CYCLES IN 3-CONNECTED CUBIC PLANAR GRAPHS II

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ABSTRACT. We show that in a 3-connected cubic planar graph a cycle can be found through any given 19 vertices. This is unlikely to be the best possible. However there do exist 3-connected cubic planar graphs for which there are 24 vertices which do not lie on a cycle.

1. Introduction.

In [**3**] it was shown that any **17** vertices in a **3**-connected cubic planar graph lie on some cycle. The largest k for which every set of k vertices lies on cycle in such graphs is at most **23**. There are several graphs with **24** vertices which do not lie on a cycle. Examples can be found in Bosák [**1**] and Grünbaum and Walther [**2**]. We give another example in Figure 1.4.

In this paper we show that k is at least 19. The proof technique used is basically that of [**3**]. We now know a little more about the hamiltonicity of the graphs involved. For instance, the following two theorems have been proved recently.

THEOREM 1.1. *If G is a 3-connected cubic bipartite planar graph on n vertices, then for $n \leq 64$, G is hamiltonian.*

The proof of this result is in [**4**]. □

THEOREM 1.2. *If G is a 3-connected cubic planar graph on n vertices, then for $n \leq 36$, G is hamiltonian.*

There are precisely six non-isomorphic non-hamiltonian cyclically 3-connected cubic planar graphs on 38 vertices. □

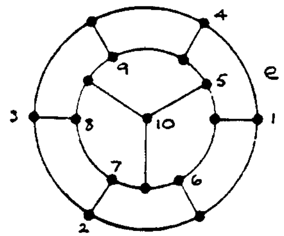
This result is proved in [**5**]. The six non-hamiltonian graphs referred to are precisely those of [**1**].

The proofs of both theorems rely on computer generated knowledge of "small" graphs. Using various reduction techniques, larger graphs are shown to be hamiltonian by extending hamiltonian cycles from the smaller graphs. The model for these proofs can be found in [**7**].

We need the following result from [**3**]. (It is stated incorrectly there.) The graphs of the theorem are shown in Figure 1.1.

THEOREM 1.3. *Let G be a 3-connected cubic planar graph. Let $A \subseteq VG$ be such that $|A| \leq 12$ and let $e' \in EG$. Then there is a cycle C in G with $A \subseteq VC$ and $e' \notin EC$ unless there is a contraction $\phi : G \to T$ with $\phi(A) \supseteq \{1,2,3,4,5,6,7,8,9,10\}$ and $\phi(e') = e$, or a contraction $\psi : G \to D_i$, for $i = 1,2,\ldots,7$ with $\phi(A) = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ and $\phi(e') = e$.* □

$|A| = 10$



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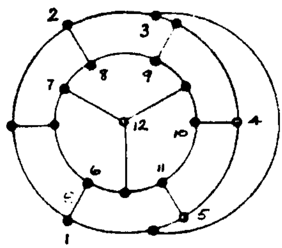
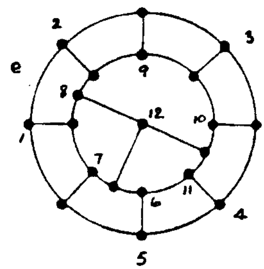
$|A| = 12$

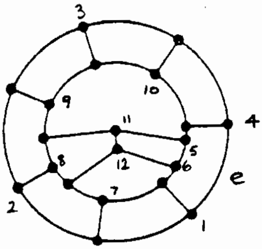


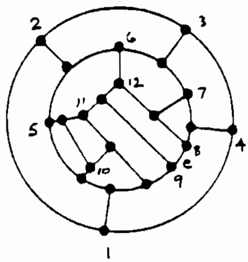
D1



D2

Figure 1.1





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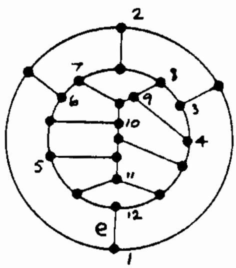
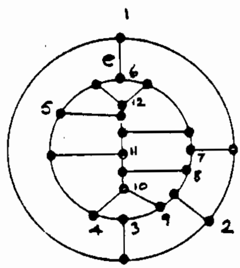
D_4





