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Spectral Conditions for the Reconstructibility of a Graph

C. D. GODSIL AND B. D. MCKAY

Department of Mathematics, Melbourne University, Parkville, Victoria 3052, Australia

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Various conditions on the eigenvalues and eigenvectors of a graph are found to be sufficient for its reconstructibility. In particular, a graph is reconstructible if all but at most one of its eigenvalues are simple and have eigenvectors not orthogonal to **c**, where **c** is the vector with each entry equal to one.

1. INTRODUCTION

Let G be a graph with vertex set $\{1, 2,..., n\}$. We will assume throughout that G has no loops or multiple edges, although most of our results will readily generalize. The *adjacency matrix* of G is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if i is adjacent to j and $a_{ij} = 0$ otherwise. The *characteristic polynomial* of G is defined to be $G(x) = \det(xI - A)$. The complement of G is denoted by $\bar{G}$.

For $1 \leqslant i \leqslant n$, G_i denotes the subgraph of G formed by deleting the vertex i. The adjacency matrix of G_i will be denoted by A_i. Any property or object associated with G which can be determined from knowledge only of the subgraphs G_i $(1 \leqslant i \leqslant n)$ will be called *reconstructible.*

2. WALKS IN A GRAPH

A *walk of length* r $(r \geqslant 0)$ in a graph X is a sequence $v_0, v_1,..., v_r$ of vertices of X in which consecutive vertices are adjacent. The walk $v_0, v_1,..., v_r$ *starts* at v_0 and is said to *return* k *times* if exactly k of the vertices $v_1, v_2,..., v_r$ are equal to v_0. It is called *closed* if $v_r = v_0$.

Suppose S is an arbitrary subset of the walks in a graph. For each $r \geqslant 0$ we define $S^{(r)}$ to be the number of elements of S with length r, and $S(x) = \sum_{r=0}^{\infty} S^{(r)} x^r$ to be the *generating function* for S. In particular, if i is a vertex

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of the graph G, $W_{G,i}$ denotes the set of walks in G starting at i and $C_{G,i}$ denotes the set of closed walks in G starting at i.

The proof of the following lemma is due to the referee, and replaces our considerably longer proof.

LEMMA 2.1.

$$C_{G,i}(x) = \frac{1}{x} G_i\left(\frac{1}{x}\right) \Big/ G\left(\frac{1}{x}\right).$$

Proof. $C_{G,i}(x)$ is the ith diagonal entry of $(I - xA)^{-1} = (1/x)((1/x) I - A)^{-1}$. The result follows immediately on noting that the cofactor of this element in $(1/x) I - A$ is just $G_i(1/x)$. ∎

Let W_G be the set of all walks in G. The following result is due to Cvetković [1].

LEMMA 2.2.

$$W_G(x) = \frac{1}{x}\left[(-1)^n \frac{\bar{G}(-1/x - 1)}{G(1/x)} - 1\right].$$

We can now obtain the analogue of Lemma 2.1 for arbitrary walks.

LEMMA 2.3.

$$W_{G,i}(x) = \frac{1}{x}\left[(-1)^n \frac{G_i(1/x)}{G(1/x)}\left\{\frac{\bar{G}(-1/x - 1)}{G(1/x)} + \frac{\bar{G}_i(-1/x - 1)}{G_i(1/x)}\right\}\right]^{1/2}.$$

Proof. Let N be the set of walks in G which start at i and never return. Then clearly

$$W_{G,i}(x) = C_{G,i}(x)\, N(x)$$

and (1)

$$W_G(x) - W_{G_i}(x) = C_{G,i}(x)\, N(x)^2.$$

Eliminating $N(x)$ and applying Lemmas 2.1 and 2.2 gives the stated result. ∎

THEOREM 2.4. *$C_{G,i}(x)$ and $W_{G,i}(x)$ are reconstructible.*

Proof. Since Tutte [4] has shown that $G(x)$ and $\bar{G}(x)$ are reconstructible, the theorem follows at once from Lemmas 2.1 and 2.3. Note that the choice of square root in Lemma 2.3 is decided by the non-negativity of the coefficients of $W_{G,i}(x)$. ∎

3. Reconstruction of the Eigenspaces of a Graph

Let $\sigma(A)$ denote the set of eigenvalues of A. If $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m$ is a complete set of orthonormal eigenvectors associated with λ, we denote by Z_λ the $n \times m$ matrix with ith column $\mathbf{z}_i$. The matrices $Z_\lambda Z_\lambda^T$ ($\lambda \in \sigma(A)$) form a complete set of orthogonal idempotents for the spectral decomposition of A, and we have

$$A^r = \sum_{\lambda \in \sigma(A)} \lambda^r Z_\lambda Z_\lambda^T \qquad \text{for} \quad r = 0, 1, 2, \ldots . \tag{2}$$

Let $\mathbf{e}_i$ denote the column n-vector with ith entry one and all other entries zero, and let $\mathbf{c}$ denote the column n-vector with each entry one. For $\lambda \in \sigma(A)$ and $1 \leqslant i \leqslant n$ define $\mathbf{w}_\lambda(i) = Z_\lambda^T \mathbf{e}_i$ and $\mathbf{c}_\lambda = Z_\lambda^T \mathbf{c}$.

