ON THE SPECTRAL CHARACTERISATION OF TREES

Brendan D. McKay

ABSTRACT: In this paper, strong negative results are obtained concerning the feasibility of identifying a tree by the spectral properties of certain associated matrices. In particular, we show that, in a precise sense, hardly any tree is identified by the characteristic polynomial of its distance matrix, thus disproving a conjecture of Edelberg, Garey and Graham [4]. At the other extreme, we prove constructively that every tree is uniquely determined by the spectrum of every polynomial function of its adjacency matrix and the diagonal matrix of vertex degrees.

1. *Introduction*

During the past 20 years, a considerable amount of research has been done on the relationship between the structure of a graph and the spectral properties of its adjacency matrix and other associated matrices. Surveys of this research can be found in [3] or [6]. It is well known that the spectrum of the adjacency matrix does not always characterise a graph, the smallest pair of "cospectral" graphs having only 5 vertices [2]. More recently, A.J. Schwenk [11] has proved that "hardly any" tree is thus characterised; the proportion dropping to zero as the number of vertices increases. In 1975, Godsil and McKay [7] showed that this result is still true if we require cospectral trees to also have cospectral complements. In this paper we demonstrate, aided by a computer, that (for example) we may also require the trees to have cospectral linegraphs and cospectral distance matrices. We then approach the characterisation problem from the opposite direction and show by an algorithm that, if the spectrum of every polynomial function of the adjacency matrix and the diagonal vertex degrees matrix is given, we can in fact determine the tree.

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2. *Matrix Theory*

All matrices will have entries from the real numbers. Suppose M is an $n \times n$ matrix. The transpose of M will be denoted by M' and the (i, k)th entry of M by M_{ik}. The matrix M^+ is defined to be the $(n + 1) \times (n + 1)$ matrix $\left|\begin{array}{c|c} 0 & j' \\ \hline j & M \end{array}\right|$, where j is the (column) vector of length n with each entry equal to one. The symbols I and J denote the identity matrix and the square matrix with every entry one, respectively. In each case, the size will not be stated if it is obvious from the context. The *characteristic polynomial* of M, $\phi[M; x]$, is the value of the determinant $|xI - M|$. We will usually write $\phi[M; x]$ as $\phi[M]$, in which case the variable is assumed to be x.

LEMMA 2.1. *For any* $t \neq 0$, $\phi[M^+] = t\phi[M + \frac{1}{t}J] + (x - t)\phi[M]$.

Proof.

$$\phi[M^+] = \left|\begin{array}{c|c} x & -j' \\ \hline -j & xI-M \end{array}\right| = \left|\begin{array}{c|c} t & -j' \\ \hline -j & xI-M \end{array}\right| + \left|\begin{array}{c|c} x-t & 0 \\ \hline -j & xI-M \end{array}\right|.$$

In the first determinant on the right-hand side, add $\frac{1}{t}$ of the first row to every other row. The result is immediate.

Suppose that a, b, c and d are column vectors. The bordered matrix $\left[\begin{array}{c|c} 0 & a' \\ \hline b & M \end{array}\right]$ will be denoted by $[M|a, b]$, and the doubly bordered matrix $\left[\begin{array}{c|c|c} 0 & 0 & a' \\ \hline 0 & 0 & c' \\ \hline b & d & M \end{array}\right]$ by $[M|a, b|c, d]$.

LEMMA 2.2. *Let M and N be square matrices of order m and n respectively. Let* α, β, γ, δ *be scalars;* a, c, $\bar{a}$, $\bar{c}$ *be m-vectors and* b, d, $\bar{b}$, $\bar{d}$ *be n-vectors. Define*

$$P_1 = \left[\begin{array}{c|c|c} \alpha & a' & b' \\ \hline \bar{a} & M & 0 \\ \hline \bar{b} & 0 & N \end{array}\right] \qquad \text{and}$$

$$P_2 = \left[\begin{array}{c|c|c|c} \alpha & \gamma & a' & b' \\ \hline \delta & \beta & c' & d' \\ \hline \bar{a} & \bar{c} & M & 0 \\ \hline \bar{b} & \bar{d} & 0 & N \end{array}\right].$$

