

# ASYMPTOTIC ENUMERATION BY DEGREE SEQUENCE OF GRAPHS OF HIGH DEGREE

Brendan D. McKay

*Computer Science Department, Australian National University,  
GPO Box 4, ACT 2601, Australia*

Nicholas C. Wormald

*Department of Mathematics and Statistics, University of Auckland,  
Private Bag, Auckland, New Zealand*

## **Abstract.**

We consider the estimation of the number of labelled simple graphs with degree sequence  $d_1, d_2, \dots, d_n$  by using an  $n$ -dimensional Cauchy integral. For sufficiently small  $\epsilon$  and any  $c > \frac{2}{3}$ , an asymptotic formula is obtained when  $|d_i - d| < n^{1/2+\epsilon}$  for all  $i$  and  $d = d(n)$  satisfies  $\min\{d, n - d - 1\} \geq cn/\log n$  as  $n \rightarrow \infty$ . These conditions include the degree sequences of almost all graphs, so our result gives as a corollary the asymptotic joint distribution function of the degrees of a random graph. We also give evidence for a formula conjectured to be valid for all  $d(n)$ .

## **1. Introduction.**

We are concerned with locally restricted graphs, that is, graphs having  $n$  labelled vertices with degree sequence  $\mathbf{d} = (d_1, d_2, \dots, d_n)$ . Many results, usually asymptotic, have been obtained on the properties of random graphs in this class for various special sequences  $\mathbf{d}$ . Most, if not all, of these results are spawned directly from methods for, or results on, enumerating such graphs. However, all the progress so far has concerned graphs with relatively small degrees, in fact with  $d_i = o(n^{1/2})$  (and, of course, their complements). Our object in this article is to derive an asymptotic formula (Theorem 2) for the number of locally restricted graphs when  $d_i$  is approximately a constant times  $n$ . The range of validity of the result is such that it applies to the degree sequences of almost all graphs in the model of random graphs with edges chosen independently with probability  $p$ , for constant or slowly vanishing  $p$ .

Let  $G(\mathbf{d})$  denote the number of graphs with degree sequence  $\mathbf{d}$ . The first significant results on the value of  $G(\mathbf{d})$  were obtained by Read [18, 19]. These were exact results, whose appeal is more theoretical than computational, but which allowed Read to obtain an asymptotic formula when  $d_1 = d_2 = \dots = d_n = 3$ . General asymptotic results were then obtained for the case of bipartite graphs—see McKay [12] for a survey. Corresponding formulae for general graphs were then obtained independently by Bender and Canfield [1], Wormald [22] and Bollobás [2], all using roughly the same method. These immediately initiated the discovery of properties of random locally restricted graphs (see Wormald [22, 23, 24], Bollobás [3], Bollobás and McKay [5], McKay [9, 10, 14], Robinson and Wormald [20, 21] and Fenner and Frieze [6] for examples). The method used for the asymptotic enumeration involved the application

of inclusion-exclusion to a model of random locally restricted graphs. This approach yields a result which is only valid when each  $d_i$  grows at most very slowly with  $n$ . An alternative family of methods based on switching edges was used to study random locally restricted graphs by McKay [8] and by McKay and Wormald [15]. A similar method was used by McKay [13] to obtain the following asymptotic formula valid when  $\max_j \{d_j\} = o(E^{1/4})$  and  $E \rightarrow \infty$ :

$$G(\mathbf{d}) \sim \frac{(2E)! e^{-\lambda - \lambda^2}}{E! 2^E \prod_{i=1}^n d_i!}, \quad \text{where } E = \frac{1}{2} \sum_{i=1}^n d_i \quad \text{and} \quad \lambda = \frac{1}{4E} \sum_{i=1}^n d_i(d_i - 1), \quad (1.1)$$

a result which has recently been extended by similar methods to cover the case  $\max_j \{d_j\} = o(E^{1/3})$  (see [16]). Most recently Jerrum and Sinclair [7] have devised an effective procedure which could be used to estimate  $G(\mathbf{d})$  for any of a wide family of degree sequences which includes all regular sequences.

Our approach for larger degrees is quite different. We approximate Cauchy's formula for the coefficient of  $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$  in  $\prod_{1 \leq j < k \leq n} (1 + x_j x_k)$  using the saddle-point method. The resulting formula is valid when  $|d_i - d| < n^{1/2+\epsilon}$  for all  $i$ , where  $d = d(n)$  satisfies  $\min\{d, n - d - 1\} \geq cn/\log n$  for sufficiently small  $\epsilon > 0$  and some  $c > \frac{2}{3}$ .

In Section 2 we isolate some of the important steps in the calculations, and in Section 3 we deal with regular graphs. Then, in Section 4, the modifications required to accommodate varying degrees are given. Some of the implications of our results, and some conjectures arising from them, are discussed in Section 5. The use of our results to prove properties of random graphs is postponed to forthcoming publications.

Although we are concerned here only with labelled graphs, an appropriate property of random graphs will, when proved, extend our results to unlabelled graphs. This was done for graphs of low degree in [15]; see also [4] and [25] for the simpler regular case.

## 2. Definitions and some calculations.

Recall that  $G(\mathbf{d})$  is the number of labelled graphs with degree sequence  $\mathbf{d}$ . We write  $RG(n, d)$  for the number  $G(d, d, \dots, d)$  of regular graphs of degree  $d$  and order  $n$ .

Assume  $n \geq 2$ . We use the following notation.

$$\beta = 1 - \sqrt{\frac{n-2}{2(n-1)}},$$

$I_n$  = the  $n \times n$  identity matrix,  $J_n$  = the  $n \times n$  matrix of all ones,

$T = I_n - \beta J_n/n$  and the associated linear transformation,

$$\mathbf{y} \in \mathbb{R}^n, \quad \boldsymbol{\theta} = T\mathbf{y},$$

$$\mu_k = \sum_{j=1}^n y_j^k \quad \text{for } k \geq 0,$$

$$U_n(t) = \{ \mathbf{x} \mid |x_i| \leq t, \quad i = 1, 2, \dots, n \},$$

$$A_n(t) = \frac{2\pi^{n/2} r^{n-1}}{\Gamma(n/2)} = \text{the surface area of the } n\text{-dimensional sphere of radius } r.$$

From Taylor's Theorem with remainder, we have

**Lemma 1.** *Let  $0 \leq \lambda \leq 1$ . Then for all real  $x$ ,*

$$\begin{aligned} |1 + \lambda(e^{ix} - 1)| &= (1 - 2\lambda(1 - \lambda)(1 - \cos x))^{1/2} \\ &\leq \exp(-\frac{1}{2}\lambda(1 - \lambda)x^2 + \frac{1}{24}\lambda(1 - \lambda)x^4). \quad \blacksquare \end{aligned}$$

Other straightforward calculations give

**Lemma 2.**

(a)

$$\begin{aligned} \sum_j \theta_j &= (1 - \beta)\mu_1 \\ \sum_j \theta_j^2 &= \mu_2 - \beta(2 - \beta)\mu_1^2/n \\ \sum_j \theta_j^3 &= \mu_3 - 3\beta\mu_1\mu_2/n + \beta^2(3 - \beta)\mu_1^3/n^2 \\ \sum_j \theta_j^4 &= \mu_4 - 4\beta\mu_1\mu_3/n + 6\beta^2\mu_1^2\mu_2/n^2 - \beta^3(4 - \beta)\mu_1^4/n^3 \\ \sum_{j < k} (\theta_j + \theta_k)^2 &= (n - 2)\sum_j \theta_j^2 + (\sum_j \theta_j)^2 \\ &= (n - 2)\mu_2 \\ \sum_{j < k} (\theta_j + \theta_k)^3 &= (n - 4)\sum_j \theta_j^3 + 3\sum_j \theta_j \sum_j \theta_j^2 \\ &= (n - 4)\mu_3 + (3(1 - 2\beta) + 12\beta/n)\mu_1\mu_2 \\ &\quad + ((-6\beta + 12\beta^2 - 4\beta^3)/n - 4\beta^2(3 - \beta)/n^2)\mu_1^3 \\ \sum_{j < k} (\theta_j + \theta_k)^4 &= (n - 8)\sum_j \theta_j^4 + 4\sum_j \theta_j \sum_j \theta_j^3 + 3(\sum_j \theta_j^2)^2 \\ &= (n - 8)\mu_4 + 3\mu_2^2 + (4(1 - 2\beta) + 32\beta/n)\mu_1\mu_3 \\ &\quad - (24\beta(1 - \beta)/n + 48\beta^2/n^2)\mu_1^2\mu_2 \\ &\quad + (8\beta^2(1 - \beta)(3 - \beta)/n^2 + 8\beta^3(4 - \beta)/n^3)\mu_1^4. \end{aligned}$$

(b)  $\det(I_n - \eta J_n/n) = 1 - \eta$  for any  $\eta$ .

