# The smallest cubic graphs of girth nine

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**Abstract.**

We describe two computational methods for the construction of cubic graphs with given girth. These were used to produce two independent proofs that the $(3,9)$-cages, defined as the smallest cubic graphs of girth 9, have 58 vertices. There are exactly 18 such graphs. We also show that cubic graphs of girth 11 must have at least 106 vertices and cubic graphs of girth 13 must have at least 196 vertices.

## 1. Introduction.

A *cubic* graph is a regular graph of degree 3, and its *girth* is the length of the shortest cycle. For $g \geq 3$, the smallest cubic graphs of girth $g$ are called $(3,g)$*-cages*. Ever since the existence of cages was proved by Erdős and Sachs in 1963 [6], their properties have been intensively studied. However, the difficulty of the problem is such that not even the order $f(3,g)$ of such graphs is known for $g = 11$ or $g \geq 13$.

For a survey of work on cages prior to 1982, see Wong [12]. In particular, Wong gives references for the results $f(3,3) = 4$, $f(3,4) = 6$, $f(3,5) = 10$, $f(3,6) = 14$, $f(3,7) = 24$, $f(3,8) = 30$, $f(3,10) = 70$, and $f(3,12) = 126$. The major result of this paper is that $f(3,9) = 58$.

The nonexistence of a cubic graph of girth 9 and order 46, the smallest feasible order, was proved independently in 1973 by Bannai and Ito [2] and by Damerell [5]. The lower bound was raised to 54 in the 1970's by C. W. Evans (by hand) and confirmed by the present second author in 1978 (by computer). Both results are unpublished.

The first small example of a cubic graph of girth 9 was one of order 60 found by Foster about 1952 (see [8]). This record held for nearly 30 years, until Biggs and Hoare [3] discovered an example of order 58. A second graph of order 58 was found by Evans in 1984 [7].

The value $f(3,9) = 58$ was established by the second author in 1984 using the method that we will describe in the next section. Over the following few years, a total of 18 $(3,9)$-cages were found by the same program, but the completeness of this set could not be established until a much faster method was developed by the first and third authors. That method will be described in Section 3.

Some of the 18 cages have been independently discovered by Geoff Exoo, Dan Ashlock and Yuan Yang using various nonexhaustive searches (private communications).

Our second method is fast enough to allow us to improve the lower bounds on $f(3,11)$ and $f(3,13)$ as well. A cubic graph of girth 11 and order 112 was found by Balaban in 1973 [1]. Despite considerable effort, no smaller graph has been constructed even though the lower bound stood at only 96 until now. As the result of our computations we have improved this to $f(3,11) \geq 106$. Similarly, we have improved the lower bound on $f(3,13)$ from 192 to 196.

## 2. The first method.

In this section we will describe the first method. As it is suitable only for girth 9 (or less), we will use the example of girth 9 throughout.

Let $G$ be a cubic graph of girth 9, with $n \geq 48$ vertices. Choose any vertex $v$ of $G$, and define $V_i$ to be the set of vertices at distance $i$ from $v$, for $i \geq 0$. Clearly, $|V_1| = 3$, $|V_2| = 6$, $|V_3| = 12$, and $|V_4| = 24$. The 45 edges incident with $V_0 \cup \cdots \cup V_3$ are the edges of a tree $T$.

It will be convenient to divide the vertices into three classes. The *internal vertices* are those in $V_0 \cup \cdots \cup V_3$. The vertices in $V_4$ are *leaves*, as they are the leaves of the tree $T$, and the $n - 46$ vertices $V_5 \cup V_6 \cup \cdots$ are *external vertices*. Similarly, the edges of $G$ can be divided into four classes. The *tree edges* are the edges of $T$. Those edges between two leaves, from a leaf to an external vertex, and between two external vertices, are *lower*, *intermediate* and *upper* edges, respectively. Let $m_1$, $m_2$, and $m_3$ be the number of lower, intermediate and upper edges, respectively. Then we have the equations $2m_1 + m_2 = 48$ and $2m_3 + m_2 = 3n - 138$, which have the solutions $\max(0,\, 93 - \frac{3}{2}n) \leq m_1 \leq 24$, $m_2 = 48 - 2m_1$, $m_3 = m_1 + \frac{3}{2}n - 93$.

The general procedure is to find the lower edges, the upper edges, and the intermediate edges, in that order.

*The lower edges.*

In the case of the lower edges, we did complete isomorph reduction using, we believe, the first application of the general method described in [10]. For $m \geq 0$, let $\mathcal{L}_m$ denote the family of valid choices of $m$ lower edges. By "valid" we mean that no leaf is incident with more than two lower edges, and that the tree edges and lower edges together induce no cycle of length less than 9. The elements of $\mathcal{L}_m$ can be divided into equivalence classes under the action of the automorphism group of $T$, a group of order $3 \times 2^{22}$. Suppose that $L_m$ is a set containing exactly one member from each equivalence class. We will describe how we can construct a similar transversal of $\mathcal{L}_{m+1}$.

