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Subgraphs of random graphs with specified degrees

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Abstract. If a graph is chosen uniformly at random from all the graphs with a given degree sequence, what can be said about its subgraphs? The same can be asked of bipartite graphs, equivalently 0-1 matrices. These questions have been studied by many people. In this paper we provide a partial survey of the field, with emphasis on two general techniques: the method of switchings and the multidimensional saddle-point method.

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1. Introduction

In this paper we will be concerned with *simple* graphs: those having no loops or parallel edges. Two classes of simple graphs will be considered, *generic* graphs, and *bipartite* graphs. In the former case, there are n vertices any two of which may be adjacent. In the latter case, there are two disjoint *classes* of respectively m and n vertices, and all edges must have one vertex from each set. The phrase "generic graph" is not standard, but we adopt it here for the sake of clarity. If we refer merely to "graph", we might mean either type.

The *degree* of a vertex is the number of edges incident to it, and the *degree sequence* of a graph is a list of the degrees of the vertices. In the case of generic graphs, we will denote the degree sequence by $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$. It satisfies the conditions that $0 \le d_i \le n-1$ for each i, and $\sum_i d_i$ is even. Let $\mathcal{G}(\boldsymbol{d})$ be the set of all generic graphs with degree sequence $\boldsymbol{d}$.

In the case of bipartite graphs, we will denote the degree sequence by $(\boldsymbol{s}, \boldsymbol{t})$, where $\boldsymbol{s} = (s_1, s_2, \ldots, s_m)$ are the degrees in one class and $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ are the degrees in the other class. We have the conditions $0 \le s_j \le n$ for each j, $0 \le t_k \le m$ for each k, and $\sum_j s_j = \sum_k t_k$. Let $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ be the set of all bipartite graphs with degree sequence $(\boldsymbol{s}, \boldsymbol{t})$. Examples appear in Figure 1.

In each case, stronger conditions on the degree sequence are needed before a graph with that degree sequence can be guaranteed to exist, but we will not require those conditions here.

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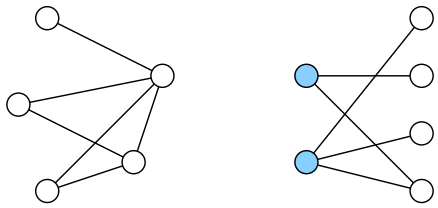

Figure 1. Members of $\mathcal{G}\big((1,2,2,4,3)\big)$ and $\mathcal{B}\big((2,3),(1,1,1,2)\big)$

If $\mathcal{G}(\boldsymbol{d}) \neq \emptyset$, which we will assume from now on, we can promote it to a probability space by assigning each element the same probability. It is this space that we refer to when we discuss a "random generic graph with degree sequence $\boldsymbol{d}$". Similarly, for a "random bipartite graph with degree sequence $(\boldsymbol{s},\boldsymbol{t})$".

There is a fair amount of literature on random graphs of these types, some of which we will cite as we go. In this incomplete survey we will focus on a particular issue: what is the probability that a specified subgraph occurs? More generally we can ask for the distribution of the number of subgraphs of a given type. Our asymptotics will be with respect to $n \to \infty$ for generic graphs, or $m, n \to \infty$ for bipartite graphs, with other parameters such as $\boldsymbol{d}$ being functions of n, or (m,n), unless otherwise specified.

Since we are dealing with uniform discrete distributions, our probability questions are just counting questions in disguise. If $\boldsymbol{X}$ is a generic graph, let $\mathcal{G}(\boldsymbol{d},\boldsymbol{X})$ denote the set of generic graphs with degree sequence $\boldsymbol{d}$ and no edge in common with $\boldsymbol{X}$. Then, if $\boldsymbol{x}$ is the degree sequence of $\boldsymbol{X}$, the probability that a random generic graph with degree sequence $\boldsymbol{d}$ has $\boldsymbol{X}$ as a subgraph is

$$\mathbf{P}_{\mathcal{G}(\boldsymbol{d})}(\boldsymbol{X}) = \frac{|\mathcal{G}(\boldsymbol{d}-\boldsymbol{x},\boldsymbol{X})|}{|\mathcal{G}(\boldsymbol{d})|}. \tag{1}$$

