VERTEX-TRANSITIVE GRAPHS WHICH ARE NOT CAYLEY GRAPHS, II

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Abstract. The Petersen graph on 10 vertices is the smallest example of a vertex-transitive graph which is not a Cayley graph. In 1983, D. Marušič asked: for what values of n does there exist such a graph on n vertices? We give several new constructions of families of vertex-transitive graphs which are not Cayley graphs and complete the proof that, if n is divisible by p^2 for some prime p, then there is a vertex-transitive graph on n vertices which is not a Cayley graph unless n is p^2, p^3, or 12.

1. Introduction

In [**12**] D. Marušič asked: for which positive integers n does there exist a vertex-transitive graph on n vertices which is not a Cayley graph? The problem of determining such numbers which are prime powers was investigated by Marušič in [**13**]. Many constructions of families of vertex-transitive, non-Cayley graphs (as we shall call them) can be found in the literature, for example see [**1**, **6**, **7**, **8**, **14**, **15**, **18**, **20**]. These constructions and partial answers to Marušič's question were summarised and extended in the first paper [**11**] of this series. That paper gave an answer for all even integers divisible by the square of some prime. In this paper we give several new constructions of families of vertex-transitive, non-Cayley graphs, and complete the proof that such a graph exists whenever n is divisible by p^2 for some prime p, unless $n = p^2$, $n = p^3$, or $n = 3p^2 = 12$.

Unless otherwise indicated, our graph-theoretic terminology will follow [**4**], and our group-theoretic terminology will follow [**21**]. If Γ is a graph, then $V\Gamma, E\Gamma$ and $\mathrm{Aut}(\Gamma)$ will denote its vertex set, its edge set, and its automorphism group, respectively. The cardinality of $V\Gamma$ is called the *order* of Γ, and Γ is said to be *vertex-transitive* if the action of $\mathrm{Aut}(\Gamma)$ on $V\Gamma$ is transitive.

For a group G and a subset $C \subset G$ such that $1_G \notin C$ and $C^{-1} = C$, the *Cayley graph* of G *relative to* C, $\mathrm{Cay}(G, C)$, is defined as follows. The vertex set of $\mathrm{Cay}(G, C)$ is G, and two vertices $g, h \in G$ are adjacent in $\mathrm{Cay}(G, C)$ if and only if $gh^{-1} \in C$. It is easy to see that $\mathrm{Cay}(G, C)$ admits a copy of G acting regularly (by right multiplication) as a group of automorphisms, and so every Cayley graph is vertex-transitive. Conversely, every vertex-transitive graph which admits a regular group of automorphisms is (isomorphic to) a Cayley-graph of that group. However, there are vertex-transitive graphs which are not Cayley graphs, the smallest example being the well-known Petersen graph. Such a graph will be called a *non-Cayley* vertex-transitive graph, and its order will be called a *non-Cayley number*. Let NC be the set of all non-Cayley numbers.

An important, but elementary, fact about non-Cayley numbers is that, for every non-Cayley number n and every positive integer k, kn is also a non-Cayley number, since the union of finitely many copies of a vertex-transitive graph Γ is a Cayley graph if and only if Γ is a Cayley graph. Thus any multiple of a member of NC is also in NC, so it is sufficient to find those members of NC whose non-trivial divisors are not members of NC. For this reason our investigations have focussed on graphs whose orders have relatively few prime divisors. This means that many well-known constructions of vertex-transitive, non-Cayley graphs (for example the Odd graphs investigated in [**7**], and the flag graphs of projective planes examined in [**8**]) and even the recent result of Watkins [**20**] (that line graphs of non-bipartite, odd valency, edge-transitive graphs are vertex-transitive, non-Cayley graphs) are not particularly helpful to us.

Vertex-transitive, non-Cayley graphs of order pq, where p and q are distinct primes have received a lot of attention in the literature, beginning with the construction given by Frucht, Graver and Watkins in [**6**] in 1971. Very recent work of D. Marušič and R. Scappelato [**14**], and of M. Y. Xu and the second author [**18**] has completed the solution to the problem of when $pq \in NC$. This work will be drawn together in Section 3 of this paper yielding:

Theorem 1. *Let p and q be primes with $q < p$. Then $pq \in NC$ if and only if one of the following holds.*

(a) q^2 *divides* $p - 1$,

(b) $p = 2q - 1 > 3$ or $p = (q^2 + 1)/2$,

(c) $p = 2^t + 1$ *and either* q *divides* $2^t - 1$ *or* $q = 2^{t-1} - 1$,

(d) $p = 2^t - 1$ *and* $q = 2^{t-1} + 1$,

(e) $p = 11$ *and* $q = 7$.

In [**11**, Section 5] (see also [**13**]) we constructed a vertex-transitive, non-Cayley graph of order p^4, where p is a prime , while in [**13**] it was shown that all vertex-transitive graphs of order p, p^2 and p^3 are Cayley graphs. Thus we were interested in orders n divisible by (at least) two distinct primes p and q, and Theorem 1 gave a complete answer in the case $n = pq$. We decided to analyse completely the case $n = p^2q$ using the information from Theorem 1 and other constructions in [**11**] as our starting point. From the work in [**11**] and [**13**], the details of which will be given in Section 4, we reduced the problem of deciding whether p^2q lies in NC to the case where q is odd, q divides $p^2 - 1$, but q^2 does not divide $p - 1$. A very delicate group theoretic analysis isolated several minimal transitive permutation groups on p^2q points as possible candidates for vertex-transitive groups of automorphisms of non-Cayley graphs. A careful examination of each of these groups led to the constructions of vertex-transitive, non-Cayley graphs of order p^2q given in Sections 5 and 6. These constructions cover all outstanding cases and yield Theorem 2 below. Happily they obviate the need to publish the group theoretic analysis which was very technical and unpleasant.

Theorem 2. *Let p and q be distinct primes. Then $p^2q \in NC$ unless $p^2q = 12$.*

One consequence of this theorem is a complete determination of membership of NC for non-square-free integers n.