Then (*i*) $\phi[P_1] = \phi[M]\phi[N|b, \bar{b}] + \phi[N]\phi[M|a, \bar{a}] - (x + \alpha)\phi[M]\phi[N]$, *and*

$$(ii)\ \phi[P_2] = \phi[M]\phi[N]\{-x^2 + (\alpha + \beta - 2\gamma - 2\delta)x + \alpha\beta - \gamma\delta\}$$
$$- (x + \alpha)\{\phi[M]\phi[N|d, \bar{d}] + \phi[N]\phi[M|c, \bar{c}]\}$$
$$- (x + \beta)\{\phi[M]\phi[N|b, \bar{b}] + \phi[N]\phi[M|a, \bar{a}]\}$$
$$+ (x + \gamma)\{\phi[M]\phi[N|d, \bar{b}] + \phi[N]\phi[M|c, \bar{a}]\}$$
$$+ (x + \delta)\{\phi[M]\phi[N|b, \bar{d}] + \phi[N]\phi[M|a, \bar{c}]\}$$
$$+ \phi[M|a, \bar{a}]\phi[N|d, \bar{d}] + \phi[M|c, \bar{c}]\phi[N|b, \bar{b}]$$
$$- \phi[M|a, \bar{c}]\phi[N|d, \bar{b}] - \phi[M|c, \bar{a}]\phi[N|b, \bar{d}]$$
$$+ \phi[M]\phi[N|b, \bar{b}|d, \bar{d}] + \phi[N]\phi[M|a, \bar{a}|c, \bar{c}].$$

Proof. Expand the determinants $\phi[P_1; x]$ and $\phi[P_2; x]$ by brute force.

3. *Matrices associated with a graph*

Let G be a graph with vertices $\{1, 2, \cdots, n\}$. We assume that $0 < n < \infty$ and that G has no loops, directed edges or multiple edges. Graph theoretic concepts not defined here can be found in Behzad and Chartrand [1]. In particular, $\bar{G}$ denotes the complement, and L(G) the linegraph of G. The *cone* G^+ of G is the graph formed by adjoining a new vertex adjacent to every vertex of G.

From the graph G, several matrices can be defined.

(1) A(G) is the *adjacency matrix* of G:

$$A_{ik} = \begin{cases} 1 \text{ if i is adjacent to k} \\ 0 \text{ otherwise.} \end{cases}$$

(2) D(G) is the *distance matrix* of G (G connected):

$D_{ik} = \partial(i, k)$, where $\partial(i, k)$ is the distance from i to k in G. By convention $\partial(i, i) = 0$.

(3) $\Lambda(G)$ is the diagonal matrix $\mathrm{diag}(d_1, d_2, \cdots, d_n)$, where d_i is the degree of vertex i.

(4) Q(G) is the (n + 1)th order matrix $[A(G) - \Lambda(G)|q, q]$, where q is the n-vector whose i-th entry is $2 - d_i$.

We shall find it convenient to write $\phi(H; x)$ instead of $\phi[A(H); x]$ for a graph H.

THEOREM 3.1. *Let G be a graph with n vertices and* $e = \binom{n}{2} - \bar{e}$ *edges. Define*

$$\phi_1(x) = \phi[A(G) + \Lambda(G); x] \qquad \text{and}$$

$$\phi_2(x) = \phi[(A(G) + \Lambda(G))^+; x]. \qquad \text{Then}$$

(i) $\phi(\bar{G}; x) = (-1)^{n+1}\{\phi(G^+; -x - 1) + x\phi(G; -x - 1)\}.$

(ii) $\phi(L(G); x) = (x + 2)^{e-n}\phi_1(x + 2).$

(iii) $\phi(L(\bar{G}); x) = (-1)^{n+1}(x + 2)^{\bar{e}-n}\{(x - n + 3)\phi_1(-x + n - 4) + \phi_2(-x + n - 4)\}.$

(iv) $\phi(\overline{L(G)}; x) = \frac{1}{4}(-1)^{e+1}(1 - x)^{e-n}\{(1 - x)\phi_2(1 - x) - (x^2 - 2x - n + 5)\phi_1(1 - x)\}.$

(v) $\phi(\overline{L(\bar{G})}; x) = \frac{1}{4}(-1)^{\bar{e}+n}(1 - x)^{\bar{e}-n}\{(x^2 - n^2 + 5n - 5)\phi_1(x + n - 3) - (x - n + 3)\phi_2(x + n - 3)\}.$

