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A COROLLARY TO PERFECT'S THEOREM

D.A. HOLTON, B.D. MC KAY AND M.D. PLUMMER

In a 3-connected graph Perfect's Theorem shows how two independent paths from a given vertex to a given subgraph may be extended to three independent paths. Here we show, again for 3-connected graphs, how two independent paths from a path with given endvertices may be extended. This can be done in one of two ways.

The need for the theorem in this paper arose in a first proof [2] of the fact that there is a cycle through any nine vertices in a 3-connected cubic graph. Subsequently an alternative proof [3] of this result has been found which is independent of the theorem. However, it is used in [1] in showing that there is a cycle through any $k+4$ vertices in a k-connected k-regular graph.

We believe that the theorem can be generalised to k-connected graphs $(k \geq 3)$ where the path (a_1,a_2) has more than two specified vertices. At this stage we have been unable to prove such a result.

Throughout we use (a,b) to denote a path between the vertices a and b, where b may equal a. If there is any doubt which a,b path we mean, then we will either write (a,c,b) to denote the a,b path which passes through c, or we will mean the a,b path which avoids all previously named vertices of the graph in question with the possible exception of a and b. When a new path is introduced it will be understood to contain no previously named vertices except possibly its endvertices.

We also use (a,b) to represent the vertices of the path (a,b) excluding a and b. It will be clear in the context which use is being made of the notation. Further, we use $[a,b)$ to denote the vertex set $(a,b) \cup \{a\}$ and we similarly define $(a,b]$ and $[a,b]$.

Two paths (a,b_1), (a,b_2) are said to be *independent* if $[a,b_1] \cap [a,b_2] = \{a\}$.

A path from a vertex a to a set B is a path (a,b) such that $(a,b] \cap B = \{b\}$.

We can now state Perfect's Theorem for k-connected graphs.

<u>Perfect's Theorem</u>. *Let* G *be a* k*-connected graph. Suppose that* (a,b_i), $i = 1,2,\ldots,k-1$ *are independent paths from* a *to* $B \subseteq VG\backslash\{a\}$ *with* $|B| \geq k$. *Then there exist* k *independent paths from* a *to* B, $k-1$ *of whose endvertices are* $\{b_i : i=1,2,\ldots,k-1\}$.

<u>Proof</u>. See [4]. □

We need some notation before stating our main result.

Let C be a set of paths in G and define $\langle C\rangle$ to be subgraph of G containing just those vertices and edges of G which are used by elements of C. Define $V(C)$ to be the set of those vertices of $\langle C\rangle$ which have degree greater than two, or are endvertices of an element of C. C is a *configuration* if C contains every path in $\langle C\rangle$ whose endvertices are in $V(C)$.

Suppose that S is a configuration in G, and that $b_1,b_2 \in V(S)$ and $a_1,a_2 \in VG\backslash V\langle S\rangle$, where $a_1 \neq a_2$ and $b_1 \neq b_2$. For $i = 1,2$, let C_i be a configuration in G such that $\{a_1,a_2,b_1,b_2\} \subseteq V(C_i)$, $V\langle C_i\rangle \cap V\langle S\rangle \subseteq V(C_i)$ and $E\langle C_i\rangle \cap E\langle S\rangle = \emptyset$. Define $C_i + S$ to be the smallest configuration in G containing both C_i and S. We say that $C_1 + S$ and $C_2 + S$ are *equivalent* if there are bijections $\phi_1 : C_1 \to C_2$ and $\phi_2 : V(C_1) \to V(C_2)$ such that

(i) $\{\phi_2(a_1), \phi_2(a_2)\} = \{a_1,a_2\}$ and $\{\phi_2(b_1), \phi_2(b_2)\} = \{b_1,b_2\}$,

(ii) ϕ_2 maps the endvertices of each $P \in C_1$ onto the endvertices of $\phi_1(P)$,

(iii) for each $v \in VG$, $v \in V(C_1) \cap V\langle S\rangle$ if and only if $v \in V(C_2) \cap V\langle S\rangle$.

For notational convenience, whenever we are dealing with a configuration $C_2 + S$ which is equivalent to $C_1 + S$, we will write z instead of $\phi_2(z)$ for all $z \in V(C_1)$.