(c) For any  $t \geq 0$ ,  $TU_n(t) \subseteq (1 + \beta)U_n(t)$  and  $T^{-1}U_n(t) \subseteq (1 - \beta)^{-1}U_n(t)$ .  $\blacksquare$

Let  $\operatorname{Re}(z)$  and  $\operatorname{Im}(z)$  denote the real and imaginary parts of  $z$ , respectively.

**Lemma 3.** *Let  $\epsilon = \epsilon(n)$  and  $\epsilon' = \epsilon'(n)$  be such that  $0 < \epsilon' < 2\epsilon < \frac{1}{6}$ . Let  $A = A(n)$  be a bounded real-valued function such that  $A(n) \geq n^{-\epsilon'}$  for sufficiently large  $n$ . Let  $B = B(n)$ ,  $C = C(n)$ ,  $\dots$ ,  $J = J(n)$  be complex-valued functions such that the ratios  $B/A, C/A, \dots, J/A$  are bounded. Suppose that  $\delta > 0$ ,  $0 < \Delta < \frac{1}{4} - \frac{1}{2}\epsilon$ , and that*

$$\begin{aligned} f(\mathbf{y}) &= \exp(-An\mu_2 + Bn\mu_3 + C\mu_1\mu_2 + D\mu_1^3/n + En\mu_4 + F\mu_2^2 \\ &\quad + G\mu_1\mu_3 + H\mu_1^2\mu_2/n + I\mu_1^4/n^2 + J\mu_1 + O(n^{-\delta})) \end{aligned}$$

is integrable for  $\mathbf{y} \in U_n(n^{-1/2+\epsilon})$ . Then, provided the error term converges to zero,

$$\begin{aligned} \int_{U_n(n^{-1/2+\epsilon})} f(\mathbf{y}) d\mathbf{y} &= \left(\frac{\pi}{An}\right)^{n/2} \exp\left(\frac{J^2}{4A} + \frac{3E + F + (C + 3B)J}{4A^2} + \frac{15B^2 + 6BC + C^2}{16A^3}\right. \\ &\quad \left. + O((n^{-1/2+6\epsilon} + n^{-\delta})Z + n^{-1+12\epsilon} + A^{-1}n^{-\Delta})\right), \end{aligned}$$

where

$$Z = \exp\left(\frac{15 \operatorname{Im}(B)^2 + 6 \operatorname{Im}(B)(\operatorname{Im}(C) + 2A \operatorname{Im}(J)) + (\operatorname{Im}(C) + 2A \operatorname{Im}(J))^2}{16A^3}\right).$$

**Proof.** For  $\rho \geq 0$ , define  $W_n(\rho) = U_n(n^{-1/2+\epsilon}) \cap \{\mathbf{y} \mid \mu_2 = \rho^2\}$ . We approach the integral by considering integration first over  $W_n(\rho)$  and then over  $\rho$ , although this is not the way we obtain the final estimate. Note first that  $W_n(\rho) = \emptyset$  if  $\rho > n^\epsilon$ .

Define  $\nu_k = \sum_{j=1}^n |y_j^k|$ . For  $\mathbf{y} \in W_n(\rho)$  and  $\rho \leq n^\epsilon$  we have

$$\begin{aligned} \nu_1 &\leq \rho n^{1/2}, & \nu_2 &\leq \rho^2, & \nu_3 &\leq \rho^2 n^{-1/2+\epsilon}, & \nu_4 &\leq \rho^2 n^{-1+2\epsilon}, \\ \nu_6 &\leq \rho^2 n^{-2+4\epsilon}, & \nu_1 \nu_2 &\leq \rho^3 n^{1/2}, & \nu_1 \nu_3 &\leq \rho^3 n^\epsilon, & \nu_1^2 \nu_2 &\leq \rho^4 n. \end{aligned}$$

In each case except  $\nu_1 \nu_3$ , the bound is achieved either when all the  $|y_j|$  are equal or when as many as possible have value  $n^{-1/2+\epsilon}$ .

We now divide the region of integration into three parts. Define

$$\begin{aligned} K_1 &= U_n(n^{-1/2+\epsilon}) \cap \{\mathbf{y} \mid 0 \leq \rho < (2A)^{-1/2}(1 - n^{-\Delta})\}, \\ K_2 &= U_n(n^{-1/2+\epsilon}) \cap \{\mathbf{y} \mid (2A)^{-1/2}(1 - n^{-\Delta}) \leq \rho \leq (2A)^{-1/2}(1 + n^{-\Delta})\}, \text{ and} \\ K_3 &= U_n(n^{-1/2+\epsilon}) \cap \{\mathbf{y} \mid (2A)^{-1/2}(1 + n^{-\Delta}) \leq \rho \leq n^\epsilon\}. \end{aligned}$$

When  $\mathbf{y} \in W_n(\rho)$ ,  $f(\mathbf{y}) = \exp(-An\rho^2 + O(A\rho^2 n^{1/2+\epsilon} + n^{-\delta}))$ . Also, the area of  $W_n(\rho)$  is at most  $A_n(\rho) = O(1)(2\pi e/n)^{n/2} \rho^{n-1}$ . Thus

$$\left| \int_{K_1} f(\mathbf{y}) d\mathbf{y} \right| \leq O(1) \left(\frac{2\pi e}{n}\right)^{n/2} \int_0^{(2A)^{-1/2}(1-n^{-\Delta})} \rho^{n-1} \exp(-An\rho^2 + O(A\rho^2 n^{1/2+\epsilon} + n^{-\delta})) d\rho.$$

Apart from the  $O(\cdot)$  term, the integrand is unimodal, with its maximum at  $\rho^2 = (n-1)/(2An)$ , so we can bound the integral by the length of its range times its maximum value, where the latter is achieved near  $\rho = (2A)^{-1/2}(1 - n^{-\Delta})$ . Using  $\log(1 - n^{-\Delta}) < -n^{-\Delta} - \frac{1}{2}n^{-2\Delta}$ , we find that

$$\left| \int_{K_1} f(\mathbf{y}) d\mathbf{y} \right| \leq \left(\frac{\pi}{An}\right)^{n/2} \exp(-n^{1-2\Delta} + O(n^{1/2+\epsilon})).$$

The same bound can be derived for the absolute value of the integral over  $K_3$ . The integral over  $K_1 \cup K_3$  will turn out to be negligible compared to that over  $K_2$ , which we now consider.

The function  $f(\mathbf{y})$  shows a lot of variation on  $W_n(\rho)$ ,  $\rho \approx (2A)^{-1/2}$ , making direct estimation of the integral difficult. Instead, we take advantage of the fact that an integral over a region symmetrical about the origin is invariant under averaging of its integrand over sign changes of the arguments.