Let $X \in L_m$, and define $p(X)$ to be the class of all pairs $\{v, w\}$ of leaves such that $v$ and $w$ have at most degree one in $X$, and have distance at least 8 in $T \cup X$. That is, $p(X)$ lists the places where a new lower edge $e$ could be added to $X$ to form a member $X + e$ of $\mathcal{L}_{m+1}$. Some of these extensions $X + e$ will be isomorphic due to the symmetries of $X$, but these are easily eliminated by using the program `nauty` [9] to

| $m$ | $\lvert L_m\rvert$ | $m$ | $\lvert L_m\rvert$ |
|---|---|---|---|
| 1 | 1 | 13 | 11898246 |
| 2 | 8 | 14 | 8803803 |
| 3 | 39 | 15 | 4089034 |
| 4 | 283 | 16 | 1155988 |
| 5 | 1785 | 17 | 204192 |
| 6 | 11279 | 18 | 26392 |
| 7 | 60642 | 19 | 2329 |
| 8 | 278533 | 20 | 135 |
| 9 | 1033389 | 21 | 0 |
| 10 | 3014371 | 22 | 0 |
| 11 | 6639096 | 23 | 0 |
| 12 | 10655541 | 24 | 0 |

**Table 1.** Number of inequivalent choices of $m$ lower edges.

find the automorphism group of $T \cup X$ and hence the equivalence classes of $p(X)$ under that group. Suppose $p'(X)$ contains exactly one member of each such equivalence class.

Suppose now that $X \in L_m$ and $e \in p'(X)$. Perhaps $X + e$ is isomorphic to $X' + e'$ for some different $X' \in L_m$ and $e' \in p'(X')$, so our next step is designed to reject such isomorphisms. Suppose we have a function $\phi$ acting on $\mathcal{L}_{m+1}$ such that $\phi(Y)$ is a nonempty subset of $Y$ satisfying these two properties: (i) $\phi(Y)$ is an orbit of lower edges under the automorphism group of $T \cup Y$; (ii) for any automorphism $\gamma$ of $T$, $\phi(Y^\gamma) = \phi(Y)^\gamma$. We implemented such a function $\phi$ by first computing a combinatorial invariant of the elements of $Y$. If there was exactly one element for which the invariant gave the greatest value, that became the sole member of $\phi(Y)$. If not, the canonical labelling feature of `nauty` was used to choose a value for $\phi(Y)$, namely the orbit of edges containing the edge whose canonical label was greatest amongst those with the greatest invariant value.

Now define $L_{m+1} = \{X + e \,|\, X \in L_m, e \in p'(X) \cap \phi(X + e)\}$. According to the theory in [10], $L_{m+1}$ contains exactly one member of each equivalence class of $\mathcal{L}_{m+1}$. The number of equivalence classes in $\mathcal{L}_m$ for $1 \le m \le 24$ appear in Table 1. The total number is 47875087, including $\mathcal{L}_0$. When this computation was repeated in 1993 as a check, it required 97 hours of cpu time on a Sun Microsystems SPARCstation SLC computer.

*The upper edges.*

For the upper edges in isolation there are few possibilities. There must be $n - 46$ vertices, maximum degree 3 or less, and no cycles shorter than 9 vertices.

The fact that $\mathcal{L}_{21} = \emptyset$ leaves no solution for $m_3$ when $n = 48$, and eliminates the possibility $m_3 = 3$ when $n = 50$. Taking this into account, the number of nonisomorphic ways to choose the upper edges for $n = 50, 52, \ldots, 58$ is $4, 17, 52, 173, 635$, respectively.

*The intermediate edges.*

Given a particular possibility for the lower edges, a good strategy would be to arrange the possibilities for the upper edges in some type of tree structure and then to scan that structure while filling in the intermediate edges. However, we did not adopt that strategy but rather considered each feasible pair of choices (upper edges, lower edges) separately. The inadequacy of this approach for $n = 58$ is immediately seen by counting the number of such pairs: there are about 2323 million. However, a fairly straightforward backtrack program that used the automorphism group of the upper edge graph was sufficient to complete the computations for $n \leq 56$.

In the case of $n = 58$, we completed the search for $m_1 \leq 8$ and $m_1 \geq 16$. Many examples of the remaining cases were also run, together with heuristic searches and attempted modifications of existing cages. For example, for every known cage $G$ and choice of root vertex $v$, we removed all intermediate and upper edges, moved or deleted up to two lower edges, then put back the upper and intermediate edges in every possible way. The result of these computations after running for about four years on several computers was that we had 18 cages each of which had been "discovered" many times over. However, the clear implication that we had the full set of cages had to remain conjectural until the method of the next section was developed.

## 3. The second method.

Our second method for generating cages arose out of ideas developed for the program `minibaum` [4]. The most important improvement was a change in the definition of canonicity that enabled more efficient processing in the case of high girth. However, we will describe the new method without assuming the reader to be familiar with `minibaum`.

As before, we will describe the operation of the program for girth 9 except where we specify otherwise.

The basic method of operation is as follows. We define a family of "representations" of the cubic graph as a list of sublists, with one sublist for each non-internal vertex constructed in a particular way. Amongst all the possible representations corresponding to different labellings of the graph, the one that is lexicographically least is an invariant of the isomorphism class and will be called the *canonical* representation. The program systematically generates representations using a simple branch and bound technique, looking for canonical representations. As usual for branch and bound programs, the efficiency depends heavily on how early we can recognise and eliminate branches of the search tree which cannot contain canonical representations.

The actual format of representations relies on *names* assigned to vertices during the construction process. Recall the classification of vertices and edges defined in the previous section. The interior vertices of the tree will not change during the whole construction process, so there is no need to give names to them. All the other vertices get a name consisting of a pair of values called the *first name* and the *second name*. The first name describes whether the vertex is an external vertex or a leaf. All the external vertices get the first name "$\infty$". The leaves get the first names $1, 2, \ldots, 24$, assigned in any order such that, for each $m$, vertex $m$ as close in $T$ to $1, \ldots, m-1$ as are any of $m+1, \ldots, 24$.

