg_G(x) = 3$. If G is not P_1-reducible in $\mathcal{Q}_3$ at the face (x, z, w, y), then G must have a cutset $\{x, w, v\}$ for some vertex v. Clearly $v \in \bar{C}$ as v lies on a face with x. But every vertex in $G \setminus \{w, v\}$ has a path to $C \setminus \{w, v\}$, and $\bar{C} \setminus \{x, w, v\}$ is connected (otherwise the $P_1(\mathcal{Q}_3)$-reduction would not be applicable at x in $\bar{C}$). Therefore $\{x, w, v\}$ is not a cutset after all. This contradiction shows that G is P_1-reducible in $\mathcal{Q}_3$. □

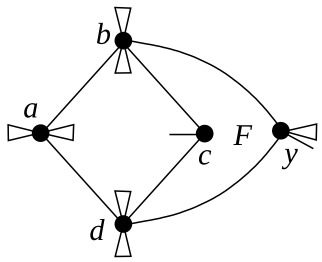

Figure 10: Configuration for Lemma 7

Lemma 7 *If a 3-connected quadrangulation G contains a minimal separating 4-cycle C such that some $c \in C$ has degree 3 in G, then G is P_1-reducible in $\mathcal{Q}_3$.*

Proof. Vertex c also has degree 3 in $\bar{C}$, since otherwise C would not be minimal. Let b and d be the vertices adjacent to c on C and a the fourth vertex on C. Now G is 3-connected, so a, b and d must each have degree at least 4 in G. Since c has degree 3, the vertices b, c, and d lie on a face $F = (b, y, d, c)$, as illustrated in Figure 10.

We can apply a $P_1(\mathcal{Q}_3)$-reduction to the face F unless there exists a cutset including c, y and one other vertex z. Then z must share a face other than F with both c and y. Clearly z cannot share a face with c other than F if $z \in G \setminus \bar{C}$, and z cannot share a face with y if $z \in \bar{C} - C$. Therefore $z \in C$, and since G is 3-connected it must be that $z = a$. Hence c and a share a face, which must lie in $\bar{C}$, contradicting the minimality of C. □

Lemma 8 *Let G be a 3-connected quadrangulation which contains a minimal separating 4-cycle C such that $\bar{C}$ is a pseudo-double wheel. Then G is P_1-reducible in $\mathcal{Q}_3$ or P_3-reducible in $\mathcal{Q}_3$ unless there is a face incident with two opposite vertices of C but no face incident with three vertices of C.*

Proof. First, suppose that $\bar{C}$ is a cube. By Lemma 7, we may assume that all vertices in C have degree at least 4 in G (which incidentally implies that no face can be incident with 3 vertices of C). The $P_3(\mathcal{Q}_3)$-reduction is valid unless the resulting graph is not 3-connected. The latter can only happen if there is a face incident with two opposite vertices of C.

Now we may assume that $\bar{C}$ is a pseudo-double wheel with strictly more than 8 vertices. Let the cycle of vertices of degree 3 be $(v_0, v_1, \ldots, v_{2k-1})$. Let y be the vertex adjacent to v_0, $v_2, \ldots, v_{2k-2}$, and z be the vertex adjacent to v_1, $v_3, \ldots, v_{2k-1}$. Without loss of generality, we assume $C = (v_0, v_{2k-1}, v_{2k-2}, y)$. By Lemma 7, we can assume that v_0, v_{2k-2} and v_{2k-1} have degree at least 4 in G. Hence the only possibility for a face incident with three vertices of C is one incident with v_0, y and v_{2k-2}.

Now consider applying a $P_1(\mathcal{Q}_3)$-reduction to the face (v_0, v_{2k-1}, z, v_1). This is valid unless the resulting graph is not 3-connected, which only happens if v_{2k-1} is on a face with v_2, v_3 or y. The first two options are clearly impossible. The third option gives a face incident with both v_{2k-1} and y, as required, and also shows that v_0, y and v_{2k-2} cannot lie on a common face. This rules out the only possibility, identified above, for a face incident with three vertices of C. $\square$