For  $k \geq 1$ , define  $\hat{\mu}_k = \hat{\mu}_k(m) = \sum_{j=m}^n y_j^k$  and  $\check{\mu}_k = \check{\mu}_k(m) = \sum_{j=1}^{m-1} y_j^k$ . Then, for  $1 \leq m \leq n+1$ , define

$$\begin{aligned} \psi_m(\mathbf{y}) &= \exp(-An\mu_2 + En\mu_4 + F\mu_2^2 + Bn\hat{\mu}_3 + C\hat{\mu}_1\mu_2 + J\hat{\mu}_1 + D\hat{\mu}_1^3/n + G\hat{\mu}_1\hat{\mu}_3 \\ &\quad + H\hat{\mu}_1^2\mu_2/n + I\hat{\mu}_1^4/n^2 + \frac{1}{2}B^2n^2\check{\mu}_6 + \frac{1}{2}(C\mu_2 + J)^2\check{\mu}_2 + B(C\mu_2 + J)n\check{\mu}_4 \\ &\quad + \frac{9}{2}D^2\check{\mu}_2\hat{\mu}_1^4/n^2 + (3BD\check{\mu}_4 + 3(C\mu_2 + J)D\check{\mu}_2/n)\hat{\mu}_1^2) \end{aligned}$$

and

$$\bar{\psi}_m = \frac{1}{2}(\psi_m(y_m) + \psi_m(-y_m)).$$

Further define  $\eta = \frac{3}{2} - 6\epsilon$ . Then we have

$$\int_{U_n(n^{-1/2+\epsilon})} \bar{\psi}_m(\mathbf{y}) d\mathbf{y} = \int_{U_n(n^{-1/2+\epsilon})} \psi_m(\mathbf{y}) d\mathbf{y} \quad (2.1)$$

and, for  $\mathbf{y} \in U_n(n^{-1/2+\epsilon})$ ,

$$\bar{\psi}_m(\mathbf{y}) = \psi_{m+1}(\mathbf{y}) \exp(O(n^{-\eta})) \quad (2.2)$$

uniformly over  $m$ , since  $\frac{1}{2}(e^x + e^{-x}) = \exp(\frac{1}{2}x^2 + O(x^4))$  for small  $x$ . Also,

$$f(\mathbf{y}) = \psi_1(\mathbf{y}) \exp(O(n^{-\delta})). \quad (2.3)$$

Because of possible cancellation, we cannot integrate (2.2) accurately for arbitrary complex functions so we turn first to integration of  $\psi_{n+1}(\mathbf{y})$ . In  $K_2$  we have  $\mu_2 = (2A)^{-1}(1 + O(n^{-\Delta}))$ , so

$$\psi_{n+1}(\mathbf{y}) = \exp(h(\mathbf{y}))(1 + R(\mathbf{y})),$$

where

$$h(\mathbf{y}) = -An\mu_2 + En\mu_4 + \frac{1}{4}F/A^2 + \frac{1}{2}B^2n^2\mu_6 + \frac{1}{16}(C + 2AJ)^2/A^3 + \frac{1}{2}B(C + 2AJ)n\mu_4/A$$

and

$$R(\mathbf{y}) = O(A^{-1}n^{-\Delta}).$$

The integral of  $\psi_{n+1}$  over  $U_n(n^{-1/2+\epsilon})$  differs from that over  $K_2$  by at most

$$\left(\frac{\pi}{An}\right)^{n/2} \exp(-n^{1-2\Delta} + O(n^{1/2+\epsilon})),$$

as in the estimation of the integral of  $f$  over  $K_1 \cup K_3$ . Furthermore,

$$\begin{aligned} & \int_{U_n(n^{-1/2+\epsilon})} \exp(h(\mathbf{y})) d\mathbf{y} \\ &= \exp\left(\frac{F}{4A^2} + \frac{(C + 2AJ)^2}{16A^3}\right) \\ & \quad \times \left(\int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \exp(-Anx^2 + Enx^4 + B(\frac{1}{2}C/A + J)nx^4 + \frac{1}{2}B^2n^2x^6) dx\right)^n \\ &= \exp\left(\frac{F}{4A^2} + \frac{(C + 2AJ)^2}{16A^3}\right) \\ & \quad \times \left(\int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} e^{-Anx^2} (1 + Enx^4 + B(\frac{1}{2}C/A + J)nx^4 + \frac{1}{2}B^2n^2x^6 + O(n^{-2+12\epsilon})) dx\right)^n \\ &= \left(\frac{\pi}{An}\right)^{n/2} \exp\left(\frac{3E + F}{4A^2} + \frac{15B^2 + 6B(C + 2AJ) + (C + 2AJ)^2}{16A^3} + O(n^{-1+12\epsilon})\right), \end{aligned}$$

since

$$\int_{-\infty}^{\infty} x^{2k} e^{-Anx^2} dx = \frac{(2k)!}{k! (4An)^k} \sqrt{\frac{\pi}{An}} \quad \text{for } k \geq 0.$$

By the same argument,

$$\begin{aligned} \left| \int_{U_n(n^{-1/2+\epsilon})} R(\mathbf{y}) \exp(h(\mathbf{y})) d\mathbf{y} \right| &= O(A^{-1}n^{-\Delta}) \left( \frac{\pi}{An} \right)^{n/2} \\ &\times \exp\left( \frac{\operatorname{Re}(3E+F)}{4A^2} + \frac{\operatorname{Re}(15B^2+6B(C+2AJ)+(C+2AJ)^2)}{16A^3} + O(n^{-1+12\epsilon}) \right). \end{aligned}$$

We conclude that

$$\begin{aligned} \int_{U_n(n^{-1/2+\epsilon})} \psi_{n+1}(\mathbf{y}) d\mathbf{y} &= \left( \frac{\pi}{An} \right)^{n/2} \\ &\times \exp\left( \frac{3E+F}{4A^2} + \frac{15B^2+6B(C+2AJ)+(C+2AJ)^2}{16A^3} + O(n^{-1+12\epsilon} + A^{-1}n^{-\Delta}) \right). \end{aligned} \quad (2.4)$$

In the following, any expression  $Q^*$  denotes the expression  $Q$  with all occurrences of  $B, C, \dots, J$  replaced by their real parts. Also, all integrals will be over  $U_n(n^{-1/2+\epsilon})$  unless otherwise specified.

Since  $|\psi_1| = \psi_1^*$ , (2.1) and (2.2) imply that

$$\int |\psi_1| = \exp(O(n^{1-\eta})) \int \psi_{n+1}^*, \quad (2.5)$$

since all the integrands involved are real. We also have for  $2 \leq m \leq n+1$

$$\begin{aligned} \int |\bar{\psi}_m| &\leq \int \frac{1}{2} (|\psi_m(y_m)| + |\psi_m(-y_m)|) \\ &= \int |\psi_m| \\ &= \exp(O(n^{-\eta})) \int |\bar{\psi}_{m-1}|, \quad \text{by (2.2),} \end{aligned}$$

which implies that

$$\int |\bar{\psi}_m| \leq \exp(O(n^{1-\eta})) \int |\psi_1|$$

for  $m = 1, 2, \dots, n+1$ .

From (2.2) we now have, for  $m = 1, 2, \dots, n$ ,

$$\begin{aligned} \left| \int \bar{\psi}_m - \int \psi_{m+1} \right| &= O(n^{-\eta}) \int |\bar{\psi}_m| \\ &= O(n^{-\eta}) \int |\psi_1| \\ &= O(n^{-\eta}) \int \psi_{n+1}^*, \quad \text{by (2.5).} \end{aligned}$$

Similarly, by (2.3),

$$\begin{aligned} \left| \int f - \int \psi_1 \right| &= O(n^{-\delta}) \int |\psi_1| \\ &= O(n^{-\delta}) \int \psi_{n+1}^*. \end{aligned}$$

Thus, by (2.1),

$$\left| \int f - \int \psi_{n+1} \right| = O(n^{1-\eta} + n^{-\delta}) \int \psi_{n+1}^*.$$

That is,

$$\int f = \exp(O((n^{1-\eta} + n^{-\delta})Z')) \int \psi_{n+1}, \quad (2.6)$$

where  $Z' = |\int \psi_{n+1}^* / \int \psi_{n+1}|$ .

Inserting the value of  $Z'$  implied by (2.4), the lemma follows from (2.4), (2.6), and the fact that  $K_1 \cup K_3$  is negligible. ■

### 3. Regular graphs.

In this section we apply Lemma 3 to the enumeration of regular graphs.

**Theorem 1.** *Let  $d = d(n)$  be an integer-valued function such that, for sufficiently large  $n$ ,  $dn$  is even and  $\min\{d, n - d - 1\} > cn / \log n$  for some  $c > \frac{2}{3}$ . Then the number of regular graphs of order  $n$  and degree  $d$  is uniformly*

$$RG(n, d) = \sqrt{2}(2\pi n \lambda^{d+1} (1 - \lambda)^{n-d})^{-n/2} \exp\left(\frac{-1 + 10\lambda - 10\lambda^2}{12\lambda(1 - \lambda)} + O(n^{-\zeta})\right)$$

for any  $\zeta < \min(\frac{1}{4}, \frac{1}{2} - 1/(3c))$ , where  $\lambda = d/(n - 1)$ .