The second name of each vertex runs from "1" to "$N$", with $N = n - 22$ being the number of non-internal vertices, and is assigned in the order in which the first non-tree edge is attached to the vertex during the construction process. It will be seen from the definition of minimality that there must be an edge from the leaf with first name "1" to the leaf with first name "9". Hence, we have two complete names: "$1, 1$" and "$9, 2$". After this, the order of insertion of the edges is that the vertices are completed in order of their second names. So, because "$1, 1$" has valence 2 at this stage, the next edge to be inserted must start there. One possibility is to connect vertex "$1, 1$" to an external vertex. That external vertex would get second name "3" and therefore its full name would be "$\infty, 3$".

A strategy like this can be used to define a "representation" for an unlabelled graph. First mark an arbitrary vertex as the root and form the tree $T$. Next, give first names to the leaves in any legal order. Finally, assign second names as if the non-tree edges were inserted one by one starting at the vertex with first (and therefore also second) name "1". At vertices where more than one non-tree edge starts, one of course has a choice of which edge one wants to regard as the earlier one, possibly inducing interchanged second names for the end vertices of these edges. Once all names have been assigned, each vertex can be associated with a list consisting of its own name followed by the names of its leaf or external neighbours in order of their second names. In our example above, vertex "$1, 1$" would get the list $1, 1; 9, 2; \infty, 3$. Leaves and external vertices have lists of different length. Finally, concatenate these lists in the order of their second components, i.e. the second name of the vertex associated to each list. The resulting long list is a *representation* of the graph. It is easy to see how the graph can be reconstructed from any representation. Among all the possible representations of a graph, we will call the one that is lexicographically least the *canonical representation*.

*The bounding criteria.*

As explained above, the overall structure is a branch and bound program that constructs representations one edge at a time. For efficient operation, a number of "bounding criteria" are employed to remove useless branches of the search.

Essentially there are two types of bounding criteria: One cuts off branches because they cannot include canonical representations and one cuts off branches because they cannot include representations of graphs of the required girth.

In considering the first type of bounding criterion, it is worth noting that we do not have an "orderly" algorithm in the sense of Read [11], because it is not necessarily true that our intermediate forms are minimal. That is, given a canonical representation, it may not be true that leading portions of it are minimal representations of partly constructed graphs. This means that, unlike in Read's approach, we cannot use a simple local minimality test as a bounding criterion. Nevertheless, it is still often possible to tell that a partial representation cannot possibly lead to a canonical representation.

Suppose we can choose a (possibly different) root and a numbering of the leaves and external vertices that gives a partial representation which is smaller than the current partial representation for some leading portion consisting of vertices of degree 3. Then any complete representation in the current branch will be greater than some representation of the same graph in the other labelling, and so is not canonical. This bounding criterion can be implemented very efficiently because the first names giving a smallest possible representation with respect to the given second names can be computed in time linear in the depth of $T$.

The most important example of the second type of bounding criterion uses a simple look-ahead for the vertices which do not yet have degree 3. Each such vertex and each pair of such vertices must be able to be joined to other such vertices without creating short cycles. To make this test easy, we maintain the distance matrix for the set of those vertices not yet completed and update it as each new edge is chosen.

There are two other bounding criteria which are not as important as the first two, but which should be mentioned.

The first one deals with canonicity. It was not used for girth 9, since it improves the program only in those cases where we have a lot of leaves compared to the number of external vertices. The main idea is that the definition of canonicity favours long chains of leaves at the beginning. This means in general that if $x$ is the center of a chain of length $n_1 \leq 11$ and $y$ is the center of a chain of length $n_2$ with $n_1 < n_2$ then taking $y$ as the vertex with second name "1" gives a smaller representation than taking $x$. At some points during the computation it can be determined by a simple calculation based on the valences of the vertices that some other leaf is (or will become) better than "1,1" in this respect, and so that branch of the computation can be cut. This improvement had significant effect for girth 11.

The other criterion has to do with the girth. As long as we have at least two external vertices that are not yet connected to other vertices, the previous bounding criterion will always say that it is useful to carry on. However, it might be that the current valences of the vertices prevent completion of the graph without creating short cycles. Using a separate program, we determined all valence sequences with up to 34 vertices of maximum valence 3, but at most 12 exactly 3, that allow graphs with girth 9 or more. During the execution of the nine-cage constructor it was then always observed whether the set of valence deficiencies belonged to this set.

Concerning the effectiveness of the conditions it can be said that the first two are much more effective than the second two.

*The implementation.*

For the implementation of the algorithm some techniques developed for `minibaum` [4] were used.

So, for example, following the observation that canonicity checking is most effective and cheap in the lower region of the generation tree and ineffective and expensive in the upper region, we checked canonicity only until 2/3 of the edges were inserted. Unlike in `minibaum`, the exact position of the boundary is not very critical, since most of the time is spent in the lower regions of the generation tree.

We also adapted some techniques from `minibaum` that make implementation of the first bounding criterion much faster. Suppose that we have two partial representations $r_1$ and $r_2$, where $r_1$ is for the current point in the search, and $r_2$ corresponds to some different labelling that we wish to compare $r_1$ against. If $r_1$ is greater than $r_2$ in some leading segment that includes only completed vertices, we can cut this branch as described earlier. If $r_2$ is greater than $r_1$ in some leading segment that includes only completed vertices, then relabellings beginning with that leading segment of $r_2$ cannot be less than successors of $r_1$ so we do not need to compare against $r_2$ anymore in this branch of the search.