(vi) *If G is a tree,* $\phi[D(G); x] = -\frac{1}{4}x^n\{\phi[Q(G); \frac{2}{x}] + (n - 1 - \frac{2}{x})\phi[A(G) - \Lambda(G); \frac{2}{x}]\}.$

(vii) *If* $\bar{G}$ *has diameter 2,* $\phi[D(\bar{G}); x] = \phi(G^+; x + 1) - x\phi(G; x + 1).$

Proof. Noting that $A(G^+) = A(G)^+$, part (i) comes immediately from Lemma 2.1, with $t = -1$. Since $D(\bar{G}) = J + A(G) - I$ when $\bar{G}$ has diameter 2, part (vii) follows similarly.

Let B be the incidence matrix of G. Then $B'B = A(L(G)) + 2I$ and $BB' = A(G) + \Lambda(G)$. Since $\phi[B'B] = x^{e-n}\phi[BB']$, [9], we obtain (ii). Part (iii) is obtained by applying part (ii) to $\bar{G}$ and using part (i).

To prove (iv), let $C = [j \mid B]$, where B is the incidence matrix of B and j is a vector of ones. Then $CC' = J + A(G) + \Lambda(G)$ and $C'C = \left[\begin{array}{c|c} n & 2j' \\ \hline 2j & A(L(\bar{G}))+2I \end{array}\right]$. A tedious calculation using (i) and (ii) produces the equation (iv). Applying this equation to $\bar{G}$ gives (v).

Part (vi) can be easily derived from Fact 4 of [8].

4. *The major results*

Let T be a rooted tree with vertices $\{0, 1, \cdots, n\}$. We will always assume in such cases that vertex 0 is the root. The n-vertex forest formed by removing the root of T will be denoted T*.

Define $\Lambda^*(T)$ to be the $n \times n$ diagonal matrix $\mathrm{diag}(d_1, d_2, \cdots, d_n)$ where d_i is the degree of vertex i in T (not T*). Similarly, define two n-vectors $a^*(T)$ and $q^*(T)$ thus:

$$a_i^*(T) = \begin{cases} 1 & \text{if } i \text{ is adjacent to the root in } T \\ 0 & \text{otherwise} \end{cases}$$

$$q_i^*(T) = 2 - d_i \quad (1 \le i \le n).$$

Note that $A(T) = [A(T^*)|a^*(T), a^*(T)]$.

Let T_1 and T_2 be the rooted trees shown in Figure 1.

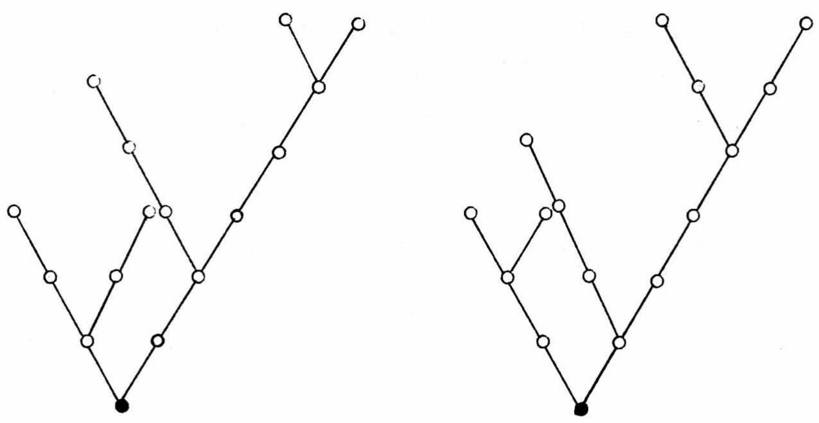


Figure 1.

The following lemma can be established by a direct computation.

LEMMA 4.1. *Let z and j denote the 15-vectors with each entry 0 and 1, respectively. Let e_1, f_1, g_1, $h_1 \in \{z, j, a^*(T_1), q^*(T_1)\}$, not necessarily* *distinct, and let* e_2, f_2, g_2, h_2 *be the corresponding elements of* $\{z, j, a^*(T_2), q^*(T_2)\}$. *Let* $k \in \{-1, 0, 1\}$.