**Proof.**  $RG(n, d)$  is the coefficient of  $x_1^d x_2^d \cdots x_n^d$  in  $\prod_{1 \leq j < k \leq n} (1 + x_j x_k)$ . By Cauchy's Formula we have

$$RG(n, d) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{j < k} (1 + z_j z_k)}{z_1^{d+1} z_2^{d+1} \cdots z_n^{d+1}} dz_1 dz_2 \cdots dz_n,$$

where each integral is around a simple closed contour enclosing the origin. We will use circles of radius  $r = \sqrt{\lambda/(1 - \lambda)}$  centred at the origin, which gives

$$\begin{aligned} RG(n, d) &= \frac{1}{(2\pi r^d)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{j < k} (1 + r^2 e^{i(\theta_j + \theta_k)})}{e^{id(\theta_1 + \theta_2 + \cdots + \theta_n)}} d\theta_1 d\theta_2 \cdots d\theta_n \\ &= \frac{(1 + r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{j < k} (1 + \lambda(e^{i(\theta_j + \theta_k)} - 1))}{e^{id(\theta_1 + \theta_2 + \cdots + \theta_n)}} d\theta_1 d\theta_2 \cdots d\theta_n. \end{aligned} \quad (3.1)$$

In order to estimate the values of the integral, we will first show that most of its value arises when all the  $\theta_j$  are clustered near 0, or are all clustered near  $\pm\pi$ .

Fix  $0 < t \leq \pi/4$  and  $\epsilon > 0$ . As necessary, we will assume that  $\epsilon$  is sufficiently small. Let  $J_1$  be the contribution to (3.1) from all those  $\boldsymbol{\theta}$  for which either  $n_0 n_2 \geq n^{1+\epsilon}$ ,  $\binom{n_1}{2} \geq n^{1+\epsilon}$  or  $\binom{n_3}{2} \geq n^{1+\epsilon}$ , where  $n_0, n_1, n_2$  and  $n_3$  are the numbers of  $\theta_j$  in the regions  $[-t, t]$ ,  $[t, \pi - t]$ ,  $[\pi - t, \pi] \cup [-\pi, -\pi + t]$  and  $[-\pi + t, -t]$ , respectively. By Lemma 1 we have, for some  $c_1 > 0$ ,

$$\begin{aligned} |J_1| &\leq \frac{(1 + r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} (1 - 2\lambda(1 - \lambda)(1 - \cos(2t)))^{n^{1+\epsilon}/2} (2\pi)^n \\ &= \frac{(1 + r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} O(\exp(-c_1 n^{1+\epsilon/2})). \end{aligned} \quad (3.2)$$

Over the region of integration not covered by  $J_1$  we have  $n_1 = O(n^{1/2+\epsilon})$ ,  $n_3 = O(n^{1/2+\epsilon})$ , and either  $n_0 = O(n^\epsilon)$  or  $n_2 = O(n^\epsilon)$ . The latter two are essentially equivalent, since (3.1)

is invariant under the transformation  $\theta_j \mapsto \theta_j + \pi$  ( $j = 1, 2, \dots, n$ ). Thus we can assume  $n_2 = O(n^\epsilon)$  without loss of generality, and double the result.

Now suppose  $0 < t \leq \pi/8$ . Define  $S_0 = S_0(\boldsymbol{\theta})$ ,  $S_1 = S_1(\boldsymbol{\theta})$ ,  $S_2 = S_2(\boldsymbol{\theta})$  thus:

$$\begin{aligned} S_0 &= \{j \mid |\theta_j| \leq t\} \\ S_1 &= \{j \mid t < |\theta_j| \leq 2t\} \\ S_2 &= \{j \mid 2t < |\theta_j| \leq \pi\}. \end{aligned}$$

Define  $s_0 = |S_0|$ ,  $s_1 = |S_1|$  and  $s_2 = |S_2|$ . To avoid parts of the integral counted in  $J_1$ , we can assume that  $s_1 + s_2 = O(n^{1/2+\epsilon})$ . Let  $J_2(s_2)$  be the contribution to (3.1) of all  $\boldsymbol{\theta}$  with  $|S_2(\boldsymbol{\theta})| = s_2$  and  $s_1 = O(n^{1/2+\epsilon})$ . The modulus of the integrand can be bounded using

$$|1 + \lambda(e^{i(\theta_j + \theta_k)} - 1)| \leq \begin{cases} \exp(-\frac{1}{2}\lambda(1-\lambda)(\theta_j + \theta_k)^2 + \frac{1}{24}\lambda(1-\lambda)(\theta_j + \theta_k)^4), & \text{if } j, k \in S_0 \cup S_1, \\ \sqrt{1 - 2\lambda(1-\lambda)(1 - \cos t)}, & \text{if } j \in S_0, k \in S_2, \\ 1, & \text{otherwise.} \end{cases}$$

The first two bounds come from Lemma 1, the second being the largest value which can occur in the stated range. Let  $\alpha$  denote  $-\log(\sqrt{1 - 2\lambda(1-\lambda)(1 - \cos t)})$ . Then the modulus of the integrand in (3.1) is bounded above by

$$\begin{aligned} &\exp\left(-\frac{1}{2}\lambda(1-\lambda) \sum_{\substack{j < k \\ j, k \in S_0 \cup S_1}} (\theta_j + \theta_k)^2 + \frac{1}{24}\lambda(1-\lambda) \sum_{\substack{j < k \\ j, k \in S_0 \cup S_1}} (\theta_j + \theta_k)^4 - \alpha s_0 s_2\right) \\ &\leq \exp\left(-\frac{1}{2}\lambda(1-\lambda)(n - s_2 - 2) \sum_{j \in S_0 \cup S_1} \theta_j^2 + \frac{1}{3}\lambda(1-\lambda)(n - s_2 - 1) \sum_{j \in S_0 \cup S_1} \theta_j^4 - \alpha s_2(n - O(n^{1/2+\epsilon}))\right), \end{aligned}$$

since

$$\sum_{1 \leq j < k \leq l} (x_j + x_k)^2 \geq (l-2) \sum_{j=1}^l x_j^2 \quad \text{and} \quad \sum_{1 \leq j < k \leq l} (x_j + x_k)^4 \leq 8(l-1) \sum_{j=1}^l x_j^4 \quad (3.3)$$

for all  $x_1, x_2, \dots, x_l$ . If  $0 < \delta < \frac{1}{4}$ ,  $\delta$  fixed, then as  $m \rightarrow \infty$ ,

$$\begin{aligned} &\int_{-2t}^{2t} \exp(-mx^2 + \frac{2}{3}m(1+o(1))x^4) dx \\ &\leq (1 + O(m^{-1+4\delta})) \int_{-m^{-1/2+\delta}}^{m^{-1/2+\delta}} e^{-mx^2} dx + 4t \exp(-m^{2\delta} + O(m^{-1+4\delta})) \\ &= \sqrt{\frac{\pi}{m}} (1 + O(m^{-1+4\delta})), \end{aligned}$$

since  $\exp(-mx^2 + \frac{2}{3}mx^4)$  is maximised at  $x = m^{-1/2+\delta}$  if  $m^{-1/2+\delta} \leq |x| \leq 2t \leq \pi/4$  and  $m$  is sufficiently large. Allowing a factor of  $n^{s_1+s_2} = \exp(O(n^{1/2+\epsilon} \log n))$  for the number of choices of  $S_0, S_1$  and  $S_2$ , we get, with  $\delta = \frac{1}{6}$  and some  $c_2 > 0$ ,

$$\begin{aligned} |J_2(s_2)| &\leq \frac{(1+r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} \left(\frac{2\pi}{\lambda(1-\lambda)(n-s_2-2)}\right)^{(n-s_2)/2} \exp(-\alpha s_2(n - O(n^{1/2+\epsilon})) + O(n^{2/3})) \\ &= \frac{(1+r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} \left(\frac{2\pi}{\lambda(1-\lambda)n}\right)^{n/2} \exp(-\alpha s_2(n - O(n^{1/2+\epsilon})) + O(n^{2/3})), \end{aligned}$$



and so

$$\sum_{s_2=1}^{O(n^{1/2+\epsilon})} |J_2(s_2)| \leq \frac{(1+r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} \left( \frac{2\pi}{\lambda(1-\lambda)n} \right)^{n/2} \exp(-c_2 n / \log n). \quad (3.4)$$