Finally, if those leading segments of $r_1$ and $r_2$ are the same, we need to continue comparing successors of $r_1$ against successors of $r_2$ but we can save computation by noting that the leading segment of $r_2$ is the same for all its successors. These ideas can be implemented very effectively using a forest of rooted trees; please see [4] for details.

*The results.*

For girth 9, the running times of the algorithm (corrected to Sun SPARCstation SLC) were 0.6s, 10s, 5.3m, 3.3h, 6.5d, 259d for $n = 48, \ldots, 58$, respectively. The results were that the nonexistence of $(3, 9)$-cages below 58 vertices was confirmed, and that the 18 known graphs with 58 vertices were proved to constitute the full set.

We have also used the same method to search for cubic graphs of order 60 and girth 9, but the search was not completed. We found 466 such graphs, of which seven have diameter 5.

For girth 11, we searched as far as 104 vertices without finding any graphs. Representative times were 70s for $n = 98$ and 2.5 years for $n = 104$. For girth 13, the cases $n \leq 194$ were completed in about 30 minutes.

At this point we wish to thank our universities for our intensive use of their computing resources.

**Note added in proof.**

The results of this paper have recently been verified by a non-orderly program developed by Wendy Myrvold and Brendan McKay. The same program has further established that $f(3, 11) = 112$ and $f(3, 13) \geq 202$. Details will appear elsewhere.

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**Appendix. The $(3,9)$-cages.**

We now present the 18 cubic graphs of order 58 and girth 9. They will be called $G_1,\ldots,G_{18}$, where $G_1$ is the graph of Biggs and Hoare [3] and $G_2$ is the graph of Evans [7]. In each graph $G$, the labelling is chosen such that 1–2–$\cdots$ – –58–1 is a Hamiltonian cycle $H$. For the automorphism group $\mathrm{Aut}(G)$ we give a set of generators, the order, and the number of orbits. We also give some eigenvalues of the adjacency matrix, namely the smallest, the second largest, and all integers except 3. The superscript indicates multiplicity.

Statistics about 9-cycles are also given, namely the counts of 9-cycles using each vertex, and the total number. Again, superscripts indicate multiplicities.

All the graphs have diameter 6. In the catalogue below we give the counts per vertex of the number of vertices at distance 6.

As noted, all the graphs are Hamiltonian. In fact, every path of length 7 can be extended to a Hamiltonian cycle. This is not true for length 8 except for $G_3$ and $G_{14}$, though for paths of length 8 whose ends are adjacent it is true for all but $G_5$ and $G_{10}$.

**Cage $G_1$.**

$E(G_1) = E(H) \cup \{$1-9, 2-27, 3-42, 4-13, 5-47, 6-55, 7-34, 8-20, 10-39, 11-52, 12-31, 14-22, 15-36, 16-54, 17-41, 18-48, 19-29, 21-44, 23-57, 24-40, 25-33, 26-53, 28-37, 30-56, 32-45, 35-50, 38-46, 43-51, 49-58$\}$

$\mathrm{Aut}(G_1) = \langle$(3 27)(4 28)(5 29)(6 19)(7 20)(11 39)(12 38)(13 37)(14 36)(17 54)(18 55)(21 34)(22 35)(23 50)(24 51)(25 43)(26 42)(30 47)(31 46)(32 45)(33 44)(40 52)(41 53)(48 56)(49 57), (2 58)(3 49)(4 50)(5 51)(6 52)(7 11)(8 10)(12 34)(13 35)(14 36)(18 41)(19 40)(20 39)(21 38)(22 37)(23 28)(24 29)(25 30)(26 56)(27 57)(31 33)(42 48)(43 47)(44 46)(53 55)$\rangle$; order 4; 20 orbits

Interesting eigenvalues: $-2.75468412^1, -1^1, 1^1, 2.38867825^1$

Nine-cycles per vertex: $9^4, 10^4, 11^{12}, 12^{12}, 13^8, 14^{10}, 15^4, 16^4$; 80 altogether

Vertices at distance six: $0^{22}, 1^{22}, 2^{10}, 3^2, 4^2$

**Cage $G_2$.**

$E(G_2) = E(H) \cup \{$1-9, 2-52, 3-30, 4-39, 5-48, 6-34, 7-15, 8-44, 10-27, 11-49, 12-32, 13-38, 14-56, 16-51, 17-26, 18-42, 19-58, 20-36, 21-29, 22-50, 23-40, 24-33, 25-46, 28-55, 31-43, 35-53, 37-45, 41-54, 47-57$\}$

$\mathrm{Aut}(G_2) = \langle$(1 8)(2 44)(3 43)(4 42)(5 18)(6 19)(7 58)(11 27)(12 28)(13 55)(14 56)(15 57)(16 47)(17 48)(20 34)(21 33)(22 24)(25 50)(26 49)(29 32)(30 31)(35 36)(37 53)(38 54)(39 41)(45 52)(46 51)$\rangle$; order 2; 31 orbits

Interesting eigenvalues: $-2.61300743^1, 2.44195688^1$

Nine-cycles per vertex: $13^4, 14^{14}, 15^{20}, 16^{16}, 17^3, 18^1$; 97 altogether

Vertices at distance six: $0^{12}, 1^{28}, 2^{14}, 3^4$

**Cage $G_3$.**

$E(G_3) = E(H) \cup \{$1-9, 2-16, 3-43, 4-50, 5-38, 6-55, 7-46, 8-27, 10-22, 11-32, 12-48, 13-37, 14-28, 15-54, 17-33, 18-39, 19-47, 20-29, 21-52, 23-44, 24-56, 25-49, 26-40, 30-57, 31-42, 34-51, 35-45, 36-58, 41-53$\}$