Similarly, if $\boldsymbol{X}$ is a bipartite graph with classes of size m and n, let $\mathcal{B}(\boldsymbol{s},\boldsymbol{t},\boldsymbol{X})$ denote the set of bipartite graphs with degree sequence $(\boldsymbol{s},\boldsymbol{t})$ and no edge in common with $\boldsymbol{X}$. Then, if $(\boldsymbol{x},\boldsymbol{y})$ is the degree sequence of $\boldsymbol{X}$, the probability that a random bipartite graph with degree sequence $(\boldsymbol{s},\boldsymbol{t})$ has $\boldsymbol{X}$ as a subgraph is

$$\mathbf{P}_{\mathcal{B}(\boldsymbol{s},\boldsymbol{t})}(\boldsymbol{X}) = \frac{|\mathcal{B}(\boldsymbol{s}-\boldsymbol{x},\boldsymbol{t}-\boldsymbol{y},\boldsymbol{X})|}{|\mathcal{B}(\boldsymbol{s},\boldsymbol{t})|}. \tag{2}$$

Largely due to the different techniques that have been fruitful, we divide the discussion into two parts. In Section 2, we consider the case where the degrees are low, such as when they are bounded. By complementation, this also applies when the degrees are almost as large as possible. In Section 3, we consider the case where the degrees are something like a constant fraction of the number of vertices.

2. Sparse graphs

For this section we consider random generic or bipartite graphs whose degrees do not grow very quickly with the size of the graph.

Define $d_{\max} = \max\{d_1, \dots, d_n\}$, and similarly $s_{\max}$, $t_{\max}$, $x_{\max}$ and $y_{\max}$. For integer $k \geq 0$, we write $(a)_k = a(a-1)\cdots(a-k+1)$.

The most celebrated technique is called the *configuration model* or *pairing model*. The popular version of it was introduced by Bollobás [2], though the concept has an older history, see [33]. We describe it for generic graphs of degree sequence $\boldsymbol{d}$; an obvious variant works the same for bipartite graphs.

Consider n disjoint *cells* $v_1, \dots, v_n$, where cell v_i is a set of d_i *points*. This makes $2E = \sum_i d_i$ points in total (recall that $\sum_i d_i$ must be even). Choose a random *pairing* (partition of the points into E pairs), where each of the $(2E)!/(E!\,2^E)$ possible pairings are equally likely. A pairing P is *simple* if each pair involves two different cells, and no two pairs involve the same two cells. In that case we can make a graph $\boldsymbol{G}(P)$ whose vertices are $v_1, \dots, v_n$ and whose edges are those $v_j v_k$ such that there is a pair involving v_j and v_k. Clearly $\boldsymbol{G}(P) \in \mathcal{G}(\boldsymbol{d})$.

The key feature of pairings is that each graph in $\mathcal{G}(\boldsymbol{d})$ corresponds to exactly $\prod_j d_j!$ simple pairings. Therefore, a random simple pairing yields a random graph in $\mathcal{G}(\boldsymbol{d})$ (i.e., with the uniform distribution). If $P_2(\boldsymbol{d}, \boldsymbol{X})$ is the probability that a random pairing is simple and avoids the graph $\boldsymbol{X}$, then

$$|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X})| = \frac{(2E)!}{E!\,2^E \prod_j d_j!}\, P_2(\boldsymbol{d}, \boldsymbol{X}).$$

The other value $|\mathcal{G}(\boldsymbol{d})|$ required by (1) is just the special case of $\boldsymbol{X} = \emptyset$, where $\emptyset$ is the graph with no edges. So the subgraph probability problem reduces to the sometimes easier calculation of the probabilities $P_2(\boldsymbol{d}, \boldsymbol{X})$.

If $d_{\max}$ is at most slowly increasing, for example if $d_{\max} = O((\log n)^{1/3})$, then $P_2(\boldsymbol{d}, \boldsymbol{X})$ can be estimated under mild additional conditions on $\boldsymbol{d}$ and $\boldsymbol{X}$ using inclusion-exclusion or the method of moments, see Bollobás and McKay [3]. For $d_{\max} = O(1)$ (refer to Janson [14] for necessary and sufficient conditions), $P_2(\boldsymbol{d}, \emptyset)$ is bounded above 0, which has an immediate dramatic consequence: every event that is asymptotically unlikely or certain for random pairings is also asymptotically unlikely or certain (respectively!) for random generic graphs with degree sequence $\boldsymbol{d}$. A great many theorems are based on this observation, and the equivalent observation for bipartite graphs, see Wormald [33] and Janson et al. [15] for summaries.