Now define  $J_3(h)$  to be the contribution to (3.1) of those  $\boldsymbol{\theta}$  such that  $|\theta_j| \leq n^{-1/2+\epsilon}$  for  $n-h$  values of  $j$  and  $n^{-1/2+\epsilon} < |\theta_j| \leq 2t$  for  $h$  values of  $j$ . Following the last computation in the case  $s_2 = 0$ , but using  $\delta = \epsilon/4$ , we find for some  $c_3 > 0$  that

$$\begin{aligned} |J_3(h)| &\leq \frac{(1+r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} n^h \left( \int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \exp\left(-\frac{1}{2}\lambda(1-\lambda)(n-2)x^2 + \frac{1}{3}\lambda(1-\lambda)(n-1)x^4\right) dx \right)^{n-h} \\ &\quad \times \left( 2 \int_{-2t}^{2t} \exp\left(-\frac{1}{2}\lambda(1-\lambda)(n-2)x^2 + \frac{1}{3}\lambda(1-\lambda)(n-1)x^4\right) dx \right)^h \\ &\leq \frac{(1+r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} \left( \frac{2\pi}{\lambda(1-\lambda)(n-2)} \right)^{n/2} \exp\left(-\frac{1}{2}\lambda(1-\lambda)n^{2\epsilon}h + O(n^\epsilon) + O(h \log n)\right), \end{aligned}$$

and so

$$\sum_{h=1}^n |J_3(h)| \leq \frac{(1+r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} \left( \frac{2\pi}{\lambda(1-\lambda)n} \right)^{n/2} \exp(-c_3 n^\epsilon). \quad (3.5)$$

Finally, consider  $J_3(0)$ . The numerator of the integrand can be expanded using

$$1 + \lambda(e^x - 1) = \exp\left(\lambda x + \frac{1}{2}\lambda(1-\lambda)x^2 + \frac{1}{6}\lambda(1-\lambda)(1-2\lambda)x^3 + \frac{1}{24}\lambda(1-\lambda)(1-6\lambda+6\lambda^2)x^4 + O(x^5)\right) \quad (3.6)$$

to obtain

$$\begin{aligned} J_3(0) &= \frac{(1+r^2)^{\binom{n}{2}}}{(2\pi r^d)^n} \int_{U_n(n^{-1/2+\epsilon})} \exp\left(-\frac{1}{2}\lambda(1-\lambda)\sum_{j<k}(\theta_j + \theta_k)^2 - \frac{1}{6}i\lambda(1-\lambda)(1-2\lambda)\sum_{j<k}(\theta_j + \theta_k)^3 \right. \\ &\quad \left. + \frac{1}{24}\lambda(1-\lambda)(1-6\lambda+6\lambda^2)\sum_{j<k}(\theta_j + \theta_k)^4 + O(\sum_{j<k}|\theta_j + \theta_k|^5)\right) d\boldsymbol{\theta}. \end{aligned}$$

Now apply the transformation  $\boldsymbol{\theta} = T\mathbf{y}$  described in Section 2. By Lemma 2(c), the region of integration is essentially unchanged. Since  $\beta = (1 - 1/\sqrt{2})(1 + O(n^{-1}))$ , we can take  $\epsilon$  sufficiently small to obtain

$$\begin{aligned} J_3(0) &\sim \frac{(1+r^2)^{\binom{n}{2}}}{(2\pi r^d)^n \sqrt{2}} \int_{U_n(n^{-1/2+\epsilon})} \exp\left(-\frac{1}{2}\lambda(1-\lambda)(n-2)\mu_2 \right. \\ &\quad \left. - \frac{1}{6}i\lambda(1-\lambda)(1-2\lambda)((n-4)\mu_3 + (3(\sqrt{2}-1) + O(n^{-1}))\mu_1\mu_2 + O(n^{-1})\mu_1^3) \right. \\ &\quad \left. + \frac{1}{24}\lambda(1-\lambda)(1-6\lambda+6\lambda^2)((n-8)\mu_4 + 3\mu_2^2 + O(1)\mu_1\mu_3 + O(n^{-1})\mu_1^2\mu_2 + O(n^{-2})\mu_1^4) \right. \\ &\quad \left. + O(n^{-1/2+5\epsilon})\right) d\mathbf{y} \\ &= (2\pi\lambda^{d+1}(1-\lambda)^{n-d}n)^{-n/2} 2^{-1/2} \exp\left(\frac{-1+10\lambda-10\lambda^2}{12\lambda(1-\lambda)} + O(n^{-\zeta})\right), \end{aligned} \quad (3.7)$$

by Lemma 3, where  $\zeta$  is defined as in the theorem statement. From (3.2), (3.4), (3.5) and (3.7) we find that

$$\begin{aligned} RG(n, d) &= 2J_3(0)(1 + O(\exp(-c_3 n^\epsilon))) \\ &= \sqrt{2}(2\pi\lambda^{d+1}(1-\lambda)^{n-d}n)^{-n/2} \exp\left(\frac{-1+10\lambda-10\lambda^2}{12\lambda(1-\lambda)} + O(n^{-\zeta})\right), \end{aligned}$$

as required.  $\blacksquare$

**Corollary 1.** *The total number of regular graphs of order  $n$  is*

$$RG(n) \sim \frac{2^{n^2/2} \sqrt{2e}}{\pi^{n/2} n^{n/2}} a(n),$$

where

$$a(n) = \begin{cases} \sum_{-\infty}^{\infty} e^{-(j+1/2)^2} \approx 1.77227050, & \text{if } n \text{ is even,} \\ \sum_{-\infty}^{\infty} e^{-4j^2} \approx 1.03663150, & \text{if } n \equiv 1 \pmod{4}, \\ \sum_{-\infty}^{\infty} e^{-(2j+1)^2} \approx 0.73600570, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

**Proof.** If  $t = o(n^{1/3})$ , then

$$RG\left(n, \frac{1}{2}(n-1) + t\right) \sim \frac{2^{n^2/2} \sqrt{2e}}{\pi^{n/2} n^{n/2}} e^{-t^2},$$

by Theorem 1. The regular graphs with  $t \neq o(n^{1/3})$  are negligible in comparison, by Theorem 1 for bounded  $\lambda$  and crude bounds for the extremes. The corollary follows on summing over those  $t$  for which  $d = \frac{1}{2}(n-1) + t$  is an integer and  $dn$  is even. ■

#### 4. Non-regular graphs.

In this section, we generalize Theorem 1 to allow non-regular graphs. The proof is similar in spirit to that of Theorem 1, so we concentrate on presenting the parts that are particularly different. In Section 5 we will recast this theorem in another form and give an intuitive partial justification.

**Theorem 2.** *Let  $d = d(n)$  and  $\delta_j = \delta_j(n)$ ,  $1 \leq j \leq n$  be such that  $\min\{d, n-d-1\} > cn/\log n$  for some  $c > \frac{2}{3}$ ,  $\sum_{j=1}^n \delta_j = 0$ ,  $\delta_j = O(n^{1/2+\epsilon})$  uniformly over  $j$  for sufficiently small fixed  $\epsilon > 0$ ,  $d_j = d + \delta_j$  is an integer for  $j = 1, 2, \dots, n$ , and  $dn$  is an even integer. Then the number of labelled graphs of order  $n$  with degree sequence  $\mathbf{d} = (d_1, d_2, \dots, d_n)$  is uniformly*

$$G(n, \mathbf{d}) = \sqrt{2} (2\pi n \lambda^{d+1} (1-\lambda)^{n-d})^{-n/2} \exp\left(\frac{5}{6} - \frac{1}{6}R - R(n-R)\gamma_2 - R^2\gamma_2^2 + n(R^2 - \frac{2}{3}R^3)\gamma_4 + \frac{2}{3}R^2(1-2\lambda)n\gamma_3 + O(n^{-\zeta})\right)$$

for any  $\zeta < \min(\frac{1}{4}, \frac{1}{2} - 1/(3c))$ , where  $\lambda = \lambda(n) = d/(n-1)$ ,  $\lambda_j = \lambda_j(n) = \delta_j/(n-1)$ ,  $R = 1/(2\lambda(1-\lambda))$  and  $\gamma_k = \sum_{j=1}^n \lambda_j^k$  for  $k > 0$ .

