TOPICS IN COMPUTATIONAL GRAPH THEORY

BY

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PREFACE

This thesis is concerned with two problems in computational Graph Theory.

The first problem is the design of an algorithm for canonically labelling a graph and for finding generators for its automorphism group. The emphasis here is on the power of the algorithm for solving practical problems, rather than the theoretical niceties of the algorithm. We succeed in developing an algorithm whose implementation is probably the most powerful practical graph isomorphism program yet devised.

The second problem considered here is the construction of an exhaustive list of vertex-transitive graphs with 19 or fewer vertices. This is accomplished with the aid of a large number of theoretical tools, some of which are developed here for the first time and may be of independent interest.

All results not attributed to another author are new. However there are several people whose suggestions and encouragements played a far from trivial part in the conduct of this research. Particular thanks are due to my supervisor D.A. Holton and to C.D. Godsil. I would also like to thank Professor R.G. Stanton for his generous support during my visit to the University of Manitoba in 1978.

CONTENTS

CHAPTER ONE

INTRODUCTION

In this chapter we present a selection of the definitions and elementary results which will be required for use in later chapters.

1·1 Basic Notation

Throughout this thesis V will denote the set $\{1, 2, \cdots, n\}$, where $n \geq 1$. The empty set is denoted by $\emptyset$. A single-element set $\{x\}$ (a *singleton*) will generally be abbreviated to x if no confusion is likely.

If X is a set, then $|X|$ denotes its cardinality. A relation $\leq$ on X is called a *linear ordering* of X if for all $x, y, z \in X$ we have (i) $x \leq x$, (ii) $x \leq y$ or $y \leq x$, (iii) if $x \leq y$ and $y \leq x$ then $x = y$, and (iv) if $x \leq y$ and $y \leq z$ then $x \leq z$. Let Z be a set whose elements are finite sequences of elements of X (the length may vary). Then the *lexicographic ordering of* Z *induced by* $\leq$ is the linear ordering $\leq$ defined as follows.
If $\underset{\sim}{x} = (x_1, x_2, \cdots, x_k) \in Z$ and $\underset{\sim}{y} = (y_1, y_2, \cdots, y_\ell) \in Z$ then $\underset{\sim}{x} \leq \underset{\sim}{y}$ if and only if either of the following holds.

(i) For some t, $1 \leq t \leq \min(k, \ell)$, we have $x_i = y_i$ for $i < t$ and $x_t < y_t$.

(ii) $x_i = y_i$ for $1 \leq i \leq k$ and $\ell \geq k$.

If X is a linearly ordered set and Y is a non-empty finite subset of X, then min Y and max Y denote the values of the smallest and the largest elements of Y with respect to $\leq$, respectively. For notational convenience we write $\min \emptyset = \infty$, where ∞ is a symbol with the property of being larger than anything it is compared with.

If M is a matrix, then M_{ij} denotes the (i, j)-th entry of M, M^T denotes the transpose of M and, if M is square, tr M denotes the trace of M. If i and j are integers, (i, j) denotes the greatest common divisor of i and j, while $i|j$ indicates that i is a divisor of j. Finally, log always denotes the natural logarithm.

1·2 Graphs

A *graph* G is a pair (V(G), E(G)), where V(G) is a finite set whose elements are called *vertices* or *points* of G, and E(G) is a set of unordered pairs of distinct elements of V(G), called *edges*. In a few special cases we will also allow E(G) to contain singletons from V(G), called *loops*. However, unless otherwise stated, our graphs do not have loops. An edge $\{x, y\} \in E(G)$ will commonly be abbreviated to xy. The *end-vertices* x and y of an edge xy are said to be *adjacent* or *joined*.

The set of all graphs G with V(G) = V will be denoted by $\underset{\sim}{G}(V)$.

The *order* of a graph G is the cardinality of V(G). A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If, in addition, V(H) = V(G) then H is a *spanning subgraph* of G. Another special type of subgraph H is that *induced by* V(H). In this case $E(H) = \{xy \in E(G) | x, y \in V(H)\}$. In general we will make no notational distinction between subsets of V(G) and the subgraphs of G which they induce.

If $x \in V(G)$, then N(x, G) denotes the set (or induced subgraph) $\{y \in V(G) | xy \in E(G)\}$ and $\overline{N}(x, H)$ denotes the set $V(G) \setminus (\{x\} \cup N(x, G))$. The *complement* $\overline{G}$ of G has $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy | x, y \in V(\overline{G}), x \neq y, xy \notin E(G)\}$. It follows that $N(x, G) = \overline{\overline{N}(x, G)}$ and $\overline{N}(x, \overline{G}) = \overline{N(x, G)}$. The *degree* of a vertex $x \in V(G)$ is the cardinality of N(x, G). If every vertex of G has the same degree k, we say that G is *regular of degree* k. If G is regular and the subgraphs N(x, G) and $\overline{N}(x, G)$ are empty or regular for each $x \in V(G)$, then G is called *strongly regular*.

A *path* of *length* $r \geq 0$ in a graph G is a sequence $x_0, x_1, \cdots, x_r$ of distinct vertices of H, such that $x_{i-1}x_i \in E(G)$ for $1 \leq i \leq r$. The *distance* $\partial(x, y)$ between $x, y \in V(G)$ is defined to be the length of the shortest path, if any, whose first and last entries are x and y. If there is no such path, $\partial(x, y) = \infty$ by convention. More generally, if $X, Y \subseteq V(G)$, the *distance* between X and Y in G is $\partial(X, Y) = \min\{\partial(x, y) | x \in X, y \in Y\}$. The *diameter* of G is $\max\{\partial(x, y) | x, y \in V(G)\}$. If G has finite diameter it is called *connected*, otherwise it is *disconnected*.

Two graphs G and H are said to be *isomorphic*, written $G \cong H$, if there is a bijection $\phi: V(G) \to V(H)$ such that $xy \in E(G)$ if and only if $\phi(x)\phi(y) \in E(G)$. It is important to realise that $\underset{\sim}{G}(V)$ contains all the graphs G with V(G) = V, not just representatives of the different isomorphism types.

Several special types of graphs are important enough to warrant names. The *complete* graph K_n has $V(K_n) = V$ and $E(K_n) = \{xy | x, y \in V, x \neq y\}$. The *empty* graph is the complement $\overline{K}_n$ of K_n, and thus has no edges. A *polygon* C_n is a connected regular graph of degree 2. A subgraph isomorphic to a complete graph is also called a *clique* and one isomorphic to a polygon is called a *cycle*.

Let G be any graph. The *linegraph* L(G) has V(L(G)) = E(G) and $E(L(G)) = \{e_1e_2 | e_1, e_2 \in E(G) \text{ and } |e_1 \cap e_2| = 1\}$. The *switching graph* Sw(G) has $V(Sw(G)) = V(G) \times \{0, 1\}$ and $E(Sw(G)) = \{(x, i)(y, j) | i = j \text{ and } xy \in E(G) \text{ or } i \neq j \text{ and } xy \in E(\overline{G})\}$. If G has n vertices, then Sw(G) has 2n vertices and is regular of degree n − 1.

Switching graphs have an important association with a relationship known as switching equivalence [40]. Two graphs G and H with V(G) = V(H) are *switching equivalent* if V(G) can be partitioned into disjoint non-empty subsets V_1 and V_2 such that

$$E(H) = \{xy \in E(G) | x, y \in V_1 \text{ or } x, y \in V_2\}$$
$$\cup \{xy \notin E(G) | x \in V_1 \text{ and } y \in V_2\}.$$

Some of the basic properties of switching equivalence are summarised in the following theorem.

1·3 THEOREM (a) *Switching equivalence is an equivalence relation.*

(b) *Each equivalence class containing graphs of odd order contains exactly one graph whose vertices all have even degree.*

(c) *Each equivalence class contains at least one graph with a vertex of degree zero.*

(d) *Two graphs* G *and* H *are switching equivalent if and only if* $Sw(G) \cong Sw(H)$.

Proof: See [40] for (a) - (c) and [14] for (d). □

Let G and H be graphs. A number of binary products can be used to construct a new graph from G and H. The simplest is the *disjoint union* $G \cup H$, for which we assume $V(G) \cap V(H) = \emptyset$. This is defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Any graph isomorphic to the disjoint union of m graphs isomorphic to G will be denoted by mG. Three other products each have vertex set $V(G) \times V(H)$. The *tensor product* $G * H$ has $E(G * H) = \{(x_1, y_1)(x_2, y_2) | x_1x_2 \in E(G), y_1y_2 \in E(H)\}$. The *cartesian product* $G \times H$ has $E(G \times H) = \{(x_1, y_1)(x_2, y_2) | x_1 = x_2 \text{ and } y_1y_2 \in E(H),$ or $y_1 = y_2$ and $x_1x_2 \in E(G)\}$. The *lexicographic product* G[H] has

$E(G[H]) = \{(x_1, y_1)(x_2, y_2) | x_1x_2 \in E(G), \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(H)\}$. Some of the elementary properties of these three operations are given in the next lemma.

1·4 LEMMA (a) $G * H \cong H * G$

(b) $G \times H \cong H \times G$

(c) $G[H] \not\cong H[G]$ (*in general*).

(d) $\overline{G}[\overline{H}] = \overline{G[H]}$

(e) $\overline{K}_m \times H = \overline{K}_m[H] = mH$ □

Let G be any graph. A *(vertex-)cutset* of G is a subset of V(G) whose removal from G leaves a disconnected graph or a single vertex. An *edge-cutset* of G is a subset of E(G) with a similar property. The *(vertex-)connectivity* κ and the *edge-connectivity* η are defined to be the size of a smallest cutset or a smallest edge-cutset, respectively.

Let G be any graph, and let V_1 and V_2 be disjoint non-empty subsets of V(G). Let $F = \{xy \in E(G) | x \in V_1, y \in V_2\}$. We say that V_1 and V_2 are *completely joined* if $|F| = |V_1||V_2|$, *trivially joined* if $|F| = 0$ or $|F| = |V_1||V_2|$ and *non-trivially joined* if $0 < |F| < |V_1||V_2|$. We say that V_1 and V_2 are *equitably joined* if there are constants k_1 and k_2 such that each vertex in V_1 is adjacent to exactly k_1 vertices in V_2, and each vertex in V_2 is adjacent to exactly k_2 vertices in V_1. Note that $k_1|V_1| = k_2|V_2|$ in this case. We also say that V_1 is equitably joined to itself if it induces a regular subgraph of G.

Let $G \in \underset{\sim}{G}(V)$. The *adjacency matrix* of G is the n×n matrix $A = A(G)$, where $A_{ij} = 1$ if $\{i, j\} \in E(G)$ and $A_{ij} = 0$ otherwise.

1·5 Partitions

A *partition* of the set V is a set of disjoint non-empty subsets of V whose union is V. An *ordered partition* of V is a sequence $(V_1, V_2, \cdots, V_r)$, such that $\{V_1, V_2, \cdots, V_r\}$ is a partition of V. The set of all partitions of V and the set of all ordered partitions of V will be denoted by $\Pi(V)$ and $\underset{\sim}{\Pi}(V)$ respectively. For notational economy we also define $\Pi^*(V) = \Pi(V) \cup \underset{\sim}{\Pi}(V)$.

The elements of a partition (or ordered partition) $\pi \in \Pi^*(V)$ are usually called its *cells*. A *trivial* cell of π is a cell of cardinality one; the element of such a cell is said to be *fixed* by π. If every cell of π is trivial, then π is a *discrete* partition, while if there is only one cell, π is the *unit* partition.

If $\pi_1, \pi_2 \in \Pi^*(V)$, we write $\pi_1 \simeq \pi_2$ if π_1 and π_1 have the same cells, in some order. We say that π_1 is *finer* than π_2, denoted $\pi_1 \leq \pi_2$, if every cell of π_1 is a subset of some cell of π_2. Under the same circumstances, π_2 is *coarser* than π_1. It is well known that the set $\Pi(V)$ forms a lattice under the partial order $\leq$. This means that, given $\pi_1, \pi_2 \in \Pi^*(V)$ there is a unique coarsest partition $\pi_1 \wedge \pi_2 \in \Pi(V)$ such that $\pi_1 \geq \pi_1 \wedge \pi_2$ and $\pi_2 \geq \pi_1 \wedge \pi_2$, and a unique finest partition $\pi_1 \vee \pi_2 \in \Pi(V)$ such that $\pi_1 \leq \pi_1 \vee \pi_2$ and $\pi_2 \leq \pi_1 \vee \pi_2$. Each cell of $\pi_1 \wedge \pi_2$ is a non-empty intersection of a cell of π_1 and a cell of π_2. Each cell of $\pi_1 \vee \pi_2$ is a minimal non-empty subset of V which is both a union of cells of π_1 and a union of cells of π_2.

Let $\pi \in \Pi^*(V)$. Then $\text{fix}(\pi)$ is the set of elements of V which are fixed by π. The *support* of π is the set $\text{supp}(\pi) = V \setminus \text{fix}(\pi)$. The set of *minimum cell representatives* of π is $\text{mcr}(\pi) = \{\min V_i \mid V_i \in \pi\}$, where the minima are under the natural ordering of V.

Some of the elementary properties of these sets are given in the following lemma.

1·6 LEMMA *Let* $\pi_1, \pi_2 \in \Pi^*(V)$.

(a) $\text{fix}(\pi_1 \vee \pi_2) = \text{fix}(\pi_1) \cap \text{fix}(\pi_2)$

(b) $\text{fix}(\pi_1 \wedge \pi_2) \supseteq \text{fix}(\pi_1) \cup \text{fix}(\pi_2)$

(c) $\text{supp}(\pi_1 \vee \pi_2) = \text{supp}(\pi_1) \cup \text{supp}(\pi_2)$

(d) $\text{supp}(\pi_1 \wedge \pi_2) \subseteq \text{supp}(\pi_1) \cap \text{supp}(\pi_2)$

(e) $\text{mcr}(\pi_1 \vee \pi_2) \subseteq \text{mcr}(\pi_1) \cap \text{mcr}(\pi_2)$

(f) $\text{mcr}(\pi_1 \wedge \pi_2) = \text{mcr}(\pi_1) \cup \text{mcr}(\pi_2)$ □

Let $\pi = (V_1, V_2, \cdots, V_r) \in \underset{\sim}{\Pi}(V)$. For each $x \in V$ define $u(x, \pi) = i$, where $x \in V_i$. If $\pi_1, \pi_2 \in \underset{\sim}{\Pi}(V)$ then we say that π_1 and π_2 are *consistent* if, for any $x, y \in V$, $u(x, \pi_1) < u(y, \pi_1)$ implies that $u(x, \pi_2) \leq u(y, \pi_2)$. As a relation, consistency is symmetric but not transitive. If $\pi_1 \leq \pi_2$ and π_1 and π_2 are consistent, we indicate this by writing $\pi_1 \leqq \pi_2$ or $\pi_2 \geqq \pi_1$. The relation $\leqq$ is transitive but not symmetric. A *partition nest* is a sequence $[\pi_1, \pi_2, \cdots, \pi_r]$, where $\pi_i \in \underset{\sim}{\Pi}(V)$ for $1 \leq i \leq r$, and $\pi_1 \geqq \pi_2 \geqq \cdots \geqq \pi_r$. The use of square brackets will always indicate that an enclosed sequence of ordered partitions is a partition nest.

1·7 Groups

The most elementary properties of groups will be assumed, as they can be found in any book on group theory, for example Hall [17]. We are only concerned here with finite groups.

The trivial (single-element) group will be denoted by 1. The cyclic group of order n will be denoted by Z_n. If p is a prime, a group whose order is a power of p is called a *p-group*, and a subgroup which is a p-group is called a *p-subgroup*. If p^m is the highest power of p which divides the order of a group Γ, then a subgroup of Γ of order p^m is a *Sylow p-subgroup* of Γ. The set of all Sylow p-subgroups of Γ will be denoted by $Syl_p(\Gamma)$. The following theorem is due to Sylow.

1·8 THEOREM *Let* p^m *be the highest power of a prime* p *which divides the order of a group* Γ. *Then* Γ *has at least one subgroup of each of the orders* $p, p^2, \cdots, p^m$. *In particular* $Syl_p(\Gamma) \neq \emptyset$. *Furthermore, any two members of* $Syl_p(\Gamma)$ *are conjugate in* Γ, *and every p-subgroup of* Γ *is contained in some member of* $Syl_p(\Gamma)$. □

If Γ and Λ are groups, then $\Gamma \otimes \Lambda$ denotes the direct product of Γ and Λ. The next lemma follows easily from Theorem 1·8.

1·9 LEMMA *If* Γ *and* Λ *are groups and* p *is prime, then*

$$Syl_p(\Gamma \otimes \Lambda) = \{P \otimes Q \mid P \in Syl_p(\Gamma),\ Q \in Syl_p(\Lambda)\}.$$ □

If Ω is a subset or a set of subsets of a group Γ, then the subgroup $\langle\Omega\rangle$ of Γ *generated* by Ω is the smallest subgroup of Γ which contains each element of Ω. If $\Lambda \leq \Gamma$ then the *normaliser* $N_\Gamma(\Lambda)$ of Λ in Γ is the largest subgroup of Γ of which Λ is a normal subgroup.

1·10 LEMMA *Let* Γ *be any group. Let* $P, Q \in Syl_p(\Gamma)$ *such that* $P \neq Q$ *and* $|P \cap Q|$ *is maximal. Then any conjugates of* $P \cap Q$ *in* Γ *which lie in* P *are conjugate in* $N_\Gamma(P)$.

Proof: See Lemma 7·4·7 in [15]. □

If Γ is a group, $\gamma, \delta \in \Gamma$ and $\Omega \subseteq \Gamma$, we use γ^δ as an abbreviation for $\delta^{-1}\gamma\delta$ and define $\Omega^\gamma = \{\omega^\gamma \mid \omega \in \Omega\}$.

1·11 Permutation Groups

Unless otherwise indicated, proofs of all the results mentioned in Sections 1·11 - 1·16 can be found in [44].