Finally we define the configurations W, X and Y as in Figure 1, where $b_3 \notin \{b_1,b_2\}$ and where, in Y, $u_1 = b_1$ and $u_1 = b_2$ are allowed.

<u>Theorem</u>. *Let* G *be a 3-connected graph and* S *a configuration in* G *such that* $|V\langle S\rangle| \geq 3$.

If G *contains the configuration* W, *then* G *contains a configuration equivalent to* X *or* Y.

<u>Proof</u>. Let $Q = V\langle S\rangle$. By Perfect's Theorem, there exists a path (a_1,y_1) from a_1 to $y_1 \in (Q\backslash\{b_1\}) \cup (a_2,b_2)$. If $y_1 \in Q\backslash\{b_1,b_2\}$, then we have a configuration equivalent to X.

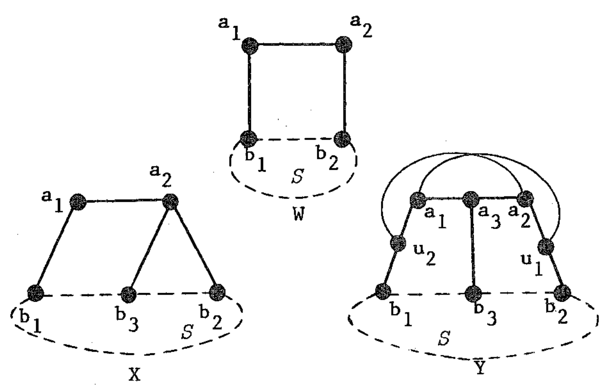

Figure 1

Case 1: $y_1 = b_2$. Using Perfect's Theorem again we see that there exists a path (a_2,y_2) from a_2 to $y_2 \in Q\backslash\{b_2\} \cup (a_1,b_1) \cup (a_1,b_2)$. If $y_2 \in Q\backslash\{b_1,b_2\}$, then we have a configuration equivalent to X.

1.1: $y_2 = b_1$. Since $\{b_1,b_2\}$ is not a cutset, there must exist a path (α_1,β_1) in G such that $\alpha_1 \in [a_1,b_1) \cup (a_1,a_2) \cup [a_2,b_2) \cup (a_1,b_2) \cup (a_2,b_1)$ and $\beta_1 \in Q\backslash\{b_1,b_2\}$. For $\alpha_1 \in [a_1,b_1) \cup [a_2,b_2) \cup (a_1,b_2) \cup (a_2,b_1)$ we have a configuration equivalent to X and for $\alpha_1 \in (a_1,a_2)$ we have a configuration equivalent to Y.

1.2: $y_2 \in (a_1,b_1)$. Again $\{b_1,b_2\}$ is not a cutset so we have a path (α_1,β_1) in G, such that $\alpha_1 \in [a_1,b_1) \cup (a_1,a_2) \cup [a_2,b_2) \cup (a_1,b_2) \cup (a_2,y_2) \cup (b_1,y_2]$ and $\beta_1 \in Q\backslash\{b_1,b_2\}$.

1.2.1: If $\alpha_1 \notin (b_1,y_2]$, then we argue as in Case 1.1 with (a_2,b_1) adjusted where necessary to (a_2,y_2,b_1).

1.2.2: Suppose $\alpha_1 \in (b_1,y_2]$. Amongst all configurations equivalent to this one, we choose a configuration which minimises $|[\alpha_1,y_2]|$. Since $\{\alpha_1,b_2\}$ is not a cutset, there is a path (α_2,β_2) in G such that $\alpha_2 \in [a_1,y_2) \cup (a_1,a_2) \cup [a_2,b_2) \cup (a_1,b_2) \cup (a_2,y_2) \cup (\alpha_1,y_2]$ and $\beta_2 \in (Q\backslash\{b_1\}) \cup (\alpha_1,b_1) \cup (\alpha_1,\beta_1)$.

If $\beta_2 \in (Q\backslash\{b_1,b_2\}) \cup (\alpha_1,\beta_1)$, then we proceed as in Case 1.2.1 or we contradict the minimality of $|[\alpha_1,y_2]|$.