**Proof.** Throughout this section,  $\omega$  will denote any expression of the form  $a\epsilon$  with the constant  $a$  possibly different at each appearance.

We will begin with a technical lemma. Define  $r = \sqrt{\lambda/(1-\lambda)}$ , and, for  $1 \leq j, k \leq n$ ,  $f_j = 2R\lambda_j + 4\lambda R^2\lambda_j^2$ ,  $r_j = r(1 + f_j)$ ,  $\eta_{jk} = f_j + f_k + f_j f_k$  and  $\lambda_{jk} = r_j r_k / (1 + r_j r_k)$ .

**Lemma 4.** *Under the conditions of the theorem,*

(a)

$$\begin{aligned} \lambda_{jk} &= \frac{r^2(1 + \eta_{jk})}{1 + r^2(1 + \eta_{jk})} = 1 - \frac{1 - \lambda}{1 + \lambda\eta_{jk}} \\ &= \lambda + \lambda_j + \lambda_k + 2R\lambda_j\lambda_k(1 - 2\lambda) - 4\lambda R^2(\lambda_j + \lambda_k)\lambda_j\lambda_k \\ &\quad - 4\lambda^2 R^2(\lambda_j + \lambda_k)(\lambda_j - \lambda_k)^2 + O(n^{-2+\omega}), \end{aligned}$$

(b) for any  $\boldsymbol{\theta}$ ,

$$i \sum_{j < k} \lambda_{jk}(\theta_j + \theta_k) - i \sum_j (d + \delta_j)\theta_j = \sum_j C_1(j)\theta_j, \text{ where } C_1(j) = O(n^{-1/2+\omega}),$$

(c) for any  $\boldsymbol{\theta}$ ,

$$\begin{aligned} - \sum_{j < k} \frac{1}{2} \lambda_{jk}(1 - \lambda_{jk})(\theta_j + \theta_k)^2 &= -\frac{1}{2} \lambda(1 - \lambda) \sum_{j < k} (\pi_j + \pi_k)^2 + g(\boldsymbol{\theta}), \text{ where} \\ \pi_j &= \theta_j + \frac{1}{2} P \lambda_j (\theta_j + \sum_k \theta_k/n) + (W \lambda_j^2 - R \gamma_2/n) \\ &\quad + \sum_k (W \lambda_k^2 + X \lambda_j \lambda_k + Y \gamma_2/n) \theta_k / (2n), \\ P &= 2(1 - 2\lambda)R, \quad W = -\frac{1}{2}R^2, \\ X &= (3 - 20\lambda + 20\lambda^2)R^2, \quad Y = \frac{1}{2}(-1 + 8\lambda - 8\lambda^2)R^2, \end{aligned} \tag{4.1}$$

and  $g$  is a quadratic form all of whose coefficients are  $O(n^{-3/2+\omega})$ .

**Proof.** In (b) we note that  $\sum_j \lambda_j = 0$ . In (c) we note that

$$\begin{aligned} \lambda_{jk}(1 - \lambda_{jk}) &= -\lambda(1 - \lambda)(1 + (1 - 2\lambda)\eta_{jk} + \lambda(-2 + 3\lambda)\eta_{jk}^2 + O(\eta_{jk}^3)) \\ &= -\lambda(1 - \lambda)(1 + P(\lambda_j + \lambda_k) - 2R(\lambda_j^2 + \lambda_k^2) \\ &\quad + 4(1 - 6\lambda + 6\lambda^2)R^2 \lambda_j \lambda_k) + O(n^{-3/2+\omega}). \end{aligned}$$

The rest is straightforward checking.  $\blacksquare$

We return to the proof of Theorem 2. Noting that  $\lambda = r^2/(1 + r^2)$ , and using  $z_j = r_j e^{i\theta_j}$  we have, in place of (3.1),

$$G(n, \mathbf{d}) = F(\mathbf{d}) \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{j < k} (1 + \lambda_{jk}(e^{i(\theta_j + \theta_k)} - 1))}{\exp(i \sum_j (d + \delta_j)\theta_j)} d\boldsymbol{\theta}, \tag{4.2}$$

where

$$F(\mathbf{d}) = \frac{\prod_{j < k} (1 + r_j r_k)}{(2\pi)^n \prod_j r_j^{d + \delta_j}}.$$

To show that the parts of the integral with  $\max_j \{|\theta_j|\}$  large can be ignored, we amend the argument in Section 3. We denote the  $\epsilon$  in that section by  $\epsilon_1$ , and will later need to ensure that  $\epsilon$  and  $\epsilon_1$  are chosen to make both  $\epsilon_1$  and  $\epsilon/\epsilon_1$  sufficiently small. Defining  $J_1$  as before (but concerning the integral in (4.2)), we find now that

$$|J_1| \leq F(\mathbf{d}) O(\exp(-c_1 n^{1+\epsilon_1/2})) \tag{4.3}$$

for some  $c_1 > 0$  in place of (3.2). As before, we assume  $n_2 = O(n^{\epsilon_1})$  and double the resulting value of the integral.

Define  $S_0, S_1, S_2, s_0, s_1, s_2$  as before, and again assume  $s_1 + s_2 = O(n^{1/2+\epsilon_1})$ . The modulus of the integrand in (4.2) can be bounded, in view of Lemma 1, by

$$\exp\left(-\frac{1}{2} \sum_{\substack{j < k \\ j, k \in S_0 \cup S_1}} \lambda^-(1 - \lambda^-)(\theta_j + \theta_k)^2 + \frac{1}{24} \sum_{\substack{j < k \\ j, k \in S_0 \cup S_1}} \lambda^+(1 - \lambda^+)(\theta_j + \theta_k)^4 - \alpha s_0 s_1\right),$$

where

$$\lambda^-(1 - \lambda^-) = \min_{j,k \in S_0 \cup S_1} \lambda_{jk}(1 - \lambda_{jk}) = \lambda(1 - \lambda) + O(n^{-1/2+\omega}),$$

$$\lambda^+(1 - \lambda^+) = \max_{j,k \in S_0 \cup S_1} \lambda_{jk}(1 - \lambda_{jk}) = \lambda(1 - \lambda) + O(n^{-1/2+\omega}),$$

and

$$\alpha = \min_{\substack{j \in S_0 \\ k \in S_2}} \left( -\log \sqrt{1 - 2\lambda_{jk}(1 - \lambda_{jk})(1 - \cos t)} \right).$$

Arguing along a line similar to that leading to (3.4),

$$|J_2(s_2)| \leq F(\mathbf{d}) \left( \frac{2\pi}{\lambda(1 - \lambda)n} \right)^{n/2} \exp(-\alpha s_2(n - O(n^{1/2+\epsilon_1})) + O(n^{2/3})). \quad (4.4)$$

A little more precision is required in dealing with  $J_3(h)$ . This time, we bound the logarithm of the modulus of the integral in (4.2) by

$$Q(\boldsymbol{\theta}) + \frac{1}{24} \sum_{j < k} \lambda_{jk}(1 - \lambda_{jk})(\theta_j + \theta_k)^4, \quad (4.5)$$

where

$$Q(\boldsymbol{\theta}) = -\frac{1}{2} \sum_{j < k} \lambda_{jk}(1 - \lambda_{jk})(\theta_j + \theta_k)^2.$$