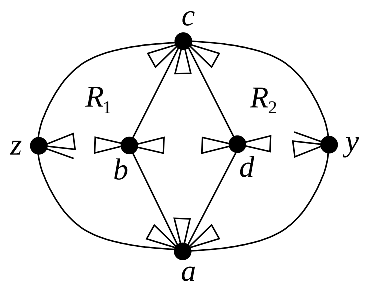

Figure 11: Configuration in the proof of Theorem 3

Proof of Theorem 3. Let G be a 3-connected quadrangulation that is not a pseudo-double wheel, yet not P_1-reducible in $\mathcal{Q}_3$ or P_3-reducible in $\mathcal{Q}_3$. By Lemmas 4 and 5, G contains a separating 4-cycle. Moreover, by Lemmas 6 and 8, each minimal separating 4-cycle C has a face F_C incident with two opposite vertices of C but no face incident with three vertices of C. We will show that this leads to a contradiction.

Since $\bar{C}$ is a pseudo-double wheel, the face F_C is disjoint from the disk containing $\bar{C}$. Without loss of generality, suppose that $C = (a, b, c, d)$ and $F_C = (a, y, c, z)$. The graph $G \setminus (\bar{C} \setminus C)$ can be written as $R_1 \cup R_2$, where $R_1 \cap R_2 = \{a, c\}$. We may assume, by exchanging y and z if necessary, that R_1 is contained in a disc bounded by the cycle (a, z, c, b) and R_2 is contained in a disc bounded by (a, d, c, y), respectively. See Figure 11, where F_C is shown as the unbounded face.

Now choose C so that the number of vertices in the smallest of these two parts is *minimized* (over all minimal separating 4-cycles). Without loss of generality, suppose that R_1 is the smallest part. Since z has degree at least 3, (a, z, c, b) is not a face and so is a separating 4-cycle. Therefore there is a minimal separating 4-cycle C' contained in R_1. Now $\bar{C}'$ cannot be R_1 because F_C would be incident with three of its vertices. In the same way that C defined R_1 and R_2, C' defines two parts $R_1{}'$ and $R_2{}'$. Either $F_{C'}$ is a face of R_1, or $F_{C'} = F_C$, but in either case one of $R_1{}'$ and $R_2{}'$ is a proper subgraph of R_1. This contradicts the minimality of R_1, proving the theorem. □

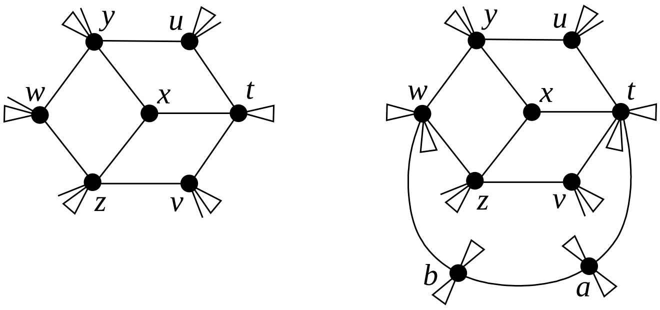

Figure 12: Configurations in the proof of Theorem 4

Proof of Theorem 4. Let G be a simple 3-connected quadrangulation, other than a pseudo-double wheel, that has no separating 4-cycle. By Lemma 4, G has a vertex of degree 3 adjacent to two vertices, y and z, of degree at least 4, as in the left part of Figure 12. We know from Lemma 5 that a $P_1(\mathcal{Q}_3)$-reduction can be applied to the face (x, z, w, y) (that is, without losing 3-connectivity). Suppose that a $P_1(\mathcal{Q}_4)$-reduction is not valid at the same place; that is, the reduced graph has a separating 4-cycle. Then there is a path of length 3 (other than those apparent in the left part of the figure) from t to w. Such a path cannot use u, y, v or z without G having a separating 4-cycle, so it must involve two additional vertices a and b, as in the right part of the figure. The existence of this path implies that t has degree at least 4, so we can consider applying a $P_1(\mathcal{Q}_4)$-reduction to the face (x, t, v, z) instead. As before, this is successful unless there is a path of length 3 from y to v. Such a path cannot use u, t, z or w without G having a separating 4-cycle, but it must also cross the path t, a, b, w. Hence, by planarity, it either has the form y, a, c, v or y, c, b, v, where c is yet another vertex. However, both these possibilities imply a separating 4-cycle in G, namely (y, u, t, a) and (w, z, v, b), respectively. So, in any case G is P_1-reducible in $\mathcal{Q}_4$ at x. □