Theorem 3. *Let n be a positive integer which is divisible by the square of some prime p. Then $n \in NC$ unless $n = p^2$ or $n = p^3$ or $n = 12$.*

2. Preliminaries

A transitive permutation group G on a set Ω is said to *preserve a partition* $\Sigma = \{B_1, \ldots, B_r\}$ of Ω if, for all $g \in G$ and all $B_i \in \Sigma$, the image $B_i^g \in \Sigma$. The partitions $\Sigma = \{\Omega\}$ and $\Sigma = \{\{\alpha\} \,|\, \alpha \in \Omega\}$ are said to be *trivial* and all others are said to be *nontrivial*; G is said to be *imprimitive* if it preserves a nontrivial partition and otherwise is said to be *primitive* on Ω. A block B_i of a partition preserved by G is called a *block of imprimitivity* for G in Ω and has the obvious defining property that, for all $g \in \Omega, B_i^g = B_i$ or $B_i^g \cap B_i = \phi$. A partition Σ preserved by G is also called a *block system* for G in Ω. For such a partition G^Σ denotes the permutation group on Σ induced by G, and $G_{(\Sigma)}$ denotes the subgroup of all elements of G which fix setwise all blocks of Σ. If $K \leq G$ fixes setwise a subset $B \subseteq \Omega$ then K^B denotes the permutation group on B induced by K, and $K_{(B)}$ denotes the subgroup of K which fixes B pointwise.

For a graph $\Gamma = (V\Gamma, E\Gamma)$ with vertex set $V\Gamma$ and edge set $E\Gamma$, and for a partition Σ of $V\Gamma$, the *quotient graph* Γ_Σ of Γ with respect to Σ is the graph with vertex set Σ such that two blocks $B, B' \in \Sigma$ are adjacent in Γ_Σ if and only if there exist $\alpha \in B$, $\alpha' \in B'$ with $\{\alpha, \alpha'\} \in E\Gamma$. For a subset B of $V\Gamma$ we denote the subgraph of Γ induced on B by $\bar{B}$, that is $\bar{B}$ has vertex set B and edge set $\{\{\alpha, \beta\} \in E\Gamma \,|\, \alpha, \beta \in B\}$. For $\alpha \in V\Gamma, \Gamma(\alpha) = \{\beta \,|\, \{\alpha, \beta\} \in E\Gamma\}$ denotes the set of *neighbours* of α in Γ and, for $B \subseteq V\Gamma, \Gamma(B) = \bigcup\{\Gamma(\alpha) \,|\, \alpha \in B\}$.

For two graphs Γ, Δ of orders n and m respectively, the *lexicographic product* $\Gamma[\Delta]$ of Γ and Δ is the graph of order mn obtained by taking n vertex disjoint copies of Δ, labelling the copies $\{\Delta_\gamma \,|\, \gamma \in \Gamma\}$ by the vertices of Γ in some way, and then joining all vertices of Δ_γ to all vertices of $\Delta_{\gamma'}$ by edges if and only if $\{\gamma, \gamma'\} \in E\Gamma$. Clearly $\mathrm{Aut}(\Gamma[\Delta])$ contains the wreath product $\mathrm{Aut}(\Delta)\, wr\, \mathrm{Aut}(\Gamma)$, but may of course be larger (for example, the complete graph $K_{ab} \cong K_a[K_b]$ admits the full symmetric group S_{ab} as a group of automorphisms, not only its subgroup $S_b\, wr\, S_a$). If Γ and Δ are Cayley graphs then clearly $\Gamma[\Delta]$ is also a Cayley graph.

For a group G and a prime p, $O_p[G]$ denotes the largest normal p-subgroup of G.

3. Vertex-transitive, non-Cayley graphs of order pq, where p and q are distinct primes.

Let p and q be primes, with $q < p$. In this section we discuss membership of pq in NC. In 1971 R. Frucht, J. Graver and M. Watkins [**6**] gave a construction of a family of vertex-transitive, non-Cayley graphs of order $2p$ when $p \equiv 1$ (mod 4), and it was shown in 1979 by B. Alspach and R. J. Sutcliffe [**2**] that $2p \in NC$ if and only if $p \equiv 1$ (mod 4) (that is $q^2 = 4$ divides $p - 1$). So, suppose that p and q are both odd. In 1982 B. Alspach and T. D. Parsons [**1**] defined metacirculant graphs as follows:

A graph Γ of order pq is called a (q, p)-*metacirculant* if it admits a vertex-transitive group of automorphisms of the form $\langle g, h \,|\, g^p = h^b = 1, h^{-1}gh = g^a \rangle$ where $\langle g \rangle$ has q orbits of length p on vertices which are permuted cyclically by h, $1 \le a < p$, and $b = \mathrm{lcm}\{o(a \ mod\ p), q\}$, where $o(a \ mod\ p)$ denotes the least positive integer m such that $a^m \equiv 1\ (mod\ p)$. Their paper [**1**] contains a comprehensive investigation of metacirculant graphs and gives necessary and sufficient conditions on a (q,p)-metacirculant graph Γ for it to be a non-Cayley graph (see [**1**, Theorem 12]). In particular a (vertex-transitive) non-Cayley (q,p)-metacirculant graph exists if and only if q^2 divides $p-1$.

Quite recently, D. Marušič and R. Scapellato [**14**] showed that a vertex-transitive, non-Cayley graph of order pq which admits a transitive imprimitive group of automorphisms is either a (p,q)-metacirculant, or admits $SL(2,2^t)$ as a vertex-transitive group of automorphisms, where $p = 2^t + 1$ is a Fermat prime and q divides $2^t - 1$. A classification of all vertex-primitive graphs of order pq (those whose group acts primitively on the vertex set) was completed by M. Y. Xu and the second author in [**18**] including information on which of the graphs were non-Cayley. The values of q, p and $\mathrm{Aut}(\Gamma)$ for vertex-primitive, non-Cayley graphs Γ of order pq are summarised in Table 1 below. In each case several graphs Γ exist and more details of these are given in [**18**].

The proof of Theorem 1 follows from the discussion above and the nature of p and q in Table 1. Note that the non-Cayley graphs which are metacirculants or are associated with $SL(2,2^t)$ correspond to cases (a) and (c) respectively, while the other cases can be read off from Table 1: the pairs $(19,3)$ $(61,31)$, and $(7,5)$ occur in (a), (b) and (d) respectively, and the pairs $(r^2+1, r+1)$ with r a power of 2, occur in case (c).

The paper [**1**] gave precise conditions under which a (q,p)-metacirculant graph was a non-Cayley graph, and similarly full information is available in [**18**] about which vertex-primitive graphs of order pq are non-Cayley graphs. However, although information about which of the graphs studied by Marušič and Scapellato (and admitting $SL(2,2^t)$) are non-Cayley graphs can be extracted from the proof of their main theorem in [**15**], it is not stated explicitly. We give here a definition of the family of graphs studied by Marušič and Scapellato and determine which are non-Cayley graphs in the case where the order is pq. We note that the full automorphism groups of all symmetric graphs in the family were determined in [**17**].