A *permutation* γ of the set V is a bijection from V to itself. The image of $x \in V$ under γ will be denoted by x^γ. An *m-cycle* $(m \geq 2)$ is a permutation of the form $(v_1\ v_2\ \cdots\ v_m)$, where elements of V not mentioned are mapped onto themselves. A 2-cycle is also called a *transposition*. The set of all permutations of V forms a group of order n! under function composition, called the *symmetric group* S_n. A *permutation group* of *degree* n is a subgroup of S_n. The group $\Gamma \leq S_n$ is *transitive* if, for each $x, y \in V$, there is some $\gamma \in \Gamma$ such that $x^\gamma = y$. If $W \subseteq V$ and $\gamma \in S_n$ define $W^\gamma = \{x^\gamma | x \in W\}$. If $\Gamma \leq S_n$, $W \subseteq V$ and $W^\gamma = W$ for each $\gamma \in \Gamma$, then Γ induces a group $\Gamma|_W$ of permutations of W. If also $\Gamma|_W$ is transitive then W is an *orbit* of Γ and we say that Γ *acts transitively* on W. The orbits of Γ are disjoint, and so are the cells of a partition $\theta(\Gamma) \in \Pi(V)$. More generally, if Ω is a subset or a set of subsets of S_n we define $\theta(\Omega) = \theta(\langle\Omega\rangle)$. The next lemma follows easily from the definitions.

1·13 LEMMA *If* Ω, $\Phi \subseteq S_n$ *then* $\theta(\Omega \cup \Phi) = \theta(\Omega) \vee \theta(\Phi)$. □

An orbit of size r can be called an *r-orbit*. A *1-orbit* will also be called a *trivial orbit*. A point $x \in V$ which is in a trivial orbit of Γ is said to be *fixed* by Γ. The set of all points fixed by Γ is denoted by fix(Γ). In other words, $\mathrm{fix}(\Gamma) = \mathrm{fix}(\theta(\Gamma))$. We can similarly define $\mathrm{fix}(\Omega) = \mathrm{fix}(\theta(\Omega))$, $\mathrm{supp}(\Omega) = \mathrm{supp}(\theta(\Omega))$ and $\mathrm{mcr}(\Omega) = \mathrm{mcr}(\theta(\Omega))$ if Ω is any subset or set of subsets of S_n. The next lemma follows from Lemmas 1·6 and 1·13.

1·14 LEMMA *Let* Ω, $\Phi \subseteq S_n$. *Then*

(a) $\mathrm{fix}(\Omega \cup \Phi) = \mathrm{fix}(\Omega) \cap \mathrm{fix}(\Phi)$,

(b) $\mathrm{supp}(\Omega \cup \Phi) = \mathrm{supp}(\Omega) \cup \mathrm{supp}(\Phi)$, and

(c) $\mathrm{mcr}(\Omega \cup \Phi) \subseteq \mathrm{mcr}(\Omega) \cap \mathrm{mcr}(\Phi)$. □

If Γ, $\Lambda \leq S_n$ have disjoint support, the *direct sum* of Γ and Λ is the group $\Gamma \oplus \Lambda = \langle \Gamma \cup \Lambda \rangle$. Clearly $\Gamma \oplus \Lambda$ is isomorphic as an abstract group to $\Gamma \otimes \Lambda$.

Let $\Gamma \leq S_n$. A *block* of Γ is a subset $W \subseteq V$ such that for every $\gamma \in \Gamma$, either $W^\gamma = W$ or $W^\gamma \cap W = \emptyset$. Obviously $\emptyset$, V and every singleton are blocks; any other blocks are called *non-trivial*. If Γ is transitive and W is a block of Γ, then the different sets W^γ, for $\gamma \in \Gamma$, form the cells of a partition of V which is called a *block-system* for Γ, *non-trivial* if W is a non-trivial block. If Γ is transitive and has no non-trivial blocks it is called *primitive*, otherwise it is called *imprimitive*.

A permutation $\gamma \in S_n$ is defined according to its action on V, but it is also convenient to define an action of γ on other objects which involve V. We have already defined the action of γ on subsets of V, for example. Other important cases are as follows.

(i) If $\pi \in \Pi^*(V)$, π^γ is formed by replacing each cell V_i with V_i^γ (*in situ* if $\pi \in \underset{\sim}{\Pi}(V)$).

(ii) If $G \in \underset{\sim}{G}(V)$ then $G^\gamma \in \underset{\sim}{G}(V)$ has $E(G^\gamma) = \{x^\gamma y^\gamma \mid xy \in E(G)\}$.

Other cases will be defined when they are first required, but in every case the idea is the same. Each element $x \in V$ is simply replaced by x^γ wherever it occurs in the object under consideration.

Let $\Gamma \leq S_n$ and let Ω be any set such that an action of each $\gamma \in \Gamma$ is defined on each element of Ω. Ω need not be closed under this action. Then the *stabiliser* of Ω in Γ is the group

$\Gamma_\Omega = \{\gamma \in \Gamma | \omega^\gamma = \omega$ for each $\omega \in \Omega\}$. Elements of Γ_Ω are said to *fix* Ω. The most important cases of this construction are as follows.

(i) (*point-wise stabiliser*)

If $W \subseteq V$ then $\Gamma_W = \{\gamma \in \Gamma | x^\gamma = x$ for each $x \in W\}$. If $W = \{x_1, x_2, \cdots, x_r\}$ we will also write Γ_W as $\Gamma_{x_1, x_2, \cdots, x_r}$.

(ii) (*set-wise stabiliser*)

If $W \subseteq V$ then $\Gamma_{\{W\}} = \{\gamma \in \Gamma | W^\gamma = W\}$.

(iii) (*partition stabiliser*)

If $\pi \in \Pi^*(V)$ has cells $V_1, V_2, \cdots, V_r$ then $\Gamma_\pi = \{\gamma \in \Gamma | V_i^{\gamma} = V_i$ for $1 \le i \le r\}$. Note that this is not the same as $\Gamma_{\{\pi\}} = \{\gamma \in \Gamma | \pi^\gamma = \pi\}$, unless $\pi \in \underset{\sim}{\Pi}(V)$.

(iv) (*automorphism group*)

If $G \in \underset{\sim}{G}(V)$, then the *automorphism group* of G is the group $\text{Aut}(G) = (S_n)_{\{G\}} = \{\gamma \in S_n | G^\gamma = G\}$. We will discuss this group in more depth later.

A group $\Gamma \le S_n$ is *semi-regular* if $\Gamma_x = 1$ for each $x \in V$, and *regular* if it is semi-regular and transitive.

Proofs of each part of the following theorem may be found in [44].

1.15 THEOREM *Let* $\Gamma, \Lambda \le S_n$ *where* Γ *is transitive but* Λ *need not be. Let* W *be an orbit of* Λ, *and let* $\{B_1, B_2, \cdots, B_r\}$ *be a block system for* Γ.

(a) *If* $\gamma \in S_n$ *then* W^γ *is an orbit of* Λ^γ. *In particular, if* $\gamma \in N_{S_n}(\Lambda)$, W^γ *is an orbit of* Λ.

(b) *For any* $x \in W$, $|\Lambda_x||W| = |\Lambda|$.

(c) *If* $P \in \mathrm{Syl}_p(\Lambda)$ *for some prime* p, *then every shortest orbit of* P *in* W *has length* p^m, *where* p^m *is the highest power of* p *which divides* $|W|$.

(d) *Both* $|B_1|$ *and* r *are divisors of* n.

(e) *If* $\Phi \trianglelefteq \Gamma$, *then* $\theta(\Phi)$ *is a block-system for* Γ.

(f) *The permutation group on* $\{B_1, B_2, \cdots, B_r\}$ *induced by the action of* Γ *is transitive.*

(g) $\Gamma_{\{B_1\}}$ *acts transitively on* B_1.

(h) *If* $\Gamma_1 \leq \Phi \leq \Gamma$, *then the orbit of* Φ *which contains* 1 *is a block for* Γ.

(i) $\mathrm{fix}(\Gamma_1)$ *is a block for* Γ. □

Now let $\Gamma \leq S_n$ be transitive. The set $J(\Gamma)$ consists of those subgroups $\Lambda \leq \Gamma$ such that

(i) $1 < \Lambda \leq \Gamma_1$, and

(ii) $N_\Gamma(\Lambda)$ acts transitively on $\mathrm{fix}(\Lambda)$.

The most useful theorem for the identification of members of $J(\Gamma)$ is due to Jordan. (See [44] for a proof.)

1·16 THEOREM *Let* $\Gamma \leq S_n$ *be transitive, and let* Λ *be a non-trivial subgroup of* Γ_1 *which is conjugate in* Γ_1 *to any of its conjugates in* Γ *which lie in* Γ_1. *Then* $\Lambda \in J(\Gamma)$. □

If Γ is regular, then $J(\Gamma) = \emptyset$ obviously. If Γ is transitive, but not regular, then Γ_1 itself and any non-trivial Sylow p-subgroups of Γ_1 are in $J(\Gamma)$. Another useful family of members of $J(\Gamma)$ is defined in the next theorem.

1·17 THEOREM *Let* Γ *be transitive and let* $\mathrm{Syl}_p(\Gamma_1) \neq \{1\}$ *for some prime* p. *Then* $\Lambda = \langle \mathrm{Syl}_p(\Gamma_1)\rangle \in J(\Gamma)$.

Proof: Obviously $1 < \Lambda \leq \Gamma_1$. Furthermore, if $\Lambda^\gamma \subseteq \Lambda$ for some $\gamma \in \Gamma$, then $\Lambda^\gamma = \langle\{P^\gamma | P \in \mathrm{Syl}_p(\Gamma_1)\}\rangle = \Lambda$, by Theorem 1·8. Therefore $\Lambda \in J(\Gamma)$ by Theorem 1·16. □

1·18 COROLLARY *Under the conditions of the theorem,* fix(Λ) *is a block for* Γ.

Proof: Let $\Phi = N_\Gamma(\Lambda)$. Then $\Gamma_1 \leq \Phi \leq \Gamma$, since $\Lambda \trianglelefteq \Gamma_1$ obviously. Therefore fix(Λ) is a block for Γ, by Theorems 1·15(h) and 1·17. □

The next theorem is due to C. E. Praeger [37].

1·19 THEOREM *Let* $\Gamma \leq S_n$ *be transitive and let* $1 < \Lambda \leq \Gamma$ *have the property that for any* $\gamma \in \Gamma$, Λ *and* Λ^γ *are conjugate in* $\langle\{\Lambda, \Lambda^\gamma\}\rangle$. *Then* $|\mathrm{fix}(\Lambda)| \leq \frac{1}{2}(n - 1)$. □

If $\Lambda \leq \Phi \leq \Gamma$ are groups, we say that Λ is *weakly closed in* Φ *with respect to* Γ if for each $\gamma \in \Gamma$, $\Lambda^\gamma \leq \Phi$ if and only if $\Lambda^\gamma = \Lambda$.

1·20 THEOREM *Let* $\Gamma \leq S_n$ *be transitive and let* $1 < P \in \mathrm{Syl}_p(\Gamma_1)$ *for some prime* p. *If* $1 < \Lambda \leq P$ *and* Λ *is weakly closed in* P *with respect to* Γ, *then* $|\mathrm{fix}(\Lambda)| \leq \frac{1}{2}n$. □

Proof: Suppose that $|\mathrm{fix}(\Lambda)| > \frac{1}{2}n$. Let $\gamma \in \Gamma$ and $\Phi = \langle\{\Lambda, \Lambda^\gamma\}\rangle$. Then $|\mathrm{fix}(\Phi)| \geq 1$, so that $\Phi \leq \Gamma_x$ for some $x \in V$. By Theorem 1·8, there are $Q \in \mathrm{Syl}_p(\Phi)$ and $\phi \in \Phi$ such that $\Lambda \leq Q$ and $\Lambda^\gamma \leq Q^\phi$. But then Λ^ϕ and Λ^γ are both in Q^ϕ and hence in any conjugate of P which contains Q^ϕ. Therefore $\Lambda^\phi = \Lambda^\gamma$ by the weak closure condition. But then $|\mathrm{fix}(\Lambda)| \leq \frac{1}{2}(n - 1)$ by Theorem 1·19, contradicting the assumption that $|\mathrm{fix}(\Lambda)| > \frac{1}{2}n$. □

1·21 Transitive graphs

A graph G is *transitive* if Aut(G) is transitive, and *edge-transitive* if Aut(G) acts transitively on E(G). A *t-arc* in G is a sequence $(x_0, x_1, \cdots, x_t)$ of vertices of G such that $x_{i-1}x_i \in E(G)$ for $1 \le i \le t$ and $x_{i-1} \ne x_{i+1}$ for $1 \le i < t$. The *arc transitivity* of G is the maximum value of t such that Aut(G) acts transitively on the t-arcs of G. A discussion of arc-transitivity can be found in [5]. Clearly 1-arc transitivity is the same as transitivity. A 2-arc transitive graph is also called *symmetric*; such a graph is clearly also edge-transitive. Some of the elementary properties of the various forms of transitivity are summarised in the next theorem.

1·22 THEOREM *Let* G *and* H *be graphs.*

(a) $\mathrm{Aut}(\overline{G}) = \mathrm{Aut}(G)$. *In particular* $\overline{G}$ *is transitive if and only if* G *is transitive.*

(b) *If* G *is edge-transitive then* L(G) *is transitive.*

(c) *If* G *and* H *are transitive, then* $G \times H$, $G * H$ *and* G[H] *are transitive.*

(d) *If* G *is transitive and disconnected then* $G = mH$ *for some* $m \ge 2$ *and some connected transitive graph* H.

(e) *Let* G *be transitive with diameter* Δ. *Define* G <u>*plus diagonals*</u> *to be the graph* D(G) *where*

$V(D(G)) = V(G)$ *and*

$E(D(G)) = E(G) \cup \{xy \mid x, y \in V(G), \partial(x, y) = \Delta\}$.

Then D(G) *is transitive.* □

A rich source of transitive graphs is the Cayley graph construction. Let Γ be a group, and let Ω be a subset of Γ such that

(i) Ω does not contain the identity of Γ, and

(ii) $\gamma \in \Omega$ if and only if $\gamma^{-1} \in \Omega$, for all $\gamma \in \Gamma$.

The *Cayley graph* of Γ with *connection set* Ω is the graph $H = C(\Gamma, \Omega)$ with

$$V(H) = \Gamma, \text{ and}$$

$$E(H) = \{\{\gamma, \gamma\omega\} | \gamma \in \Gamma, \omega \in \Omega\}.$$

H is a transitive graph on which Γ acts (by left multiplication) as a regular subgroup of Aut(H). Conversely, if Aut(H) contains a regular subgroup Γ then H is (isomorphic to) a Cayley graph of Γ.

Transitive graphs which are not Cayley graphs are comparatively rare, but they do exist. See Appendix 2 for some examples. In order to algebraically represent all transitive graphs we can generalise the Cayley graph construction as in the next theorem, first published by Teh [41].

1·23 THEOREM *Let Γ be any group. Let $\Lambda \leq \Gamma$ and $\Omega \subseteq \Gamma$ satisfy the conditions*

(i) $\Lambda\Omega\Lambda$ *does not contain the identity of* Γ, *and*

(ii) $\gamma \in \Omega$ *if and only if* $\gamma^{-1} \in \Omega$, *for* $\gamma \in \Gamma$.

Define the graph $H = C(\Gamma, \Lambda, \Omega)$ *as follows:*

$$V(H) = \{\gamma\Lambda | \gamma \in \Gamma\}$$

$$E(H) = \{\{\gamma\Lambda, \delta\Lambda\} | \gamma, \delta \in \Gamma, \gamma^{-1}\delta \in \Lambda\Omega\Lambda\}$$

Then H *is a transitive graph for which* Aut(H) *contains a transitive (not-necessarily faithful) representation of* Γ. *Conversely, if* Γ *is a transitive subgroup of* Aut(H) *then* $H \cong C(\Gamma, \Lambda, \Omega)$ *for suitable choices of* Λ *and* Ω. □

1·24 Algorithms

Algorithms in this thesis are given in an informal a manner as is possible without loss of rigor. Execution commences at the command marked (1) and proceeds as directed until the command *stop* is encountered. The only special symbol is the assignment operator $\leftarrow$, which indicates that the expression on the right of the operator is to be evaluated and the resulting value assigned to the variable on the left. When we are describing the operation of the algorithm, *Step* (i) refers to the set of commands starting at that marked (i) and finishing with the command preceding that marked (i + 1).

CHAPTER TWO

A NEW GRAPH LABELLING ALGORITHM

In this chapter we will discuss the design of an algorithm for canonically labelling a vertex-coloured graph and for finding a small set of generators for the automorphism group of the graph. This algorithm is a descendant of the one described in McKay [26], which in turn was descended from an algorithm first developed in McKay [28]. Other algorithms which are related to ours in some respects have been devised by Mathon [25], Arlazarov, Zuev, Uskov and Faradzev [21] and Bayer and Proskurowski [3]. However we believe that the algorithm we will present here, or more precisely the implementation which we will discuss in Chapter 3, is the most powerful which is presently in use. It has been successfully applied to difficult graphs of order greater than 600 (see Chapter 3) and to rather easier graphs with around 3000 vertices.

2·1 Canonical Labelling Maps

A *canonical labelling map* is a map $\mathcal{C} : \underset{\sim}{G}(V) \times \underset{\sim}{\Pi}(V) \to \underset{\sim}{G}(V)$, such that for any $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ and $\gamma \in S_n$ we have

(C1) $\mathcal{C}(G, \pi) \cong G$

(C2) $\mathcal{C}(G^\gamma, \pi^\gamma) = \mathcal{C}(G, \pi)$

(C3) If $\mathcal{C}(G, \pi^\gamma) = \mathcal{C}(G, \pi)$, then $\pi^\gamma = \pi^\delta$ for some $\delta \in \mathrm{Aut}(G)$.

The main use of a canonical labelling map is to solve various graph isomorphism problems as indicated in the following theorem.

2·2 THEOREM *Let* G_1, $G_2 \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ *and* $\delta \in S_n$. *Then* $\mathcal{C}(G_1, \pi) = \mathcal{C}(G_2, \pi^\gamma)$ *if and only if there is a permutation* $\delta \in S_n$ *such that* $G_2 = G_1^{\delta}$ *and* $\pi^\gamma = \pi^\delta$.