5 The connection to radial graphs

In this section we describe a completely different construction method for simple quadrangulations. Given a connected graph H embedded in the sphere, the *radial graph* G is defined as follows: the vertices of G are the vertices and faces of H, while the edges of G correspond to vertex-face incidences of H. Multiplicities are significant. For a vertex of H and a face f of H there are k edges of G with endpoints v and f, where k is the number of times that v appears in the walk that bounds f. The embedding of G is induced in the obvious way from that of H. The definition implies that the planar dual of H has the same radial graph. Figure 13 shows a planar graph and its radial graph. The dual of the radial graph of H is commonly called the *medial graph* of H.

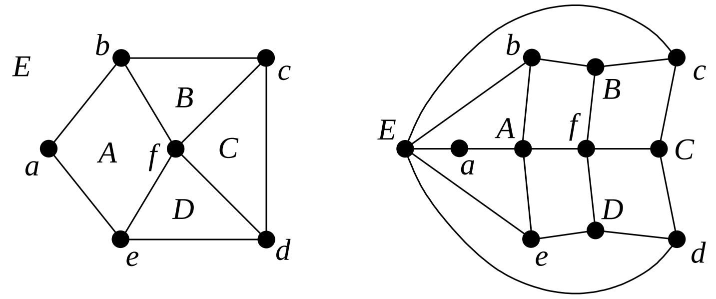

Figure 13: A plane graph and its radial graph

It is easy to see that the radial graph is a quadrangulation. Conversely, given a connected quadrangulation G of the sphere, we can construct the planar graphs H of which G is the radial graph. Recall that G is bipartite. Define the vertices of H to be the vertices in one of the colour classes of G, with two of them being adjacent if they lie on the same face of G. As before, multiplicities are significant and the embedding of H is induced from that of G in the obvious way. Choosing the other colour class of G gives the dual of H, which also has G as its radial graph.

Theorem 5 *Let H be a connected planar graph and let G be its radial graph. Then*
(i) G is a simple quadrangulation if, and only if, H is loop-free and 2-connected (but not necessarily simple);
(ii) G is a simple quadrangulation with minimum degree 3 if, and only if, H is loop-free and 2-connected (but not necessarily simple) with no vertices of degree 2 or faces of size 2;
(iii) G is a (simple) 3-connected quadrangulation if, and only if, H is simple, 2-connected, and 3-edge-connected;
(iv) G is a (simple) 3-connected quadrangulation without separating 4-cycles if, and only if, H is simple and 3-connected.

Proof. Since G is bipartite, it cannot have loops. Multiple edges in G correspond to 1-cuts or loops in H, and vice-versa. Vertices of degree 1 in G correspond to loops that bound faces or vertices of degree 1 in H, whereas vertices of degree 2 in G correspond to faces of size 2 or vertices of degree 2 in H. A 2-cut in G corresponds to a 2-edge cut or a 2-cycle in H, and vice versa. Finally, separating 4-cycles in G correspond to 2-cuts in H, and vice-versa.

Parts (i), (iii) and (iv) have previously appeared in [5], [7] and [11], respectively. □

Theorem 5 gives an alternate way to generate our classes of quadrangulations. For instance, by (iv), generating members of $\mathcal{Q}_4$ is equivalent to generating simple 3-connected planar graphs. This class may also be generated using Tutte's synthesis of 3-connected graphs [15], which we now state. We say that a graph H is obtained from a graph G by *splitting a vertex* if G is obtained from H by contracting an edge with both ends of degree at least three that belongs to no triangle of H. A *wheel* is a graph obtained from a cycle by adding a new vertex joined to every vertex of the cycle. Note that the radial graph of a wheel is a pseudo-double wheel.

Theorem 6 *A graph G is simple and 3-connected if an only if it can be obtained from a wheel by repeatedly applying the following two operations:*

(a) adding an edge between two distinct nonadjacent vertices, and

(b) splitting a vertex.