$\mathrm{Aut}(G_3) = \langle$(3 16)(4 17)(5 33)(6 32)(7 11)(8 10)(12 46)(13 45)(14 44)(15 43)(18 50)(19 49)(20 25)(21 26)(22 27)(23 28)(24 29)(30 56)(31 55)(34 38)(35 37)(39 51)(40 52)(41 53)(42 54)(47 48), (2 9)(3 8 16 10)(4 27 17 22)(5 28 33 23)(6 14 32 44)(7 15 11 43)(12 42 46 54)(13 31 45 55)(18 21 50 26)(19 52 49 40)(20 51 25 39)(24 38 29 34)(30 35 56 37)(36 57)(41 47 53 48), (1 41 48)(2 40 12)(3 26 11)(4 27 32)(5 28 33)(6 29 34)(7 30 51)(8 31 50)(9 42 49)(10 43 25)(13 16 39)(14 17 38)(15 18 37)(19 36 54)(20 35 55)(21 45 56)(22 44 24)(46 57 52)(47 58 53)$\rangle$; order 24; 5 orbits

Interesting eigenvalues: $-2.55907566^3, -2^1, -1^3, 1^1, 2.40835635^3$

Nine-cycles per vertex: $15^{36}, 16^{18}, 18^4$; 100 altogether

Vertices at distance six: $0^{18}, 1^{24}, 2^{12}, 3^4$

**Cage $G_4$.**

$E(G_4) = E(H) \cup \{$1-9, 2-44, 3-20, 4-37, 5-13, 6-47, 7-53, 8-23, 10-30, 11-50, 12-41, 14-26, 15-58, 16-32, 17-52, 18-40, 19-28, 21-49, 22-33, 24-39, 25-45, 27-35, 29-55, 31-46, 34-42, 36-51, 38-56, 43-54, 48-57$\}$

$\mathrm{Aut}(G_4) = \langle$(1 9)(2 10)(3 11)(4 50)(5 49)(6 48)(7 57)(8 58)(12 20)(13 21)(14 22)

(15 23)(16 24)(17 39)(18 40)(19 41)(25 32)(26 33)(27 34)(28 42)(29 43)(30 44)
(31 45)(37 51)(38 52)(53 56)(54 55), (1 18)(2 17)(3 16)(4 32)(5 33)(6 34)(7 42)(8 41)
(9 40)(10 39)(11 24)(12 23)(13 22)(14 21)(15 20)(19 58)(25 50)(26 49)(27 48)(28 57)
(29 56)(30 38)(31 37)(35 47)(36 46)(43 53)(44 52)(45 51)⟩; order 4; 16 orbits
Interesting eigenvalues: $-2.75987259^1, -1^1, 1^2, 2.45953031^1$
Nine-cycles per vertex: $10^8, 11^4, 12^{14}, 13^{16}, 15^4, 16^{10}, 18^2$; 84 altogether
Vertices at distance six: $0^{16}, 1^{18}, 2^{22}, 3^2$

**Cage $G_5$.**
$E(G_5) = E(H) \cup \{$1-9, 2-27, 3-17, 4-46, 5-38, 6-52, 7-33, 8-42, 10-22, 11-37, 12-54, 13-45, 14-32, 15-50, 16-41, 18-55, 19-34, 20-44, 21-51, 23-31, 24-47, 25-40, 26-53, 28-36, 29-43, 30-56, 35-48, 39-57, 49-58$\}$
$\mathrm{Aut}(G_5)$ = ⟨(1 13)(2 14)(3 15)(4 50)(5 51)(6 52)(7 53)(8 54)(9 12)(10 11)(16 17)
(18 41)(19 40)(20 39)(21 38)(22 37)(23 36)(24 35)(25 34)(26 33)(27 32)(28 31)
(29 30)(42 55)(43 56)(44 57)(45 58)(46 49)(47 48), (1 26)(2 27)(3 28)(4 36)(5 37)
(6 11)(7 12)(8 54)(9 53)(10 52)(13 33)(14 32)(15 31)(16 30)(17 29)(18 43)(19 44)
(22 51)(23 50)(24 49)(25 58)(34 45)(35 46)(40 57)(41 56)(42 55)(47 48)⟩;
order 4; 17 orbits
Interesting eigenvalues: $-2.75990702^1, -1^2, 2.45806583^1$
Nine-cycles per vertex: $10^4, 11^8, 12^{14}, 13^{12}, 14^8, 15^4, 16^6, 18^2$; 84 altogether
Vertices at distance six: $0^{16}, 1^{34}, 2^8$

**Cage $G_6$.**
$E(G_6) = E(H) \cup \{$1-9, 2-37, 3-48, 4-13, 5-28, 6-34, 7-19, 8-51, 10-30, 11-40, 12-22, 14-52, 15-43, 16-31, 17-25, 18-39, 20-46, 21-55, 23-35, 24-50, 26-58, 27-45, 29-54, 32-47, 33-57, 36-44, 38-53, 41-49, 42-56$\}$
$\mathrm{Aut}(G_6)$ = ⟨(1 4)(2 3)(5 9)(6 8)(10 28)(11 27)(12 26)(13 58)(14 57)(15 56)(16 55)
(17 21)(18 20)(22 25)(23 24)(29 30)(31 54)(32 53)(33 52)(34 51)(35 50)(36 49)
(37 48)(38 47)(39 46)(40 45)(41 44)(42 43)⟩; order 2; 30 orbits
Interesting eigenvalues: $-2.73094906^1, -1^1, 2.45704109^1$
Nine-cycles per vertex: $10^4, 11^6, 12^{12}, 13^8, 14^{14}, 15^6, 16^4, 17^2, 18^2$; 86 altogether
Vertices at distance six: $0^{11}, 1^{36}, 2^{11}$