When $d_{\max}$ increases more quickly with n, the same methods do not suffice to estimate $P_2(\boldsymbol{d}, \boldsymbol{X})$. For example, the terms of the inclusion-exclusion expansion cancel too precisely to allow estimation of their sum. An alternative method is required, which is where the method of switchings comes in.

The basic idea behind the method of switchings is the following: given two finite sets A, B and a relation R (in this context called a *switching operation*) between them, then the ratio of the average number of elements of B related to each element of A to the average number of elements of A related to each element

of B is the same as the ratio of $|B|$ to $|A|$. This idea can be applied to the problem of subgraph probabilities in two different ways. In the first approach graphs with a given degree sequence are manipulated directly. In the second approach, switchings are used to analyse pairings.

We first consider the direct application of switching to subgraph probabilities. Following [20], we generalise the notation $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X})$ to $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y})$, where $\boldsymbol{Y}$ is a subgraph of $\boldsymbol{X}$: $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y})$ is the set of all generic graphs $\boldsymbol{G} \in \mathcal{G}(\boldsymbol{d})$ such that the intersection of $\boldsymbol{G}$ and $\boldsymbol{X}$ (considered as sets of edges) is exactly $\boldsymbol{Y}$. We can see that $\mathcal{G}(\boldsymbol{d}) = \bigcup_{\boldsymbol{Y} \subseteq \boldsymbol{X}} \mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y})$, $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}) = \mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{\emptyset})$, and $|\mathcal{G}(\boldsymbol{d}-\boldsymbol{x}, \boldsymbol{X})| = |\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{X})|$. Therefore,

$$\mathbf{P}_{\mathcal{G}(\boldsymbol{d})}(\boldsymbol{X}) = \Bigg(\sum_{\boldsymbol{Y} \subseteq \boldsymbol{X}} \frac{|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y})|}{|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{X})|} \Bigg)^{-1}. \tag{3}$$

Let $e = ab$ be an edge of $\boldsymbol{X}$ that is not an edge of $\boldsymbol{Y}$. We can define a relation between $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y} \cup ab)$ and $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y})$ using the switching operation shown in Figure 2. If the left diagram appears in a graph $\boldsymbol{G} \in \mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y} \cup ab)$, $ac, bd \notin \boldsymbol{G}$, and $ac, cd, bd \notin \boldsymbol{X}$, then replacing it by the right diagram produces a graph in $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y})$.

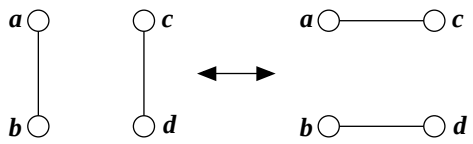

Figure 2. A simple switching operation

By bounding the number of switching operations that can apply to $\boldsymbol{G}$, and similarly bounding the number of ways of coming back from $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y})$, we obtain bounds on the ratio of $|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y} \cup ab)|$ to $|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y})|$. Combining all such ratios to obtain the relative sizes of $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{Y})$ for all $\boldsymbol{Y} \subseteq \boldsymbol{X}$, we can finally apply (3) to get $\mathbf{P}_{\mathcal{G}(\boldsymbol{d})}(\boldsymbol{X})$.

The following is a consequence of Theorems 2.9 and 2.10 of McKay [20]. Define $\Delta = d_{\max}(d_{\max} + x_{\max})$ and $X = \frac{1}{2}\sum_i x_i$.

Theorem 2.1 ([20]). *If $\Delta X = o(E)$ then*

$$\mathbf{P}_{\mathcal{G}(\boldsymbol{d})}(\boldsymbol{X}) = \frac{\prod_{j=1}^{n} (d_j)_{x_j}}{2^X (E)_X} \big(1 + O(\Delta X/E)\big).$$

In the bipartite case, we can use the same switching operation provided a and c are in opposite vertex classes. Define $\Delta' = (s_{\max}+t_{\max})(s_{\max}+t_{\max}+x_{\max}+y_{\max})$ and $X = \sum_j x_j = \sum_k y_k$.

