Computer Reconstruction of Small Graphs

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ABSTRACT

The Reconstruction Conjecture is established for graphs with nine vertices.

1. INTRODUCTION

The purpose of this note is to report recent computational work on graphical reconstruction. Surveys of theoretical work on this subject can be found in [2] and [4]. For each $n \geq 2$, let $\mathcal{G}_n$ denote the family of all graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$ and let I_n denote the set $\{1, 2, \ldots, n\}$. If $G \in \mathcal{G}_n$ and $i \in I_n$, define G_i to be the subgraph of G formed by removing the vertex v_i. We shall consider three conjectures.

Conjecture 1. Let $G, H \in \mathcal{G}_n$, where $n > 2$. If $G_i \cong H_i$ for each $i \in I_n$, then $G \cong H$. (This conjecture is commonly called the Reconstruction Conjecture.)

Conjecture 2. Let $G, H \in \mathcal{G}_n$, where $n > 3$. Suppose that for any $i, j \in I_n$ there exists k, $l \in I_n$ such that $G_i \cong H_k$ and $H_j \cong G_l$. Then $G \cong H$. (This conjecture was first made by Harary [3].)

Conjecture 3. Let $G, H \in \mathcal{G}_n$, where $n > 5$. Suppose that for any $i, j \in I_n$ there exists $k, l \in I_n$ such that G_i is isomorphic to either H_k or $\bar{H}_k$ and H_j is isomorphic to either G_l or $\bar{G}_l$. Then G is isomorphic to either H or $\bar{H}$.

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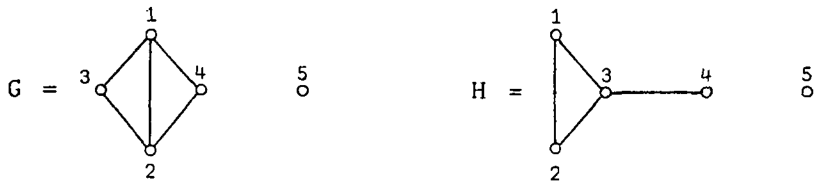

FIGURE 1. Largest counterexample to conjecture 3 for $n \leq 5$.

If Conjecture 3 holds for some fixed $G \in \mathcal{G}_n$ and arbitrary $H \in \mathcal{G}_n$, it also holds for $\bar{G}$, since $(\bar{G})_i = \overline{(G_i)}$ for any $i \in I_n$. This fact allows Conjecture 3 to be tested on the computer in just over half the time required to test Conjecture 2, thus justifying our introduction of Conjecture 3.

The largest counterexamples to Conjecture 3 for $n \leq 5$ is shown in Figure 1. It is easily verified that $G_1 \cong G_2 \cong H_1 \cong H_2 \cong \bar{H}_5$, $G_3 \cong G_4 \cong H_4$, and $G_5 \cong \bar{H}_3$.

2. RESULTS

Conjecture 1 has been previously verified by Stockmeyer for $3 \leq n \leq 8$ and by McKay and Godsil for $n = 9$ (both unpublished). In this note we report the following stronger result.

Theorem. Conjectures 1, 2, and 3 are true for $6 \leq n \leq 9$.

It is only necessary to verify Conjecture 3, since it is clearly stronger than either of the other conjectures. The computation is now briefly described for $n = 9$.

First, a linear ordering on $\mathcal{G}_n$ was devised, using the degrees of the vertices and the canonically labelled adjacency matrix of each graph [6]. Relative to this ordering, a list of the 137,352 9-vertex graphs with $G \leq \bar{G}$ was prepared from the graphs generated by Baker, Dewdney, and Szilard [1]. For each graph G of this list, generators for the automorphism group were found, and for one value of i from each orbit, the graph $\min(G_i, \bar{G}_i)$ was constructed and canonically labelled. This required the canonical labelling algorithm of [6] to be applied over 1.1 million times to 8-vertex graphs. Isomorphic graphs were then eliminated from the set thus associated with each 9-vertex graph. Finally, a search revealed that no two of these sets were the same. The total execution time (CDC Cyber 70 model 73) for $n = 9$ was about 71 min.

An incidental outcome of the computation was the production of those graphs with 8 or 9 vertices having pseudosimilar vertices [5]. Vertices v_i and v_j of G are *pseudosimilar* if $G_i \cong G_j$ but there is no automorphism of G taking v_i onto v_j. There are 44 such graphs with 8 vertices and 454 with 9 vertices. A listing of these is available from the author.

References

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