**Cage $G_7$.**
$E(G_7) = E(H) \cup \{$1-9, 2-45, 3-38, 4-54, 5-13, 6-48, 7-25, 8-31, 10-19, 11-40, 12-34, 14-28, 15-51, 16-37, 17-43, 18-55, 20-47, 21-29, 22-36, 23-53, 24-44, 26-39, 27-56, 30-42, 32-52, 33-46, 35-57, 41-49, 50-58$\}$
No nontrivial automorphisms
Interesting eigenvalues: $-2.66844317^1, 2.45387438^1$
Nine-cycles per vertex: $12^5, 13^6, 14^{19}, 15^{18}, 16^7, 17^3$; 93 altogether
Vertices at distance six: $0^{17}, 1^{20}, 2^{19}, 3^2$

**Cage $G_8$.**
$E(G_8) = E(H) \cup \{$1-9, 2-33, 3-39, 4-25, 5-53, 6-47, 7-19, 8-42, 10-51, 11-37, 12-46, 13-21, 14-41, 15-26, 16-32, 17-56, 18-38, 20-29, 22-34, 23-50, 24-44, 27-36, 28-58, 30-52, 31-45, 35-54, 40-49, 43-55, 48-57$\}$

$\mathrm{Aut}(G_8) = \langle(1\ 58)(2\ 57)(3\ 56)(4\ 55)(5\ 54)(6\ 35)(7\ 36)(8\ 27)(9\ 28)(10\ 29)(11\ 20)(12\ 21)(15\ 41)(16\ 40)(17\ 39)(18\ 38)(19\ 37)(22\ 46)(23\ 45)(24\ 44)(25\ 43)(26\ 42)(30\ 51)(31\ 50)(32\ 49)(33\ 48)(34\ 47)\rangle$; order 2; 31 orbits
Interesting eigenvalues: $-2.63455451^1, 2.44090449^1$
Nine-cycles per vertex: $13^2, 14^{25}, 15^{19}, 16^{11}, 18^1$; 95 altogether
Vertices at distance six: $0^{32}, 1^{18}, 2^8$

**Cage $G_9$.**
$E(G_9) = E(H) \cup \{$1-9, 2-52, 3-23, 4-13, 5-34, 6-18, 7-46, 8-38, 10-29, 11-20, 12-55, 14-43, 15-37, 16-31, 17-25, 19-51, 21-36, 22-47, 24-40, 26-56, 27-35, 28-44, 30-49, 32-54, 33-41, 39-50, 42-58, 45-53, 48-57$\}$
No nontrivial automorphisms
Interesting eigenvalues: $-2.66107880^1, -1^1, 2.44829202^1$
Nine-cycles per vertex: $13^6, 14^{15}, 15^{19}, 16^{15}, 17^3$; 96 altogether
Vertices at distance six: $0^5, 1^{22}, 2^{26}, 3^4, 4^1$

**Cage $G_{10}$.**
$E(G_{10}) = E(H) \cup \{$1-9, 2-41, 3-34, 4-28, 5-17, 6-38, 7-45, 8-22, 10-53, 11-27, 12-37, 13-48, 14-42, 15-31, 16-24, 18-54, 19-47, 20-40, 21-29, 23-35, 25-50, 26-44, 30-57, 32-52, 33-46, 36-56, 39-51, 43-55, 49-58$\}$
$\mathrm{Aut}(G_{10}) = \langle(1\ 16)(2\ 24)(3\ 23)(4\ 22)(5\ 8)(6\ 7)(9\ 17)(10\ 18)(11\ 19)(12\ 47)(13\ 48)(14\ 49)(15\ 58)(20\ 27)(21\ 28)(25\ 41)(26\ 40)(31\ 57)(32\ 56)(33\ 36)(34\ 35)(37\ 46)(38\ 45)(39\ 44)(42\ 50)(43\ 51)(52\ 55)(53\ 54)\rangle$; order 2; 30 orbits
Interesting eigenvalues: $-2.73540427^1, 2.42255680^1$
Nine-cycles per vertex: $8^2, 9^4, 10^{14}, 11^{15}, 12^9, 13^8, 14^4, 16^2$; 73 altogether
Vertices at distance six: $0^{44}, 1^{14}$

**Cage $G_{11}$.**
$E(G_{11}) = E(H) \cup \{$1-9, 2-15, 3-38, 4-49, 5-20, 6-55, 7-34, 8-41, 10-22, 11-48, 12-54, 13-28, 14-43, 16-33, 17-47, 18-40, 19-29, 21-44, 23-32, 24-39, 25-56, 26-50, 27-35, 30-58, 31-52, 36-45, 37-53, 42-51, 46-57$\}$
No nontrivial automorphisms
Interesting eigenvalues: $-2.74601966^1, 2.42365543^1$
Nine-cycles per vertex: $8^1, 9^4, 10^{12}, 11^{10}, 12^{15}, 13^7, 14^7, 16^2$; 75 altogether
Vertices at distance six: $0^{42}, 1^{16}$