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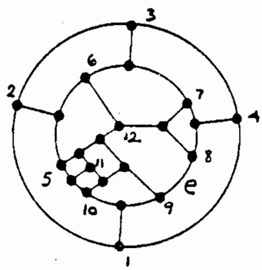
D_6



D_7

Figure 1.1 (continued)

An *a-edge* is an edge which is on every hamiltonian cycle in a graph. The edges e of Figure 1.1 are all a-edges. A *b-edge* is an edge which is on no hamiltonian cycle in a graph. Figure 1.2 shows the unique smallest 3-connected cubic planar graph which has a b-edge (both (0,3) and (4,7) are b-edges). There are 7 graphs with a b-edge on 26 vertices.

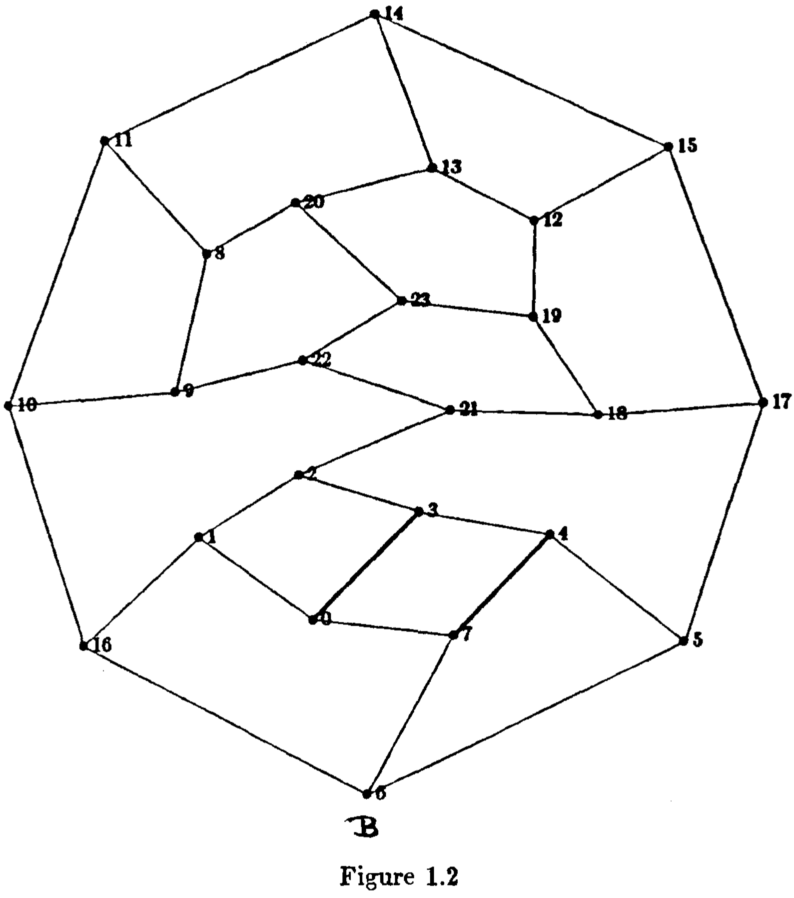


Figure 1.2

If E is a cutset of a graph G consisting of three independent edges, then the graphs G_1, G_2 obtained from G by contracting a component of $G - E$ to a vertex are called *3-cut reductions* of G. These 3-cut reductions are shown diagrammatically in Figure 1.3.

The graph B of Figure 1.2 has the property that there is no cycle through the b-edge (0,3) and all of the vertices 1,2,4,5,6,7,8,10,12,13,14,