The quadratic form  $Q$  causes some difficulty here since the  $\lambda_{jk}$  may vary by  $O(n^{-1/2+\omega})$ . To avoid this, we take a step towards diagonalising  $Q(\boldsymbol{\theta})$  by making use of the transformation (4.1). (It is rather curious that the further transformation  $\boldsymbol{\pi} = T\mathbf{y}$  does not seem to give a useful result because  $y_j$  is not so closely related to  $\pi_j$ .) From Lemma 4(c), we have

$$\begin{aligned} Q(\boldsymbol{\theta}) &= -\frac{1}{2} \lambda(1 - \lambda) \sum_{j < k} (\pi_j + \pi_k)^2 + O(h^2 n^{-3/2+\omega} + n^{\omega+\epsilon_1}) \\ &\leq -\frac{1}{2} \lambda(1 - \lambda)(n - 2) \sum_j \pi_j^2 + O(h^2 n^{-3/2+\omega} + n^{\omega+\epsilon_1}) \end{aligned}$$

by (3.3). Also, from (4.1) we have

$$\pi_j^2 = \theta_j^2(1 + P\lambda_j) + O(n^{-3/2+\omega})(n^{1/2+\epsilon_1} + h),$$

and so

$$Q(\boldsymbol{\theta}) \leq -\frac{1}{2} \lambda(1 - \lambda)(n - 2) \sum_j \theta_j^2 + O(n^{\omega+\epsilon_1} + hn^{-1/2+\omega}).$$

We also have

$$\frac{1}{24} \sum_{j < k} \lambda_{jk}(1 - \lambda_{jk})(\theta_j + \theta_k)^4 \leq \frac{1}{3} \lambda(1 - \lambda)(n + O(n^{1/2+\epsilon})) \sum_j \theta_j^4,$$

by (3.3). The argument in Section 3 now gives, in place of (3.5),

$$|J_3(h)| \leq F(\mathbf{d}) \left( \frac{2\pi}{\lambda(1 - \lambda)(n - 2)} \right)^{n/2} \exp\left( \left( -\frac{1}{2} \lambda(1 - \lambda)n^{2\epsilon_1} + n^{-1/2+\omega} \right) h + O(n^{\omega+\epsilon_1}) \right). \quad (4.6)$$

From this point we assume that  $\epsilon/\epsilon_1$  is sufficiently small that the exponent  $\omega + \epsilon_1$  in (4.6) is less than  $3\epsilon_1/2$ .

We turn now to the evaluation of  $J_3(0)$ , in which  $\theta_j = O(n^{-1/2+\epsilon_1})$  uniformly over  $j$ . In view of (3.6), the linear terms in the Taylor series expansion of the logarithm of the integrand in (4.2) are given by Lemma 4(b), and the quadratic terms by Lemma 4(c). Let  $V$  denote the matrix of the linear transformation defined in (4.1), so that  $\boldsymbol{\pi} = V\boldsymbol{\theta}$ . Then Gaussian elimination gives

$$\begin{aligned}\det V &= (1 + O(n^{-1/2+\omega})) \prod_j (1 + \frac{1}{2}P\lambda_j + W\lambda_j^2 - R\gamma_2/n) \\ &= (1 + O(n^{-1/2+\omega})) \exp(2W\gamma_2).\end{aligned}\tag{4.7}$$

Noting that  $\pi_j = \theta_j(1 + O(n^{-1/2+\omega}))$ , we find that the summation in Lemma 4(b) is

$$\sum_j C_1(j)\pi_j + O(n^{-1/2+\omega}),\tag{4.8}$$

and the parts of the the logarithm of the integrand of (4.2) not included in Lemma 4 are

$$\begin{aligned}-iC_2 \sum_{j<k} (1 + O(n^{-1/2+\omega})) (\theta_j + \theta_k)^3 + C_3 \sum_{j<k} (\theta_j + \theta_k)^4 + O(n^{-1/2+\omega}) \\ = -iC_2 \sum_{j<k} (1 + O(n^{-1/2+\omega})) (\pi_j + \pi_k)^3 \\ + C_3 \sum_{j<k} (\pi_j + \pi_k)^4 + O(n^{-1/2+\omega}) + \Delta,\end{aligned}\tag{4.9}$$

where

$$C_2 = \frac{1}{6}\lambda(1-\lambda)(1-2\lambda), \quad C_3 = \frac{1}{24}\lambda(1-\lambda)(1-6\lambda+6\lambda^2),$$

and  $\Delta$  contains miscellaneous terms like  $\mu_2 \sum \lambda_j \pi_j$ .

We now transform by  $\boldsymbol{\pi} = T\boldsymbol{y}$  as in Section 2, and then apply a result similar to Lemma 3. Note that  $C_1(j)$  is independent of  $\boldsymbol{\pi}$ , and hence the terms  $C_1(j)y_j$  are subsumed into the error terms during the averaging process in the proof of Lemma 3. Similarly,  $\Delta$  is negligible. The result, from (4.7), (4.8), (4.9) and Lemma 4 is a value of  $J_3(0)$  which dominates (4.3), (4.4) and (4.6). Assuming that  $\epsilon_1$  is sufficiently small, we now have

$$G(n, \mathbf{d}) = RG(n, d) \frac{\sum_{j<k} (1 + \lambda\eta_{jk})}{\prod_j (1 + f_j)^{d+\delta_j}} \exp(-2W\gamma_2 + O(n^{-\zeta}))\tag{4.10}$$

for any  $\zeta$  satisfying the conditions of the theorem. To estimate the value of (4.10), we make the following calculations, where the errors are of magnitude  $O(n^{-1/2+\omega})$ .

$$\begin{aligned}\sum_{j<k} f_j &\approx 2\lambda R^2(n-1)\gamma_2, & \sum_{j<k} f_j^2 &\approx 2R^2(n-1)\gamma_2 + 8\lambda R^3 n\gamma_3 + 8\lambda^2 R^4 n\gamma_4, \\ \sum_{j<k} f_j^3 &\approx 4R^3 n\gamma_3 + 24\lambda R^4 n\gamma_4, & \sum_{j<k} f_j^4 &\approx 8R^4 n\gamma_4, \\ \sum_{j<k} f_j f_k &\approx -2R^2\gamma_2 + 8\lambda^2 R^4 \gamma_2^2, & \sum_{j<k} f_j^2 f_k &\approx 8\lambda R^4 \gamma_2^2, \\ \sum_{j<k} f_j^3 f_k &\approx 0, & \sum_{j<k} f_j^2 f_k^2 &\approx 8R^4 \gamma_2^2.\end{aligned}$$

Note that the above sums are over both  $j$  and  $k$  with  $1 \leq j < k \leq n$ , even if  $k$  doesn't appear

in the summand. From these expressions we find, to the same degree of accuracy,

$$\begin{aligned}\sum_{j<k} \eta_{jk} &\approx 4\lambda R^2(n-1)\gamma_2 - 2R^2\gamma_2 + 8\lambda^2 R^4\gamma_2^2, \\ \sum_{j<k} \eta_{jk}^2 &\approx 4R^2(n-1)\gamma_2 + 16\lambda R^3 n\gamma_3 + 16\lambda^2 R^4 n\gamma_4 + 8R^4\gamma_2^2 \\ &\quad + 32\lambda R^4\gamma_2^2 - 4R^2\gamma_2 + 16\lambda^2 R^4\gamma_2^2, \\ \sum_{j<k} \eta_{jk}^3 &\approx 8R^3 n\gamma_3 + 48\lambda R^4 n\gamma_4 + 48\lambda R^4\gamma_2^2 + 48R^4\gamma_2^2, \\ \sum_{j<k} \eta_{jk}^4 &\approx 16R^4 n\gamma_4 + 48R^4\gamma_2^2.\end{aligned}$$

Hence

$$\log\left(\sum_{j<k}(1 + \lambda\eta_{jk})\right) = 2\lambda R^2(n\lambda - 1)\gamma_2 - \frac{16}{3}\lambda^3 R^3 n\gamma_3 + 4\lambda^4 R^4 n\gamma_4 - R^2\gamma_2^2 + O(n^{-1/2+\omega}).$$

Also,

$$\begin{aligned}\sum_j \log(1 + f_j)^{d+\delta_j} &= (n-1) \sum_j (\lambda + \lambda_j) \log(1 + 2R\lambda_j + 4\lambda R^2\lambda_j^2) \\ &= 2\lambda R^2(n-1)\gamma_2 - \frac{4}{3}\lambda R^3(1 - 3\lambda + 6\lambda^2)n\gamma_3 \\ &\quad + \frac{4}{3}\lambda R^4(1 - 4\lambda + 6\lambda^2)n\gamma_4 + O(n^{-1/2+\omega}),\end{aligned}$$

and so the theorem now follows from (4.10).  $\blacksquare$

Theorem 2 and its proof are sufficiently complex to justify some independent checking. One check we can offer is to sum our expression for  $G(n, \mathbf{d})$  over all degree sequences of graphs with  $\lambda N$  edges, where  $\lambda$  is constant and  $N = \binom{n}{2}$ . This should yield a close approximation to the total number  $\binom{N}{\lambda N}$  of graphs with  $\lambda N$  edges.