Proof: The existence of δ as required implies that $\mathcal{C}(G_1, \pi) = \mathcal{C}(G_2, \pi^\gamma)$ by Property C2. Suppose conversely that $\mathcal{C}(G_1, \pi) = \mathcal{C}(G_2, \pi^\gamma)$. By Property C1, $G_2 = G_1^{\beta}$ for some $\beta \in S_n$. Therefore $\mathcal{C}(G_2, \pi^\gamma) = \mathcal{C}(G_1^{\beta}, \pi^\gamma) = \mathcal{C}(G_1, \pi^{\gamma\beta^{-1}})$, by Property C2. Since $\mathcal{C}(G_1, \pi) = \mathcal{C}(G_2, \pi^\gamma)$, there is some $\alpha \in \mathrm{Aut}(G_1)$ such that $\pi^{\gamma\beta^{-1}} = \pi^{\alpha}$, by Property C3, and so $\pi^\gamma = \pi^{\alpha\beta}$. But $\alpha \in \mathrm{Aut}(G_1)$, and so $G_2 = G_1^{\beta} = G_1^{\alpha\beta}$. □

The isomorphism problem described in Theorem 2·2 can be thought of as that of testing vertex-coloured graphs for isomorphism. Given $|\pi|$ colours, we colour those vertices of G_1 which lie in the i-th cell of G_1 with the i-th colour, for $1 \le i \le |\pi|$. We then similarly colour the vertices of G_2 in accordance with π^γ. This will use the same colours with the same frequency. Theorem 2·2 now says that $\mathcal{C}(G_1, \pi) = \mathcal{C}(G_2, \pi^\gamma)$ if and only if there is a colour-preserving isomorphism from G_1 to G_2.

The most important case is, of course, when π is the unit partition (V), in which case Property C3 holds trivially. However we will maintain the more general setting we have created, since the added complications will only be slight.

2·3 Equitable Partitions

For $G \in \underset{\sim}{G}(V)$, $v \in V$ and $W \subseteq V$, define $d_G(v, W)$ to be the number of elements of W which are adjacent in G to v. The subscript G will normally be suppressed. We will say that $\pi \in \Pi^*(V)$ is *equitable* (with respect to G) if, for all V_1, $V_2 \in \pi$ (not necessarily distinct) and $v_1, v_2 \in V_1$, we have $d(v_1, V_2) = d(v_2, V_2)$. Some of the elementary properties of equitable partitions are studied in McKay [26]. For our purposes here we need only recall that the equitable members of $\Pi(V)$ form a lattice which is closed under $\vee$. Since the discrete partition is always equitable, it follows that for every $\pi \in \Pi^*(V)$ there is a unique coarsest equitable partition $\xi(\pi) \in \Pi(V)$ which is finer than π.

One of our first concerns in this chapter will be to study an efficient procedure for computing $\xi(\pi)$ from π.

2·4 The Refinement Procedure

The algorithm we give here is a descendant of one first described in McKay [26]. It actually turns out to be a generalization of an algorithm of Hopcroft ([19], see also [16]) for minimizing the number of states in a finite automaton, although it was not derived from the latter.

The algorithm accepts a graph $G \in \underset{\sim}{G}(V)$, an ordered partition $\pi \in \underset{\sim}{\Pi}(V)$ and a sequence $\alpha = (W_1, W_2, \cdots, W_M)$ of distinct cells of π. The result is an ordered partition $\boldsymbol{R}(G, \pi, \alpha) \in \underset{\sim}{\Pi}(V)$. Under suitable conditions on α, to be discussed below, $\boldsymbol{R}(G, \pi, \alpha) \approx \xi(\pi)$.

2·5 ALGORITHM *Compute* $\boldsymbol{R}(G, \pi, \alpha)$ *given* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ *and*

$\alpha = (W_1, W_2, \cdots, W_M) \subseteq \pi$.

(1) $\tilde{\pi} \leftarrow \pi$

$m \leftarrow 1$

(2) If ($\tilde{\pi}$ is discrete *or* $m > M$) *stop*: $\boldsymbol{R}(G, \pi, \alpha) = \tilde{\pi}$

$W \leftarrow W_m$

$m \leftarrow m + 1$

$k \leftarrow 1$

{Suppose $\tilde{\pi} = (V_1, V_2, \cdots, V_r)$ at this point}

(3) Define $(X_1, X_2, \cdots, X_s) \in \underset{\sim}{\Pi}(V_k)$ such that for any $x \in X_i$, $y \in Y_j$ we have $d(x, W) < d(y, W)$ if and only if $i < j$.

If $(s = 1)$ go to (4)

Let t be the smallest integer such that $|X_t|$ is maximum $(1 \le t \le s)$.

If$(W_j = V_k$ for some j $(m \le j \le M))$ $W_j \leftarrow X_t$

For $1 \le i < t$ set $W_{M+i} \leftarrow X_i$

For $t < i \le s$ set $W_{M+i-1} \leftarrow X_i$

$M \leftarrow M + s - 1$

Update $\tilde{\pi}$ by replacing the cell V_k *in situ* with the cells $X_1, X_2, \cdots, X_s$ in that order.

(4) $k \leftarrow k + 1$

If $(k \le r)$ go to (3)

Go to (2) □

2·6 THEOREM *For any* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$, $\boldsymbol{R}(G, \pi, \pi) \simeq \xi(\pi)$.

Proof: (a) The value of $M - m$ is decreased in Step (2) and is only increased when $\tilde{\pi}$ is made strictly finer. Therefore the algorithm is certain to terminate.

(b) By definition, $\xi(\pi) \le \pi$, so $\xi(\pi) \le \tilde{\pi}$ at Step (1). Now suppose that $\xi(\pi) \le \tilde{\pi}$ before some execution of Step (3). Since W is a cell of some partition coarser than $\xi(\pi)$ (some earlier value of $\tilde{\pi}$), it is a union of cells of $\xi(\pi)$. Since $\xi(\pi)$ is equitable, we must have that $\xi(\pi) \le \tilde{\pi}$ after the execution of Step (3). Therefore, by induction, $\xi(\pi) \le \boldsymbol{R}(G, \pi, \pi) \le \pi$ when the algorithm stops.

(c) Suppose that $\boldsymbol{R}(G, \pi, \pi)$ is not equitable. Then for some $V_1, V_2 \in \boldsymbol{R}(G, \pi, \pi)$ there are $x, y \in V_1$ such that

$d(x, V_2) \neq d(y, V_2)$. Since $\tilde{\pi}$ is made successively finer by the algorithm, x and y must always be in the same cell of $\tilde{\pi}$.

(d) At step (1), V_2 is contained in some element of α. Hence V_2 must sometime be contained in W for an execution of Step (3).

(e) Since x and y are never separated, $d(x, W) = d(y, W)$. But $d(x, V_2) \neq d(y, V_2)$, and since W is a union of cells of $\mathcal{R}(G, \pi, \pi)$, there is at least one other cell V_3 of $\mathcal{R}(G, \pi, \pi)$ contained in W for which $d(x, V_3) \neq d(y, V_3)$. Since V_2 and V_3 are different cells of $\mathcal{R}(G, \pi, \pi)$ they must be separated at some execution of Step (3). At least one of them, say V_2 will then be contained in some new element of α.

(f) Since the argument in (e) can clearly be repeated indefinitely, the algorithm never stops, contradicting (a). Therefore our assumption that $\mathcal{R}(G, \pi, \pi)$ is not equitable must be false, which proves that $\mathcal{R}(G, \pi, \pi) \simeq \xi(\pi)$. □

An important advantage that Algorithm 2·5 has over previous algorithms for computing $\xi(\pi)$ is that α can sometimes be chosen to be a proper subset of π. One method of choosing α is described in the next theorem.

2·7 THEOREM *Let* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ *and suppose that there is some equitable partition* π' *which is coarser than* π. *Choose* $\alpha \subseteq \pi$ *such that for any* $W \in \pi'$, *we have* $X \subseteq W$ *for at most one* $X \in \pi \setminus \alpha$. *Then* $\mathcal{R}(G, \pi, \alpha) \simeq \xi(\pi)$.

Proof: (a) By the same arguments as in Theorem 2·6, the algorithm will eventually stop, and $\xi(\pi) \leq \mathcal{R}(G, \pi, \alpha) \leq \pi$.

(b) Suppose that $\mathcal{R}(G, \pi, \alpha)$ is not equitable. Then for some $V_1, V_2 \in \mathcal{R}(G, \pi, \alpha)$ there are $x, y \in V_1$ such that $d(x, V_2) \neq d(y, V_2)$.

Since $\mathcal{R}(G, \pi, \alpha) \leq \pi'$, and π' is equitable, there is at least one other cell V_3 of $\mathcal{R}(G, \pi, \alpha)$ such that $d(x, V_3) \neq d(y, V_3)$.

(c) If V_2 and V_3 are in different cells of π, the defined relationship between π, α and π' ensures that at least one of them, say V_2, is contained in some cell of α at step (1). We can then take up the proof of Theorem 2·6 at step (d), and conclude that $\mathcal{R}(G, \pi, \alpha) \approx \xi(\pi)$.

(d) On the other hand, V_2 and V_3 may be in the same cell of π. Since they are different cells of $\mathcal{R}(G, \pi, \alpha)$ they must be separated at step (3) of the algorithm. At least one of them, say V_2, will then be contained in some new element of α. We can now take up the proof of Theorem 2·6 at step (e) and conclude as before that $\mathcal{R}(G, \pi, \alpha) \approx \xi(\pi)$. □

One application of Theorem 2·7 occurs when G is regular and π has more than one cell. The unit partition π_0 is equitable, and so we can choose α to be π less any one cell. This will be particularly time-saving if $\pi = (v, V \backslash v)$ for some v, in which case we can use $\alpha = (v)$.

A much more important application of Theorem 2·7 will be described in Section 2·9.

Two very useful properties of Algorithm 2·5 are stated in the next lemma. Both of them are immediate consequences of the definition of the algorithm.

2·8 LEMMA *Let* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$, α *an ordered subset of* π *and* $\gamma \in S_n$. *Then*

(a) $\mathcal{R}(G, \pi, \alpha) \leqq \pi$, *and*

(b) $\mathcal{R}(G^\gamma, \pi^\gamma, \alpha^\gamma) = \mathcal{R}(G, \pi, \alpha)^\gamma$. □

2·9 Partition nests

Let $\pi = (V_1, V_2, \cdots, V_k) \in \underset{\sim}{\Pi}(V)$ and let $v \in V_i$ for some i. If $|V_i| = 1$ define $\pi \circ v = \pi$. If $|V_i| > 1$ define $\pi \circ v = (V_1, \cdots, V_{i-1}, v, V_i\backslash v, V_{i+1}, \cdots, V_k)$. Also define $\pi \perp v = \mathcal{R}(G, \pi \circ v, (v))$.

Given $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ and a sequence $\underset{\sim}{v} = v_1, v_2, \cdots, v_{m-1}$ of distinct elements of V, we define the *partition nest derived from* G, π *and* $\underset{\sim}{v}$ to be $[\pi_1, \pi_2, \cdots, \pi_m]$, where

(a) $\pi_1 = \mathcal{R}(G, \pi, \pi)$, and

(b) $\pi_i = \pi_{i-1} \perp v_{i-1}$, for $2 \le i \le m$.

It follows from Theorems 2·6 and 2·7 that each π_i is equitable. Define $\underset{\sim}{N}(V)$ to be the set of all partition nests derived from some $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ and vector $\underset{\sim}{v}$ of distinct elements of V.

2·10 The basic search tree

Let $G \in \underset{\sim}{G}(V)$ and $\pi \in \underset{\sim}{\Pi}(V)$. Then the *search tree* $T(G, \pi)$ is the set of all partition nests $\nu = [\pi_1, \pi_2, \cdots, \pi_m] \in \underset{\sim}{N}(V)$ such that ν is derived from G, π and a sequence $v_1, v_2, \cdots, v_{m-1}$ where, for $1 \le i \le m - 1$, v_i is an element of the first non-trivial cell of π_i which has the smallest size. This definition implies that $|\pi_i| < |\pi_{i+1}|$ for $1 \le i < m$.

The elements of $T(G, \pi)$ will be referred to as *nodes*. The *length* $|\nu|$ of a node ν is the number of partitions it contains. If $\nu = [\pi_1, \pi_2, \cdots, \pi_m]$ is a node then $\nu^{(i)}$ denotes the node $[\pi_1, \pi_2, \cdots, \pi_i]$, for $1 \le i \le m$. Thus $\nu^{(m)} = \nu$. If $m \ge 2$ then ν is called a *successor* of $\nu^{(m-1)}$. Similarly, ν is a *descendant* of $\nu^{(i)}$ (and $\nu^{(i)}$ is an *ancestor* of ν) if $1 \le i < m$. The *root mode* $[\pi_1]$ is an ancestor of every node other than itself. The set of all nodes equal to or descended from a node ν constitutes the *subtree of* $T(G, \pi)$

rooted at ν, and is denoted by $T(G, \pi, \nu)$. If the last partition in a node is discrete, ν will be called a *terminal node*.

Suppose that ν_1 and ν_2 are distinct nodes, neither of which is a descendant of the other. Then for some i, $\nu_1^{(i)} = \nu_2^{(i)}$ but $\nu_1^{(i+1)} \neq \nu_2^{(i+1)}$. The node $\nu_1^{(i+1)}$ will be denoted by $\nu_1 - \nu_2$ and $\nu_2^{(i+1)}$ by $\nu_2 - \nu_1$.

The natural linear ordering of V can be used to provide an ordering < of the nodes of $T(G, \pi)$. Let ν_1 and ν_2 be distinct nodes. If ν_1 is an ancestor of ν_2 then $\nu_1 < \nu_2$. If neither of ν_1 or ν_2 is an ancestor of the other, there is a node $[\pi_1, \pi_2, \cdots, \pi_m]$ and vertices $v_1 \neq v_2$ such that $\nu_1 - \nu_2 = [\pi_1, \pi_2, \cdots, \pi_m, \pi_m \perp v_1]$ and $\nu_2 - \nu_1 = [\pi_1, \pi_2, \cdots, \pi_m, \pi_m \perp v_2]$. Then we have $\nu_1 < \nu_2$ if $v_1 < v_2$. If $\nu_1 < \nu_2$, we say that ν_1 is *earlier* than ν_2, and that ν_2 is *later* than ν_1.

Some of the obvious properties of this ordering of $T(G, \pi)$ are listed in the next lemma.

2·11 LEMMA *Let* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ *and* $\nu_1, \nu_2, \nu_3 \in T(G, \pi)$. *Then*

(a) *Exactly one of* $\nu_1 < \nu_2$ $\nu_1 = \nu_2$ *and* $\nu_2 < \nu_1$ *is true.*

(b) *If* $\nu_1 < \nu_2$ *and* $\nu_2 < \nu_3$ *then* $\nu_1 < \nu_3$.

(c) *If* $\nu_1 < \nu_2$, $\nu_1' \in T(G, \pi, \nu_1)$ *and* $\nu_2' \in T(G, \pi, \nu_2)$ *then* $\nu_1' < \nu_2'$, *except possibly if* ν_1 *is an ancestor of* ν_2.

If $\nu_1 \neq \nu_2$ *and neither of* ν_1 *and* ν_2 *is an ancestor of the other, then* $\nu_1 < \nu_2$ *if and only if*

$\nu_1 - \nu_2 < \nu_2 - \nu_1$. □

Given $G \in \underset{\sim}{G}(V)$ and $\pi \in \underset{\sim}{\Pi}(V)$ we can generate the elements of $T(G, \pi)$ in the order given by <, with the simple backtrack algorithm given below.

2·12 ALGORITHM *Generate* $T(G, \pi)$ *in the order earliest to latest, given* $G \in \underset{\sim}{G}(V)$ *and* $\pi \in \underset{\sim}{\Pi}(V)$.

(1) $k \leftarrow 1$

$\pi_1 \leftarrow \mathcal{R}(G, \pi, \pi)$

Output $[\pi_1]$

(2) If (π_k is discrete) go to (4)

$W_k \leftarrow$ first non-trivial cell of π_k of the smallest size

(3) If ($W_k = \emptyset$) go to (4)

$v \leftarrow \min W_k$

$W_k \leftarrow W_k \setminus v$

$\pi_{k+1} \leftarrow \pi_k \perp v$

$k \leftarrow k + 1$

Output $[\pi_1, \pi_2, \cdots, \pi_k]$

Go to (2)

(4) $k \leftarrow k - 1$

If ($k \geq 1$) go to (3)

Stop: All the nodes of $T(G, \pi)$ have been output in the proper order. □

2·13 Group actions on $T(G, \pi)$

If $\nu = [\pi_1, \pi_2, \cdots, \pi_m] \in \underset{\sim}{N}(V)$ and $\gamma \in S_n$, then we can define $\nu^\gamma = [\pi_1{}^\gamma, \pi_2{}^\gamma, \cdots, \pi_m{}^\gamma]$. Obviously $\nu^\gamma \in \underset{\sim}{N}(V)$. The property of Algorithm 2·5 described in Lemma 2·8 has immediate consequences for $T(G, \pi)$, as we describe in the next theorem.

2·14 THEOREM *Let* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ *and* $\gamma \in S_n$.

(a) $T(G^\gamma, \pi^\gamma) = T(G, \pi)^\gamma$.

(b) *If* $\nu \in T(G, \pi)$, *then* $T(G^\gamma, \pi^\gamma, \nu^\gamma) = T(G, \pi, \nu)^\gamma$. □

The map from $T(G, \pi)$ to $T(G, \pi)^\gamma$ will not in general preserve the ordering $<$.

We will be particularly interested in permutations $\gamma \in S_n$ such that $G^\gamma = G$ and $\pi^\gamma \in \pi$. In other words, $\gamma \in \mathrm{Aut}(G)_\pi$. If $\nu_1, \nu_2 \in T(G, \pi)$ and $\nu_2 = \nu_1{}^\gamma$ for some $\gamma \in \mathrm{Aut}(G)_\pi$ we write $\nu_1 \sim \nu_2$ and say that ν_1 and ν_2 are *equivalent*. By Theorem 2·14, $\sim$ is an equivalence relation on $T(G, \pi)$. If ν is a terminal node of $T(G, \pi)$ then ν is called an *identity node* if there is no earlier node of $T(G, \pi)$ which is equivalent to ν.