However, our method is more efficient in the sense that it generates fewer duplicates. To see that, let us restate Theorem 4 in terms of simple 3-connected graphs, as a variation of Theorem 6 for planar graphs. We say that a planar graph H is obtained from a planar graph G by a *restricted addition* if H is obtained from G by adding an edge joining nonadjacent vertices x and z, where for some vertex y the vertices x, y, z occur on the boundary of a face of G in the order listed. We say that a planar graph H is obtained from a planar graph G by a *restricted split* if G is obtained from H by contracting an edge that belongs to no triangle of H, has one end of degree three, and the other end of degree at least three. Theorem 4 implies the following strengthening of Theorem 6 for planar graphs.

Corollary 1 *A planar graph G is simple and 3-connected if an only if it can be obtained from a wheel by repeatedly applying the operations of restricted addition and restricted split.*

Seymour [14] generalized Theorem 5 by showing that (except for a few well-defined exceptions), the graph G can be generated as in Theorem 5 from any simple 3-connected minor of itself. It is natural to ask whether the corresponding strengthening holds for planar graphs. Unfortunately, that is not true. Let G be the 6-sided prism (the Cartesian product of C_6 and K_2), and let H be obtained from G by joining two diagonally opposite vertices on the boundary of one of the faces of length six. Then clearly H cannot be obtained from G by applying the operations of Corollary 1.

6 Theoretical enumeration

We are not aware of any theoretical enumeration of isomorphism types of planar quadrangulations. However, there are some enumerations of labelled planar quadrangulations and in this section we will add to them.

Recall that a *rooted* embedded graph is one in which a flag (a vertex, an edge incident with that vertex, and a face incident with that edge) has been distinguished. Rooted embedded graphs are generally easier to enumerate analytically than isomorphism types, since the rooting eliminates automorphisms.

Theorem 7 *Let $r_i(n)$, $i = 1, 2, 4$, be the number of rooted (simple) quadrangulations with n vertices in class $\mathcal{Q}_i$. Then*
(a) For $n \geq 4$,

$$r_1(n) = \frac{12\,(3n-10)!}{(n-4)!\,(2n-4)!}.$$

(b) For $i = 1, 2$, define $R_i(x) = \sum_{n=4}^{\infty} r_i(n) x^{n-4}$. Then

$$R_2(x) = \frac{1-2x-x^2}{1+2x+x^2}\left(\frac{1-x}{1+x} R_1\Big(\frac{x(1-x)}{1+x}\Big) - 1\right).$$

(c) For $n \geq 8$,

$$r_4(n) = 2(-1)^n + \sum_{i=0}^{n-5} \frac{(i+2)^2(i+1)(-2)^i}{2(n-4)} \binom{2n-5}{n-i-5}.$$

Proof. For parts (a) and (c), see [5] and [11], respectively. We will prove part (b) here.

We will draw rooted quadrangulations with the root face on the outside and the root vertex and edge indicated by an arrowhead. *Internal* vertices are those not on the root face.

If G is a rooted simple quadrangulation (class $\mathcal{Q}_1$), we can successively remove internal vertices of degree 2 (using $P_0(\mathcal{Q}_1)$-reductions) until there are no internal vertices of degree 2 remaining. A simple induction shows that the resulting rooted (simple) quadrangulation is independent of the order in which the internal vertices of degree 2 are removed; we will call it the *frame* of G.

Define $F(y) = \sum_{i\geq 0} f_i\, y^i$, where f_i is the number of rooted simple quadrangulations with i internal vertices, all internal vertices having degree at least 3. (These are the possible frames.) Also define $S(y) = \sum_{i\geq 0} s_i\, y^i$, where s_i is the number of *square-framed* quadrangulations with i internal vertices, which are the rooted simple quadrangulations whose frame is the square. Then an arbitrary rooted simple quadrangulation consists of a frame with each internal face filled in according to some square-framed quadrangulation. Since this can be done in exactly one way, and having i internal vertices implies that there are $i+1$ internal faces, we have

$$R_1(y) = \sum_{i\geq 0} f_i\, S(y)^{i+1} y^i = S(y)F(yS(y)). \qquad (1)$$

We next write $F(y)$ in terms of $R_2(y)$. Rooted simple frames have no internal vertices of degree 2, but can have 0, 1, 2 or 4 vertices of degree 2 on the root face. Clearly $F(y) = F_0(y) + 4F_1(y) + 2F_2(y) + F_4(y)$, where $F_i(y)$ counts the frames with one of the possible positions for i vertices of degree 2. Clearly $F_0(y) = R_2(y)$ and $F_4(y) = 1$.