Lemma 3.1. *For $1 \leqslant i \leqslant n$ and $r \geqslant 0$ we have*

$$C_{G,i}^r = \sum_{\lambda \in \sigma(A)} \lambda^r \mathbf{w}_\lambda(i)^T \mathbf{w}_\lambda(i) \tag{3}$$

and

$$W_{G,i}^r = \sum_{\lambda \in \sigma(A)} \lambda^r \mathbf{w}_\lambda(i)^T \mathbf{c}_\lambda . \tag{4}$$

Proof. Both claims follow immediately from Eq. (2) and the fact that $C_{G,i}^r = \mathbf{e}_i^T A^r \mathbf{e}_i$ and $W_{G,i}^r = \mathbf{e}_i^T A^r \mathbf{c}$. ∎

Theorem 3.2. *For each eigenvalue λ, the numbers $\mathbf{w}_\lambda(i)^T \mathbf{w}_\lambda(i)$ and $\mathbf{w}_\lambda(i)^T \mathbf{c}_\lambda$ are reconstructible* ($1 \leqslant i \leqslant n$).

Proof. For $r \geqslant 0$ and $1 \leqslant i \leqslant n$ we have shown in Theorem 2.4 that $C_{G,i}^r$ and $W_{G,i}^r$ are reconstructible. Since $\sigma(A)$ is also reconstructible, by [4], we can uniquely solve Eqs. (3) and (4) for the numbers $\mathbf{w}_\lambda(i)^T \mathbf{w}_\lambda(i)$ and $\mathbf{w}_\lambda(i)^T \mathbf{c}_\lambda$. ∎

Corollary 3.3. *Let λ be a simple eigenvalue of A with associated eigenvector $\mathbf{z}_\lambda$. Then the entries of $\mathbf{z}_\lambda$ can be reconstructed up to sign. If $\mathbf{z}_\lambda$ is not orthogonal to $\mathbf{c}$, then $\mathbf{z}_\lambda$ is reconstructible.*

Proof. The entries of $\mathbf{z}_\lambda$ are the numbers $\mathbf{w}_\lambda(i)$, and so the square of each entry is reconstructible by Theorem 3.2. Furthermore $\mathbf{z}_\lambda$ is orthogonal to $\mathbf{c}$ if, and only if, all of the numbers $\mathbf{w}_\lambda(i)^T \mathbf{c}_\lambda$ are zero. If this is not the case, we can reconstruct $\mathbf{z}_\lambda$, since $\mathbf{c}_\lambda$ is then a non-zero constant. ∎

Let $\mathbb{Q}$ denote the rational numbers, and for an algebraic number t let $\mathbb{Q}(t)$ denote the extension of $\mathbb{Q}$ by t.

COROLLARY 3.4. *Let G be connected and let μ be the greatest eigenvalue of A. If λ is any eigenvalue of A which is algebraically conjugate to μ over $\mathbb{Q}$, then the associated eigenvector $\mathbf{z}_\lambda$ is reconstructible.*

Proof. By the Perron–Frobenius theorem, μ is a simple eigenvalue of A and $\mathbf{z}_\mu$ can be chosen to have positive entries from $\mathbb{Q}(\mu)$. Hence $\mathbf{z}_\mu$ is reconstructible, by Corollary 3.3. Now let α be an automorphism of $\mathbb{Q}(\mu)$ over $\mathbb{Q}$ which takes μ onto λ. Then α takes $\mathbf{z}_\mu$ element-wise onto $\mathbf{z}_\lambda$, since A is rational. ∎

We can now prove our main result.

THEOREM 3.5. *Suppose that all but at most one of the eigenvalues of A are simple, with the corresponding eigenvectors not orthogonal to* $\mathbf{c}$. *Then G is reconstructible.*

Proof. By Corollary 3.3, eigenvectors not orthogonal to $\mathbf{c}$ and corresponding to simple eigenvalues can be reconstructed. Any remaining eigenspace is the orthogonal complement of the space spanned by these vectors.