Then $\phi[M_1|e_1, f_1] = \phi[M_2|e_2, f_2]$ *and*

$$\phi[M_1|e_1, f_1|g_1, h_1] = \phi[M_2|e_2, f_2|g_2, h_2],$$

where $M_i = A(T_i^*) + k\Lambda^*(T_i)$, $(i = 1, 2)$.

We are now able to present our major theorem. Suppose that S and T are rooted trees with m + 1 and n + 1 vertices respectively (m, n ≥ 1). The *coalescence* S•T of S and T is the (m + n + 1)-vertex tree formed by identifying the roots of S and T. The rooted trees S and T are called *limbs* of S•T.

THEOREM 4.2. *Let* $S_i = S \cdot T_i$ $(i = 1, 2)$, *where S is any rooted tree with 2 or more vertices, and* T_1, T_2 *are the rooted trees of Figure 1. Then* S_1 *and* S_2 *are not isomorphic, but*

(i) $\phi(S_1) = \phi(S_2)$; $\phi(\bar{S}_1) = \phi(\bar{S}_2)$.

(ii) $\phi(L(S_1)) = \phi(L(S_2))$; $\phi(L(\bar{S}_1)) = \phi(L(\bar{S}_2))$.

(iii) $\phi(\overline{L(S_1)}) = \phi(\overline{L(S_2)})$; $\phi(\overline{L(\bar{S}_1)}) = \phi(\overline{L(\bar{S}_2)})$.

(iv) $\phi[D(S_1)] = \phi[D(S_2)]$; $\phi[D(\bar{S}_1)] = \phi[D(\bar{S}_2)]$.

Proof. The trees S_1 and S_2 are not isomorphic, since S_2 has the rooted tree of Figure 2, more times as a limb than does S_1.

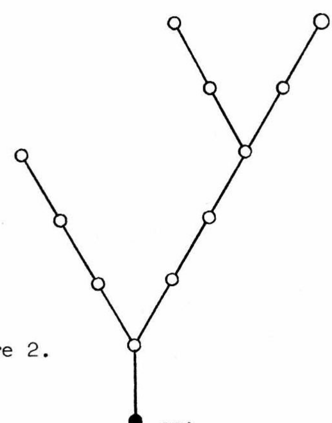


Figure 2.

Parts (i)-(iv) can be proved by elementary calculation from Lemma 2.2, Theorem 3.1 and Lemma 4.1. We illustrate the method by proving the first part of (ii). Let δ be the degree of the root of S.

By Theorem 3.1, $\phi(L(S_1); x - 2) = \frac{1}{x}\phi[A(S_1) + \Lambda(S_1); x]$.

But $A(S_1) + \Lambda(S_1) = \left[\begin{array}{c|c|c} \delta+2 & a_1' & a_S' \\ \hline a_1 & \Gamma_1 & 0 \\ \hline a_S & 0 & \Gamma_S \end{array}\right]$, where $a_1 = a^*(T_1)$, $a_S = a^*(S)$, $\Gamma_1 = A(T_1^*) + \Lambda^*(T_1)$, and $\Gamma_S = A(S^*) + \Lambda^*(S)$.

Hence, by Lemma 2.2, $\phi(L(S_1); x - 2) = \frac{1}{x}\{\phi[\Gamma_1]\phi[\Gamma_S|a_S, a_S] + \phi[\Gamma_1|a_1, a_1]\phi[\Gamma_2] - (x + \delta + 2)\phi[\Gamma_1]\phi[\Gamma_S]\}$, with a similar formula for $\phi(L(S_2); x - 2)$.

However, by Lemma 4.1, $\phi[\Gamma_1] = \phi[\Gamma_2]$ and $\phi[\Gamma_1|a_1, a_1] = \phi[\Gamma_2|a_2, a_2]$, from which we have $\phi(L(S_1)) = \phi(L(S_2))$ immediately.

The following lemma (see [10] and [11]) actually only requires that T_1 and T_2 have the same size.

LEMMA 4.3. *For $m \geq 1$, $i = 1, 2$, let $p_i(m)$ be the proportion of trees on m vertices which have T_i as a limb.*

Then (i) $p_1(m) = p_2(m)$ for all m, and

(ii) $p_1(m) \to 1$ as $m \to \infty$.