We need to sum  $G(n, \mathbf{d})$  over all sequences  $\mathbf{d}$  such that  $\sum_j \delta_j = 0$ . It well known from random graph theory that only a vanishingly small part of the sum is lost if we restrict  $\delta$  to  $U_n(n^{-1/2+\epsilon})$ . Now define  $\bar{\gamma}_k = \sum_{j=1}^{n-1} \lambda_j^k$ , so that  $\lambda_n = -\bar{\gamma}_1$ ,  $\gamma_2 = \bar{\gamma}_2 + \bar{\gamma}_1^2$ ,  $\gamma_3 = \bar{\gamma}_3 - \bar{\gamma}_1^3$  and  $\gamma_4 = \bar{\gamma}_4 + \bar{\gamma}_1^4$ . Let  $\exp h(\gamma_2, \gamma_3, \gamma_4)$  denote the exponential in Theorem 2. We can write the sum of this over  $\mathbf{d}$  subject to  $\sum_j \delta_j = 0$  as the sum over  $(d_1, d_2, \dots, d_{n-1})$  with  $d_n$  determined by  $\lambda_n = -\bar{\gamma}_1$ . We can approximate this sum with the integral

$$(n-1)^{n-1} \int_{U_{n-1}(n^{-1/2+\epsilon})} \exp h(\bar{\gamma}_2 + \bar{\gamma}_1^2, \bar{\gamma}_3 - \bar{\gamma}_1^3, \bar{\gamma}_4 + \bar{\gamma}_1^4) d\bar{\boldsymbol{\lambda}},$$

where  $\bar{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ . (Since this is only a checking calculation, we will not attempt to justify this approximation.) Transforming by  $\bar{\boldsymbol{\lambda}} = \bar{T}\boldsymbol{\xi}$ , where  $\bar{T} = I_{n-1} + \bar{\beta}J_{n-1}/(n-1)$ ,  $\bar{\beta} = 1 - n^{-1/2}$ , the integral becomes approximately

$$\begin{aligned}\det \bar{T} \int_{U_{n-1}(n^{-1/2+\epsilon})} \exp(h(\bar{\mu}_2, \bar{\mu}_3 - 3\bar{\mu}_1\bar{\mu}_2/n + (2n^{-2} - n^{-3/2})\bar{\mu}_1^3, \\ \bar{\mu}_4 - 4\bar{\mu}_1\bar{\mu}_3/n + 6\bar{\mu}_1^2\bar{\mu}_2/n^2 + (n^{-2} - 3n^{-3})\bar{\mu}_1^4) + O(n^{-\delta})) d\boldsymbol{\xi},\end{aligned}$$

where  $\bar{\mu}_k = \sum_{j=1}^{n-1} \xi_j^k$ . We can now apply Lemma 3 (noting that it is still valid when  $I(n) = O(n)$  and  $D(n) = O(\sqrt{n})$ ) and use  $\det \bar{T} = n^{-1/2}$  from Lemma 2(b). The result is as expected.

## 5. Some conjectures.

Theorem 2 has an interesting probabilistic interpretation. Define  $d(n)$ ,  $\lambda$  and  $\gamma_2$  as in Theorem 2. Generate a random graph of order  $n$  by choosing each of the  $\binom{n}{2}$  possible edges independently with probability  $\lambda$ . This will generate each labelled graph with  $E = nd/2$  edges with the same probability  $\lambda^E(1-\lambda)^{\binom{n}{2}-E}$ . (Of course, other graphs may be generated as well.) Each of the events “vertex  $j$  has degree  $d_j$ ” occurs with probability  $\binom{n-1}{d_j}\lambda^{d_j}(1-\lambda)^{n-d_j-1}$ . If we (falsely) suppose that those events are independent, we arrive at the naive estimate  $G(n, \mathbf{d}) \approx \check{G}(n, \mathbf{d})$ , where

$$\check{G}(n, \mathbf{d}) = (\lambda^\lambda(1-\lambda)^{1-\lambda})^{\binom{n}{2}} \prod_{j=1}^n \binom{n-1}{d_j}.$$

The interesting thing about this estimate is that the relative error  $G(n, \mathbf{d})/\check{G}(n, \mathbf{d})$  can be cast in the following form, which depends only on  $\gamma_2$  for the ranges covered by both (1.1) and Theorem 2.

**Theorem 3.** *Let  $\mathbf{d} = \mathbf{d}(n) = (d_1, d_2, \dots, d_n)$  be a graphical degree sequence for each  $n$ . Define  $d$ ,  $\delta_j$  and  $\gamma_2$  as in Theorem 2. Suppose that one of the following is true:*

- (i)  $\max\{d_j\} = o(n^{1/3}d^{1/3})$ ,  $\max\{|\delta_j|\} = o(\min\{n^{1/8}d^{5/8}, n^{1/6}d^{1/2}\})$ , and  $dn \rightarrow \infty$ .
- (ii)  $\max\{|\delta_j|\} = O(n^{1/2+\epsilon})$  and  $\min\{d, n-d-1\} > cn/\log n$  for sufficiently small  $\epsilon$  and some  $c > \frac{2}{3}$ .

Then

$$G(n, \mathbf{d}) \sim \sqrt{2} \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\lambda^2(1-\lambda)^2}\right) \check{G}(n, \mathbf{d}).$$

**Proof.** Case (i) is a strengthening of Equation (1.1) which will be proved in [16] using methods similar to those of [13]. Case (ii) follows from Theorem 2. ■

Cases (i) and (ii) cover three parts of the spectrum of average degrees: (i) for the low and high extremes, (ii) for the middle part near  $n/2$ . We cannot resist the temptation to conjecture that similar claims hold for the other parts of the spectrum as well.

**Conjecture 1.** *For some absolute constant  $\epsilon > 0$ , the conclusion of Theorem 3 holds for  $0 < d < n-1$  provided that  $\max_j\{|\delta_j|\} = o(n^\epsilon \min\{d, n-d-1\}^{1/2})$  and  $n \min\{d, n-d-1\} \rightarrow \infty$ . ■*

The condition on  $\delta_j$  in Conjecture 1 holds easily for regular graphs of any degree and in this case we can investigate the truth of the conjecture experimentally. McKay [11] has computed the actual values of  $RG(n, d)$  for  $1 \leq d \leq 4$ ,  $n \leq 50$  and for  $1 \leq d \leq n-2$ ,  $n \leq 21$ . Careful numerical extrapolation of these numbers not only supports Conjecture 1, but suggests the following stronger conjecture. A conjecture which is consistent for bounded  $d$  was made in [11].

**Conjecture 2.** Let  $d = d(n)$  satisfy  $1 \leq d \leq n - 2$  with  $dn$  always an even integer. Then

$$RG(n, d) = \sqrt{2} \binom{n-1}{d}^n (\lambda^\lambda (1-\lambda)^{1-\lambda})^{\binom{n}{2}} \exp\left(\frac{1}{4} - \frac{3c-1}{12cn} - \frac{23c^2-20c+6}{24c^2n^2} + O(n^{-3})\right)$$

uniformly as  $n \rightarrow \infty$ , where  $\lambda = d/(n-1)$  and  $c = \lambda(1-\lambda)(n-1)$ . ■

Theorem 3 leads to a simple probabilistic model for the degree sequences of random graphs. For the details, and some applications, see [17].

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