The following theorem is fundamental to our treatment of group actions on $T(G, \pi)$.

2·15 THEOREM *Let* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ *and* $\gamma \in \mathrm{Aut}(G)_\pi$. *Then*

(a) $T(G, \pi)^\gamma = T(G, \pi)$.

(b) *If* $\nu \in T(G, \pi)$, *then* $T(G, \pi, \nu^\gamma) = T(G, \pi, \nu)^\gamma$.

(c) *If* $\nu_1, \nu_2 \in T(G, \pi)$, $\nu_1 < \nu_2$ *and* $\nu_1 \sim \nu_2$, *then* $T(G, \pi, \nu_2 - \nu_1)$ *contains no identity nodes.*

Proof: Assertions (a) and (b) are immediate consequences of Theorem 2·14, so we consider only assertion (c). If $\nu_1 \sim \nu_2$, there is some $\gamma \in \mathrm{Aut}(G)_\pi$ such that $\nu_2 = \nu_1{}^\gamma$. But then $\nu_2 - \nu_1 = (\nu_1 - \nu_2)^\gamma$ and so $T(G, \pi, \nu_2 - \nu_1) = T(G, \pi, \nu_1 - \nu_2)^\gamma$ by (b). However $\nu_1 < \nu_2$ and so $\nu_1 - \nu_2 < \nu_2 - \nu_1$, by Lemma 2·11. Therefore, every terminal node in $T(G, \pi, \nu_2 - \nu_1)$ is equivalent to an earlier terminal node in $T(G, \pi, \nu_1 - \nu_2)$, which proves (c).

2·16 Indicator functions

Let Δ be any linearly ordered set. An *indicator function* is a map
$$\Lambda : \underset{\sim}{G}(V) \times \underset{\sim}{\Pi}(V) \times \underset{\sim}{N}(V) \to \Delta$$
such that $\Lambda(G^\gamma, \pi^\gamma, \nu^\gamma) = \Lambda(G, \pi, \nu)$ for any $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$, $\nu \in T(G, \pi)$ and $\gamma \in S_n$.

Given one indicator function Λ, we can define another indicator function $\underset{\sim}{\Lambda}$ by:

$\underset{\sim}{\Lambda}(G, \pi, \nu) = (\Lambda(G, \pi, \nu^{(1)}), \Lambda(G, \pi, \nu^{(2)}), \cdots, \Lambda(G, \pi, \nu^{(k)}))$,

where $k = |\nu|$, with the lexicographic ordering induced from the ordering of Λ.

2·17 Definition of $\mathcal{C}(G, \pi)$

If $\nu = [\pi_1, \pi_2, \cdots, \pi_m]$ is a terminal node of $T(G, \pi)$ then π_m is a discrete ordered partition, by definition. This means that π_m defines an ordering of the elements of V. We can define a graph $G(\nu)$ isomorphic to G by relabelling the vertices of G in the order that they appear in π_m. More precisely, if $\pi_m = (v_1|v_2| \cdots |v_n)$, and $\delta \in S_n$ is the permutation taking v_i onto i for $1 \leq i \leq n$, then $G(\nu) = G^{\delta}$. The following lemma is an immediate consequence of the definitions.

2·18 LEMMA *If* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$, $\gamma \in S_n$ *and* $\nu \in T(G, \pi)$ *is a terminal node, then* $G(\nu^{\gamma}) = G(\nu)$ *if and only if* $\gamma \in Aut(G)$.

Proof: Let $\nu = [\pi_1, \pi_2, \cdots, \pi_m]$, where $\pi_m = (v_1|v_2| \cdots |v_n)$, and take the permutation $\delta \in S_n$ which takes v_i onto i for $1 \leq i \leq n$. Then $G(\nu) = G^{\delta}$ by definition. Also by definition, $\pi_m^{\gamma} = (v_1^{\gamma}|v_2^{\gamma}| \cdots |v_n^{\gamma})$, and so $G(\nu^{\gamma}) = G^{\gamma^{-1}\delta}$. Therefore $G(\nu) = G(\nu^{\gamma})$ if and only if $G^{\delta} = G^{\gamma^{-1}\delta}$, which is possible if and only if $\gamma \in Aut(G)$. □

Our next requirement is a linear ordering of $\underset{\sim}{G}(V)$. Any such ordering will do, but it will be convenient for us to use an ordering defined using the adjacency matrices of elements of $\underset{\sim}{G}(V)$. Given $G \in \underset{\sim}{G}(V)$ we can define an integer $n(G)$ by writing down the elements of the adjacency matrix in a row-by-row fashion, and interpreting the result as an n^2-bit binary number. If $G_1, G_2 \in \underset{\sim}{G}(V)$ we can then define $G_1 \le G_2$ if and only if $n(G_1) \le n(G_2)$.

We can at last define $\mathcal{C}(G, \pi)$. Let $X(G, \pi)$ be the set of all terminal nodes of $T(G, \pi)$. Choose an arbitrary (but fixed) indicator function Λ. Let $\Lambda^* = \max\{\underset{\sim}{\Lambda}(G, \pi, \nu) \mid \nu \in X(G, \pi)\}$. Then we define $\mathcal{C}(G, \pi) = \max\{G(\nu) \mid \nu \in X(G, \pi) \text{ and } \underset{\sim}{\Lambda}(G, \pi, \nu) = \Lambda^*\}$.

2·19 THEOREM *$\mathcal{C}$ is a canonical labelling map.*

Proof: We show that $\mathcal{C}$ has Properties C1 - C3 (Section 2·1). Property C1 is true because $G(\nu) \cong G$ for any $\nu \in X(G, \pi)$. Now let $\gamma \in S_n$. By Theorem 2·14 $T(G^\gamma, \pi^\gamma) = T(G, \pi)^\gamma$ and so $X(G^\gamma, \pi^\gamma) = X(G, \pi)^\gamma$. Also, by the definition of indicator function, $\underset{\sim}{\Lambda}(G^\gamma, \pi^\gamma, \nu^\gamma) = \underset{\sim}{\Lambda}(G, \pi, \nu)$ for any $\nu \in X(G, \pi)$. Finally, by the definition of $G(\nu)$, we find that $G^\gamma(\nu^\gamma) = G(\nu)$. Therefore $\mathcal{C}$ has Property C2.

In order to prove Property C3 we must recall Lemma 2·8(a). Together with the fact that any $\nu \in X(G, \pi)$ is a partition nest, this implies that $\mathcal{C}(G, \pi) = G^\delta$ for some $\delta \in S_n$ such that $\pi^\delta = \pi$.

Now suppose that $\mathcal{C}(G, \pi^\gamma) = \mathcal{C}(G, \pi)$ for some $\gamma \in S_n$. Since $\mathcal{C}$ satisfies Property C2, $\mathcal{C}(G, \pi^\gamma) = \mathcal{C}(G^{\gamma^{-1}}, \pi)$. Therefore there are $\alpha, \beta \in S_n$ such that $\pi^\alpha = \pi^\beta = \pi$, $\mathcal{C}(G, \pi^\gamma) = G^{\gamma^{-1}\alpha}$ and $\mathcal{C}(G, \pi) = G^\beta$. The assumption that $\mathcal{C}(G, \pi^\gamma) = \mathcal{C}(G, \pi)$ thus implies that $G^{\gamma^{-1}\alpha} = G^\beta$ and so $\beta\alpha^{-1}\gamma \in \mathrm{Aut}(G)$. Finally, $\pi^{\beta\alpha^{-1}\gamma} = \pi^\gamma$ since $\pi^\beta = \pi^\alpha = \pi$. Therefore $\mathcal{C}$ has Property C3. □

An elementary means of computing $\mathcal{C}(G, \pi)$ is now apparent. Using Algorithm 2·12 we can generate every element of $X(G, \pi)$. We can then identify those $\nu \in X(G, \pi)$ for which $\underset{\sim}{\Lambda}(G, \pi, \nu)$ is maximum and so find $\mathcal{C}(G, \pi)$ from its definition. It is not necessary to store all of $X(G, \pi)$ simultaneously; its elements can be processed as they are generated and then discarded. However, this process is not practical for use with a great many graphs because of the size of $X(G, \pi)$. One problem is with graphs having large automorphism groups. Since Aut(G) acts semi-regularly on $X(G, \pi)$, $|X(G, \pi)|$ must be a multiple of $|\text{Aut}(G)|$, and so can be impossibly large, even for moderate n. Secondly, there are graphs for which $|X(G, \pi)|$ is very large, even if $|\text{Aut}(G)|$ is small. We will meet some of these graphs in the next chapter.

The method which we will use to attack these difficulties is a process of pruning $T(G, \pi)$. Let us say that $\nu \in X(G, \pi)$ is a *canonical* node if $\mathcal{C}(G, \pi) = G(\nu)$. Obviously, any part of $T(G, \pi)$ can be ignored if the remainder is known to contain a canonical node. Our guiding light is the following theorem, which is already implicit in the foregoing.

2·20 THEOREM *Let* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$, *and* $\Lambda^* = \max\{\underset{\sim}{\Lambda}(G, \pi, \nu) | \nu \in X(G, \pi)\}$. *Let* $X^*(G, \pi)$ *be any subset of* $X(G, \pi)$ *which contains those identity nodes* ν *for which* $\underset{\sim}{\Lambda}(G, \pi, \nu) = \Lambda^*$. *Then* $X^*(G, \pi)$ *contains a canonical node.* □

In the terms of Theorem 2·20 our aim will be to reduce the size of $X^*(G, \pi)$ as much as possible. We will reduce the number of elements of $X^*(G, \pi)$ which are not identity nodes by searching for automorphisms of G and employing any we find to delete subtrees of $T(G, \pi)$. We will reduce the number of identity nodes in $X^*(G, \pi)$ by using Λ.

2·21 Using automorphisms to prune $T(G, \pi)$

The existence of one or more automorphisms of G can be inferred during the generation of $T(G, \pi)$ in at least two different ways.

(1) We may find two terminal nodes ν_1, $\nu_2 \in X(G, \pi)$ such that $G(\nu_1) = G(\nu_2)$.

(2) We can sometimes infer the presence of automorphisms from the structure of an equitable partition.

The first case is the more important and will be treated first. The second case can wait until Section 2·24.

Suppose then that during the generation of $T(G, \pi)$ we encounter a terminal node $\nu_2 \in X(G, \pi)$, compute $G(\nu_2)$, and discover that it is the same as $G(\nu_1)$ for some earlier terminal node ν_1. Since ν_1 and ν_2 are terminal nodes, there is a unique permutation $\gamma \in S_n$ such that $\nu_2 = \nu_1^{\gamma}$. It then follows from Lemma 2·18 that $\gamma \in \mathrm{Aut}(G)$. We will call γ an *explicit* automorphism.

Once we have found an explicit automorphism there are several ways we can put it to work. These are based on Theorem 2·15. The immediate outcome of Theorem 2·15 is that we may ignore the remainder of the subtree $T(G, \pi, \nu_2 - \nu_1)$. However, we can do better than that. Since $\mathrm{Aut}(G)$ is a group, not only γ but all its powers are in $\mathrm{Aut}(G)$. Moreover, if we have found several automorphisms of G, any permutation which is generated by these is also in $\mathrm{Aut}(G)$. The following scheme for handling this mass of information is not always the best, but has been found to work very well in many circumstances.

Let $\varepsilon \in X(G, \pi)$ be the earliest terminal node. We will need the following lemma.

2·22 LEMMA *Let* $\nu_1 < \nu_2 \in X(G, \pi)$. *Then* $|\varepsilon - \nu_2| \leq |\nu_1 - \nu_2|$.

Proof: If $|\nu_1 - \nu_2| < |\varepsilon - \nu_2|$, then $\nu_2 \in T(G, \pi, \varepsilon - \nu_1)$, which contradicts the assumption that $\nu_1 < \nu_2$. □

We next introduce an auxiliary partition $\theta \in \Pi(V)$. We initially set θ to the discrete partition of V, and whenever we obtain an explicit automorphism γ, we update $\theta \leftarrow \theta \vee \theta(\gamma)$. This means, by Lemma 1·13, that θ is at every stage the orbit partition of the group generated by all the explicit automorphisms so far discovered. It also means that $\theta \leq \theta(\mathrm{Aut}(G)_{\pi_m})$, where $[\pi_1, \pi_2, \cdots, \pi_m]$ is any common ancestor of all the terminal nodes we have yet considered. This is because a permutation taking one node to another fixes their common ancestors.

Now consider a node $\nu = [\pi_1, \pi_2, \cdots, \pi_m]$ which is an ancestor of ε. Because of the definition of ε, ν is also an ancestor of all the terminal nodes generated so far. Let $W = \{v_1, v_2, \cdots, v_k\}$ be the first non-trivial cell of smallest size of π_m, where $v_1 < v_2 < \cdots < v_k$. Since $\theta \leq \pi_m$, θ induces a partition of W. Now the successors of ν, in the order earliest to latest, are $\nu(v_1), \nu(v_2), \cdots, \nu(v_k)$, where $\nu(v_i) = (\pi_1 \geq \pi_2 \geq \cdots \geq \pi_m \geq \pi_m \perp v_i)$. If $v_i < v_j$ are in the same cell of θ, there is some automorphism γ, generated by the explicit automorphisms so far discovered, such that $\nu(v_j) = \nu(v_i)^\gamma$. Therefore we can exclude the subtree $T(G, \pi, \nu(v_j))$ from further examination. There are two ways of doing this. The first is that, as we generate successive subtrees $T(G, \pi, \nu(v_1))$, $T(G, \pi, \nu(v_2))$, $\cdots$ we only consider those for which $v_i \in \mathrm{mcr}(\theta)$. The second is that, upon discovering an explicit automorphism γ during the generation of $T(G, \pi, \nu(v_i))$, and updating θ, we check to see if it is still true that $v_i \in mcr(\theta)$. If not, we have found proof (namely γ) that $T(G, \pi, \nu(v_i))$ only contains terminal nodes equivalent to the terminal nodes of some subtree we have already examined. Therefore we can return at once to ν and consider $\nu(v_{i+1})$.

The technique just described often allows us to jump all the way back to an ancestor ν of ε after only generating one terminal node of a subtree rooted at a successor of ν. Unfortunately this is not always possible, for example when a new terminal node is not recognized as being equivalent to an earlier one. It will also be possible (due to the use of Λ - see later) for a whole subtree to be ignored without knowing it to be equivalent to anything else. In order to put our automorphisms to work in such cases we have devised the following scheme.

Firstly, we maintain a store $\underset{\sim}{S}$ which contains $(fix(\gamma), mcr(\gamma))$ for every explicit automorphism γ so far discovered (or some subset of them). Then, with each non-terminal node $\nu \in T(G, \pi)$ we associate a set $W(\nu) \subseteq V$. The first time (if any) we encounter ν in the search of $T(G, \pi)$, $W(\nu)$ is set equal to the first smallest non-trivial cell of π_m, where $\nu = [\pi_1, \pi_2, \cdots, \pi_m]$. The next time we encounter ν (if any), we redefine $W(\nu) \leftarrow W(\nu) \cap mcr(\gamma_1) \cap mcr(\gamma_2) \cap \cdots \cap mcr(\gamma_r)$, where $\gamma_1, \gamma_2, \cdots, \gamma_r$ are those previously encountered explicit automorphisms which fix ν. From then on we can ignore subtrees $T(G, \pi, \nu(v))$ for which $v \notin W(\nu)$. This is justified by Lemma 1·14. The reasons for deferring the modification of $W(\nu)$ until the second encounter with ν are (i) that the subtree rooted at the earliest successor of ν has to be examined anyway (since the smallest element of $W(\nu)$ before the modification remains in $W(\nu)$ after the modification) and (ii) that there is ofter no second encounter with ν (we may find an automorphism allowing us to jump back to an ancestor of ν). The next lemma shows that we can determine whether γ fixes ν by looking at $fix(\gamma)$.

2·23 LEMMA *Let* γ *be an explicit automorphism. Let* $\nu = [\pi_1, \pi_2, \cdots, \pi_m] \in T(G, \pi)$ *be derived from* G, π *and* $v_1, v_2, \cdots, v_{m-1}$. *Then* γ *fixes* ν *if and only if* $\{v_1, v_2, \cdots, v_{m-1}\} \subseteq fix(\gamma)$.

Proof: The necessity is obvious. To prove the sufficiency we use induction on the ancestors of ν. We know that γ fixes π_1, because π_1 is an ancestor of the two equivalent terminal nodes via which γ was discovered. Now suppose that $\{v_1, v_2, \cdots, v_{m-1}\} \subseteq fix(\gamma)$ and that γ fixes $(\pi_1 \geq \pi_2 \geq \cdots \geq \pi_r)$ for some r $(1 \leq t \leq m - 1)$. Thus $\gamma \in Aut(G)_{\pi_r}$. Furthermore, γ fixes v_r and π_{r+1} is the coarsest equitable partition finer than π_r which fixes v_r. Therefore γ fixes π_{r+1}. □

There is one other circumstance under which we may wish to change $W(\nu)$. If we find two equivalent terminal nodes ν_1, ν_2 where $\nu_2 = \nu_1^{\gamma}$ and where ν is the longest common ancestor of ν_1 and ν_2, we can set $W(\nu) \leftarrow W(\nu) \cap mcr(\gamma)$.

2·24 Implicit automorphisms

There are occasions when we can infer the presence of one or more automorphisms without generating any of them explicitly. These are based on the following lemma.

2.25 LEMMA *Let* $G \in \underset{\sim}{G}(V)$ *and let* $\pi \in \underset{\sim}{\Pi}(V)$ *be equitable with respect to* G. *If* π *has* m *non-trivial cells and either* $n \leq |\pi| + 4$, $n = |\pi| + m$ *or* $n = |\pi| + m + 1$, *then* $\pi_1 = \theta(\mathrm{Aut}(G)_{\pi_1})$ *for any equitable* $\pi_1 \leq \pi$.