Theorem 2.2 ([20]). *If $\Delta' X = o(E)$ then*

$$\mathbf{P}_{\mathcal{B}(\boldsymbol{s},\boldsymbol{t})}(\boldsymbol{X}) = \frac{\prod_{j=1}^{m} (s_j)_{x_j} \prod_{k=1}^{n} (t_k)_{y_k}}{(E)_X} \big(1 + O(\Delta' X/E)\big).$$

For both Theorems 2.1 and 2.2, the exact bounds given in [20] can be useful even when the error term is not vanishing.

Since [20], two improvements to this method have been found. As first shown by McKay and Wormald [26] in a slightly different context, the counting is substantially easier if the more complex switching operation of Figure 3 is used.

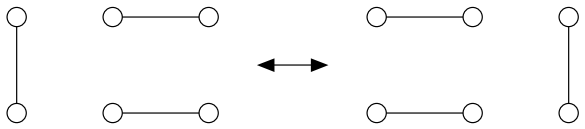

Figure 3. A better switching operation

The other improvement, introduced by Lieby, McKay, McLeod and Wanless [18], is a rearrangement of the calculation. Let the edges of $\boldsymbol{X}$ be $e_1, e_2, \ldots, e_X$ and define $\boldsymbol{X}_j$ to be the graph with edges $\{e_1, \ldots, e_j\}$, $0 \le j \le X$. For $j \ge 1$, we have $\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}_{j-1}, \boldsymbol{X}_{j-1}) = \mathcal{G}(\boldsymbol{d}, \boldsymbol{X}_j, \boldsymbol{X}_{j-1}) \,\dot\cup\, \mathcal{G}(\boldsymbol{d}, \boldsymbol{X}_j, \boldsymbol{X}_j)$, and so

$$\mathbf{P}_{\mathcal{G}(\boldsymbol{d})}(\boldsymbol{X}) = \frac{|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{X})|}{|\mathcal{G}(\boldsymbol{d}, \emptyset, \emptyset)|} = \prod_{j=1}^{X} \biggl(1 + \frac{|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}_j, \boldsymbol{X}_{j-1})|}{|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}_j, \boldsymbol{X}_j)|}\biggr)^{-1},$$

assuming all the denominators are nonzero. The ratio of $|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}_j, \boldsymbol{X}_{j-1})|$ to $|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X}_j, \boldsymbol{X}_j)|$ can be obtained by analysing a switching, as before. This method avoids the problematic sum in (3), and also allows the ordering of the edges of $\boldsymbol{X}$ to be tuned to optimise the precision of the answer.

As we mentioned, the other way to apply switchings is to use them to analyse the pairing model. Recall that the task is to estimate the probability $P_2(\boldsymbol{d}, \boldsymbol{X})$ that a random pairing is simple and avoids $\boldsymbol{X}$. The basic idea is to classify pairings according to their non-simple parts (such as double pairs or pairs hitting $\boldsymbol{X}$). Then switching operations are used to estimate the relative sizes of these classes. This was first done by McKay [22] for generic graphs and McKay [21] for bipartite graphs; we summarise the main theorems below. Define E, Δ and Δ' as before.

Theorem 2.3 ([21, 22]).

(a) Suppose $d_{\max} \ge 1$ and $\Delta = o(E^{1/2})$. Then, as $n \to \infty$,

$$|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X})| = \frac{(2E)!}{E!\, 2^E \prod_{j=1}^n d_j!} \exp\biggl(-\frac{\sum_{j=1}^n d_j(d_j-1)}{4E} - \frac{\bigl(\sum_{j=1}^n d_j(d_j-1)\bigr)^2}{16E^2} - \frac{\sum_{jk \in \boldsymbol{X}} d_j d_k}{2E} + O(\Delta^2/E)\biggr).$$

(b) Suppose $s_{\max} \geq 1$ *and* $\Delta' = o(E^{1/2})$*. Then, as* $m, n \to \infty$*,*

$$|\mathcal{B}(\boldsymbol{s},\boldsymbol{t},\boldsymbol{X})| = \frac{E!}{\prod_{j=1}^{m} s_j! \prod_{k=1}^{n} t_k!} \exp\biggl(-\frac{\sum_{j=1}^{n} s_j(s_j-1) \sum_{k=1}^{n} t_k(t_k-1)}{2E^2} - \frac{\sum_{jk\in\boldsymbol{X}} s_j t_k}{E} + O(\Delta'^2/E)\biggr).$$

In the above, the notion $\sum_{jk\in\boldsymbol{X}}$ means a sum over unordered pairs $\{j,k\}$ such that jk is an edge of $\boldsymbol{X}$, with j being in the first class for the bipartite case.