Definition 3.1. Let $p = 2^t + 1$ and let q be a divisor of $2^t - 1$. Let S be a subset of $\mathbb{Z}_q \setminus \{0\}$ such that $-S = \{-s \,|\, s \in S\} = S$, let $U \subseteq \mathbb{Z}_q$, and let w be a primitive element of $GF(2^t)$. Then $\Gamma = X(t,q,S,U)$ is the graph with vertex set

$$V\Gamma = PG(1,2^t) \times \mathbb{Z}_q$$

where $PG(1,2^t) = GF(2^t) \cup \{\infty\}$ denotes the projective line over $GF(2^t)$, such that

Vertex-primitive, non-Cayley graphs Γ of order pq, where q and p are primes and $3 \le q < p$.

p	q	$\mathrm{Aut}(\Gamma)$
p	$\frac{p+1}{2}$	S_{p+1}
7	5	S_7
7	5	$PSL(4,2)$
r^2+1	$r+1$	$P\Gamma Sp(4,r)$
$\frac{q^2+1}{2}$	q	$P\Sigma L(2,q^2)$
19	3	$PGL(2,19)$
61	31	$PGL(2,61)$
11	7	$\mathrm{Aut}\ M_{22}$
p	$\frac{p+1}{2}$	$PSL(2,p)$ or $PGL(2,p)$
$2^d \mp 1$	$2^{d-1} \pm 1$	$\Omega^{\pm}(2d,2)$

Table 1.

the neighbours of $(\infty, r) \in V\Gamma$ are

$$\Gamma((\infty,r)) = \{(\infty, r+s) \mid s \in S\} \cup \{(y, r+u) \mid y \in GF(2^t), u \in U\}$$

and the neighbours of $(x,r) \in GF(2^t) \times \mathbb{Z}_q \subseteq V\Gamma$ are

$$\Gamma((x,r)) = \{(x, r+s) \mid s \in S\}$$

$$\cup \{(\infty, r-u) \mid u \in U\} \cup \{(x+w^i, -r+u+2i) \mid i \in \mathbb{Z}_q, u \in U\}.$$

Proposition 3.2. *Suppose that $p = 2^t+1$ and q dividing 2^t-1 are primes. Each of the graphs $X(t,q,S,U)$ admits $SL(2,2^t)$ acting vertex-transitively, and $X(t,q,S,U)$ is a non-Cayley graph if and only if U is a proper non-empty subset of $\mathbb{Z}_q$.*

Proof. That $\Gamma = X(t,q,S,U)$ admits $SL(2,2^t)$ acting vertex-transitively was shown in [**14**]. Suppose that Γ is a Cayley graph. Then $\mathrm{Aut}(\Gamma)$ has a subgroup R

of order pq acting regularly in $V\Gamma$. Since q does not divide $p-1$, R must be cyclic whence in particular Γ is a metacirculant. The proof of the theorem in [**15**] shows that this implies that U is empty or equal to $\mathbb{Z}_q$. Conversely if U is empty or equal to $\mathbb{Z}_q$ then Γ is disconnected, or a lexicographic product of (Cayley) graphs of order q and p, respectively, whence Γ is a Cayley graph.

4. Membership of p^2q in NC: a discussion.

Let p and q be distinct primes. If either p or q is 2 then it was shown in [**11**, Theorem 2] that $p^2q \in NC$ unless $p^2q = 12$, while we know that $12 \notin NC$ by the enumeration of vertex-transitive graphs of order 12 in [**10**]. Thus we may assume that p and q are both odd. If q^2 divides $p-1$ then by [**1**], see Theorem 1, $pq \in NC$, whence $p^2q \in NC$. On the other hand if q does not divide p^2-1 then the construction given in [**11**, see Theorem 2] shows that $p^2q \in NC$. Thus, membership of p^2q in NC has been determined unless p and q are both odd primes and either q divides $p+1$, or q, but not q^2, divides $p-1$. In Section 5 we give a construction of a non-Cayley, vertex-transitive graph of order p^2q in the case where q divides $p+1$, p and q both odd, and in Section 6 we give a construction of such a graph in the case where q, but not q^2, divides $p-1$, q odd. These constructions, together with the discussion above, complete the proof of Theorem 2. We complete this section by proving Theorem 3.

Proof of Theorem 3. Let n be a positive integer divisible by p^2 for some prime p. By [**13**], $p^2 \notin NC$, $p^3 \notin NC$, and by [**10**], $12 \notin NC$, so assume that $n \neq p^2, p^3, 12$. If p^4 divides n then $n \in NC$ by [**11**], so assume this is not the case. Then n is divisible by a prime q distinct from p. If $p^2q \neq 12$ then it follows from Theorem 2 that $n \in NC$, so assume that $p^2q = 12$ divides n, $n \neq 12$. Then by [**11**, Theorem 2(c)], $n \in NC$.

5. Vertex-transitive, non-Cayley graphs of order p^2q where q divides $p+1$.

We give two general constructions of classes of graphs with the properties in the title of this section. The first construction, based on the extraspecial group of order p^3 and exponent p, where p is an odd prime, is simple to define, but the proof that it is non-Cayley is very technical. We have also a construction of a non-Cayley vertex-transitive graph of order p^2q where q^2 divides $p+1$. This latter construction (while not essential to the proof that $p^2q \in NC$ when q divides $p+1$ and p is odd) is interesting in that it is very similar to the Alspach and Parsons construction of metacirculant graphs, while the graphs it produces are not actually metacirculant. So we have decided to give both constructions. First we present the construction based on the extraspecial p-group P of exponent p and order p^3. P has the following presentation.

$$P = \langle\, a, b, c \,|\, a^p = b^p = c^p = [a, c] = [b, c] = 1, c = [a, b] \,\rangle$$

We record the following facts about P.

Lemma 5.1.

(a) For all i, j we have $b^j a^i = a^i b^j c^{-ij}$.

(b) Let q divide $p+1$. The group P has an automorphism x of order q such that $a^x = b$ and $b^x = a^{-1}b^v c^w$ for some v, w.

(c) Let $Q = \langle x \rangle, H = \langle a \rangle < P$, and let $G = PQ$ be the semidirect product of P by Q, with Q acting on P as in (b). Then, in its action on the coset space $[G : H]$ by right multiplication, G preserves the block system $\Sigma = \{B_i \,|\, i \in Z_q\}$ where $B_i = \{Hyx^i \,|\, y \in P\}$ is the set of right H-cosets contained in Px^i; moreover for all i, $B_{i+1} = B_i x$.

Proof. (a) This can be proved by direct computation.