Proof: Let $\pi_1 = \{V_1, V_2, \cdots, V_k\}$, where $|V_i| > 1$ for $1 \leq i \leq m$ and $|V_i| = 1$ for $m < i \leq k$. Since π_1 is equitable, there is a set of numbers e_{ij} $(1 \leq i,j \leq k)$ such that each vertex in V_i is adjacent to e_{ij} vertices in V_j. Counting the edges between V_i and V_j we find that

$$|V_i|e_{ij} = |V_j|e_{ji}. \tag{*}$$

Since $0 \leq e_{ij} \leq |V_j|$, (*) implies that $e_{ij} = 0$ or $e_{ij} = |V_j|$ whenever $(|V_i|, |V_j|) = 1$.

If $\pi_1{}^\gamma = \pi_1$ for some $\gamma \in S_n$ and $(|V_i|, |V_j|) = 1$, the permutation γ will preserve the set of edges between V_i and V_j. Therefore, in determining whether or not $\gamma \in \mathrm{Aut}(G)$ we can ignore such edges. In particular we can ignore any edge incident with a vertex in a trivial cell.

If π satisfies the requirements of the theorem, there are seven possibilities for the sizes of the non-trivial cells of π_1. We will treat these separately.

(a) $|V_i| = 2$ for $1 \leq i \leq m$.

Let $V_i = \{v_i, w_i\}$ for $1 \leq i \leq m$. For $1 \leq i < j \leq m$ there are four possibilities for the edges between V_i and V_j. Either there are no such edges, all possible such edges or two such edges. In the last case the edges are either $\{v_i, v_j\}$ and $\{w_i, w_j\}$ or $\{v_i, w_j\}$ and $\{w_i, v_j\}$. Therefore the permutation $\gamma = (v_1\ w_1)(v_2\ w_2)\cdots(v_m\ w_m)$ is in Aut(G), and since $\theta(\gamma) = \pi_1$, $\pi_1 = \theta(\mathrm{Aut}(G)_{\pi_1})$.

(b) $|V_1| = 3$, $|V_i| = 2$ for $2 \le i \le m$.

Let $V_1 = \{v_1, w_1, x_1\}$ and $V_i = \{v_i, w_i\}$ for $2 \le i \le m$. Since $(2, 3) = 1$ we can ignore the edges between V_1 and V_i $(2 \le i \le m)$. Furthermore, V_1 itself either contains no edges or a triangle. Therefore the permutation $\gamma = (v_1\ w_1\ x_1)(v_2\ w_2)(v_3\ w_3)\cdots(v_m\ w_m)$ is in Aut(G).

(c) $m = 1$ and $|V_1| = 3$, 4 or 5.

In any of these cases the required result is a simple corollary of the fact that all regular graphs with 3, 4 or 5 vertices are transitive.

(d) $m = 2$, $|V_1| = 4$ and $|V_2| = 2$.

(e) $m = 2$, $|V_1| = |V_2| = 3$.

Each of these cases is easily settled by considering every possibility for the edges inside or between V_1 and V_2. □

The most commonly occurring case of Lemma 2·25 is when $n = |\pi| + m$, which corresponds to π_1 only having cells of size 1 or 2.

Lemma 2·25 can be put to several uses. The most immediate application is that whenever we encounter a node $\nu = [\pi_1, \pi_2, \cdots, \pi_m]$ for which π_m satisfies the requirements of Lemma 2·25, we can infer that all the terminal nodes descended from ν are equivalent, and so at most one of them is an identity node (the earliest one, if any). A less direct technique is to store the pair $(\mathrm{fix}(\pi_m), \mathrm{mcr}(\pi_m))$ in the list $\underset{\sim}{S}$, along with the similar pairs derived from explicit automorphisms. It can then become useful in pruning later parts of the search tree.

2·26 Eliminating identity nodes

The techniques of the last few sections are generally quite efficient in removing terminal nodes which are not identity nodes. However, there are occasions when the number of identity nodes is unmanageably large. Examples of these will be given in the next chapter. Some of these can be eliminated by means of an indicator function Λ.

Suppose that during the search of $T(G, \pi)$ we maintain a node variable ρ. When the first terminal node ε is generated, we initialize $\rho \leftarrow \varepsilon$. Thereafter we update $\rho \leftarrow \nu$ whenever we find a terminal node ν such that $\underset{\sim}{\Lambda}(G, \pi, \nu) > \underset{\sim}{\Lambda}(G, \pi, \rho)$ or $\underset{\sim}{\Lambda}(G, \pi, \nu) = \underset{\sim}{\Lambda}(G, \pi, \rho)$ and $G(\nu) > G(\rho)$. The definition of $\mathcal{C}(G, \pi)$ ensures that by the time we have finished searching $T(G, \pi)$ we have $G(\rho) = \mathcal{C}(G, \pi)$, provided the set of terminal nodes examined includes all the identity nodes. Now suppose that at some instant during our search we have $\rho = [\pi_1, \pi_2, \cdots, \pi_m]$ and encounter a node $\nu = [\pi'_1, \pi'_2, \cdots, \pi'_k]$, not necessarily terminal. Let $r = \min(m, k)$. Then, if $\underset{\sim}{\Lambda}(G, \pi, \nu^{(r)}) < \underset{\sim}{\Lambda}(G, \pi, \rho^{(r)})$, the definition of an indicator function tells us that $\underset{\sim}{\Lambda}(G, \pi, \nu') < \underset{\sim}{\Lambda}(G, \pi, \rho)$ for every terminal node ν' of $T(G, \pi, \nu)$. Therefore we can safely ignore $T(G, \pi, \nu)$ without miscalculating $\mathcal{C}(G, \pi)$.

The efficiency of this technique depends mainly on two factors. One is the power of $\underset{\sim}{\Lambda}$ in distinguishing between non-equivalent nodes. This, of course, can only be improved by changing Λ, which will generally involve a power/computation-time trade-off. The other factor depends on the initial labelling of G. Suppose that we wish to search the subtree $T(G, \pi, \nu)$. We do this by successively searching the subtrees $T(G, \pi, \nu_1)$, $T(G, \pi, \nu_2)$, $\cdots$, $T(G, \pi, \nu_r)$, where $\nu_1, \nu_2, \cdots, \nu_r$ are the successors of ν, in the order earliest to latest. We can use the information provided by Λ by ignoring the subtree $T(G, \pi, \nu_i)$ if $\Lambda(G, \pi, \nu_i) < \Lambda(G, \pi, \nu_j)$ for some $j < i$. The number of subtrees which are thus ignored could vary from none (if the $\Lambda(G, \pi, \nu_i)$ are in non-decreasing order) to the maximum number possible (if $\Lambda(G, \pi, \nu_i) \le \Lambda(G, \pi, \nu_1)$ for $1 \le i \le r$). While there is no efficient way of ensuring that the best case always occurs we can arrange for the worst case to be very unlikely. The simplest way of doing this (but not the one we will adopt) is to label G in a random fashion before commencing the generation of $T(G, \pi)$. A precise statistical analysis of how this effects the overall efficiency would be very difficult, but a rough idea can perhaps be gained from the following two theorems. We will use $E(X)$ to denote the expectation of a random variable X, and $P(x)$ to denote the probability of an event x. The first theorem suggests that the number of ignored subtrees will not usually be much less than the maximum number possible.

2·27 THEOREM *Let* $\delta_1 < \delta_2 < \cdots < \delta_k$ *be elements of a linearly ordered set* Δ. *Let* $m_1, m_2, \cdots, m_k$ *be positive integers, and put* $\ell = m_1 + m_2 + \cdots + m_k$. *Let* $x_1, x_2, \cdots, x_\ell$ *be elements of* Δ, *exactly* m_j *of which are equal to* δ_j *for* $1 \le j \le k$. *Now permute the* x_i *at random to get* $x^{(1)}, x^{(2)}, \cdots, x^{(\ell)}$, *each of the* $\ell!$ *possible permutations being equally likely. For* $1 \le i \le \ell$, *<u>mark</u>* $x^{(i)}$ *if* $x^{(i)} \ge x^{(j)}$ *for* $j < i$, *but* $x^{(i)} \ne \delta_k$. *Let* M *be the number of marked elements. Then* $E(M) = \sum_{j=1}^{k-1} \frac{m_j}{1 + m_{j+1} + m_{j+2} + \cdots + m_k}$, *where the sum is taken as* 0 *if* $k = 1$.

In particular, if $m_j = m$ *for* $1 \le j \le k$, *then*

$$E(M) = \sum_{j=1}^{k-1} \frac{m}{mj + 1} \le \log(2k).$$

Proof: By the additivity of expectation,

$$E(M) = \sum_{i=1}^{\ell} P(x_i \text{ is marked})$$
$$= \sum_{j=1}^{k} m_j p_j, \text{ where } p_j = P(\text{a given element equal to } \delta_j \text{ is marked})$$
$$= \sum_{j=1}^{k-1} m_j p_j, \text{ since } p_k = 0.$$

Now suppose $x^{(i)} = \delta_j$, where $j \neq k$. Let $x_{(1)}, x_{(2)}, \cdots, x_{(t)}$ be the $t = m_{j+1} + m_{j+2} + \cdots + m_k$ elements greater than δ_j. Then each of the $(t + 1)!$ possible relative orders in which the elements $x^{(i)}, x_{(1)}, \cdots, x_{(t)}$ occur in the sequence $x^{(1)}, x^{(2)}, \cdots, x^{(\ell)}$ are equally likely, but only the $t!$ orders for which $x^{(i)}$ is first result in $x^{(i)}$ being marked. Therefore $p_j = \frac{t!}{(t+1)!} = \frac{1}{t+1}$, as required. □

The second theorem concerns the number of different values of $\Lambda(G, \pi, \nu_i)$ amongst those $T(G, \pi, \nu_i)$ which are not ignored. It therefore has a bearing on the number of identity nodes which are excluded by means of Λ.

2·28 THEOREM *Under the conditions of Theorem* 2·27, *let* N *be the number of different values amongst the marked elements. Then*

$$E(N) = \sum_{j=1}^{k-1} \frac{m_j}{m_j + m_{j+1} + \cdots + m_k}, \text{ where the sum is } 0 \text{ if } k = 1.$$
$$\leq \log \ell$$

In particular, if $m_i = m$ *for* $1 \leq i \leq k$, *then*

$$E(N) = \sum_{j=2}^{k} \frac{1}{j} \leq \log k.$$

Proof: The proof of the exact expression for $E(N)$ is nearly the same as the proof of Theorem 2·27 and so will be omitted.

We will prove the bound $E(N) \leq \log \ell$ by induction on k. It is obviously true for $k = 1$. Now let $k > 1$.

Then $E(N) = \sum_{j=1}^{k-1} \frac{m_j}{m_j + m_{j+1} + \cdots + m_k} = \frac{m_1}{\ell} + \sum_{j=2}^{k-1} \frac{m_j}{m_j + m_{j+1} + \cdots + m_k}$

$\leq \frac{m_1}{\ell} + \log(\ell - m_1)$, by the induction hypothesis

$= \log \ell + \frac{m_1}{\ell} + \log(1 - \frac{m_1}{\ell})$

$\leq \log \ell$, since $0 < \frac{m_1}{\ell} < 1$. □

An alternative to this technique for using Λ is to compute $\Lambda(G, \pi, \nu_i)$ for $1 \leq i \leq r$ and then only search $T(G, \pi, \nu_i)$ for those ν_i for which $\Lambda(G, \pi, \nu_i)$ is the largest. This is undoubtedly the best approach in many cases. However we are not adopting this method because it severely degrades the average-case behaviour. This is because the discovery of automorphisms frequently allows us to reject a subtree $T(G, \pi, \nu_i)$ without ever computing ν_i.

The theorems above relate to the effect of performing an initial random relabelling of G. The reasons we are not adopting this approach are, firstly, that this relabelling may almost double the total execution time (for a very large random graph; see Chapter 3) and, secondly, that in order to make some of the output useful (e.g. the list of automorphisms produced) it may be necessary to translate it back to the original labelling, which is inconvenient. We will describe an alternative, but will only justify it qualitatively. A more precise analysis would be impossibly difficult to perform.

Let $\Lambda' : \underset{\sim}{G}(V) \times \underset{\sim}{\Pi}(V) \times \underset{\sim}{N}(V) \rightarrow \Delta$ be any convenient indicator function. Now devise a map $f : \Delta \rightarrow \Delta$ with the property that for pairs $x, y \in \Delta$, $x - y$ is very poorly correlated with $f(x) - f(y)$. (This is not meant to be a rigorous definition). For example, take $\Delta = [-1, 1]$ and $f(x) = \sin(10^{10}x)$; knowledge of $x - y$ tells us almost nothing about $f(x) - f(y)$ except in special circumstances. Now define $\Lambda : \underset{\sim}{G}(V) \times \underset{\sim}{\Pi}(V) \times \underset{\sim}{N}(V) \rightarrow \Delta$ by $\Lambda(G, \pi, \nu) = f(\Lambda'(G, \pi, \nu))$. The hope is that any tendency to an unfavourable ordering of the values of $\Lambda'(G, \pi, \nu_1), \cdots, \Lambda'(G, \pi, \nu_r)$ will not occur for $\Lambda(G, \pi, \nu_1), \cdots, \Lambda(G, \pi, \nu_r)$. However, as we have stated, there is little hope of an exact statistical analysis. The best we can say is that the computational experience is favourable.

2·29 Storage of identity nodes

Up to this point we have been tacitly assuming that we are keeping a record of all those identity nodes so far generated, so that we can recognize later terminal nodes which are equivalent to any of them. In practice this can cause a severe storage problem, since the number of identity nodes can be very large, even if we don't count those which are eliminated by use of an indicator function. Therefore it is necessary to put a limit on the number of identity nodes (strictly, terminal nodes not known to be equivalent to an earlier node) to be stored. The optimum strategy is not clear. On the one hand, storing more identity nodes improves our chances of detecting automorphisms, which can be put to use as we have seen. On the other hand, testing two terminal nodes for equivalence is quite time consuming (especially for large graphs), and having to do a lot of these tests would have a very bad effect on the overall execution time.

The technique which we have adopted, without a great deal of theoretical justification, is to store two identity nodes at a time. The earliest terminal node ε is always stored. The other terminal node (which may be the same as the first) is our best guess so far at the identity node corresponding to $\mathcal{C}(G)$. This is the node ρ

referred to in Section 2·26. We also permit the algorithm to search for terminal nodes equivalent to ε, with the aim of using the automorphisms thus discovered to shorten the total amount of work. This will sometimes degrade the performance somewhat, but on the average it works very well.

We are now able to summarize the way in which terminal nodes are processed. Suppose that we have just created a node ν, not necessarily terminal, which is not an ancestor of ε (i.e. is later than ε).

The node ρ and the partition θ have the same interpretation as before. Suppose that ν is the node $[\pi_1, \pi_2, \cdots, \pi_m]$ so that $|\nu| = k$. Also define $m = |\varepsilon|$ and $r = |\rho|$, and define variables as follows.

hh: If π_k satisfies the requirements of Lemma 2·25, then hh is the smallest value of i, $1 \le i \le k$, for which π_i satisfies these requirements.
Otherwise, hh = k.

ht: This is the smallest value of i, $1 \le i \le m$, for which all the terminal nodes descended from or equal to $\varepsilon^{(i)}$ have been shown to be equivalent.

h: The longest common ancestor of ε and ν is $\nu^{(h)}$.

v: $\pi_{h+1} = \pi_h \perp v$

hb: The longest common ancestor of ρ and ν is $\nu^{(hb)}$.

hzb: This is the maximum value of i, $1 \le i \le \min\{k, r\}$, such that $\underset{\sim}{\Lambda}(G, \pi, \nu^{(i)}) = \underset{\sim}{\Lambda}(G, \pi, \rho^{(i)})$.

By *returning to* $\nu^{(i)}$ we mean backtracking in the search tree to $\nu^{(i)}$ and proceeding with the next successor of $\nu^{(i)}$ not yet generated, if any. If there are no such successors, we return to $\nu^{(i-1)}$, and so forth. "*Return to* $\nu^{(0)}$" is equivalent to "*stop*".

Now suppose we have just created $\nu = \nu^{(k)}$. Let $\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}(G, \pi, \nu)$

(1) If ($k > m$ *or* $\underset{\sim}{\Lambda} \neq \underset{\sim}{\Lambda}(G, \pi, \varepsilon^{(k)})$
and ($k > r$ *or* $\underset{\sim}{\Lambda} < \underset{\sim}{\Lambda}(G, \pi, \rho^{(k)})$), go to (B).

(2) If ν is non-terminal, proceed to search $T(G, \pi, \nu)$.

(3) If ($k > m$ *or* $\underset{\sim}{\Lambda} \neq \underset{\sim}{\Lambda}(G, \pi, \varepsilon)$) go to (4).
If the permutation γ taking ε onto ν is an automorphism, go to (A).

(4) If ($k > r$ *or* $\underset{\sim}{\Lambda} < \underset{\sim}{\Lambda}(G, \pi, \rho)$ *or*
($\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}(G, \pi, \rho)$ *and* $G(\nu) < G(\rho)$)) go to (B).
If ($\underset{\sim}{\Lambda} > \underset{\sim}{\Lambda}(G, \pi, \rho)$ *or* ($\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}(G, \pi, \rho)$ *and* $G(\nu) > G(\rho)$)) set $\rho \leftarrow \nu$ then go to (B).
If ($\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}(G, \pi, \rho)$ *and* $G(\nu) = G(\rho)$), let γ be the permutation taking ρ onto ν and go to (A).

(A) {At this stage we have found an automorphism γ.}

(A·1) Add ($fix(\gamma)$, $mcr(\gamma)$) to $\underset{\sim}{S}$ (if there is room)
Set $\theta \leftarrow \theta \vee \theta(\gamma)$.

(A·2) If $v \notin mcr(\theta)$, return to $\nu^{(h)}$.
Otherwise, return to $\nu^{(hb)}$.

(B) {At this stage we have a terminal node ν not known to be equivalent to an earlier terminal node.}

(B·1) If $hh < k$, add ($fix(\pi_{hh})$, $mcr(\pi_{hh})$) to $\underset{\sim}{S}$ (if there is room).