The special case $P_2(\boldsymbol{d},\boldsymbol{\emptyset})$, needed for estimating $|\mathcal{G}(\boldsymbol{d})|$ was improved McKay and Wormald [27] to cover generic graphs with $d_{\max} = o(E^{1/3})$, and by Greenhill, McKay and Wang [12] to cover bipartite graphs with $s_{\max}t_{\max} = o(E^{2/3})$. An example of a switching operation used by these papers is shown in Figure 4, where the shaded ovals represent the cells of the pairing.

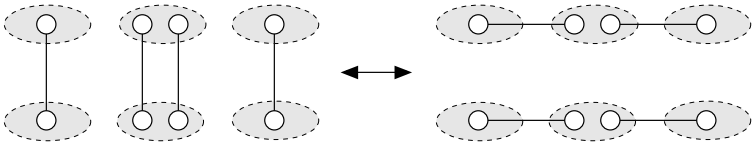

Figure 4. Removing a double pair from a pairing

The distribution of cycle counts in random regular graphs has been studied quite a lot. For fixed or very slowly increasing degree, the counts of fixed length cycles are asymptotically Poisson and independent, as shown by Bollobás [2] and Wormald [32]. Counts of longer cycles were studied by Garmo [9]. By using switching operations specifically tailored for the purpose, McKay, Wormald and Wysocka [28] found the joint distribution of the counts of cycles up to length g in a random regular graph of order n and degree d, whenever $(d-1)^{2g-1} = o(n)$. Gao and Wormald [8] found the central part of the distribution of the number of cycles of length g under the weaker condition $d = o(n^{2/(3g-2)})$, as a special case of a theory (developed in [7]) that allows asymptotic normality of the counts of many small subgraphs to be inferred from certain higher moments.

Perhaps the deepest result of this nature was that of Robinson and Wormald [30, 31] who showed that almost all regular graphs of fixed degree $d \geq 3$ are hamiltonian. The somewhat easier problem of extending this to all $d \geq 3$ was achieved later in [5, 17].

Counts of perfect matchings in the regular cases of $\mathcal{G}(\boldsymbol{d})$ and $\mathcal{B}(\boldsymbol{s},\boldsymbol{t})$ for small degree were studied by Bollobás and McKay [3]. The expectation is also found in [3] in the bipartite case for extreme degrees ($m = n$ and degree at least $n - n^{1-\varepsilon}$) using enumeration results for Latin rectangles [10]. A similar calculation for extreme-degree generic graphs could easily be done starting with the results in [29].

The furthest reach of the switching method to higher vertex degrees was achieved by Krivelevich, Sudakov, Vu and Wormald [17], who determined several almost-sure properties of random regular graphs of degree $o(n)$.

Ben-Shimon and Krivelevich [1] used switchings to study the number of edges spanned by a set of vertices, or between two sets of vertices, in regular graphs of degree $o(n^{1/2})$.

3. Dense graphs

The methods of the previous section are most suitable when the graph degrees are relatively small. The exception is that Theorems 2.1 and 2.2 can provide the probability of very small subgraphs for higher degrees in some cases.

Define a new parameter $\lambda \in [0,1]$, which we call the *density*. For generic graphs, $\lambda = E/\binom{n}{2}$. For bipartite graphs, $\lambda = E/(mn)$.

It will be worth comparing the subgraph probabilities in $\mathcal{G}(\boldsymbol{d})$ and $\mathcal{B}(\boldsymbol{s},\boldsymbol{t})$ to the probabilities in similar binomial random graph models. Let $\mathcal{G}_{n,p}$ be the probability space of random generic graphs with n vertices and edge probability p (i.e., each possible edge is present with independent probability p), and let $\mathcal{B}_{m,n,p}$ be the similar space of random bipartite graphs with vertex classes of size m and n. Intuition suggests that subgraph probability $\mathbf{P}_{\mathcal{G}_{n,\lambda}}(\boldsymbol{X}) = \lambda^X$ may be a rough approximation to those in $\mathbf{P}_{\mathcal{G}(\boldsymbol{d})}(\boldsymbol{X})$, and similarly for $\mathbf{P}_{\mathcal{B}_{m,n,\lambda}}(\boldsymbol{X}) = \lambda^X$ versus $\mathbf{P}_{\mathcal{B}(\boldsymbol{s},\boldsymbol{t})}(\boldsymbol{X})$.