COROLLARY 4.4. *Let $p(m)$ be the proportion of trees T on m vertices which are characterised (amongst trees) by the characteristic polynomials of any or all of the following matrices. Then $p(m) \to 0$ as $m \to \infty$.*

(i) $A(T)$	*(ii)* $A(\bar{T})$
(iii) $A(L(T))$	*(iv)* $A(L(\bar{T}))$
(v) $A(\overline{L(T)})$	*(vi)* $A(\overline{L(\bar{T})})$
(vii) $D(T)$	*(viii)* $D(\bar{T})$.

For the matrix A(T) only, this result was first proved by Schwenk [11]. The matrix $A(\bar{T})$ was added by Godsil and McKay [7]. The other additions are new. In particular, it was not previously known whether $\phi[D(T)]$ characterised T, for the solution of which problem the author grudgingly accepted the $100 prize offered by R.L. Graham. A computer search has found that the only trees with 18 or fewer vertices having cospectral distance matrices are those given by Theorem 4.2.

5. *Functions of A(T) and Λ(T)*

The generality of Lemma 4.1 allows quite a few extra matrices to be added to the list of Corollary 4.4, but we have merely selected a few with simple interpretations. Unless we can find a simple matrix whose characteristic polynomial determines the tree, there seems to be little motivation for extending the list. Nevertheless, there is certainly room for research into more general classes of matrices and their relationship to graph structure. As a small contribution in this direction, we look at the characteristic polynomials of functions of A(T) and Λ(T) for a tree T.

To begin with, note that by Lemma 2.1 and Theorem 3.1, the characteristic polynomials of each of the matrices of Corollary 4.4 are determined by the characteristic polynomials of the matrices A(T), J + A(T), A(T) + Λ(T), J + A(T) + Λ(T) and D(T). The next lemma [5] shows that each of these matrices is a polynomial function of A(T) and Λ(T).

LEMMA 5.1. *Let T be a tree on n vertices. For each non-negative integer r define the n × n matrix $D_r(T)$ by*

$$(D_r(T))_{ik} = \begin{cases} 1 \text{ if } \partial(i, k) = r \\ 0 \text{ otherwise.} \end{cases}$$

Then $D_1(T) = A(T)$,

$$D_2(T) = A(T)^2 - \Lambda(T), \text{ and}$$

$$D_{r+1}(T) = A(T)D_r(T) - (\Lambda(T) - I)D_{r-1}(T), \quad r \geq 2.$$

COROLLARY 5.2. *$D(T)$ and J are polynomial functions of $A(T)$ and $\Lambda(T)$.*

Proof. $D(T) = \sum_{r=1}^{n} rD_r(T)$, and $J = \sum_{r=0}^{n} D_r(T)$.

This observation might tempt one to suspect that any polynomial functions of $A(T)$ and $\Lambda(T)$ might be added to the list of matrices in Corollary 4.4. In fact, the situation is quite the opposite.

THEOREM 5.3. *Let S and T be any two non-isomorphic trees. Then there is a two-variable real polynomial p such that $\phi[p(A(S), \Lambda(S))] \neq \phi[p(A(T), \Lambda(T))]$.*

Proof. We establish the theorem by giving a constructive proof of the contrapositive. Specifically, we give an algorithm for reconstructing T from $\psi(T)$, where $\psi(T)$ is the family of two-variable real polynomials p such that $p(A(T), \Lambda(T))$ has non-zero trace. Since the trace of a matrix is the sum of its eigenvalues, it is obviously determined by the characteristic polynomial.

Let T be a tree with n vertices ($n \geq 1$). For each integer m ($0 \leq m \leq n$) define ε_m to be any real (one-variable) polynomial such that for integers ℓ ($0 \leq \ell \leq n$, $\ell \neq m$), $\varepsilon_m(\ell) = 0$, but that $\varepsilon_m(m) = 1$. Finally, let Π indicate the set of real polynomials in two non-commuting variables.

ALGORITHM 5.4. *Reconstruct T from $\psi = \psi(T)$.*

(1) Define $\alpha, \lambda \in \Pi$ by $\alpha(x, y) = x$, $\lambda(x, y) = y$ for all x, y.