(B·2) Return to $\nu^{(i)}$, where $i = \min\{hh-1, \max\{ht-1, hzb\}\}$. □

The only feature in the foregoing informal algorithm which we have not already justified is the use of the variable ht in Step (B·2). What we want to do in Step (B·2) is to return to the longest ancestor ν_i of ν which may conceivably have a terminal descendant which is either equivalent to ε or improves on ρ as the "best canonical label so far". All the terminal nodes in $T(G, \pi, \nu^{(hh)})$ are known to be equivalent to ν, so we can assume that $i < hh$. Furthermore, if $i > hzb$, none of the descendants of $\nu^{(i)}$ can improve on ρ. Finally, if $i \geq ht$, and one of the descendants of $\nu^{(i)}$ was equivalent to ε then $\nu^{(i)}$ would be equivalent to $\varepsilon^{(i)}$. However, all the terminal nodes descended from $\varepsilon^{(i)}$ are equivalent, and so all those descended from $\nu^{(i)}$ are equivalent, giving a contradiction.

2·30 We will now give a complete formal description of the whole algorithm.

Notes: (i) *lab* and *dig* are boolean variables. If *lab* = false, ρ is not used, and the algorithm only searches for terminal nodes equivalent to ε. We will show in Theorem 2·33 that useful information about Aut(G) is still obtained. If *dig* = true, the algorithm will not use Lemma 2·25, and will be valid for digraphs and graphs with loops (for which Lemma 2·25 does not hold).

(ii) The variable ν refers everywhere to the node $[\pi_1, \pi_2, \cdots, \pi_k]$. It thus changes value if π_i $(1 \leq i \leq k)$ or k changes value.

(iii) $L \geq 1$ is an integer specifying a limit on the number of pairs (fix(x), mcr(x)) to be stored at one time. The result computed by the algorithm is independent of the choice of L, although the efficiency in general may not be.

(iv) $P \subseteq \underset{\sim}{\Pi}(V)$ is the set of all ordered partitions of V which satisfy the requirements of Lemma 2·25.

(v) We are assuming for convenience that $\Lambda(G, \pi, \nu)$ is real in value. If this is not the case replace

$$\text{"qzb} \leftarrow \Lambda_k - zb_k\text{"} \quad \text{by}$$

$$qzb \leftarrow \begin{cases} -1 & \text{if } \Lambda_k < zb_k \\ 0 & \text{if } \Lambda_k = zb_k \\ 1 & \text{if } \Lambda_k > zb_k \end{cases}$$

2·31 ALGORITHM

(1) $k \leftarrow size \leftarrow 1$

$h \leftarrow hzb \leftarrow index \leftarrow \ell \leftarrow 0$

$\theta \leftarrow$ discrete partition of V

$\pi_1 \leftarrow \boldsymbol{R}(G, \pi, \pi)$

If ($\pi_1 \in P$ *and* not *dig*) $hh \leftarrow 1$, otherwise $hh \leftarrow 2$

If (π_1 is discrete) go to (18)

$W_1 \leftarrow$ first smallest cell of π_1

$v_1 \leftarrow \min W_1$

$e_1 \leftarrow 0$

$\Lambda_1 \leftarrow 0$

(2) $k \leftarrow k + 1$

$\pi_k \leftarrow \pi_{k-1} \perp v_{k-1}$

$\Lambda_k \leftarrow \Lambda(G, \pi, \nu)$

If ($h = 0$) go to (5)

If ($hzf = k - 1$ *and* $\Lambda_k = zf_k$) $hzf \leftarrow k$

If (not *lab*) go to (3)

$qzb \leftarrow \Lambda_k - zb_k$

If ($hzb = k - 1$ *and* $qzb = 0$) $hzb \leftarrow k$

If ($qzb > 0$) $zb_k \leftarrow \Lambda_k$

(3) If (hzb = k *or* (*lab and* qzb ≥ 0)) go to (4)

Go to (6)

(4) If (π_k is discrete) go to (7)

$W_k \leftarrow$ first smallest cell of π_k

$v_k \leftarrow \min W_k$

If (*dig or* $\pi_k \notin P$) hh ← k + 1

$e_k \leftarrow 0$

Go to (2)

(5) $zf_k \leftarrow zb_k \leftarrow \Lambda_k$

Go to (4)

(6) k' ← k

k ← min(hh − 1, max(ht − 1, hzb))

If (k' = hh) go to (13)

$\ell \leftarrow \min(\ell + 1, L)$

$\Lambda_\ell \leftarrow \text{mcr}(\pi_{hh})$

$\Phi_\ell \leftarrow \text{fix}(\pi_{hh})$

Go to (12)

(7) If (h = 0) go to (18)

If (k ≠ hzf) go to (8)

Define $\gamma \in S_n$ by $\varepsilon^\gamma = \nu$

If ($G^\gamma = G$) go to (10)

(8) If (not *lab or* qzb < 0) go to (6)

If (qzb > 0 *or* k < $|\rho|$) go to (9)

If ($G(\nu) > G(\rho)$) go to (9)

If ($G(\nu) < G(\rho)$) go to (6)

Define $\gamma \in S_n$ by $\nu^\gamma = \rho$

Go to (10)

(9) $\rho \leftarrow \nu$

$qzb \leftarrow 0$

$hb \leftarrow hzb \leftarrow k$

$zb_{k+1} \leftarrow \infty$

Go to (6)

(10) $\ell \leftarrow \min(\ell + 1, L)$

$\Omega_\ell \leftarrow mcr(\gamma)$

$\Phi_\ell \leftarrow fix(\gamma)$

If $(\theta^\gamma = \theta)$ go to (11)

Output γ

If $(tvc \in mcr(\theta))$ go to (11)

$k \leftarrow h$

Go to (13)

(11) $k \leftarrow hb$

(12) If $(e_k = 1)$ $W_k \cap \Omega_\ell$

(13) If $(k = 0)$ *Stop*

If $(k > h)$ go to (17)

If $(k = h)$ go to (14)

$h \leftarrow k$

$tvc \leftarrow \min(W_k)$

$tvh \leftarrow tvc$

(14) If (v_k and tvh are in the same cell of θ) index $\leftarrow$ index + 1

$v_k \leftarrow \min\{v \in W_k | v > v_k\}$

If $(v_k = \infty)$ go to (16)

If $(v_k \notin mcr(\theta))$ go to (14)

(15) hh ← min(hh, k + 1)

hzf ← min(hzf, k)

If (not *lab* *or* hzb < k) go to (2)

hzb ← k

qzb ← 0

Go to (2)

(16) If ($|W_k|$ = index *and* ht = k + 1) ht ← k

size ← size × index

index ← 0 ← (A)

k ← k − 1

Go to (13)

(17) If ($e_k = 0$) set $W_k \leftarrow W_k \cap \Omega_i$ for each i, $1 \le i \le \ell$,

such that $\{v_1, v_2, \cdots, v_{k-1}\} \subseteq \Phi_i$

$e_k \leftarrow 1$

$v_k \leftarrow \min\{v \in W_k | v > v_k\}$

If ($v_k \neq \infty$) go to (15)

k ← k − 1

Go to (13)

(18) h ← ht ← hzf ← k

$zf_{k+1} \leftarrow \infty$

$\varepsilon \leftarrow \nu$ ← (B)

k ← k − 1

If (not *lab*) go to (13)

$\rho \leftarrow \nu$

hzb ← hb ← k + 1

$zb_{k+2} \leftarrow \infty$

qzb ← 0

Go to (13)

2·32 Consider the stage during the execution of Algorithm 2·31 that we pass the point marked B (in Step (18)). At this instant define $K = k - 1$ and $w_i = v_i$ $(1 \le i \le K)$.

Now let $\Gamma^{(0)} = \Gamma = \mathrm{Aut}(G)_\pi$, and define $\Gamma^{(i)} = \Gamma_{\{w_1, w_2, \cdots, w_i\}}$ (point-wise stabiliser) for $1 \le i \le K$. Since ε is a terminal node, the coarsest equitable partition which is finer than π and fixes $w_1, w_2, \cdots, w_k$ is discrete. Therefore $\Gamma^{(K)} = 1$.

2·33 THEOREM *During the execution of Algorithm* 2·31, *each time we pass point* A (*in Step* (16)) *or point* B (*in Step* (18)) *the following are true:*

(i) index = $|\Gamma^{(k-1)}|/|\Gamma^{(k)}|$ (point A only)

(ii) size = $|\Gamma^{(k-1)}|$

(iii) $\theta = \theta(\Gamma^{(k-1)})$

(iv) $\Gamma^{(k-1)} = \langle Y \rangle$, *where* Y *is the set of all automorphisms "output" up to the present stage* (*in Step* (10)).

(v) $|Y| \le n - |\theta|$

Proof: The theorem follows readily from the theory that we have already discussed, so we will only describe briefly how this needs to be assembled.

Point B is only passed once, when ε is created, and $k = K + 1$ at this stage. Point A is then passed K times, at which stages k has the values $K, K - 1, \cdots, 1$ in that order.

We prove the theorem by backward induction on k. For $k = K + 1$ it is obvious. Now assume it for k', for some k', $2 \le k' \le K + 1$, and let $k = k' - 1$.

Consider $\nu = [\pi_1, \pi_2, \cdots, \pi_k]$. The successors of ν, in the order earliest to latest are $\nu_1, \nu_2, \cdots, \nu_m$ where $\nu_i = \nu(w_i)$, and $W_k = \{w_1, w_2, \cdots, w_m\}$. The previous time we passed point A (or B) was when we completed our examination of the subtree $T(G, \pi, \nu_1)$. We now claim that, for $1 \leq i \leq m$, by the time we have completed examination of $T(G, \pi, \nu_i)$, w_i is in the same cell of θ as ν_1 if and only if $\nu_i \sim \nu_1$.

Suppose on the contrary that there is an earliest ν_i for which our assertion is not true. If ν_i is not equivalent to ν_1 then w_i and w_1 are obviously in different cells of θ, since θ is the orbit partition of some subgroup of $Aut(G)_{\pi_k}$. On the other hand, if $\nu_i \sim \nu_1$, $T(G, \pi, \nu_i)$ contains one or more terminal nodes equivalent to ε. The nature of the algorithm is such that if one of these nodes is generated, it will be recognized as being equivalent to ε, and if it is not generated this will only be because it has been shown to be equivalent to an earlier terminal node. Furthermore, implicit automorphisms are never used to reduce W_k, and during the examination of $T(G, \pi, \nu_i)$, if any, the only stored pairs (Φ_j, Ω_j) which are used to reduce any W_r have $w_i \in \Phi_j$. Therefore, either w_i is already in the same cell of θ as w_1 or we are sure to discover some automorphism γ such that $\nu_i^{\gamma} < \nu_i$. By the induction hypothesis w_i^{γ} is the same of θ as w_1, and so the update $\theta \leftarrow \theta \vee \theta(\gamma)$ merges the cells of θ containing w_1 and w_i, contrary to hypothesis. Note also that we have just proved that $\gamma \in Y$.

We have thus concluded that the cell of θ containing w_1 is the orbit of $\Gamma^{(k-1)}$ containing w_1. Since $\theta = \theta(Y)$ by construction, and $\Gamma^{(k)} \leq \langle Y \rangle$ by the original induction hypothesis, we must have $\Gamma^{(k-1)} = \langle Y \rangle$, since $\langle Y \rangle$ contains a full set of cosetrepresentatives for $\Gamma^{(k)}$ in $\Gamma^{(k-1)}$. This proves that $\theta = \theta(\Gamma^{(k-1)})$. The variable index merely counts the number of elements in the cell of θ containing w_1, so claims (i) and (ii) follow immediately.

Claim (v) follows from the simple observation that the number of cells of θ starts at n and decreases by at least one for each new element of Y. □

In closing we note a few simple properties of the set of generators of Γ found by Algorithm 2·31. These are essentially the same as those given in Theorems 36 - 38 in [27] and the proofs given there apply with only notational changes. Let Y be the full set of automorphisms "output" by Algorithm 2·31, and let $\Gamma = \mathrm{Aut}(G)$.

2·34 THEOREM (1) Y *does not contain any element of the form* $\gamma\delta$, *where* $\gamma, \delta \in \Gamma$, $\mathrm{supp}(\gamma) \cap \mathrm{supp}(\delta) = \emptyset$ *and* $\gamma \neq (1) \neq \delta$.

(2) *Suppose that for some subset* $Y^* \subseteq Y$, *we have* $\langle Y^* \rangle = \Lambda^{(1)} \oplus \Lambda^{(2)}$, *where* $\Lambda^{(1)}$ *and* $\Lambda^{(2)}$ *are non-trivial subgroups of* Γ. *Then* $Y^* = Y^{(1)} \cup Y^{(2)}$ *where* $Y^{(1)} \cap Y^{(2)} = \emptyset$, $\langle Y^{(1)} \rangle = \Lambda^{(1)}$ *and* $\langle Y^{(2)} \rangle = \Lambda^{(2)}$.

(3) *Suppose that for some subset* $W \subseteq V$ *the point-wise stabiliser* Γ_W *has exactly one non-trivial orbit. Then some subset of* Y *generates a conjugate of* Γ_W *in* Γ. □

CHAPTER THREE

IMPLEMENTATION CONSIDERATIONS

In this chapter we will discuss some of the problems that arise in the implementation of Algorithm 2·31 and how these have been approached. We will then examine the theoretical and empirical performance of our implementation. Finally, we will mention a few of practical uses to which our implementation has been put. The notation we have devised in Chapter 2 will continue to apply here.

3·1 Time versus storage

The program described in McKay [28] worked so efficiently for many classes of graphs that the practical limit on the size of graph that could be processed was set by the amount of storage available, rather than by execution time considerations. Consequently the present implementation places considerably more emphasis on storage conservation, in some places to the slight detriment of time efficiency.

The variable types used by Algorithm 2·31 include graphs, sets, partitions and partition nests. We will now describe the data structures used in our implementation for each of these variable types.

3·2 Partition nests

Let $\nu = [\pi_1, \pi_2, \cdots, \pi_k] \in \underset{\sim}{N}(V)$. Then ν can be represented by two arrays a and b of length n as follows. Define $\pi_0 = (V)$.

(i) The array a contains the elements of V in any order consistent with π_k. Precisely, if $u(a(i), \pi_k) < u(a(j), \pi_k)$ then $i < j$, for any $i, j \in V$.

(ii) Each entry of b is an integer in the interval $[0, n + 1]$ chosen thus:

(a) If $u(a(i), \pi_k) = u(a(i + 1), \pi_k)$, then $b(i) = n + 1 \quad (1 \le i \le n - 1)$.

(b) If $u(a(i), \pi_{j-1}) = u(a(i + 1), \pi_{j-1})$ but $u(a(i), \pi_j) < u(a(i + 1), \pi_j)$, then $b(i) = j \quad (1 \le j \le k, 1 \le i \le n - 1)$.

(c) $b(n) = 0$.

The three main operations on a partition nest that are required by Algorithm 2·31 can be performed as follows.

(1) *To determine* π_j $(1 \le j \le k)$: Let $i_1 < i_2 < \cdots < i_r$ be all the values of i such that $b(i) \le j$. Define $i_0 = 0$. Then $\pi_j = (V_1, V_2, \cdots, V_r)$, where $V_\ell = \{a(i) | i_{\ell-1} + 1 \le i \le i_\ell\}$.

(2) *To replace* ν *by* $\nu^{(j)}$ $(1 \le j < k)$: Change each $b(i) > j$ to $n + 1$, for $1 \le i \le n$.

(3) *To extend* ν *by cell subdivision*: Suppose we wish to update ν to $[\pi_1, \pi_2, \cdots, \pi_{k+1}]$, where π_{k+1} is formed from π_k by subdividing a cell $V_i \in \pi_k$ into disjoint subsets $W_1, W_2, \cdots, W_s$. The elements of V_i are $a(j), a(j + 1), \cdots, a(j + t - 1)$ for some j, where $t = |V_i|$. Permute these t elements of a into any order consistent with π_{k+1} and then set the appropriate $t - 1$ elements of b to $k + 1$ (so that the result is a correct representation of $[\pi_1, \pi_2, \cdots, \pi_{k+1}]$).

3·3 Unordered partitions

The only unordered partition used by Algorithm 2·31 is θ. For any $v \in V$ let θ_v denote the cell of θ containing v and let $p(v) = \min \theta_v$. Clearly θ can be uniquely represented by the array p, and most of the necessary questions about θ can be answered very quickly by reference to p. For example, if $v, w \in V$ then v and w are in the same cell of θ if and only if $p(v) = p(w)$, and $v \in mcr(\theta)$ if and only if $p(v) = v$.

This representation of θ suffers from the disadvantage that updates of the form $\theta \leftarrow \theta \vee \theta(\gamma)$, for $\gamma \in S_n$, are quite expensive in terms of computation time. This problem has been considerably alleviated by the use of a second array q which "chains together" the elements of each cell. More precisely, if $i \in mcr(\theta)$, then $\theta_i = \{i, q(i), q(q(i)), q(q(q(i))), \cdots\}$, where the sequence terminates on the term before the first zero.

Suppose that we wish to merge the cells $\theta_i \neq \theta_j$ of θ, where we can assume that $p(i) < p(j)$. This operation can easily be performed as follows.

(a) $i' \leftarrow i$

(b) Repeat $i' \leftarrow q(i')$ until $q(i') = 0$.

(c) $q(i') \leftarrow j' \leftarrow p(j)$

(d) Repeat $p(j') \leftarrow p(i)$ and $j' \leftarrow q(j')$ until $j' = 0$.

The representation we have chosen for θ may not be the most efficient possible but, since we know of no graphs for which our implementation of Algorithm 2·31 spends more than a small fraction of the total time in manipulating θ, we have felt no need to improve it.

3·4 Sets

The sets used by Algorithm 2·31 are all subsets of V, namely W_i, Φ_i and Ω_i for each i. These can be represented in the computer by *bit-vectors*. A bit-vector is a set of n (generally contiguous) machine bits designated bit(1) to bit(n). A set $W \subseteq V$ can be represented by a bit-vector with bit(i) = 1 if $i \in W$ and bit(i) = 0 otherwise ($1 \le i \le n$). The most obvious advantage of this representation is its storage economy. The other main advantage is that many elementary set operations (such as intersection) and relational tests (such as subset) can be done very quickly using the bit-wise boolean operations available on most machines. On the other hand testing whether $i \in W$ can be annoyingly awkward, especially if the bit-vector extends over more than one machine word, since several arithmetic operations may be required to locate bit(i).