(b) Since P is the relatively free 2 generator, exponent p, class 2 group, all maps $\{a, b\} \to P$ lift to homomorphisms $P \to P$. In particular Aut P has a quotient $GL(2, p)$ and hence P has an automorphism x of order q which maps a to $a^x = b$, and b to $b^x = a^u b^v c^w$ for some u, v, w. Since q divides $p + 1$, x centralizes $\langle c \rangle$, the centre of P, whence $c = [a^x, b^x] = [b, a^u b^v c^w] = [b, a]^u = c^{-u}$. Hence $b^x = a^{-1}b^v c^w$.

(c) The group P is a normal subgroup of G and Σ is the set of P-orbits in $[G : H]$. Hence Σ is a block system for G. Clearly $B_{i+1} = B_i x$ for all $i \in Z_q$.

Now we can define the graph $\Gamma = A(P, q)$.

Construction 5.2. The graph $\Gamma = A(P, q)$, where p, q are odd primes and q divides $p+1$, is defined to have vertex set $V\Gamma = [G : H]$, and cosets Hy and Hz are adjacent in Γ, where $y, z \in G$, if and only if either

(a) $yz^{-1} \in P \setminus H$, or

(b) $yz^{-1} \in HxH \cup Hx^{-1}H$

where G, H, P and x are as in Lemma 5.1.

Note that, since p and q are odd primes and q divides $p + 1$, we have $p \geq 5$. Let $\alpha = H \in V\Gamma$, and let $A = \mathrm{Aut}(\Gamma)$ be the (full) automorphism group of Γ. We begin by making a few observations about Γ and A.

Lemma 5.3.

(a) We have $G \leq A$, so Γ is vertex-transitive.

(b) For $\alpha = H$ we have $\Gamma(\alpha) \cap B_1 = \{Hxa^j \,|\, j \in Z_p\}$, and $\Gamma(\alpha) \cap B_{q-1} = \{Hb^j x^{-1} \,|\, j \in Z_p\}$, the sets of H-cosets in HxH and $Hx^{-1}H$ respectively.

(c) The automorphism group A preserves the block system Σ, and the permutation group A^Σ induced by A on Σ satisfies $Z_q \leq A^\Sigma \leq D_{2q}$.

(d) The set of fixed points of H in B_0 is $fix_{B_0}(H) = \{Hc^i \mid i \in Z_p\}$, a set of size p, and for all $i \in Z_p, \Gamma(Hc^i) \cap B_{q-1} = \{Hc^i b^j\ x^{-1} \mid j \in Z_p\}$, the set of all H-cosets in $Hc^i x^{-1} H$. If $\gamma = Hc^i b^j \in B_0 \setminus fix_{B_0} H$, that is $j \neq 0$, then $\Gamma(\gamma) \cap B_{q-1}$ consists of one point from each of the p orbits of H in B_{q-1}.

(e) The subgroup K of A fixing each of the B_i setwise acts faithfully on B_0.

Proof. (a) Since $\Gamma(\gamma)$ is a union of orbits of $G_\gamma = H$ in $V\Gamma$, G preserves adjacency in Γ, whence $G \leq A$ and A is vertex-transitive on Γ.

(b) This is immediate, noting that $Hx^{-1}a^j = Hb^j x^{-1}$.

(c) If δ, δ' lie in the same block of Σ, say B_i, then $|\Gamma(\delta) \cap \Gamma(\delta')| \geq |\Gamma(\delta) \cap \Gamma(\delta') \cap B_i| = p^2 - 2$, while if δ, δ' lie in different blocks of Σ then $|\Gamma(\delta) \cap \Gamma(\delta')| \leq 2p$ if $q \geq 5$, and $\leq 3p$ if $q = 3$. Since $p \geq 5, 3p < p^2 - 2$, so the relation of being in the same block of Σ is preserved by A. Since the quotient graph of Γ modulo Σ is a cycle C_q, we have $Z_q = \langle x^\Sigma \rangle \leq A^\Sigma \leq \mathrm{Aut}(C_q) = D_{2q}$.

(d) Each point of B_0 is of the form $\gamma = Hb^j c^i$, and H fixes γ if and only if $\gamma = Hb^j c^i a = Hab^j c^{i-j}$ (by Lemma 5.1(a)) $= Hb^j c^{i-j}$, that is, $\gamma = Hc^i$. For $\gamma = Hb^j c^i, \Gamma(\gamma) \cap B_{q-1} = (\Gamma(\gamma) \cap B_{q-1})\, b^j c^i = \{Hb^k x^{-1} b^j c^i \mid k \in Z_p\}$. If $j = 0$ this set is $\{Hb^l c^i x^{-1} \mid l \in Z_p\} = \{Hc^i x^{-1} a^l \mid l \in Z_p\}$ as claimed. To obtain the result for nonzero j observe that $Hb^l c^i x^{-1} b = Hb^l c^i b^x x^{-1} = Hb^l c^i (a^{-1} b^v c^w) x^{-1} = Hc^i a^{-1} b^l c^l b^v c^w x^{-1}$ (by Lemma 5.1) $= Hb^{l+v} c^{i+l+w} x^{-1}$. So $\Gamma(Hbc^i) \cap B_{q-1} = \big(\Gamma(Hc^i) \cap B_{q-1}\big)\, b = \{Hb^{l+v} c^{i+l+w} x^{-1} \mid l \in Z_q\}, \Gamma(Hb^2 c^i)\ \cap B_{q-1} = \{Hb^{l+2v} c^{i+2(l+w)} x^{-1} \mid l \in Z_q\}$, and in general $\Gamma(Hb^j c^i) \cap B_{q-1} = \{Hb^{l+jv} c^{i+j(l+w)} x^{-1} \mid l \in Z_q\}$ which for $j \neq 0$ consists of one point from each H-orbit in B_{q-1}.

(e) Let L be the subgroup of K fixing B_0 pointwise. Then L fixes $Hc^i \in fix_{B_0}(H)$ for each $i \in Z_q$, and hence L fixes setwise each H-orbit $\Gamma(Hc^i) \cap B_{q-1}, i \in Z_p$, in B_{q-1}. Also, as L fixes $Hc^i b^j$, $j \neq 0$, L fixes the intersection of $\Gamma(Hc^i b^j) \cap B_{q-1}$ with each H-orbit in B_{q-1} and, by (d) it follows that L fixes B_{q-1} pointwise. It follows that L fixes pointwise each B_i, $i \in Z_q$, whence $L = 1$.

Next we study the p-subgroups of K and find another A-invariant partition of $V\Gamma$.

Lemma 5.4.

(a) The subgroup P is a Sylow p-subgroup of K.

(b) Let $C = fix_{B_0}(H)$. Then $\Delta = \{C^g \mid g \in A\}$ is a partition of $V\Gamma$ preserved by A. Further $P = O_p(K)$, and Δ is the set of orbits of $Z(P) = \langle c \rangle$, a normal subgroup of A.