Set $\Xi \leftarrow \{(K_1, \iota)\}$ where K_1 is the rooted tree with one vertex, and $\iota \in \Pi$ has $\iota(x, y) = 1$ for all x, y.

(2) If $\lambda \notin \psi$, go to step (6).

If $\lambda^2 - \lambda \notin \psi$, go to step (7).

(3) Select any $(S, \sigma) \in \Xi$ such that $\eta \in \psi$, where $\eta = \sigma\varepsilon_1(\lambda)$.

Define $\xi \rightarrow \alpha\eta\alpha$

$\lambda \leftarrow \lambda - \eta - \xi$

$\alpha \leftarrow (1 - \eta)\alpha(1 - \eta)$

$\Delta \leftarrow \{(R, \pi) \in \Xi | \pi\xi \in \psi\}$

$\Xi \leftarrow \Xi \backslash \Delta$.

(4) Choose any $(R, \pi) \in \Delta$ and delete it from Δ.

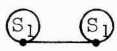
(5) For each k ($0 \le k < n$) such that $\pi\varepsilon_k(\xi) \in \psi$, add $(Q, \pi\varepsilon_k(\xi))$ to Ξ, where Q is the rooted tree formed from R and k copies of S by joining the root of each copy of S by an edge to the root of R and rooting at the root of R.

If Δ is empty, to to step (2); otherwise, to go step (4).

(6) There is exactly one $(S, \delta) \in \Xi$ such that S has n vertices. T is isomorphic (as an unrooted tree) to S.

Stop.

(7) Let $(S_1, \sigma_1), \cdots, (S_t, \sigma_t)$ be the elements of Ξ such that $\sigma_i\lambda \in \psi$.

If t = 1, T is 

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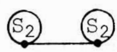
If t = 2, T is 

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Erratum: S_1 and S_2

Stop.

Let $V = \{1, 2, \cdots, n\}$ be the vertex set of T. For any $U \subseteq V$, the *characteristic matrix* $\chi(U)$ of U will be defined to be the diagonal matrix $\mathrm{diag}(u_1, u_2, \cdots, u_n)$, where $u_i = 1$ if $i \in U$, $u_i = 0$ otherwise. Suppose F is a spanning subforest of T which has no two-vertex components and let ev(F) be the set of its end-vertices. The following lemmas can easily be verified.

(a) $\mathrm{tr}(\chi(U)) = |U|$ for any $U \subseteq V$, where tr denotes the trace.

(b) If $U_1, U_2 \subseteq V$, $\chi(U_1 \cap U_2) = \chi(U_1)\chi(U_2)$.

(c) $\chi(\mathrm{ev}(F)) = \varepsilon_1(\Lambda(F))$.

(d) If $W \subseteq \mathrm{ev}(F)$, then $A(F)\chi(W)A(F) = \mathrm{diag}(w_1, w_2, \cdots, w_n)$, where w_i is the number of vertices in W adjacent in F to vertex i.

(e) Let G be the spanning subforest of F formed by removing the edges incident with vertices in W, where $W \subseteq \mathrm{ev}(F)$.

Then $A(G) = (I - \chi(W))A(F)(I - \chi(W))$, and

$$\Lambda(G) = \Lambda(F) - \chi(W) - A(F)\chi(W)A(F).$$

Rather than presenting a general proof of the correctness of Algorithm 5.4, it will be more instructive to work through an example. Suppose that T is the tree in Figure 3, so that $n = 7$, and define $A = A(T)$, $\Lambda = \Lambda(T)$.

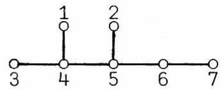


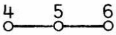
Figure 3.

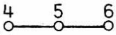
Step 1. $\alpha(A, \Lambda) = A$, $\lambda(A, \Lambda) = \Lambda$, and $\Xi = \{(K_1, \iota)\}$, where $\iota(A, \Lambda) = I = \chi(V)$.

Step 2. Both of the polynomials λ and $\lambda^2 - \lambda$ are in ψ.

Step 3. $S = K_1$, $\sigma = \iota$. Thus $\eta(A, \Lambda) = \mathrm{diag}(1, 1, 1, 0, 0, 0, 1)$ by (c).