3·5 Graphs

Algorithm 2·31 requires the input graph G and, for reasonably efficient operation, requires the graph variable $G(\rho)$. From the great number of possible ways of representing these graphs in the computer, we have chosen an adjacency matrix representation because of its greater storage economy. More precisely, G is stored as a list of n bit-vectors representing N(1, G), N(2, G), ···, N(n, G), and so requires around n^2 bits of storage. Since Algorithm 2·31 is valid also for digraphs, it is clearly not possible to reduce this storage requirement in general. However if the program was only intended to be applied on graphs with very low degree, a different sort of representation would save space, and probably time as well.

3·6 Efficiency of Algorithm 2·5

Algorithm 2·5 can easily be implemented using the data structures above. We will now consider the efficiency which can be achieved in such an implementation, but first we need to consider an associated sorting problem.

Suppose that we have an array a(1), a(2), ···, a(m) taking values from V and a map f: V → {0, 1, 2, ···, k}. We wish to sort the values of the array a so that f(a(1)) ≤ f(a(2)) ≤ ··· ≤ f(a(m)). This can be done by the following algorithm, using an auxiliary array c(0), c(1), ···, c(k). The time requirement of the algorithm is clearly O(m + k).

(1) c(j) ← 0 for 0 ≤ j ≤ k
    c(f(a(i))) ← c(f(a(i))) + 1 for 1 ≤ i ≤ m
    i ← 1
    For j = 0, 1, ···, k do i' ← i + c(j), c(j) ← i and i ← i'
    i ← 1

(2) x ← a(i)

(3) If (x < 0) go to (4)
    i' ← c(f(x))
    c(f(x)) ← c(f(x)) + 1
    x' ← a(i')
    a(i') ← -x
    x ← x'
    Go to (3)

(4) i ← i + 1
    If (i ≤ m) go to (2)
    a(i) ← -a(i) for 1 ≤ i ≤ m □

The following complexity result was suggested by a related result in Gries [16]. For the necessary definitions, refer back to Section 2·9.

3·7 THEOREM *For any* $G \in \underset{\sim}{G}(V)$, $\pi \in \underset{\sim}{\Pi}(V)$ *and distinct* $v_1, v_2, \cdots, v_{m-1} \in V$, *the derived partition nest* $[\pi_1, \pi_2, \cdots, \pi_m]$ *can be computed in* $O(n^2 \log n)$ *time, assuming an implementation in which* $d(v, W)$ *can be computed in time proportional to* $|W|$, *for any* $v \in V$, $W \subseteq V$.

Proof: It is obvious that the time occupied in the computation of $\pi_i \circ v_i$ for $1 \leq i \leq m - 1$ and in Step (1) of Algorithm 2·5 will be easily $O(n^2 \log n)$. Since each execution of Step (2) of Algorithm 2·5 requires only a fixed amount of time and leads to an execution of Step (3), we are justified in restricting our attention to Step (3).

For any given W, the necessary r executions of Step (3) can be performed in $O(n|W|)$ time. This follows from the assumption about the computation of $d(v, W)$ and from the algorithm in Section 3·6. Therefore the total time for the computation of $[\pi_1, \pi_2, \cdots, \pi_m]$ is $O(n^2 \log n + n\Sigma|W|)$, where the sum is over all sets assigned to W during any execution of Step (2) (for any execution of Algorithm 2·5).

Let $x \in V$ and consider the real variable q_x, defined at any point of time during any execution of Algorithm 2·5 by $q_x = h_x + \log_2 \ell_x$ Here h_x is the number of sets containing x which have been previously assigned to W during an execution of Step (2), plus the number of sets W_j $(m \leq j \leq M)$ which contain x, plus one for the set $\{x\} = \{v_i\}$ created by the operation $\pi_i \circ v_i$, if it exists and has not already been counted. Also ℓ_x is the current size of the cell of $\tilde{\pi}$ which contains x. Note that h_x, ℓ_x and q_x are variables which frequently change value during Algorithm 2·5.

The value of q_x clearly remains constant or decreases between different executions of Algorithm 2·5. The only other place where it can change is during Step (3), when h_x remains fixed while ℓ_x decreases, or h_x increases by one. In the latter case ℓ_x decreases by at least a factor of two, so that q_x does not increase. Therefore q_x is non-increasing throughout the computation, implying that its last value is bounded above by its first, which is bounded above by $2 + \log_2 n$. Therefore the final value $\bar{h}_x$ of h_x is at most $2 + \log_2 n$.

We conclude that the total time required for the computation of $[\pi_1, \pi_2, \cdots, \pi_m]$ is $O(n^2 \log n + n \sum_{x \in V} \bar{h}_x) = O(n^2 \log n)$, as required. □

For our particular choice of data structures, and our particular implementation environment, we have found that the fastest way to compute d(v, W) for $n/30 \leq |W| \leq n$ approximately is to represent W as a bit-vector and to count the number of one-bits in the bit-vector representing $N(v, G) \cap W$. Although this technique (used for $|W| > 1$) appears to reduce the total time in "the majority" of cases, it has the unfortunate side-effect of invalidating the premises of Theorem 3·7. The best replacement for the bound $O(n^2 \log n)$ which we have been able to prove is $O(n^3)$. Since the time required for the computation of d(v, W) is now essentially independent of $|W|$, Step (3) of Algorithm 2·5 can be simplified by using t = 1. This is especially convenient if the sequence α is represented as a set of pointers to the array a (see Section 3·2).

3·8 Efficiency of Algorithm 2·31

Let $T^*(G, \pi)$ be the portion of the search tree $T(G, \pi)$ which is examined by Algorithm 2·31. Let m_1 be the number of terminal nodes of $T^*(G, \pi)$ which are equivalent to the earliest terminal node ε (including ε itself). Let m_2 be the number of nodes of $T^*(G, \pi)$ which are not equivalent to ε and which do not have any descendants in $T^*(G, \pi)$. Let L be the constant defined in Section 2·30. Then the total time required by Algorithm 2·31 is $O(m_1 n^2 \log n + m_2 n^2(L + \log n))$, under the conditions of Theorem 3·7, where m_2 may depend on L. For our implementation, this must be increased to $O(n^3(m_1 + m_2) + m_2 n^2 L)$. By Theorem 2·33, $m_1 \leq n$, but we have not found any reasonable bound on m_2. It varies in a very complicated manner with the initial labelling of the input graph and the value of L.

3·9 Other implementation details

Algorithm 2·31 has been implemented on a Cyber 170 computer, mainly in Fortran. Because of the difficulty in manipulating bit-vectors efficiently in Fortran, several small subroutines are coded in assembler language.

The indicator function Λ is evaluated by the subroutine which implements Algorithm 2·5. It is formed by taking cell sizes, relative vertex degrees and other information which is computed in the course of Algorithm 2·5, and merging these into a single integer value in a "pseudo-random" fashion (see Section 2·28).

A technique which produced considerable improvements in efficiency in some cases involves the updating of the graph $G(\rho)$ when ρ is updated. The computation of $G(\rho)$ is quite time-consuming (up to about 6 seconds for $n = 1000$), so this computation is delayed for as long as possible, in case it is not necessary.

3·10 Storage requirements

Let m be the number of machine-words required to hold a bit-vector of size n. Let K be the maximum length of a node of $T^*(G, \pi)$. Obviously $K \le n$, but very much smaller values are normal. Define L as before. The total amount of storage required by our implementation, ignoring a minor amount independent of n, is $2mn + 10n + m + (m + 4)K + 2mL$ words. This figure includes 2mn words for the storage of G and $G(\rho)$. If *lab* = false (see Algorithm 2·31), the storage requirement can be reduced by $mn + 2n$ words.

3·11 Experimental performance

In Figure 3·1 we give the execution time required for several families of graphs. In each description below, β gives the approximate slope of the curve in the region $50 \le n \le 200$. Although the results of Section 3·8 predict a value of $\beta \ge 4$, even when $m_2 = 0$, the experimental value of β is less than 3 in each of these classes.

(i) E : empty graph on n vertices ($\beta = 2{\cdot}8$).

(ii) Q : m-dimensional cube, where $n = 2^m$ ($\beta = 2{\cdot}3$).

(iii) C : random circulant graph of degree 10 ($\beta = 2{\cdot}2$). This is defined by $V(G) = V$ and $E(G) = \{xy \mid |x - y| \in W (\mathrm{mod}\ n)\}$, where W is a random subset of $\{1, 2, \cdots, [(n - 1)/2]\}$ of size 5.

(iv) R_6 : "random" regular graph of degree 6 ($\beta = 2{\cdot}9$). There is no known practical algorithm for randomly generating regular

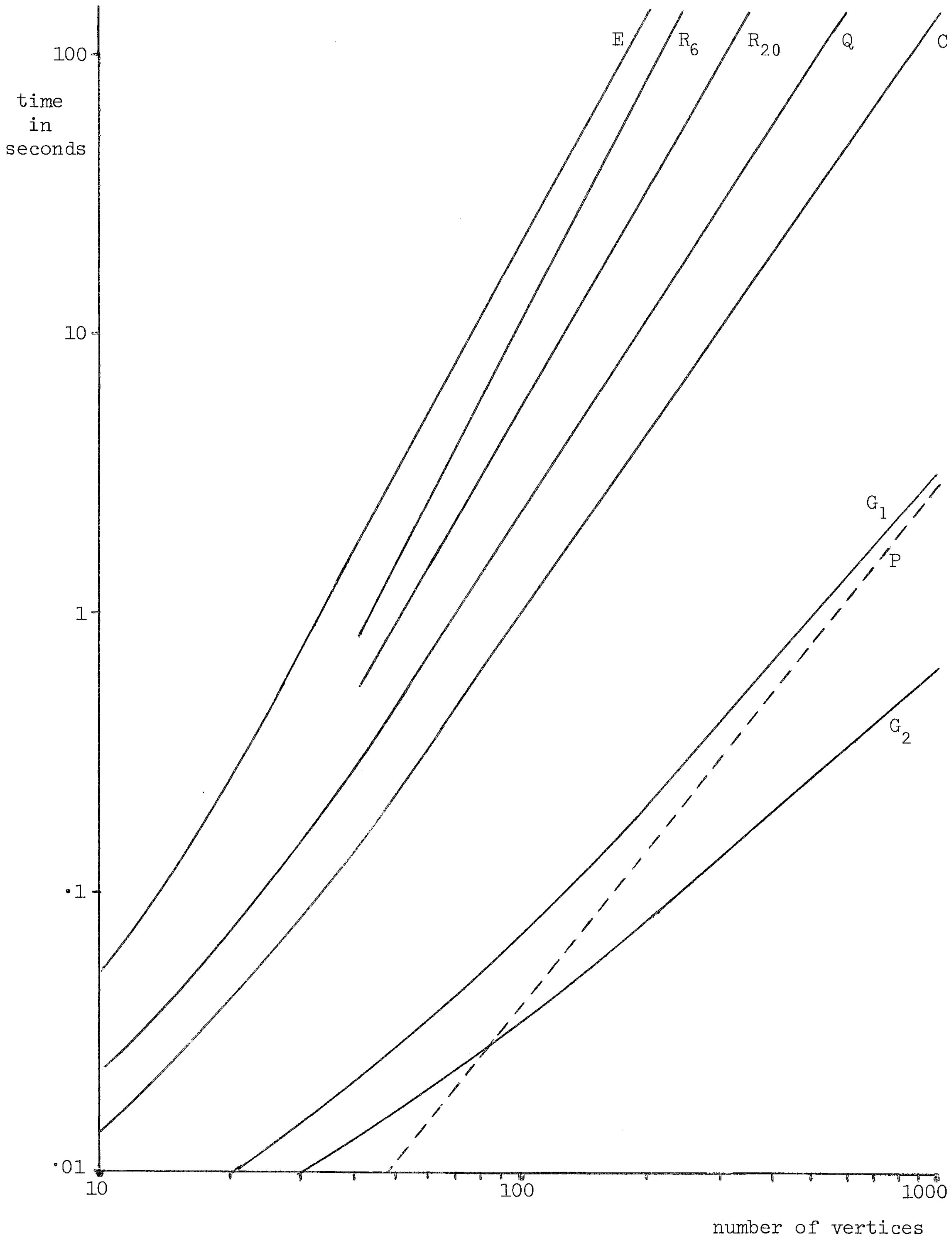

100
time
in
seconds
10
1
·1
·01
10
100
1000
number of vertices
E
R6
R20
Q
C
G1
P
G2

Figure 3·1

graphs so that each graph appears with equal frequency. The graphs represented by the curve R_6 were made by randomly generating three permutations γ_1, γ_2 and γ_3 such that $x^\delta \neq x$, $\delta \in \{\gamma_1{}^2, \gamma_2{}^2, \gamma_3{}^2\}$, and $x^{\gamma_i} \neq x^{\gamma_j}$, $1 \leq i < j \leq 3$, for each $x \in V$. Define G by $V(G) = V$ and $E(G) = \{xx^{\gamma_i} | x \in V, 1 \leq i \leq 3\}$. For $n \geq 40$ all those graphs constructed had trivial automorphism groups, and produced search trees with maximum depth 2.

(v) R_{20}: same as R_6 but with degree 20 ($\beta = 2{\cdot}6$).

(vi) G_1 : random graph. Each possible edge is independently chosen or not chosen with probability $\frac{1}{2}$. The dashed line marked P in Figure 5·1 gives the average time required for the computation of $G(\rho)$ for some ρ. At least one such step is essential for any program which computes $\mathcal{C}(G, \pi)$ from G using an adjacency matrix representation. Therefore Figure 5·1 suggests that the performance of our program is close to optimal for large random graphs.

(vii) G_2 : same as (vi) but with *lab* = false.

3·12 Harder Examples

We have also tested our program on a number of graphs which have traditionally been regarded as difficult cases for graph isomorphism programs.

(i) The strongly regular graphs with 25 vertices required between 0·1 and 2·4 seconds, with the average time being 1·0 seconds.

(ii) A strongly regular graph G with 35 vertices can be formed from a Steiner Triple System (STS) with 15 points. The vertices of G are the blocks of the STS, and two vertices are adjacent if the corresponding blocks overlap. For the 80 graphs so formed, our program required between 0·3 and 7 seconds, with an average of 4·8 seconds. Most of these graphs have a trivial automorphism group.

(iii) Certain strongly regular graphs G with n vertices can be extended to graphs E(G), having 2n + 2 vertices, which are *2-level regular*. See Mathon [24] for the necessary definitions. There are good theoretical reasons to expect 2-level regular graphs to be particularly difficult to process, and this is borne out by experience. The graphs A_{60} and B_{60} (60 vertices; see [24]) required 79 and 180 seconds respectively, while the graphs A_{72} - D_{72} (72 vertices) required about 500 seconds each.

3·13 Design isomorphism

A *design* D (also known as a *hypergraph*) is a pair of sets (P, B), where B is a collection of subsets of P. The elements of P are called *points* and the elements of B are called *blocks*. Two designs $D_1 = (P_1, B_1)$ and $D_2 = (P_2, B_2)$ are *isomorphic* if there are bijections $f_1 : P_1 \to P_2$ and $f_2 : B_1 \to B_2$ such that $x \in X$ implies $f_1(x) \in f_2(X)$ for all $x \in P_1$, $X \in B_1$.

Given a design D = (P, B) we can construct a graph G = G(D), where $V(G) = P \cup B$ and $E(G) = \{xX | x \in P, X \in B, x \in X\}$. It is easy to prove ([6] , [30]) that two designs $D_1 = (P_1, B_1)$ and $D_2 = (P_2, B_2)$

are isomorphic if and only if there is an isomorphism $f : G(D_1) \to G(D_2)$ such that $f(P_1) = P_2$ and $f(B_1) = B_2$. Therefore Algorithm 2·31 can be used for design isomorphism.

If D is a balanced incomplete block-design (BIBD) then G(D) is known to present difficulties for many graph isomorphism programs, and ours is no exception. Two 50-vertex graphs G(D), named A_{50} and B_{50} in [24], required about 60 seconds each. In another experiment [33], we established the isomorphism of six BIBDs with 36 points and 36 blocks (so n = 72) using about 6·6 seconds of machine time each. The smallness of this figure is principally due to the reasonably rich automorphism groups of the designs.

A much more difficult problem was posed by two BIBDs, D_1 and D_2, with 126 points and 525 blocks [30]. This problem was solved using an ancestor of Algorithm 2·31 implemented on an IBM 370/168 computer. The graph $G(D_2)$ was processed in 582 seconds, and has an automorphism group of size 756000 with two orbits (the points and the blocks). The graph $G(D_1)$ was similarly tackled, but the execution had not finished before it was aborted after 1200 seconds. We then constructed the strongly regular graphs S_1 and S_2 of order 525 and degree 144 whose edges are the intersecting blocks of D_1 and D_2. The graph S_2 had a transitive automorphism group of order 756000 (running time 66 seconds) and S_1 had an automorphism group of order 1000 and three orbits (running time 461 seconds). The orbits of S_1 were then used to provide an initial partitioning of the vertices of $G(D_1)$ into four cells (the point cell and three block cells). It was then processed in 227 seconds, and found to have an automorphism group of order 1000. Thus D_1 and D_2 are not isomorphic.

3·14 Hadamard equivalence

Let M_1 and M_2 be two m×n matrices with ±1 entries. We say that M_1 and M_2 are *Hadamard equivalent* if M_2 can be obtained from M_1 by applying an element of the group Γ generated by the following operations.

p_α : Permute the rows according to $\alpha \in S_m$.

q_β : Permute the columns according to $\beta \in S_n$.

r_i : Multiply row i by -1 ($1 \le i \le m$).

c_j : Multiply column j by -1 ($1 \le j \le n$).

Suppose that M is any m×n matrix with ±1 entries. Define $G = G(M)$ to be the graph with $V(G) = \{v_i, \bar{v}_i, w_j, \bar{w}_j \mid 1 \le i \le m, 1 \le j \le n\}$ and $E(G) = \{v_i w_j, \bar{v}_i \bar{w}_j \mid 1 \le i \le m, 1 \le j \le n, M_{ij} = 1\} \cup \{v_i \bar{w}_j, \bar{v}_i w_j \mid 1 \le i \le m, 1 \le j \le n, M_{ij} = -1\}$. We will refer to the vertices v_i and $\bar{v}_i$ as *v-type vertices*. The following theorem first appeared in McKay [31].