For $\beta_2 \in (\alpha_1,b_1]$ and $\alpha_2 \in (\alpha_1,y_2]$ we again contradict the minimality of $|[\alpha_1,y_2]|$. If $\alpha_2 \in [a_1,y_2) \cup [a_2,b_2) \cup (a_1,b_2) \cup (a_2,y_2)$, we have a configuration equivalent to X. If $\alpha_2 \in (a_1,a_2)$, we have a configuration equivalent to Y.

1.3: $y_2 \in (a_1,b_2)$. The proof follows a similar pattern to that of Case 1.2.

Case 2: $y_1 \in (a_2,b_2)$. Amongst all equivalent configurations, we choose one which minimises $|[y_1,b_2]|$. We note that $|[y_1,b_2]| \neq 1$ since otherwise we apply Case 1.

By Perfect's Theorem, there exists a path (a_2,y_2) with $y_2 \in (y_1,b_2] \cup (Q\backslash\{b_1,b_2\}) \cup (a_1,b_1] \cup (a_1,y_1)$. If $y_2 \in (y_1,b_2]$, then we contradict the minimality of $|[y_1,b_2]|$ since (a_1,y_1,y_2) intersects (a_2,y_2,b_2) at y_2 with $|[y_2,b_2]| < |[y_1,b_2]|$. If $y_2 \in Q\backslash\{b_2\}$, then we argue as in Case 1.2. Hence we may assume that $y_2 \in (a_1,b_1) \cup (a_1,y_1)$.

2.1: Let $A = (a_1,y_2) \cup [a_1,a_2] \cup (a_2,y_1) \cup (a_1,y_1) \cup (a_2,y_2)$. If $y_2 \in (a_1,b_1)$, then since $\{b_1,b_2\}$ is not a cutset, there exists a path (α_1,β_1) in G. Then $\alpha_1 \in A \cup (b_1,y_2] \cup (b_2,y_1]$ and $\beta_1 \in Q\backslash\{b_1,b_2\}$.

If $\alpha_1 \in A$, then we have either configuration X or configuration Y. We thus have $\alpha_1 \in (b_1,y_2] \cup (b_2,y_1]$. By equivalence we may assume that $\alpha_1 \in (b_1,y_2]$.

Amongst all configurations which are equivalent to the surviving configuration of this case choose one which minimises $|[\alpha_1,y_2]|$.

Now $\{\alpha_1,b_2\}$ is not a cutset and so there exists a path (α_2,β_2) in G joining $\alpha_2 \in A \cup (b_2,y_1] \cup (\alpha_1,y_2]$ to $\beta_2 \in (Q\backslash\{b_1,b_2\}) \cup (\alpha_1,\beta_1) \cup [b_1,\alpha_1)$. For $\alpha_2 \in A \cup (\alpha_1,y_2]$ we argue as in Case 1.2.2 with minor modifications. Hence we have a configuration equivalent to C, D or E of Figure 2 where in C and E we may have $\beta_1 = \beta_2$, and in D we may have $\beta_2 = b_1$.

Amongst all equivalent configurations which now arise, we choose one which minimises $\ell = |[\alpha_1,y_2]| + |[\alpha_2,y_1]|$.

In G, $\{\alpha_1,\alpha_2\}$ is not a cutset. Hence there exists a path (α_3,β_3) in G joining $\alpha_3 \in A \cup (\alpha_1,y_2] \cup (\alpha_2,y_1]$ to $\beta_3 \in (Q\backslash\{b_1,b_2\}) \cup (\alpha_1,\beta_1) \cup (\alpha_2,\beta_2) \cup (\alpha_1,b_1] \cup (\alpha_2,b_2]$.

2.1.1: Suppose the configuration giving rise to minimum ℓ is equivalent to C. If $\alpha_3 \in A$, then we achieve configurations equivalent to X or Y with arguments similar to those of Case 1.2.2.

If $\beta_3 \in (\alpha_1,\beta_1]$ and $\alpha_3 \in (\alpha_1,y_2]$, then we produce a configuration equivalent to C which contradicts the minimality of ℓ.