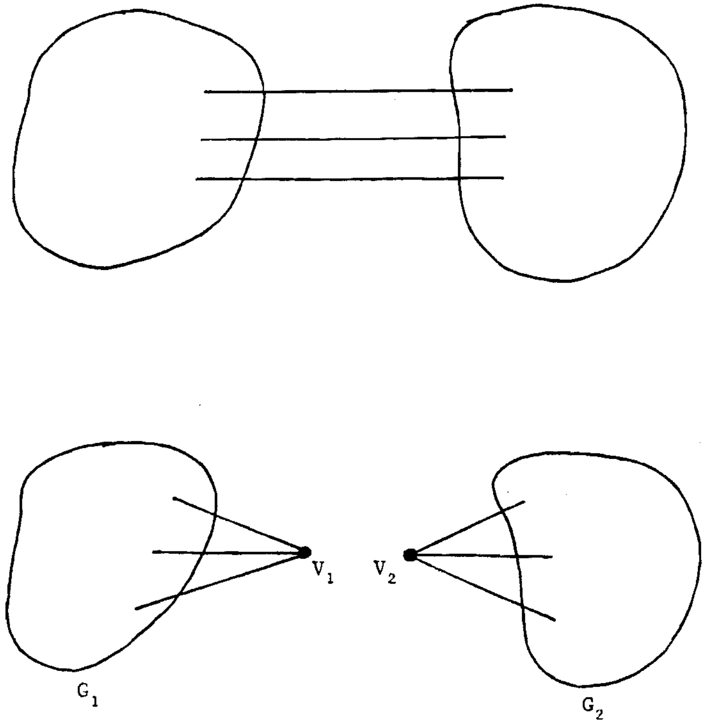


Figure 1.3

15,18,22,23. Neither is there a cycle through the *b*-edge (4,7) and all of the vertices 0,1,2,3,5,6,8,10,12,13,14,15,18,22,23. Hence we may join *B* and the graph *T* of Figure 1.1 to give a graph *H* so that (i) *B* and *T* are the 3-cut reductions of *H* and (ii) a *b*-edge of *B* corresponds to the *a*-edge of *T* (see Figure 1.4).

Now *H* has the property that it contains 24 vertices which do not lie on a cycle. These vertices are either of the two sets of 15 vertices listed in the previous paragraph and 9 of the 10 labelled vertices of *T*.

This can be seen by noting that if we avoid the edge "*e*" of Figure 1.4 we cannot find a cycle through the 9 vertices of "*T*" and an arbitrary vertex of "*B*" (Theorem 1.3). On the other hand if we use "*e*" we must use "*b*" and then we cannot find a cycle through the 15 vertices of "*B*" specified earlier.

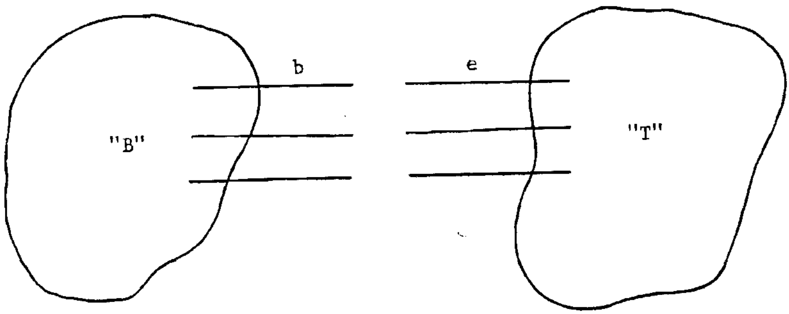


Figure 1.4

2. The main result.

In this section we prove the following theorem.

THEOREM 2.1. *Let G be a 3-connected cubic planar graph and let $A \subseteq VG$ such that $|A| \leq 19$. Then there exists a cycle C in G such that $A \subseteq VC$.*

PROOF: The method of proof closely follows that of Theorem 1 in [6]. We proceed by induction. By Theorem 1.2 we may assume that $|VG| \geq 38$.

Case 1. G is cyclically 3-connected but not cyclically 4-connected. Thus G has an edge cut E with three edges. Form the edge cut reduction defined in Section 1 to give graphs G_1 and G_2. Let $A_i = (A \cap VG_i) \cup \{v_i\}$ for $i = 1, 2$ where $v_i \in VG_i - VG$. Assume without loss of generality that $|A_1| \leq |A_2|$. Hence $|A_1| \leq 10$. Since $|VG_2| < |VG|$, there exists a cycle C_2 in G_2 through the vertices A_2. This cycle will not include one of the edges, e_2, of $EG_2 - EG$. Let e_1 be the edge of $EG_1 - EG$ which corresponds to e_2.