The strongest results of this type were proved by Greenhill and McKay [11] and McKay [24]. We will start with generic graphs and need the following additional parameters, for $\ell, m \geq 1$.

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i = \lambda(n-1) = 2E/n$$

$$\delta_j = d_j - \bar{d} + \lambda x_j \quad (1 \leq j \leq n), \qquad X_\ell = \sum_{j=1}^{n} x_j^\ell,$$

$$L = \sum_{jk \in \boldsymbol{X}} (\delta_j - x_j)(\delta_k - x_k), \qquad C_{\ell,m} = \sum_{j=1}^{n} \delta_j^\ell x_j^m.$$

Theorem 3.1 ([24]). *Let $a, b > 0$ be constants such that $a + b < \frac{1}{2}$. For some $\varepsilon > 0$, suppose that $d_j - \bar{d}$ and x_j are uniformly $O(n^{1/2+\varepsilon})$ for $1 \leq j \leq n$, and that $X = O(n^{1+2\varepsilon})$. For sufficient large n, suppose that*

$$\min\{\bar{d}, n - \bar{d} - 1\} \geq \frac{n}{3a \log n}.$$

Then, provided ε is small enough, we have

$$\mathbf{P}_{\mathcal{G}(\boldsymbol{d})}(\boldsymbol{X}) = \lambda^X \exp\biggl(\frac{(1-\lambda)X}{\lambda n} - \frac{(1+\lambda)X_2}{2\lambda n} - \frac{(1+\lambda)(1+2\lambda)X_3}{6\lambda^2 n^2} + \frac{(1-\lambda)X^2}{\lambda n^2} - \frac{L}{\lambda(1-\lambda)n^2} + \frac{C_{1,1}}{\lambda n} + \frac{(1+2\lambda)C_{1,2}}{2\lambda^2 n^2} - \frac{C_{2,1}}{2\lambda^2 n^2} + O(n^{-b}) \biggr).$$

A corollary of Theorem 3.1 is that $\mathbf{P}_{\mathcal{G}(\boldsymbol{d})}(\boldsymbol{X}) \sim \mathbf{P}_{\mathcal{G}_{n,\lambda}}(\boldsymbol{X})$ when

$$X \max_j |d_j - d| + (1-\lambda)X_2 = o\big(\lambda n\big).$$

This sufficient condition holds, for example, if $X = O(n^{1/2-2\varepsilon})$, or if $d_j - \bar{d}$ and x_j are uniformly $O(n^{\varepsilon})$ for $1 \le j \le n$ and $X = O(n^{1-2\varepsilon})$.

A special case of this result was proved by Krivelevich, Sudakov and Wormald [16], who determined the probability of induced subgraphs of $o(n^{1/2})$ vertices in random regular graphs of degree $(n-1)/2$ under some conditions on the degree sequence of the subgraph.

For bipartite graphs, a similar result holds. Define the following parameters for integers $h, \ell \ge 0$.

$$\bar{s} = \frac{1}{m}\sum_{j=1}^{m} s_j = E/m = \lambda n, \qquad \bar{t} = \frac{1}{n}\sum_{k=1}^{n} t_k = E/n = \lambda m,$$

$$\xi_j = s_j - \bar{s} + \lambda x_j \quad (1 \le j \le m), \qquad \eta_k = t_k - \bar{t} + \lambda y_k \quad (1 \le k \le n)$$

$$Z = \sum_{jk \in \boldsymbol{X}} (x_j - \xi_j)(y_k - \eta_k). \qquad Q_{h,\ell} = n^{1-h-\ell}\sum_{j=1}^{m} \xi_j^h x_j^\ell + m^{1-h-\ell}\sum_{k=1}^{n} \eta_k^h y_k^\ell$$

Theorem 3.2 ([11]). *Let $a, b > 0$ be constants such that $a + b < \frac{1}{2}$. For some $\varepsilon > 0$, suppose that m, $n \to \infty$ with $n = o(m^{1+\varepsilon})$ and $m = o(n^{1+\varepsilon})$, and further that $s_j - \bar{s}$, x_j, $t_k - \bar{t}$ and y_k are uniformly $O(n^{1/2+\varepsilon})$ for $1 \le j \le m$ and $1 \le k \le n$, and $X = O(n^{1+2\varepsilon})$. Assume*

$$\frac{(1-2\lambda)^2}{4\lambda(1-\lambda)}\left(1 + \frac{5m}{6n} + \frac{5n}{6m}\right) \le a \log n.$$