Proof. (a) Let S be a Sylow p-subgroup of K containing P. Noting that P fixes each B_i setwise, we have $P \subseteq S \subseteq K$. Then $S_\alpha \supseteq P_\alpha = H$ and hence S_α and

H have the same orbits in B_0 : p fixed points $Hc^i, i \in Z_p$, and $p-1$ orbits of length p. Hence S_α fixes setwise each H-orbit $\Gamma(Hc^i) \cap B_{q-1}, i \in Z_p$, in B_{q-1}. Let $\gamma = Hb^jc^i \in B_0 \setminus fix_{B_0}(H)$. Then $S_{\alpha\gamma}$ fixes setwise $\Gamma(\gamma) \cap B_{q-1}$ which, by Lemma 5.3, consists of one point of each H-orbit in B_{q-1}. It follows that $S_{\alpha\gamma}$ fixes B_{q-1} pointwise. Similarly it follows that $S_{\alpha\gamma}$ fixes each B_i pointwise, whence $S_{\alpha\gamma} = 1$, so $|S| = p^3$ and hence $S = P$.

(b) Set $C = fix_{B_0}(H)$. We claim that K_α fixes C setwise and K preserves the partition $\Delta = \{C^g \,|\, g \in G\}$ of $V\Gamma$. By Lemma 5.3, each point of $B_0 \setminus C$ is joined to a unique point of the H-orbit $D = \Gamma(\alpha) \cap B_{q-1}$ and α is joined to all points of D. As there are exactly p^2 edges from D to B_0 it follows that $\Gamma(D) \cap B_0 = \bigcup\{\Gamma(\beta) \cap B_0 \,|\, \beta \in D\}$ is equal to $(B_0 \setminus C) \cap \{\alpha\}$. Hence K_α, which fixes D setwise, must also fix setwise $B_0 \setminus C$ and C. Then $K_C \geq \langle K_\alpha, H\rangle > K_\alpha$ and hence C is a block of imprimitivity for K in $V\Gamma$ and K preserves $\Delta = \{C^g \,|\, g \in G\}$. In particular H fixes setwise each block of Δ contained in B_0. In fact the blocks of Δ in B_0 are the H-orbits in B_0 together with C.

Let L be the group $\langle H^k \,|\, k \in K\rangle$ generated by all p-subgroups of K which fix a point of B_0. Then L is normal in K and L fixes setwise each block of Δ contained in B_0. Suppose that L is not elementary abelian. Then the group $L^C \cong T$ induced by L on C is a nonabelian simple 2-transitive group, see [**5**, p.202], and $L \cong T^l$ is a subdirect product of $\prod\{L^{C'} \,|\, C' \in \Delta, C' \subseteq B_0\} \cong T^p$. Since K permutes the simple direct factors of L transitively, each such factor is a diagonal subgroup of a direct product T^k where $p = kl, l \geq 1, k \geq 1$. However $|T|$ is divisible by p but not p^2, and $|L|$ is divisible by p^2 (since a Sylow p-subgroup of L has p orbits of length p) but not by p^3 (since $P \not\subseteq L$). Hence $l = 2$, which contradicts $p = lk$. Thus L is an elementary abelian p-group of order p^2. As L is normal in $K, L \subseteq P$ and in fact $L = \langle a, c\rangle$ (since the H-orbits in B_0 are $\{Hc^ib^j \,|\, i \in Z_p\}$ for $j \in Z_p$). It follows that $O_p(K) \neq 1$. However, $O_p(K)$ is a normal subgroup of A, and hence $a^x = b \in L^x \subseteq O_p(K)$, that is $P \subseteq O_p(K)$. Since P is a Sylow p-subgroup of K by (a), we have $P = O_p(K)$. In particular $\langle c\rangle = Z(P)$ is normal in A and Δ is the set of $\langle c\rangle$-orbits in $V\Gamma$, so Δ is preserved by A.

Theorem 5.5. *Let $\Gamma = A(p, q)$ be the graph defined in Construction 5.2, where p and q are distinct odd primes and q divides $p + 1$. Then Γ is a vertex-transitive, non-Cayley graph.*

Proof. By Lemma 5.3, Γ is vertex-transitive. Suppose that a subgroup R of the automorphism group A is regular on $V\Gamma$. Then $|R| = p^2q$.

We first show that Q is a Sylow q-subgroup of A. By Lemma 5.3, $K \cong K^{B_0}$ is faithful on B_0, and by Lemma 5.4, K has a set of p blocks of size p in B_0. Also by Lemma 5.4, P is normal in K and consequently $K \leq AGL(1, p)\ wr\ AGL(1, p)$. Thus $|K|$ is not divisible by q and, since $A/K \leq D_{2q}$ it follows that q^2 does not divide

$|A|$. Hence Q is a Sylow q-subgroup of A and so we may assume that $Q \leq R$. Then $R = Q(R \cap P)$ and $R \cap P$ is a subgroup of P of order p^2 normalized by $Q = \langle x \rangle$. However Q acts irreducibly on $P/Z(P)$, so P has no Q-invariant subgroup of order p^2. Thus A contains no regular subgroups and so Γ is a non-Cayley graph.

Now we give our second construction of a family of vertex-transitive, non-Cayley graphs of order p^2q. As mentioned before these graphs are similar to metacirculant graphs, and we have deliberately chosen notation similar to that of [**1**] to emphasise this. There are several examples of families of this type and we give just one example.

Construction 5.6. Let p and q be odd primes such that q^2 divides $p+1$. Let $F = GF(p^2)$, the Galois field of order p^2, and let $a \in F \setminus \{0\}$ have multiplicative order q^2 and $A = \{a^{jq} \,|\, 0 \leq j < q\}$ denote the multiplicative subgroup of $F \setminus \{0\}$ generated by a^q. So $|A| = q$. Set $S_0 = -A \cup A$ of order $2q$ and $S_1 = \{a^j \,|\, 0 \leq j < q^2\}$ of order q^2. Define $\Gamma = \Gamma(q, p^2, a, S_0, S_1)$ to be the graph with vertex set

$$V\Gamma = \{v_j^i \,|\, i \in Z_q, j \in F\}$$

such that the vertex v_j^i is adjacent to vertex v_h^{i+r} if and only if either $h - j \in a^i S_r$ and $r = 0$ or 1, or $j - h \in a^{i-1} S_1$ with $r = -1$. This adjacency relation is symmetric so the graph Γ is undirected.