**Cage $G_{12}$.**
$E(G_{12}) = E(H) \cup \{$1-9, 2-41, 3-14, 4-46, 5-33, 6-18, 7-38, 8-23, 10-28, 11-52, 12-20, 13-37, 15-24, 16-50, 17-57, 19-42, 21-47, 22-31, 25-54, 26-34, 27-48, 29-43, 30-56, 32-51, 35-58, 36-44, 39-55, 40-49, 45-53$\}$
$\mathrm{Aut}(G_{12}) = \langle(1\ 48)(2\ 27)(3\ 26)(4\ 34)(5\ 33)(6\ 32)(7\ 51)(8\ 50)(9\ 49)(10\ 40)(11\ 39)(12\ 55)(13\ 54)(14\ 25)(15\ 24)(16\ 23)(17\ 22)(18\ 31)(19\ 30)(20\ 56)(21\ 57)(28\ 41)(29\ 42)(35\ 46)(36\ 45)(37\ 53)(38\ 52)(47\ 58)\rangle$; order 2; 30 orbits
Interesting eigenvalues: $-2.64207341^1, 2.45694395^1$
Nine-cycles per vertex: $13^6, 14^{20}, 15^{21}, 16^5, 17^6$; 95 altogether
Vertices at distance six: $0^{18}, 1^{20}, 2^{18}, 3^2$

**Cage $G_{13}$.**

$E(G_{13}) = E(H) \cup \{1\text{-}9, 2\text{-}33, 3\text{-}41, 4\text{-}24, 5\text{-}17, 6\text{-}45, 7\text{-}30, 8\text{-}38, 10\text{-}19, 11\text{-}53, 12\text{-}27, 13\text{-}47, 14\text{-}22, 15\text{-}58, 16\text{-}36, 18\text{-}49, 20\text{-}43, 21\text{-}32, 23\text{-}51, 25\text{-}56, 26\text{-}37, 28\text{-}42, 29\text{-}50, 31\text{-}55, 34\text{-}46, 35\text{-}52, 39\text{-}48, 40\text{-}54, 44\text{-}57\}$

No nontrivial automorphisms

Interesting eigenvalues: $-2.68120334^1, -1^1, 2.44375425^1$

Nine-cycles per vertex: $12^4, 13^8, 14^{10}, 15^{17}, 16^{15}, 17^4$; 95 altogether

Vertices at distance six: $0^6, 1^{26}, 2^{21}, 3^4, 4^1$

**Cage $G_{14}$.**

$E(G_{14}) = E(H) \cup \{1\text{-}9, 2\text{-}27, 3\text{-}41, 4\text{-}50, 5\text{-}36, 6\text{-}55, 7\text{-}20, 8\text{-}47, 10\text{-}52, 11\text{-}35, 12\text{-}42, 13\text{-}29, 14\text{-}49, 15\text{-}58, 16\text{-}24, 17\text{-}40, 18\text{-}34, 19\text{-}28, 21\text{-}43, 22\text{-}51, 23\text{-}31, 25\text{-}46, 26\text{-}54, 30\text{-}38, 32\text{-}56, 33\text{-}48, 37\text{-}45, 39\text{-}53, 44\text{-}57\}$

$\mathrm{Aut}(G_{14}) = \langle(1\ 2)(3\ 9)(4\ 10)(5\ 11)(6\ 12)(7\ 42)(8\ 41)(13\ 55)(14\ 54)(15\ 26)(16\ 25)(17\ 46)(18\ 45)(19\ 44)(20\ 43)(27\ 58)(28\ 57)(29\ 56)(30\ 32)(33\ 38)(34\ 37)(35\ 36)(39\ 48)(40\ 47)(49\ 53)(50\ 52), (1\ 14\ 33\ 36\ 39\ 26)(2\ 15\ 48\ 35\ 38\ 54)(3\ 16\ 47\ 11\ 30\ 55)(4\ 17\ 46\ 10\ 29\ 56)(5\ 40\ 25\ 9\ 13\ 32)(6\ 41\ 24\ 8\ 12\ 31)(7\ 42\ 23)(18\ 45\ 52\ 28\ 57\ 50)(19\ 44\ 51)(20\ 43\ 22)(27\ 58\ 49\ 34\ 37\ 53)\rangle$; order 12; 10 orbits

Interesting eigenvalues: $-2.59647408^1, -1^4, 2^1, 2.42304638^2$

Nine-cycles per vertex: $12^4, 14^{21}, 15^{12}, 16^{15}, 17^6$; 96 altogether

Vertices at distance six: $0^{28}, 1^{12}, 2^{18}$

**Cage $G_{15}$.**

$E(G_{15}) = E(H) \cup \{1\text{-}9, 2\text{-}22, 3\text{-}51, 4\text{-}45, 5\text{-}17, 6\text{-}28, 7\text{-}54, 8\text{-}48, 10\text{-}42, 11\text{-}19, 12\text{-}27, 13\text{-}46, 14\text{-}53, 15\text{-}23, 16\text{-}39, 18\text{-}34, 20\text{-}55, 21\text{-}30, 24\text{-}43, 25\text{-}49, 26\text{-}57, 29\text{-}37, 31\text{-}47, 32\text{-}41, 33\text{-}52, 35\text{-}58, 36\text{-}44, 38\text{-}50, 40\text{-}56\}$