If $|A_2| \leq 9$, then by Theorem 1.3 there exists a cycle C_1 in G_2 through the vertices of A_1 and avoiding e_1. Cycles corresponding to C_1 and C_2 in G can be combined to give the cycle C required by the theorem.

Suppose $|A_1| = 10$. Then we can repeat the argument of the last paragraph unless G_1 is contractible to T. In this case let C_1 be a cycle in G_1 such that $A_1 \subseteq VC_1$ and let $e_1, e_1' \in EC_1 \cup (EG_1 - EG)$ where e_1 corresponds to the a-edge in T. Further let e_2, e_2' be the edges of $EG_2 - EG$ corresponding to e_1, e_1'. Now $|A_2| = 11$, so there is a cycle C_2 in G_2 through A_2 and e_2, e_2' unless G_2 is also contractible to T.

Hence if e_2'' is the third edge of $EG_2 - EG$, then this must correspond to the a-edge in T. Thus G must be contractible to the graph J of Figure 2.1. Since J is hamiltonian there exists a cycle in G through all the vertices of A.

(The labels on the vertices of J correspond to those of T in Figure 1.1. The arrows on edges of J indicate the hamiltonian cycle.)

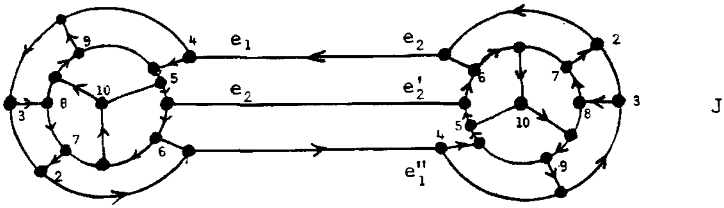


Figure 2.1

Case 2. G is cyclically 4-connected. If G contains an edge f whose end vertices are not incident with a vertex of A then perform the f-reduction of Figure 2.2. Since G' is 3-connected then there exists a cycle in G' through the vertices of A which may be extended to a cycle in G through A.

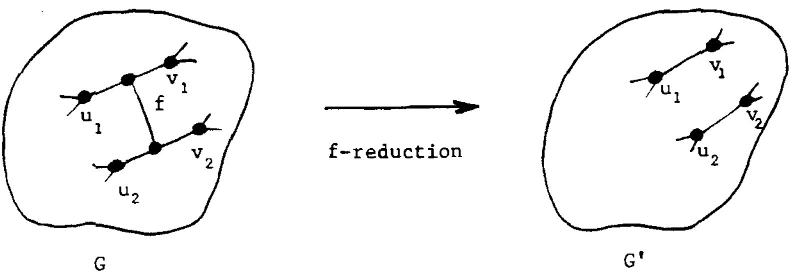


Figure 2.2

Hence we may assume that every edge of G is incident with some vertex of A. Therefore $|VG| \leq 38$. But $|VG| \geq 38$ by Theorem 1.2. Thus $|VG| = 38$ and G is bipartite. The theorem now follows since Theorem 1.1 shows that G is hamiltonian. □

REFERENCES

1. J. Bosák, *Private communication to B. Grünbaum, 1971.*
2. B. Grünbaum and H. Walther, *Shortness exponents of graphs*, J. Comb. Th. **14A** (1973), 364-385.
3. D.A. Holton, *Cycles in 3-connected cubic planar graphs*, Annals of Discrete Math. **27** (1985), 219-226.
4. D.A. Holton, B. Manvel and B.D. McKay, *Hamiltonian cycles in cubic 3-connected bipartite planar graphs, (submitted).*
5. D.A. Holton and B.D. McKay, *The smallest non-hamiltonian cubic 3-connected planar graphs have 38 vertices, (submitted).*
6. D.A. Holton, B.D. McKay, M.D. Plummer and C. Thomassen, *A nine point theorem for 3-connected graphs*, Combinatoria **2** (1982), 53-62.
7. H. Okamura, *Every simple 3-polytope of order 32 or less is hamiltonian*, J. Graph Th. **6** (1982), 185-196.

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