Theorem 5.7. *The graph $\Gamma(q, p^2, a, S_0, S_1)$, where q^2 divides $p+1$, of Construction 5.6, is a vertex-transitive, non-Cayley graph of order p^2q.*

Proof. It is easily checked that (for each $x \in F$) the maps

$$g_x : v_j^i \to v_{j+x}^i, \;\; h : v_j^i \to v_{aj}^{i+1}$$

are automorphisms of Γ and hence $\mathrm{Aut}(\Gamma)$ contains the group $\langle g_x, h \,|\, x \in F \rangle$ of order q^2p^2 acting vertex-transitively. We shall show next that the group $P = \{g_x \,|\, x \in F\}$ is a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$. Clearly P is isomorphic to the additive group of F, so P is elementary abelian of order p^2. Let T be a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$ containing P. Since P has $q < p$ orbits of length p^2 in $V\Gamma$, namely $V^i = \{v_j^i \,|\, j \in F\}, i \in Z_q$, it follows that these orbits are also orbits of T. Note that since p, q are odd primes we have $p + 1 \geq 2q^2$. Let T_0 denote the stabilizer of v_0^0 in T. Then, for $i = -1, 0, 1$, T_0 fixes setwise the set $\Gamma(v_0^0) \cap V^i$ of neighbours of v_0^0 in V^i of size $q^2, 2q, q^2$ respectively. Then, since $q^2 < p$ it follows that T_0 fixes each of these sets pointwise. Thus T_0 fixes $\Gamma(v_0^0)$ pointwise. By the connectivity of Γ it follows that $T_0 = 1$. Hence $T = P$ is a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$.

Now suppose that Γ is a Cayley graph so that $\mathrm{Aut}(\Gamma)$ contains a subgroup R of order p^2q acting regularly on $V\Gamma$. Since all Sylow p-subgroups of $\mathrm{Aut}(\Gamma)$ are conjugate we may assume that R contains P. Then by Sylow's Theorems, P is a normal subgroup of R, so both R and $\langle h\rangle \cong Z_{q^2}$ are contained in the normalizer N of P in $\mathrm{Aut}(\Gamma)$. Let Q be a Sylow q-subgroup of N containing $\langle\rangle$. Suppose that the centralizer C of P in $\mathrm{Aut}(\Gamma)$ has order prime to q. Then $Q \cong QC/C \leq N/C$. Since N acts by conjugation on P with C being the subgroup inducing the identity automorphism of P, we have $N/C \leq \mathrm{Aut}\ P = GL(2,p)$. Now the Sylow q-subgroups of $GL(2,p)$ are cyclic of order the q-part of $p+1$. So Q is cyclic and hence every subgroup of N of order q is conjugate in N to the unique subgroup of Q of order q, namely $\langle h^q\rangle$, and therefore fixes each V^i setwise. This contradicts the fact that $R \leq N$ contains a subgroup of order q which permutes $V^0, \ldots, V^{q-1}$ cyclically. Thus q divides $|C|$. Let x be an element of C of order q. Suppose that x permutes $V^0, \ldots, V^{q-1}$ cyclically. By replacing x by some power of itself if necessary we may assume that x maps V^0 to V^1 so $(v_0^0)^x = v_l^1$ for some $l \in F$. Then $\{v_{l+j}^1 \,|\, j \in aS_0\} = \Gamma(v_l^1) \cap V^1 = (\Gamma(v_0^0) \cap V^0)^x = \{v_j^0 \,|\, j \in S_0\}^x$ and $(v_j^0)^x = (v_0^0)^{g_j x} = (v_0^0)^{x g_j}$, since x centralizes P, which equals $(v_l^1)^{g_j} = v_{l+j}^1$. Thus $\{v_{l+j}^1 \,|\, j \in aS_0\} = \{v_{l+j}^1 \,|\, j \in S_0\}$ and hence $S_0 = aS_0$ which is not the case. Thus x fixes each V^i setwise. However, since the permutation group induced by P on V^i is regular and abelian it is self-centralizing, for each $i \in Z_q$. Hence the subgroup of C fixing each V^i setwise is a p-group, and this contradicts the fact that x fixes each V^i setwise. Thus no such subgroup R exists and Γ is a non-Cayley graph.

We note that our construction can be generalized as follows. Let k be minimal such that q divides $p^k - 1$ and suppose that q^2 divides $p^k - 1$. Then replacing F by the field $GF(p^k)$ we easily obtain a vertex-transitive, non-Cayley graph of order $p^k q$. This is the Alspach and Parsons construction for $k = 1$. However in view of [**11**, Theorem 2] it gives no further information about membership of NC when k is greater than 2.

6. Vertex-transitive, non-Cayley graphs of order p^2q where q, but not q^2, divides $p-1$.

In this section we give a construction of a class of graphs with the properties given in the section title. As preparation for this construction we define a group G of permutations of the set $V = \mathbb{Z}_q \times \mathbb{Z}_p \times \mathbb{Z}_p$ of p^2q elements, where p, q are odd primes such that q divides $p-1$.

Definition 6.1.

(a) Let $s \in \mathbb{Z}_p, s \neq 1$, satisfy $s^q = 1$, and define $F : \mathbb{Z}_p^q \to \mathbb{Z}_p^q$ by

$$F(x_0, x_1, \ldots, x_{q-1}) = (sx_{q-1}, x_0, x_1, \ldots, x_{q-2})$$

for all $(x_0, x_1, \ldots, x_{q-1}) \in \mathbb{Z}_p^q$.

(b) For $\mathbf{c} = (c_0, c_1, \ldots, c_{q-1}) \in \mathbb{Z}_p^q$ and $a \in \mathbb{Z}_p, a \neq 0$, define $\gamma(a; \mathbf{c}) : V \to V$ by

$$(i, x, y)^{\gamma(a;\mathbf{c})} = (i, ax + c_{i-1}, ay + c_i)$$

for all $(i, x, y) \in V$.

(c) Define $g : V \to V$ by

$$(i, x, y)^g = \begin{cases} (1, sx, y) & \text{if } i = 0 \\ (i+1, x, y) & \text{if } 1 \leq i \leq q-2 \\ (0, x, \ sy) & \text{if } i = q-1. \end{cases}$$

Lemma 6.2. *The map g and, for each $0 \neq a \in \mathbb{Z}_p$ and $\mathbf{c} \in \mathbb{Z}_p^q$, the map $\gamma(a; \mathbf{c})$ are permutations of V. Further,*

$$\gamma(a; \mathbf{c})\gamma(a'; \mathbf{c}') = \gamma(aa'; \ a'\mathbf{c} + \mathbf{c}')$$

$$g^q = \gamma(s; \mathbf{0}),$$

$$\gamma(a; \mathbf{c})^g = g^{-1}\gamma(a; \mathbf{c})g = \gamma(a; F\mathbf{c}),$$

for all $a, a' \in \mathbb{Z}_p \setminus \{0\}; \mathbf{c}, \mathbf{c}' \in \mathbb{Z}_p^q$.