$\mathrm{Aut}(G_{15}) = \langle(4\ 51)(5\ 52)(6\ 33)(7\ 34)(8\ 35)(9\ 58)(10\ 57)(11\ 56)(12\ 40)(13\ 39)(14\ 16)(17\ 53)(18\ 54)(19\ 55)(25\ 43)(26\ 42)(27\ 41)(28\ 32)(29\ 31)(36\ 48)(37\ 47)(38\ 46)(44\ 49)(45\ 50), (1\ 15)(2\ 23)(3\ 24)(4\ 25)(5\ 26)(6\ 27)(7\ 12)(8\ 13)(9\ 14)(10\ 53)(11\ 54)(16\ 58)(17\ 57)(18\ 56)(19\ 55)(33\ 41)(34\ 40)(35\ 39)(36\ 38)(42\ 52)(43\ 51)(44\ 50)(45\ 49)(46\ 48), (1\ 28\ 15\ 32)(2\ 29\ 23\ 31)(3\ 37\ 24\ 47)(4\ 38\ 43\ 48)(5\ 39\ 42\ 8)(6\ 16\ 41\ 9)(7\ 17\ 40\ 10)(11\ 54\ 18\ 56)(12\ 53\ 34\ 57)(13\ 52\ 35\ 26)(14\ 33\ 58\ 27)(19\ 55)(22\ 30)(25\ 46\ 51\ 36)(44\ 49\ 45\ 50)\rangle$; order 8; 13 orbits

Interesting eigenvalues: $-2.65866887^1, -1^2, 1^1, 2^1, 2.41573211^1$

Nine-cycles per vertex: $8^1, 10^2, 12^4, 14^{16}, 15^{32}, 16^3$; 92 altogether

Vertices at distance six: $0^{31}, 1^{16}, 2^2, 3^8, 4^1$

**Cage $G_{16}$.**

$E(G_{16}) = E(H) \cup \{1\text{-}9, 2\text{-}50, 3\text{-}38, 4\text{-}31, 5\text{-}23, 6\text{-}15, 7\text{-}45, 8\text{-}28, 10\text{-}33, 11\text{-}21, 12\text{-}39, 13\text{-}55, 14\text{-}49, 16\text{-}35, 17\text{-}41, 18\text{-}58, 19\text{-}30, 20\text{-}47, 22\text{-}52, 24\text{-}57, 25\text{-}34, 26\text{-}48, 27\text{-}40, 29\text{-}54, 32\text{-}43, 36\text{-}53, 37\text{-}46, 42\text{-}51, 44\text{-}56\}$

$\mathrm{Aut}(G_{16}) = \langle(1\ 38)(2\ 3)(4\ 50)(5\ 51)(6\ 42)(7\ 41)(8\ 40)(9\ 39)(10\ 12)(13\ 33)(14\ 32)(15\ 43)(16\ 44)(17\ 45)(18\ 46)(19\ 47)(23\ 52)(24\ 53)(25\ 54)(26\ 29)(27\ 28)(30\ 48)(31\ 49)(34\ 55)(35\ 56)(36\ 57)(37\ 58)\rangle$; order 2; 31 orbits

Interesting eigenvalues: $-2.73514876^1, -1^1, 2.39671861^1$

Nine-cycles per vertex: $8^1, 9^2, 10^{18}, 11^{22}, 12^{13}, 13^2$; 70 altogether
Vertices at distance six: $0^{56}, 1^2$

**Cage $G_{17}$.**
$E(G_{17}) = E(H) \cup \{$1-9, 2-26, 3-35, 4-21, 5-41, 6-16, 7-31, 8-52, 10-43, 11-19, 12-34, 13-50, 14-27, 15-45, 17-57, 18-38, 20-29, 22-51, 23-44, 24-32, 25-39, 28-54, 30-47, 33-56, 36-46, 37-53, 40-49, 42-55, 48-58$\}$
$\mathrm{Aut}(G_{17}) = \langle$(1 4)(2 3)(5 9)(6 8)(10 41)(11 40)(12 39)(13 38)(14 37)(15 53)(16 52)(17 51)(18 50)(19 49)(20 48)(21 58)(22 57)(23 56)(24 33)(25 34)(26 35)(27 36)(28 46)(29 47)(42 43)(44 55)(45 54), (1 16 55)(2 15 54)(3 45 53)(4 44 52)(5 43 8)(6 42 9)(7 41 10)(11 31 40)(12 30 39)(13 29 25)(14 28 26)(17 56 58)(18 33 48)(19 32 49)(20 24 50)(21 23 51)(34 47 38)(35 46 37)$\rangle$; order 6; 13 orbits
Interesting eigenvalues: $-2.69512475^1, -2^1, 1^1, 2.36233983^1$
Nine-cycles per vertex: $9^2, 10^{18}, 11^{24}, 12^{14}$; 70 altogether
Vertices at distance six: $0^{50}, 1^8$

**Cage $G_{18}$.**
$E(G_{18}) = E(H) \cup \{$1-9, 2-35, 3-49, 4-23, 5-13, 6-28, 7-42, 8-19, 10-46, 11-39, 12-33, 14-52, 15-58, 16-25, 17-48, 18-37, 20-53, 21-32, 22-45, 24-40, 26-34, 27-54, 29-38, 30-50, 31-57, 36-44, 41-56, 43-51, 47-55$\}$
$\mathrm{Aut}(G_{18}) = \langle$(1 6 49 38 16 41)(2 28 48 39 15 42)(3 29 17 40 58 7)(4 30 18 24 57 8)(5 50 37 25 56 9)(10 13 51 36 26 55)(11 14 43 35 27 47)(12 52 44 34 54 46)(19 23 31)(20 22 32)(33 53 45)$\rangle$; order 6; 12 orbits
Interesting eigenvalues: $-2.60085914^1, 2.39939037^2$
Nine-cycles per vertex: $12^7, 14^{24}, 15^{24}, 16^3$; 92 altogether
Vertices at distance six: $0^{49}, 1^6, 2^3$