Therefore, we can reconstruct the eigenspace corresponding to each eigenvalue. This enables us to reconstruct A, and hence G. ∎

We now explore a number of conditions under which the requirements of Theorem 3.5 can be shown to hold.

THEOREM 3.6. *Let $\bar{A}$ denote the adjacency matrix of $\bar{G}$. If A and $\bar{A}$ have no eigenvectors in common, then G is reconstructible.*

Proof. A has no eigenvector orthogonal to $\mathbf{c}$, since this would also be an eigenvector of $\bar{A}$. Furthermore, any non-simple eigenvalue of A would necessarily have an associated eigenvector orthogonal to $\mathbf{c}$. Hence the hypotheses of Theorem 3.5 are satisfied and so G is reconstructible. ∎

THEOREM 3.7. *Let $\mathscr{A}$ be the algebra generated by the matrices I, A, and $\bar{A}$ over $\mathbb{Q}$. Then if any element of $\mathscr{A}$ has an irreducible characteristic polynomial, G is reconstructible.*

Proof. Let $n = |V(G)|$. If $n = 1$, then G is trivially reconstructible, and so we assume $n > 1$.

Let X be an element of $\mathscr{A}$ with irreducible characteristic polynomial. If $\mathbf{x}$ is a common eigenvector for A and $\bar{A}$, it is also an eigenvector of X. Since X has an irreducible characteristic polynomial, it has n linearly independent eigenvectors and, since X is rational, these can be chosen to be all algebraically conjugate over $\mathbb{Q}$.

Since A and $\bar{A}$ are rational matrices, it also follows that the algebraic conjugates of $\mathbf{x}$ over $\mathbb{Q}$ are the eigenvectors of A and of $\bar{A}$. Thus A and $\bar{A}$ have n linearly independent eigenvectors in common and so commute. Since $\bar{A} = J - I - A$, it follows that A commutes with J. This implies that the row sums of A are equal, and hence $\mathbf{c}$ is an eigenvector of A. But $\mathbf{c}$ has no distinct algebraic conjugates over $\mathbb{Q}$, which contradicts the conclusion that all the eigenvectors of A are conjugate.

Hence A and $\bar{A}$ have no eigenvector in common, and so G is reconstructible by Theorem 3.6. ∎

In [4] Tutte defines the *idiosyncratic polynomial* of G to be the two-variable polynomial $p(x, \alpha) = \det(A + \alpha\bar{A} - xI)$, and proves that if $p(x, \alpha)$ is irreducible over $\mathbb{Q}$, then G is reconstructible. By Hilbert's Irreducibility Theorem [2], $p(x, \alpha)$ is irreducible only if the one-variable polynomial $p_\alpha(x) = p(x, \alpha)$ is irreducible for some rational α. Hence Tutte's result is a corollary to Theorem 3.7, at least for the type of graph we are considering here.

Finally, we note the following interesting consequence of the assumption that A and $\bar{A}$ have no common eigenvectors.

THEOREM 3.8. *If A and $\bar{A}$ have no common eigenvector, then no two vertex-deleted subgraphs of G are isomorphic.*

Proof. By Lemma 2.3 we know that $W_{G,i}(x)$ is determined by a knowledge of $G(x)$, $\bar{G}(x)$, $G_i(x)$, and $\bar{G}_i(x)$. Hence if G_i and G_j are isomorphic, $W_{G,i}(x) = W_{G,j}(x)$ and so by the arguments used in proving Corollary 3.3 we conclude that the ith and jth entries of each eigenvector of A are equal. This contradicts the fact that the eigenvectors of A are linearly independent. ∎

Thus, under the hypotheses of the theorem, we find that G has trivial automorphism group. This generalizes a result of Mowshowitz [3] that a graph with irreducible characteristic polynomial has trivial automorphism group.

REFERENCES

1. D. M. CVETKOVIĆ, Graphs and their spectra, *Publ. Elek. Fak. U. Beograd, Ser. Mat. Fiz.*, Nos. 354–356 (1971), 1–50.
2. D. HILBERT, Ueber die Irreducibilität ganzer rationaler Functionen mit ganzzahligen Coefficienten, *J. Reine Angew. Math.* **110** (1892), 104–179.
3. A. MOWSHOWITZ, Graphs, groups and matrices, *in* "Proceedings, 25th Summer Meeting Canadian Math. Congress, Congr. Num. IV, Utilitas Math., Winnipeg, 1971," pp. 509–522.
4. W. T. TUTTE, "All the King's Horses," University of Waterloo, Dept. of Combinatorics and Optimization, Res. Report CORR 76/37, Sept. 1976.

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