Then, provided $\varepsilon > 0$ is small enough, we have

$$\begin{aligned}\mathbf{P}_{\mathcal{B}(\boldsymbol{s},\boldsymbol{t})}(\boldsymbol{X}) = \lambda^X \exp\Bigg(&\frac{(1-\lambda)X}{2\lambda}\Big(\frac{1}{n} + \frac{1}{m}\Big) + \frac{(1-\lambda)X^2}{2\lambda mn} + \frac{Q_{1,1}}{\lambda} \\ &- \frac{(1+\lambda)Q_{0,2}}{2\lambda} - \frac{Q_{2,1}}{2\lambda^2} + \frac{(1+2\lambda)Q_{1,2}}{2\lambda^2} \\ &- \frac{(1+\lambda)(1+2\lambda)Q_{0,3}}{6\lambda^2} - \frac{Z}{\lambda(1-\lambda)mn} + O(n^{-b})\Bigg).\end{aligned}$$

A corollary of Theorem 3.2 is that $\mathbf{P}_{\mathcal{B}(\boldsymbol{s},\boldsymbol{t})}(\boldsymbol{X}) \sim \mathbf{P}_{\mathcal{B}_{m,n,\lambda}}(\boldsymbol{X})$ when

$$X \max_j |s_j - s| + (1-\lambda)\sum_j x_j^2 = o(\lambda n), \text{ and}$$

$$X \max_k |t_k - t| + (1-\lambda)\sum_k y_k^2 = o(\lambda m).$$

These extra requirements are met, for example, if $X = O(n^{1/2-2\varepsilon})$. Another interesting case is when $s_j - s$, x_j, $t_k - t$ and y_k are uniformly $O(n^{\varepsilon})$ and $X = O(n^{1-2\varepsilon})$.

In [11] and [24] we also give the probability that $\boldsymbol{X}$ is avoided, and the probability of occurrence of a specified induced subgraph. We also record the simpler formulae that are implied if the whole graph or the subgraph is regular.

We mention a sample application of Theorem 3.2. In the binomial model $\mathcal{B}_{n,n,\lambda}$, corresponding to square 0-1 matrices with each entry independently being 1 with probability λ, the expected permanent is exactly $n!\,\lambda^n$. It is interesting to see what the effect of specifying $\boldsymbol{s}$ and $\boldsymbol{t}$ (the row and column sums) is.

Corollary ([11]). *Suppose that $m = n$ and $\boldsymbol{s}, \boldsymbol{t}, \lambda$ satisfy the requirements of Theorem 3.2. Then the expected permanent of a random $n \times n$ matrix over $\{0,1\}$ with row sums $\boldsymbol{s}$ and column sums $\boldsymbol{t}$ is*

$$n!\,\lambda^n \exp\biggl(\frac{1-\lambda}{2\lambda} - \frac{\sum_j (s_j - \bar{s})^2 + \sum_k (t_k - \bar{t})^2}{2\lambda^2 n^2} + O(n^{-b})\biggr). \quad \square$$

Theorems 3.1 and 3.2 are proved by complex analysis, namely a multidimensional saddle-point calculation first demonstrated by McKay and Wormald [23] and McKay [25].

We will sketch the proof method for Theorem 3.1, based on [24]. Consider the n-variable generating function

$$F(\boldsymbol{z}) = \prod_{jk \in \bar{\boldsymbol{X}}} (1 + z_j z_k),$$

where $\bar{\boldsymbol{X}}$ is the set of all unordered distinct pairs $\{j,k\}$ that are not edges of $\boldsymbol{X}$. This function counts n-vertex graphs disjoint from $\boldsymbol{X}$ according to the degrees of their vertices. Specifically,

$$|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X})| = [z_1^{d_1} \cdots z_n^{d_n}] F(\boldsymbol{z}),$$

where the square bracket notation indicates coefficient extraction. By Cauchy's theorem this implies

$$|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X})| = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{jk \in \bar{\boldsymbol{X}}} (1 + z_j z_k)}{z_1^{d_1+1} \cdots z_n^{d_n+1}}\, dz_1 \cdots dz_n,$$

where each integral is along a simple closed contour enclosing the origin anticlockwise. Taking these contours to be circles, namely $z_j = r_j e^{i\theta_j}$ for each j, and changing variables gives