By (d) and (e), after redefining ξ, α and λ we have

$\alpha(A, \Lambda) = A($$)$ (the other vertices are isolated)

$\lambda(A, \Lambda) = \Lambda($$)$

$\xi(A, \Lambda) = \text{diag}(0, 0, 0, 2, 1, 1, 0)$.

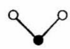
Finally we find $\Delta = \{(K_1, \iota)\}$, $\Xi = \emptyset$.

Step 4. $R = K_1$, $\pi = \iota$, $\Delta = \emptyset$.

Step 5. When $k = 0$, $\pi(A, \Lambda)\varepsilon_k(\xi(A, \Lambda)) = \chi(\{1, 2, 3, 7\})$ and so we add $(K_1, \pi\varepsilon_0(\xi))$ to Ξ.

When $k = 1$, $\pi(A , \Lambda)\varepsilon_k(\xi(A, \Lambda)) = \chi(\{5, 6\})$ and so we add (, $\pi\varepsilon_1(\xi)$) to Ξ.

When $k = 2$, $\pi(A, \Lambda)\varepsilon_k(\xi(A, \Lambda)) = \chi(\{4\})$ and so we add (, $\pi\varepsilon_2(\xi)$) to Ξ.

{Notice that for each k, the polynomial $\pi\varepsilon_k(\xi)$ selects those vertices which were adjacent to k of the end-vertices which were "isolated" at step (3). At the present stage, the elements of Ξ associate the rooted tree K_1 with the vertices 1, 2, 3 and 7, the rooted tree  with vertices 5 and 6, and the rooted tree  with vertex 4. We can think of the rooted tree associated with a particular vertex as a record of the way that particular vertex was connected to vertices that have been previously isolated. The Algorithm works by repeatedly isolating end-vertices until at most one edge remains. The polynomials α and λ keep track of the adjacency matrix and the diagonal matrix of degrees.}

Step 2. Both λ and $\lambda^2 - \lambda$ are in ψ.

Step 3. A possible choice of (S, σ) is (, σ), where $\sigma(A, \Lambda) = \chi(\{5, 6\})$. Thus $\eta(A, \Lambda) = \chi(\{6\})$, and we find

$\xi(A, \Lambda) = \text{diag}(0, 0, 0, 0, 1, 0, 0)$,

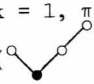
$\alpha(A, \Lambda) = A(\overset{4}{\circ}\!\!-\!\!-\!\!\overset{5}{\circ})$,

$\lambda(A, \Lambda) = \Lambda(\overset{4}{\circ}\!\!-\!\!-\!\!\overset{5}{\circ})$, and

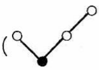
$\Delta = \{($  $, \sigma)\}$.

Step 4. $R =$ , $\pi(A, \Lambda) = \chi(\{5, 6\})$, $\Delta = \emptyset$.

Step 5. When $k = 0$, $\pi(A, \Lambda)\varepsilon_k(\xi(A, \Lambda)) = \chi(\{6\})$ and so we add (, $\pi\varepsilon_0(\xi)$) to Ξ.

When $k = 1$, $\pi(A, \Lambda)\varepsilon_k(\xi(A, \Lambda)) = \chi(\{5\})$ and so we add (, $\pi\varepsilon_1(\xi)$) to Ξ.

At this stage we have

$\Xi = \{(K_1, \pi_1)$, (, π_2), (, π_3), (, π_4), where

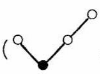
$\pi_1(A, \Lambda) = \chi(\{1, 2, 3, 7\})$,

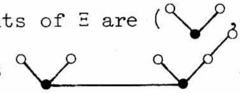
$\pi_2(A, \Lambda) = \chi(\{6\})$,

$\pi_3(A, \Lambda) = \chi(\{4\})$, and

$\pi_4(A, \Lambda) = \chi(\{5\})$.

Step 2. Since $\lambda(A, \Lambda) = \text{diag}(0, 0, 0, 1, 1, 0, 0)$, we find that $\lambda^2 - \lambda \notin \psi$.

Step 7. The required elements of Ξ are (, π_3) and (, π_4).

Consequently, T is  as required.

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Department of Mathematics,
Melbourne University,
Parkville, 3052,
Australia.