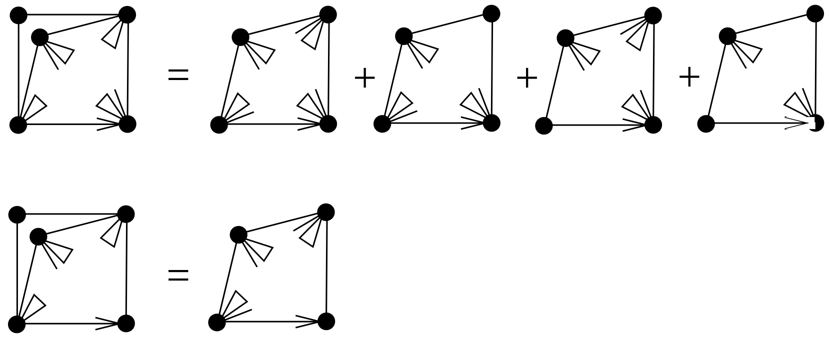

Figure 14: Recurrences for two types of frame

Figure 14 shows recurrences for $F_1(y)$ and $F_2(y)$. The first recurrence is $F_1(y) = y(F_0(y) + 2F_1(y) + F_2(y))$, while the second is $F_2(y) = yF_1(y)$. (Recall that simple quadrangulations other than the square cannot have adjacent vertices of degree 2.) Solving all these equations, we obtain

$$F(y) = 1 + \frac{1 + 2y + y^2}{1 - 2y - y^2} R_2(y). \tag{2}$$

To complete the calculation, we determine the counting series $S(y)$ for square-framed quadrangulations. A *diagonal* is an internal vertex adjacent to two opposite vertices of the outside face. Choose a fixed proper colouring of the vertices of the quadrangulation with colours black and white. By considering the last vertex removed as a square-framed quadrangulation is reduced to its frame, we see that a square-framed quadrangulation with at least one internal vertex has a diagonal. Moreover, by planarity, it cannot have both black and white diagonals. Thus we have $S(y) = 1 + S_B(y) + S_W(y)$, where the last two series count square-framed quadrangulations with black and white diagonals, respectively. If a square-framed quadrangulation G has exactly $k \geq 1$ black diagonals, these divide the interior of the quadrangulation into $k+1$ regions, each of which can be empty or (considered as quadrangulations in their own right with colouring inherited from G) contain white diagonals. Therefore we have the recurrence

$$S_B(y) = \sum_{k \geq 1} y^k (1 + S_W(y))^{k+1} = \frac{y(1 + S_W(y))^2}{1 - y(1 + S_W(y))}.$$

Since $S_B(y) = S_W(y)$ by symmetry, we infer that $S(y)$ is given by the equation

$$yS(y)^2 - (1 - y)S(y) + 1 = 0. \tag{3}$$

We can now apply (2) and (3) to (1) to obtain the relationship between $R_1(y)$ and $R_2(y)$. The version given in the theorem statement is obtained by substituting $x = yS(y)$, which (3) shows to be equivalent to $y = x(1 - x)/(1 + x)$. □

7 Computational experience

The algorithms proved in Theorems 1–4 have been used in the program `plantri` [2, 3] written by two of the present authors. The program `plantri` can generate many classes of planar graphs without explicit isomorphism testing. The generation speed (1 GHz Pentium III) is 79,000–95,000 quadrangulations per second for classes $\mathcal{Q}_2$–$\mathcal{Q}_4$ and more than 400,000 per second for class $\mathcal{Q}_1$.

The method used for isomorph rejection is the "canonical construction path" method introduced by McKay [10]. Details are in [2]; essentially, the program chooses one of the (possibly many) sequences of expansions by which each graph can be made, then rejects any graph made by other sequences. Those graphs not rejected then comprise exactly one member of each isomorphism class.

Tables 1–4 give the numbers of isomorphism classes of simple quadrangulations with n vertices and f faces in each of the classes $\mathcal{Q}_1$–$\mathcal{Q}_4$ (equivalently, the numbers of planar quartic graphs with f vertices and n faces in the corresponding dual classes). In each case $q_i(n)$ ($i = 1, \ldots, 4$) is the number of isomorphism classes in $\mathcal{Q}_i$ if orientation-reversing (reflectional) isomorphisms are permitted, whereas $q'_i(n)$ permits only isomorphisms which are orientation-preserving. The values of $q_3(n)$ up to $n = 17$ previously appeared in [4] and those up to $n = 24$ in [6].