The proofs of these facts are just straightforward computations and are omitted. It is now clear that

$$K = \{\gamma(a; \mathbf{c}) \,|\, 0 \neq a \in \mathbb{Z}_p, \mathbf{c} \in \mathbb{Z}_p^q\}$$

is a group of permutations of V, and that $G = \langle g, K \rangle$, also a permutation group on V, has K as a normal subgroup. We collect some properties of G in the following lemma.

Lemma 6.3.

(a) We have $|G| = q|K| = q(p-1)p^q$.

(b) The group G is transitive and imprimitive on V, preserving the partition $\Sigma = \{B_0, B_1, \ldots, B_{q-1}\}$ where $B_i = \{(i, x, y) \,|\, x, y \in \mathbb{Z}_p\}$.

(c) For $\alpha = (0, 0, 0) \in V$, $G_\alpha = \{\gamma(a; \mathbf{c}) \,|\, c_0 = c_{q-1} = 0\}$.

(d) The group G has a subgroup which acts regularly on V if and only if q^2 divides $p-1$.

Proof. Since g has order q modulo K, part (a) is obvious. Also part (b) is clearly true with $G_{B_i} = G_{(\Sigma)} = K$ for all i. For (c) note that $G_\alpha < G_{B_0} = K$, and $\alpha^{\gamma(a;\mathbf{c})} = \alpha$ if and only if $c_0 = c_{q-1} = 0$. Now suppose that q^2 does not divide $p-1$ and that $R < G$ acts regularly on V. Then R contains an element x, say, of order q which cyclically permutes the blocks of Σ. Since $G^\Sigma = \langle g \rangle^\Sigma$ we may choose x such that $x^\Sigma = (B_0, B_1, \ldots, B_{q-1}) = g^\Sigma$. Then $x = g\gamma$ for some $\gamma =$

$\gamma(a;\mathbf{c}) \in K$. Using Lemma 6.2 we have $\gamma(1;\mathbf{0}) = 1 = x^q = (g\gamma)^q = g^q\gamma^{g^{q-1}}\cdots\gamma^g\gamma = \gamma(sa^q; a^{q-1}F^{q-1}\mathbf{c}+a^{q-2}F^{q-2}\mathbf{c}+\cdots+aF\mathbf{c}+\mathbf{c})$. It follows that $sa^q = 1$ which implies that a has order q^2 modulo p. This contradicts the fact that q^2 does not divide $p-1$. Thus no such regular subgroup R exists. Finally suppose that q^2 divides $p-1$. Then $s = a^{-q}$ for some $a \in \mathbb{Z}_p$ and, arguing as in the previous paragraph, the element $x = g\gamma$ has order q where $\gamma = \gamma(a;(1,a,a^2,\ldots,a^{q-1}))$. Set $r_j = as^j$ for $j \in \mathbb{Z}_q$. Then if $\mathbf{d}_j = (r_j^{q-1}, r_j^{q-2},\ldots,1)$ we have $F\mathbf{d}_j = r_j\mathbf{d}_j$ and by Lemma 6.2 the element $\gamma_j = \gamma(1;\mathbf{d}_j)$ satisfies $\gamma_j^x = \gamma(1;aF\mathbf{d}_j) = \gamma(1;ar_j\mathbf{d}_j) = \gamma_j^{ar_j}$. It follows that $P = \{\gamma(1;y\mathbf{d}_0 + y'\mathbf{d}_1) = \gamma_0^y\gamma_1^{y'} \mid y,y' \in \mathbb{Z}_p\}$ is a subgroup of K of order p^2 which is normalized by x, whence $R = \langle P, x\rangle$ is a subgroup of G of order p^2q. Further it is easily checked, using part (c), that $G_\gamma \cap\ R = 1$, so R is regular on V.

Now we construct the graph $Y(p,q,s)$ with s as above.

Construction 6.4. For p,q odd primes with q dividing $p-1$, let $s \in\ \mathbb{Z}_p$ with $s \neq 1, s^q = 1$, and define $\Gamma = Y(p,q,s)$ to be the graph with vertex set $V\Gamma = V = \mathbb{Z}_q \times \mathbb{Z}_p \times \mathbb{Z}_p$ and edge set $E\Gamma = E_1 \cup E_2$, where

$$\begin{aligned} E_1 &= \{(0,x,y)(0,x+k,y+sk) \mid x,y,k \in \mathbb{Z}_p; k \neq 0\} \\ &\qquad \cup \{(i,x,y)(i,x+k,y+k) \mid 0 \neq i \in \mathbb{Z}_q; x,y,k \in \mathbb{Z}_p; k \neq 0\}, \\ E_2 &= \{(i,x,y)(i+1,y,z) \mid i \in \mathbb{Z}_q, x,y,z \in \mathbb{Z}_p\}. \end{aligned}$$

Theorem 6.5. *The graph $Y(p,q,s)$ has as automorphism group the group G defined above, whence $Y(p,q,s)$ is a vertex-transitive, non-Cayley graph if and only if $p-1$ is divisible by q but not by q^2.*

Proof. It is straightforward to check that g, and each of the $\gamma(a;\mathbf{c})$ defined in Definition 6.1 preserve the edge sets E_1 and E_2, so that $G \leq \mathrm{Aut}(\Gamma)$, where $\Gamma = Y(p,q,s)$. Let $v = (i,x,y)$ and $w = (i',x',y')$ be distinct elements of V. If $i - i' \not\equiv 0,1,-1 \pmod q$ then v and w have no common neighbours; if $i - i' \equiv 1$ or $-1 \pmod q$ then v and w have two common neighbours; if $i = i'$ then v and w have either p common neighbours (if $x = x'$ or $y = y'$) or at most $p-2$ common neighbours (otherwise). Write $v \sim w$ if v and w have p common neighbours in Γ. The reflexive, transitive closure of $\sim$ is an equivalence relation on V preserved by $\mathrm{Aut}(\Gamma)$, and having equivalence classes $B_0, B_1, \ldots, B_{q-1}$. Thus the partition $\Sigma = \{B_0, B_1, \ldots, B_{q-1}\}$ is a block system for $\mathrm{Aut}(\Gamma)$. This implies that E_1 and E_2 are fixed setwise by $\mathrm{Aut}(\Gamma)$, so we have that $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Gamma_2)$, where Γ_2 is the subgraph induced by E_2. Now, Γ_2 is the graph $C(p,q,2)$ of [**11**] (apart from an obvious relabelling) and the structure of $\mathrm{Aut}(\Gamma_2)$ appears there repeated from [**19**]; we need only identify which elements of $\mathrm{Aut}(\Gamma_2)$ preserve E_1.