$$|\mathcal{G}(\boldsymbol{d}, \boldsymbol{X})| = \frac{\prod_{jk \in \bar{\boldsymbol{X}}} (1 + r_j r_k)}{(2\pi)^n \prod_{j=1}^n r_j^{d_j}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{jk \in \bar{\boldsymbol{X}}} \bigl(1 + \lambda_{jk}(e^{i(\theta_j + \theta_k)} - 1)\bigr)}{\exp\bigl(i \sum_{j=1}^n d_j \theta_j\bigr)}\, d\boldsymbol{\theta}, \quad (4)$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ and

$$\lambda_{jk} = \frac{r_j r_k}{1 + r_j r_k} \quad (1 \le j, k \le n). \quad (5)$$

Equation (4) is valid for any positive radii $\{r_j\}$, but to facilitate estimation of the integral we choose $\{r_j\}$ so that the linear terms vanish when the logarithm of the integrand is expanded around the origin (in $\boldsymbol{\theta}$ space). This happens when

$$\sum_{k:jk\in\bar{\boldsymbol{X}}} \lambda_{jk} = d_j \quad (1 \le j \le n). \tag{6}$$

Equations 5 and 6 have a unique solution which appears to have no closed form. Under the conditions of Theorem 3.1, the solution can be expanded to sufficient accuracy in terms of λ, $\boldsymbol{d}$ and $\boldsymbol{X}$. This involves summation over small subgraphs of $\boldsymbol{X}$ and the expression is rather complex.

The integrand of (4) achieves its maximum modulus 1 at $\boldsymbol{\theta} = (0, 0, \ldots, 0)$ and $\boldsymbol{\theta} = (\pi, \pi, \ldots, \pi)$, which two points are equivalent under the symmetries of the integrand. We now define two small cubes:

$$\mathcal{R}_0 = \{\boldsymbol{\theta} : |\theta_j| \le n^{-1/2+\varepsilon}, 1 \le j \le n\},\ \mathcal{R}_\pi = \{\boldsymbol{\theta} : |\theta_j - \pi| \le n^{-1/2+\varepsilon}, 1 \le j \le n\},$$

where absolute value is taken modulo 2π. Within $\mathcal{R}_0 \cup \mathcal{R}_\pi$ we expand the logarithm of the integrand up to terms of order 4 and evaluate the integral by first diagonalising the quadratic part (recall that we chose the radii to eliminate the linear part) then integrating term by term. Outside $\mathcal{R}_0 \cup \mathcal{R}_\pi$ we split the region up into many pieces and show that in total the contribution to the integral is negligible.

In the case of empty $\boldsymbol{X}$, Barvinok and Hartigan [13] have identified the matrix (λ_{jk}), with zero diagonal, as the unique symmetric matrix which satisfies (6) and maximises the entropy function

$$-\sum_{jk} \big(\lambda_{jk} \log \lambda_{jk} + (1 - \lambda_{jk}) \log(1 - \lambda_{jk})\big).$$

For the case of $\bar{d} = \Theta(n)$, they then show that an asymptotic approximation of $|\mathcal{G}(\boldsymbol{d})|$ can be expressed as a computable function of (λ_{jk}) whenever the values of λ_{jk} are uniformly bounded away from 0 and 1. This allows for a much larger variation amongst the degrees than Theorem 3.1 allows, but at the expense of more restricted $\bar{d}$ and loss of explicitness. It is also shown in [13] under the same conditions that for a set S of $\Theta(n^2)$ edge-positions, a random graph in $\mathcal{G}(\boldsymbol{d})$ has close to $\sum_{jk\in S} \lambda_{jk}$ edges within S, with high probability.

A result similar to Theorem 3.2 for tournaments was proved by Gao, McKay and Wang [6].

4. Concluding remarks

It is clear that many gaps still remain in our understanding of this problem. For example, there is almost nothing known about the distribution of subgraph counts in $\mathcal{G}(\boldsymbol{d})$ or $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ when the degrees are $\Theta(n)$. In the intermediate range of degrees between $n^{1/2}$ and $n/\log n$, not even the precise value of $|\mathcal{G}(\boldsymbol{d})|$ is known, though there is a well-tested conjecture [27]. The same is true in the bipartite case [4]. Another missing story is that of $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ when m and n are very different.

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