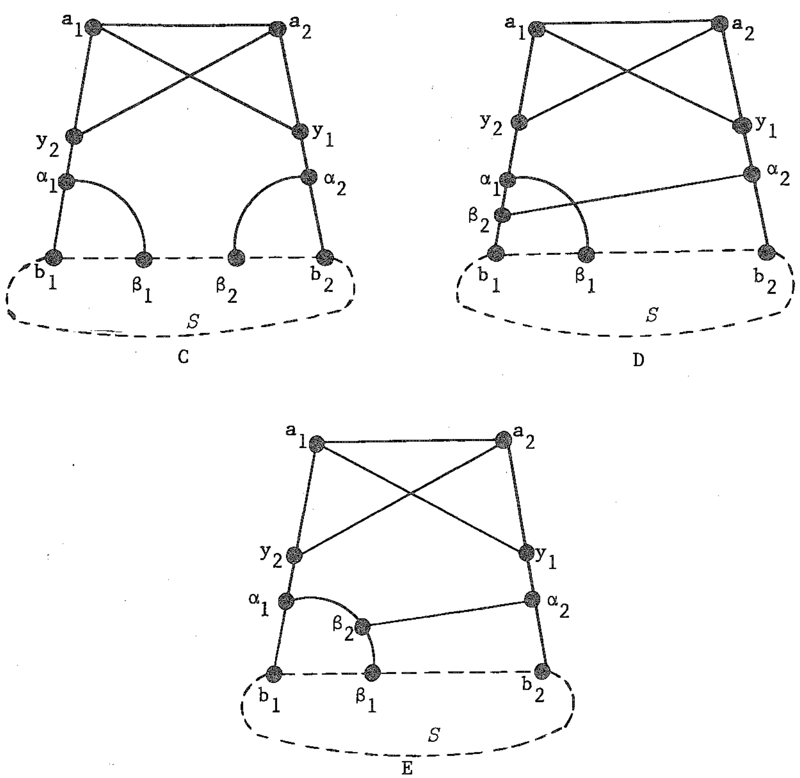

Figure 2

If $\beta_3 \in (\alpha_1, b_1]$ and $\alpha_3 \in (\alpha_2, y_1]$, then we have a configuration equivalent to D which contradicts the minimality of ℓ.

By symmetry we deal with $\beta_3 \in (\alpha_2, b_2]$.

If $\beta_3 \in Q\backslash\{b_1, b_2\}$ and $\alpha_3 \in (\alpha_1, y_2] \cup (\alpha_2, y_1]$, then the minimality of ℓ is contradicted by a configuration equivalent to C.

If $\beta_3 \in (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2)$, then the minimality of ℓ is contradicted via a configuration equivalent to C or by one equivalent to E.

2.1.2: Suppose the configuration giving rise to minimum ℓ is equivalent to E. Then we argue as in Case 2.1.1.

2.1.3: Suppose the configuration giving rise to minimum ℓ is equivalent to D. If $\alpha_3 \in A$, then by the standard arguments we achieve a configuration equivalent to either X or Y. The only minor variation occurs when $\beta_3 \in (\alpha_2, \beta_2)$ when we treat the path $(\beta_2, \beta_3, \alpha_3)$ as having origin in $(\alpha_1, b_1]$.

For all other positions of α_3 we contradict the minimality of ℓ

through some configuration which is equivalent to C, D or E. The only difficult cases here arise when $\alpha_3 \in (\alpha_1, y_2]$ and $\beta_3 \in (\alpha_2, \beta_2)$ or $\beta_3 \in (\alpha_2, b_2]$.

In both cases we get a configuration equivalent to D which contradicts the minimality of ℓ. In the first case it is necessary to map $(b_1, \beta_2, \beta_3, \alpha_3, y_2)$ to (b_1, y_2), (α_2, β_3) to (α_2, β_2) and $(\alpha_3, \alpha_1, \beta_1)$ to (α_1, β_1) and in the second case the homeomorphism takes $(b_1, \beta_2, \alpha_2, y_1)$ to (b_1, y_2), $(b_1, \beta_3, \alpha_3, y_2)$ to (b_2, y_1), (α_3, β_2) to (α_2, β_2) and $(\alpha_3, \alpha_1, \beta_1)$ to (α_1, β_1).

2.2: If $y_2 \in (a_1, y_1)$, then we achieve configurations equivalent to X or Y by arguments similar to those in Case 2.1. □

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Department of Mathematics
University of Melbourne
Parkville
Victoria

Department of Computer Science
Vanderbilt University
Nashville
U.S.A.

Mathematics Department
Vanderbilt University
Nashville
U.S.A.