Set $K^* = \mathrm{Aut}(\Gamma)_{(\Sigma)}$, the subgroup of $\mathrm{Aut}(\Gamma)$ fixing each B_i setwise. Then $K \leq K^*$. From [**11**], each element of K^* has the form $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{q-1})$ for some $\sigma_0, \sigma_1, \ldots, \sigma_{q-1} \in S_p$, where $(i,x,y)^\sigma = (i, x^{\sigma_{i-1}}, y^{\sigma_i})$ for each $(i,x,y) \in V$. For $i \neq 0$, the edge $(i,x,y)(i,x+1,y+1) \in E_1$ is mapped by σ to the pair $(i, x^{\sigma_{i-1}}, y^{\sigma_i})(i, (x+1)^{\sigma_{i-1}}, (y+1)^{\sigma_i})$. For σ to lie in K^* this pair must lie in E_1 and hence $(x+1)^{\sigma_{i-1}} - x^{\sigma_{i-1}} = (y+1)^{\sigma_i} - y^{\sigma_i}$ for all $x, y \in \mathbb{Z}_p$. For a fixed x, letting y vary over $\mathbb{Z}_p$, we see that σ_i is an affine map, that is $y^{\sigma_i} = a_i y + b_i$ for some $a_i \neq 0, b_i$, and similarly σ_{i-1} is an affine map, $x^{\sigma_{i-1}} = a_{i-1}x + b_{i-1}$ with $a_{i-1} = a_i = a$ say. Since this is true for all $i \neq 0$, we have, for $i = 0, 1, \ldots, q-1$, $y^{\sigma_i} = ay + b_i$ for some $b_i \in \mathbb{Z}_p$. It follows from Definition 6.1 that $\sigma = \gamma(a; (b_0, b_1, \ldots, b_{q-1}))$. Thus $\sigma \in K$, and hence $K^* = K$.

The quotient graph Γ_Σ of Γ modulo Σ is clearly a cycle of length q, and so $\mathbb{Z}_q \cong G^\Sigma \leq \mathrm{Aut}(\Gamma)/K \leq \mathrm{Aut}(\Gamma_\Sigma) = D_{2q}$. Thus $|\mathrm{Aut}(\Gamma) : G| \leq 2$ and $\mathrm{Aut}(\Gamma) = G$ if and only if $\mathrm{Aut}(\Gamma)/K \cong \mathbb{Z}_q$. Suppose for a contradiction that $\mathrm{Aut}(\Gamma)/K = D_{2q}$. Then there is an element $h \in \mathrm{Aut}(\Gamma)_{B_0}$ which interchanges B_i and B_{q-i} for all i. We know from [**11**] that all such elements of $\mathrm{Aut}(\Gamma_2)$ can be represented by a sequence $\sigma_0, \sigma_1, \ldots, \sigma_{q-1} \in S_p$, such that $(i,x,y)^h = (q-i, y^{\sigma_{i+1}}, x^{\sigma_i})$ for each $(i,x,y) \in V$.

For $i \neq 0$, the image of the edge $(i,x,y)(i,x+1,y+1)$ under h is an edge only if $(y+1)^{\sigma_{i+1}} - y^{\sigma_{i+1}} = (x+1)^{\sigma_i} - x^{\sigma_i}$ for all x, y. This implies that σ_i and σ_{i+1} are affine with the same linear coefficient. Since this is true for $i = 1, 2, ..., q-1$, and σ_q is just another name for σ_0, we see that there are $a, b, b' \in \mathbb{Z}_p$ such that $z^{\sigma_0} = az + b$ and $z^{\sigma_1} = az + b'$ for every z. However, the edge $(0,0,0)(0,1,s)$ is mapped by h onto $(0, b', b)(0, as+b', a+b)$, which cannot be an edge because $s^2 \neq 1$.

This is a contradiction, and hence $\mathrm{Aut}(\Gamma) = G$. By Lemma 6.3(d), $\mathrm{Aut}(\Gamma)$ has no regular subgroup (that is, Γ is a vertex-transitive, non-Cayley graph) if and only if q, but not q^2, divides $p-1$.

7. Consolidation.

It is interesting to consider how successful we have been in identifying the small members of NC. In addition to all the results mentioned in this paper so far, we have the following theorem of Miller and Praeger [**16**].

Theorem 7.1. *Let $2 < q < p$, where q and p are primes congruent to 3 (mod 4).*
(a) If q divides $p-1$ or $(q,p) = (3,11)$, then $2pq \in NC$.
(b) If q does not divide $p-1$, $pq \notin NC$, and $(q,p) \neq (3,11)$, then every vertex-transitive non-Cayley graph of order $2pq$ has a primitive automorphism group which has no transitive imprimitive subgroups.

The completion of the study of NC for order $2pq$ requires a classification of the primitive groups of degree $2pq$, which will follow from work in progress by Greg

Gamble. At this stage we will just record some preliminary ideas. Firstly, the primitive groups of degree $2pq$ in the case $2q < p$ have been found by Liebeck and Saxl [**9**]. The nonexistence of a suitable primitive group of the right order sometimes shows (from Theorem 7.1) that a number is not in NC. For example, $138 \notin NC$. Secondly, many of the groups listed in [**9**] are easily shown to yield non-Cayley vertex-transitive graphs by application of the following simple idea.

Lemma 7.2. *Let $n = 2m$, where m is odd. Let X be a vertex-transitive graph of order n such that $\mathrm{Aut}(X) \leq A_n$. Then X is not a Cayley graph.*

Proof. Every regular group of degree $2m$ contains an involution without fixed points. Such a permutation lies outside A_n if m is odd.

A useful case where Lemma 7.2 applies is when $\mathrm{Aut}(X)$ is simple. For example, the Livingston graph of order 266 has as its full automorphism group the Janko group J_1 [**3**, Theorem 13.5.1], showing that $266 \in NC$.

After collecting together the known theory, we find that we can determine for all but 11 numbers less than 1000 whether or not they belong to NC. The exceptions are 418, 429, 483, 609, 651, 759, 874, 897, 903, 957 and 987. All of these are the product of three distinct primes. The first product of four distinct primes for which membership of NC is undecided at present is 6118, and the first such number that is odd is 9867. The first unclassified product of five primes is 189658.

We close with the problem that we believe to be the key unresolved question concerning the structure of NC.

Question. *Is there a number $k > 0$ such that every product of k distinct primes is in NC?*

Since $138 \notin NC$, we know that $k \geq 4$ if k exists.

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