The number of rooted simple quadrangulations corresponding to each isomorphism class is the number of orbits of flags, which can easily be computed from the automorphism group. Since `plantri` can compute the automorphism group as a normal part of its operation, the rooted counts can be easily derived. By comparing these counts to the theoretical values given in the previous section, we could check the computations in classes $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_4$ for all the sizes shown in the tables.

We do not know of any theoretical enumeration for class $\mathcal{Q}_3$. However, the values of $q_3(n)$ up to $n = 35$ were successfully replicated using the method of Section 5.

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n	f	$q_1(n)$	$q'_1(n)$
4	2	1	1
5	3	1	1
6	4	2	2
7	5	3	3
8	6	9	10
9	7	18	21
10	8	62	83
11	9	198	298
12	10	803	1339
13	11	3378	6049
14	12	15882	29765
15	13	77185	148842
16	14	393075	770267
17	15	2049974	4054539
18	16	10938182	21743705
19	17	59312272	118237471
20	18	326258544	651370528
21	19	1815910231	3628421181
22	20	10213424233	20416662314
23	21	57974895671	115919209155
24	22	331820721234	663548898942
25	23	1913429250439	3826577783917
26	24	11109119321058	22217382001865
27	25	64901418126997	129800215435088

Table 1: Simple quadrangulations

n	f	$q_2(n)$	$q'_2(n)$
8	6	1	1
9	7	0	0
10	8	1	1
11	9	1	1
12	10	3	4
13	11	3	3
14	12	12	16
15	13	19	26
16	14	64	99
17	15	155	256
18	16	510	895
19	17	1514	2789
20	18	5146	9740
21	19	16966	32799
22	20	58782	115024
23	21	203269	401180
24	22	716607	1421170
25	23	2536201	5046539
26	24	9062402	18066772
27	25	32533568	64940825
28	26	117498072	234712099
29	27	426212952	851801048
30	28	1553048548	3104690139
31	29	5681011890	11358900851
32	30	20858998805	41710948878
33	31	76850220654	153684688127
34	32	284057538480	568079430741
35	33	1053134292253	2106188450292
36	34	3915683667721	7831185534651

Table 2: Simple quadrangulations with minimum degree 3

n	f	$q_3(n)$	$q'_3(n)$
8	6	1	1
9	7	0	0
10	8	1	1
11	9	1	1
12	10	3	4
13	11	3	3
14	12	11	15
15	13	18	25
16	14	58	92
17	15	139	234
18	16	451	803
19	17	1326	2469
20	18	4461	8512
21	19	14554	28290
22	20	49957	98148
23	21	171159	338673
24	22	598102	1188338
25	23	2098675	4180854
26	24	7437910	14840031
27	25	26490072	52904562
28	26	94944685	189724510
29	27	341867921	683384218
30	28	1236864842	2472961423
31	29	4493270976	8984888982
32	30	16387852863	32772085447
33	31	59985464681	119963084542
34	32	220320405895	440623586740
35	33	811796327750	1623555117611
36	34	3000183106119	6000283550482

Table 3: 3-connected quadrangulations

n	f	$q_4(n)$	$q'_4(n)$
8	6	1	1
9	7	0	0
10	8	1	1
11	9	1	1
12	10	2	3
13	11	2	2
14	12	9	12
15	13	11	16
16	14	37	59
17	15	79	133
18	16	249	445
19	17	671	1248
20	18	2182	4162
21	19	6692	13014
22	20	22131	43474
23	21	72405	143304
24	22	243806	484444
25	23	822788	1639388
26	24	2815119	5617205
27	25	9679205	19332596
28	26	33551192	67048051
29	27	116900081	233691112
30	28	409675567	819121608
31	29	1442454215	2884443024
32	30	5102542680	10204104900
33	31	18124571838	36247138920
34	32	64634480340	129264732757
35	33	231334873091	462661038926
36	34	830828150081	1661637913984
37	35	2993489821771	5986941546017

Table 4: 3-connected quadrangulations without separating 4-cycles