3·15 THEOREM *Let* $G_1 = G(M_1)$ *and* $G_2 = G(M_2)$. *Then* M_1 *and* M_2 *are Hadamard equivalent if and only if there is an isomorphism from* G_1 *to* G_2 *which maps the v-type vertices of* G_1 *onto those of* G_2.

Proof: Let $\bar{\Gamma}$ be the set of permutations of $V(G_1)$ generated by the following elements:

$\bar{p}_\alpha$: For each i, map v_i onto v_{i^α} and $\bar{v}_i$ onto $\bar{v}_{i^\alpha}$ ($\alpha \in S_m$).

$\bar{q}_\beta$: For each j, map w_j onto w_{j^β} and $\bar{w}_j$ onto $\bar{w}_{j^\beta}$ ($\beta \in S_n$).

$\bar{r}_i$: Interchange v_i and $\bar{v}_i$ ($1 \le i \le m$).

$\bar{c}_i$: Interchange w_j and $\bar{w}_j$ ($1 \le j \le n$).

Define ϕ to be the homomorphism from Γ into $\bar{\Gamma}$ which takes p_α onto $\bar{p}_\alpha$, q_β onto $\bar{q}_\beta$, r_i onto $\bar{r}_i$ and c_j onto $\bar{c}_j$ for each $\alpha \in S_m$, $\beta \in S_n$, $1 \le i \le m$ and $1 \le j \le n$. It is easily verified that ϕ is a group isomorphism and that $G(M_1^{\gamma}) = G(M_1)^{\gamma\phi}$ for each $\gamma \in \Gamma$. Therefore, the Hadamard equivalence of M_1 and M_2 implies the presence of an isomorphism from G_1 to G_2 which maps the v-type vertices of G_1 onto those of G_2.

Suppose conversely that there is an isomorphism θ of the required type from G_1 to G_2. Let e_1 be any edge of G_1 and let $e_2 = e_1^{\theta}$ be its image in G_2. For $k \in \{1, 2\}$, define H_k to be the subgraph of G_k induced by those vertices adjacent to either end of e_k. The structure of G_k ensures that H_k has three important properties.

(i) Exactly one of v_i and $\bar{v}_i$ is in H_k $(1 \le i \le m)$.

(ii) Exactly one of w_j and $\bar{w}_j$ is in H_k $(1 \le j \le n)$.

(iii) H_k completely determines G_k.

To explain (iii), suppose for example that $v_i w_j \in E(H_k)$. Then $\bar{v}_i\bar{w}_j \in E(G_k)$ but $v_i\bar{w}_j, \bar{v}_i w_j \notin E(G_k)$.

Since θ is an isomorphism, it maps H_1 onto H_2. By properties (i) and (ii) we can find $\bar{\gamma} \in \bar{\Gamma}$ whose restriction to H_1 is the same as that of θ. But then $\bar{\gamma}$ is an isomorphism from G_1 to G_2, by property (iii). Therefore $M_2 = M_1^{\bar{\gamma}\phi^{-1}}$. □

If M is a Hadamard matrix ($m = n$ and $M^T M = nI$) then the graph G(M) may prove exceedingly difficult for Algorithm 2·31. This was discovered when our implementation was applied to a collection of 126 Hadamard matrices of order 24, produced by C. Dibley and W.D. Wallis, in an attempt to determine the equivalence classes. Several of the graphs, having very large automorphism groups, were processed in about 300 seconds, but some of those smaller automorphism groups would require more than 1800 seconds - the program was not run to completion. These graphs are all

2-level regular in the sense of Mathon [24], but are very much harder than those given in [24], even though they have larger groups. The reason for this is that the search tree $T^*(G, \pi)$ has depth 7 or 8 (compared with 4 for the graphs in [24]), although only 2 or 3 vertices generally need to be fixed in order to eliminate any non-trivial automorphisms. This means that the automorphism group is of no use for a large part of $T^*(G, \pi)$.

Other workers (see [8] for example) have found that a count of small subgraphs (e.g. cliques) can often be used to provide an initial partitioning of the vertices of a difficult graph, which greatly speeds up a subsequent isomorphism test. Similar techniques can be used here, but they are of no use in many cases. Some of the hardest graphs amongst the 126 mentioned above have only two orbits (the v-type vertices and the others) - the initial partitioning which we were using anyway (because of Theorem 3·15). However we have devised a method based on a generalisation of the *profile* defined in [7] which can be used to refine the partitions at the immediate successors of the root node in $T^*(G, \pi)$. With this improvement, we can now process these graphs in about 20 seconds on the average. More details will be given in a future paper.

An algorithm specifically for equivalence of Hadamard matrices has been devised by Leon [22]. The details given in [22] are insufficient to permit a direct comparison with our technique, but a cursory examination suggests that Leon's technique may be competitive with ours for this particular problem.

3·16 Examples

Some examples of the automorphism group generators produced by Algorithm 2·31 are given in Appendix 3.

CHAPTER FOUR

TRANSITIVE GRAPHS - MISCELLANEOUS THEORY

In this chapter we present a miscellaneous collection of theoretical results concerning the structure of transitive graphs. Most of these results are required for use in Chapter 5. Anything not attributed to another author is new.

4·1 Lexicographic Products

Sections 4·1 - 4·5 were inspired by unpublished work by C. Godsil, who proved Theorem 4·5 (a) $\Leftrightarrow$ (e) without the use of Lemma 4·4.

A graph $G \in \underset{\sim}{G}(V)$ is called a *non-trivial lexicographic product* (NTLP) if $G = H[J]$, where H and J have at least two vertices.

A subset $W \subseteq V$ is called *externally-related* (ER) in a graph G if each vertex in $V \setminus W$ is either adjacent to every vertex of W or to no vertex of W. Subsets of size 0, 1 or n are necessarily ER, so we will call W a *non-trivial* ER subset if $2 \leq |W| < n$.

4·2 LEMMA *Let* $W_1, W_2 \subseteq V$ *be* ER. *Then*

(a) $W_1 \cap W_2$ *is* ER,

(b) *if* $W_1 \cap W_2 \neq \emptyset$ *then* $W_1 \cup W_2$ *is* ER, *and*

(c) *if* $W_1 \cap W_2 \neq \emptyset$, $W_1 \setminus W_2 \neq \emptyset$ *and* $W_1 \setminus W_2 \neq \emptyset$ *then* $W_1 \setminus W_2$ *and* $W_1 \setminus W_2$ *are* ER.

Proof: Part (a) is trivial. For part (b), any vertex not in $W_1 \cup W_2$ but adjacent to some vertex of $W_1 \cup W_2$ is adjacent to every vertex in $W_1 \cap W_2$ and therefore to every vertex of $W_1 \cup W_2$. Now consider part (c). Suppose some vertex $x \notin W_1 \setminus W_2$ is adjacent to some vertex $y \in W_1 \setminus W_2$. If $x \notin W$, then x is adjacent to every vertex in $W_1 \setminus W_2$, since W_1 is ER. Suppose that $x \in W_1 \cap W_2$, and let z and w be arbitrary vertices in $W_1 \setminus W_2$ and $W_2 \setminus W_1$ respectively. Since W_2 is ER, y is adjacent to w. Since W_1 is ER, w is adjacent to z. Finally, since W_2 is ER, z is adjacent to x. Therefore $W_1 \setminus W_2$ is ER, and similarly $W_2 \setminus W_1$ is ER. □

4·3 LEMMA *If* W *is an* ER *subset of* V, *then*

$$\mathrm{Aut}(G)_{\{W\}} = \mathrm{Aut}(W) \oplus \mathrm{Aut}(V \setminus W).$$

Proof: obvious. □

4·4 LEMMA *Let* G *be any graph with at least one non-trivial* ER *subset, such that* $\Gamma = \mathrm{Aut}(G)$ *contains no transpositions. Then a non-trivial* ER *subset of minimum size is a block for* Γ.

Proof: Let B be a non-trivial ER subset of minimum size. If $B = \{x, y\}$ then $(x\ y) \in \Gamma$ obviously, so $|B| \geq 3$. Now suppose that $B \cap B^\gamma \neq \emptyset$ for some $\gamma \in \Gamma$. Then $B = B^\gamma$, since otherwise either $B \cap B^\gamma$ or $B \setminus B^\gamma$ is a non-trivial ER subset smaller than B, by Lemma 4·2. Thus B is a block for Γ. □

4·5 THEOREM *Let* G *be a transitive graph which is neither empty nor complete. Then the following are equivalent.*

(a) G *is a* NTPL.

(b) $G = G_1[G_2]$, *where* G_1 *and* G_2 *are non-trivial and transitive.*

(c) G *has a non-trivial* ER *subset.*

(d) Aut(G) *has a non-trivial* ER *block.*

(e) Aut(G) *has an intransitive subgroup with exactly one orbit of size greater than one.*

Proof: Obviously, (b) ⇒ (a) ⇒ (c) and (d) ⇒ (e) ⇒ (c), so it will suffice to prove that (d) ⇒ (b) and (c) ⇒ (d).

Suppose that Aut(G) has a non-trivial ER block B, and let $B_1, B_2, \ldots, B_r$ be the complete block system containing B. For $i \neq j$, B_i and B_j are trivially joined, since B_i and B_j are ER. Furthermore, the subgraphs B_i are isomorphic and transitive, and Aut(G) acts transitively on $\{B_1, B_2, \ldots, B_r\}$. Therefore condition (b) is satisfied.

Suppose now that G has a non-trivial ER subset. Then, by Lemma 4·4, either condition (d) is satisfied or Aut(G) contains a transposition (x y). In the latter case, we can assume without loss of generality (replace G by $\bar{G}$ if necessary) that $N(x, G) = N(y, G)$. Now define $B = \{v \in V \mid N(v, G) = N(x, G)\}$. Then $B \neq V$, since otherwise G is empty. Therefore B is a non-trivial ER block of Aut(G). □

4·6 Vertex-connectivity

Sections 4·6 – 4·8 are adapted from Watkins [42].

Let G be a transitive graph with degree k ($1 \le k \le n - 2$) and vertex-connectivity $\kappa \ge 1$. Obviously, $\kappa \le k$. A *part* of G is a component of the subgraph $V \setminus X$, for some minimum cutset X. The parts of G of the smallest size are the *atomic parts* of G.

4·7 THEOREM *Suppose that* $\kappa < k$ *and that the atomic parts of* G *have size* a. *Then*

(1) $2 \le a \le \frac{n}{4}$.

(2) *The atomic parts of* G *are disjoint and form a block system for* Aut(G).

(3) *The minimum cutset defined by an atomic part is a union of at least two atomic parts.*

(4) $\kappa \ge k - a + 1$

Proof: See [42]. □

4·8 COROLLARY $\kappa > 2k/3$.

Proof: This follows from (1) and (4) above. □

4·9 THEOREM *If* $\kappa < k$ *and equality holds in Theorem* 4·7(4), *then* G *is a* NTLP.

Proof: Let W be an atomic part. Since the cutset determined by W has $\kappa = k - a + 1$ vertices, each vertex in W must be adjacent to every vertex in the cutset. In other words, W is ER. Since W is a block, by Theorem 4·7, G is a NTLP by Theorem 4·5. □

4·10 <u>Edge-connectivity</u>

Sections 4·10 - 4·14 are extracted from McKay [29]. Theorem 4·14 has also been proved by Lovasz [23], by very similar means.

Let G be a graph with edge-connectivity $\eta \geq 1$. For $X, Y \subseteq V$ let $e(X, Y)$ denote the number of edges of the form xy, where $x \in X$, $y \in Y$. A non-empty proper subset $W \subset V$ is an *edge-part* of G if $e(W, V \setminus W) = \eta$. The edge-parts of minimum size are called *edge-atoms*.

4·11 LEMMA *Let* X *and* Y *be edge-parts and suppose that* $A = X \cap Y$, $B = X \setminus Y$, $C = Y \setminus X$ *and* $D = V \setminus (X \cup Y)$ *are non-empty. Then* A, B, C *and* D *are edge-parts.*

Proof: Since X and Y are edge-parts,

$e(A, C) + e(A, D) + e(B, C) + e(B, D) = \eta$, and

$e(A, B) + e(A, D) + e(B, C) + e(C, D) = \eta$.

Since $\emptyset \neq A, B, C, D \neq V$,

$e(A, B) + e(A, C) + e(A, D) \geq \eta$,

$e(A, B) + e(B, C) + e(B, D) \geq \eta$,

$e(A, C) + e(B, C) + e(C, D) \geq \eta$, and

$e(A, D) + e(B, D) + e(C, D) \geq \eta$.

Adding the two equations and subtracting half the sum of the four inequalities, we obtain $e(A, D) + e(B, C) \le 0$. Consequently, $e(A, D) = e(B, C) = 0$, and the four inequalities are equalities. □

4·12 COROLLARY *Distinct edge-atoms are disjoint.* □

4·13 LEMMA *Suppose that* G *is regular with degree* k. *Let* W *be an edge-part. Then if* $|W| \le k$ *we have* $\eta = k$ *and either* $|W| = 1$ *or* $|W| = k$.

Proof: Let ℓ be the average degree of the subgraph W. Counting the edges of G adjacent to elements of W we have

$$\ell|W| = k|W| - \eta$$

$$\ge k\ell + k - \eta, \text{ since } \ell \le |W| - 1$$

Therefore $k - \eta \le 0$, since $|W| \le k$, and so $k = \eta$, since $\eta \le k$ obviously. Therefore the inequality above becomes $\ell|W| \ge \ell k$ which implies that $|W| = k$ or $\ell = 0$. In the latter case, $|W| = 1$, since edge-parts are connected. □

4·14 THEOREM *Let* G *be a connected transitive graph with degree* k. *Then* $\eta = k$.

Proof: Suppose $\eta < k$ and that W is an edge-atom of G. By Lemma 4·13, $|W| > k$. Also, W^γ is an edge-atom for any $\gamma \in \mathrm{Aut}(G)$, and so W is a block of Aut(W), by Corollary 4·12. However, the condition $|W| > \eta$ implies that the set-wise stabiliser $\mathrm{Aut}(G)_{\{W\}}$ cannot act transitively on W, since otherwise $e(W, V \setminus W)$ would be a multiple of $|W|$. This is a contradiction. □

We remark that Theorem 4·14 is also true for infinite transitive graphs, if η is defined as $\min\{e(W, V \setminus W) | W \subseteq V, 0 < |W| < \infty\}$ See McKay [29] for this and many related results.

4·15 Other connectivity results

The first theorem in this section was proved by Gardiner [11], and independently by Ashbacher [1].

4·16 THEOREM *Let* G *be a graph with* $n \geq 3$ *vertices such that for any two vertices* $v \neq w$ *we have* $N(v, G) \neq N(w, G)$ *and* $\bar{N}(v, G) \cong \bar{N}(w, G)$. *Then either* $\bar{N}(v, G)$ *is connected for each* $v \in V$ *or* Aut(G) *has a non-trivial* ER *block.* □

4·17 COROLLARY *Let* G *be a non-complete connected transitive graph. If* $\bar{N}(v, G)$ *is disconnected for some* $v \in V$, G *is a* NTLP.

Proof: If $N(v, G) = N(w, G)$ for some $v \neq w$ or $n = 2$, Aut(G) contains a transposition, and so is a NTLP by Theorem 4·5. Otherwise, G satisfies the requirements of Theorem 4·16, and is thus a NTLP by Theorem 4·5. □

4·18 Sections 4·18 to 4·21 are due to Godsil [14].

A regular graph G is called an *(s, t)-graph* if the graphs $N(v, G)$ and $N(v, \bar{G})$ have exactly s and t isolated vertices, respectively, for each $v \in V$.

4·19 THEOREM *Let* $s, t \geq 1$. *Then a regular graph* G *is an* (s, t)-graph *if and only if it is* C_5 *a switching graph of the form* $Sw(H[\bar{K}_{t+1}])$, *where* $N(v, H) \neq N(w, H)$ *for* $v \neq w$, *or the complement of such a switching graph.* □

4·20 THEOREM *Let* G *be a regular graph of degree* k *such that* $N(v, G) = N(w, G)$ *for all* $v, w \in V$, *and suppose that* $N(v, G)$ *has a component of size* c, *where* $1 < c \leq \frac{k}{2}$. *Then* $n \geq 2k + 1$. *If* $n = 2k + 1$, G *is one of the two graphs shown in Figure* 4·1. □

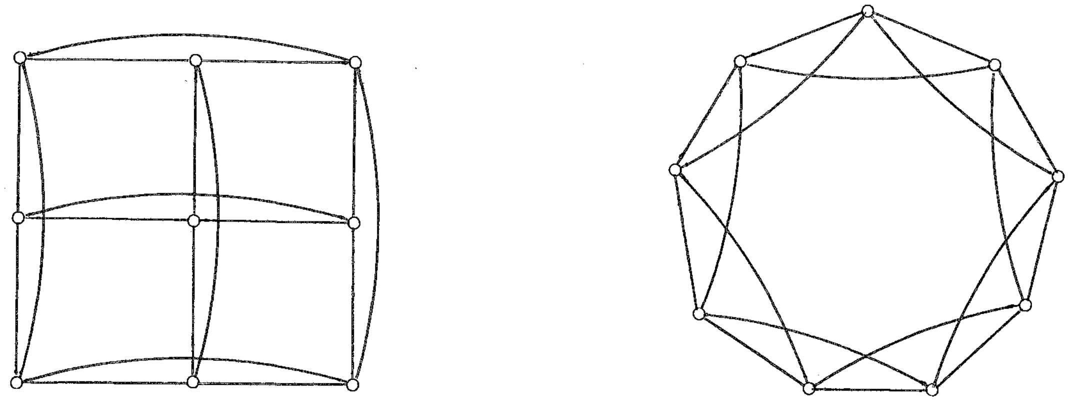


Figure 4.1

4.21 THEOREM *Let* G *be a transitive graph such that both* N(v, G) *and* N(v, $\bar{G}$) *are disconnected. Then* G *is either* $C_3 \times C_3$ *or an (s, t)-graph, with* s, t ≥ 1. □

4.22 THEOREM *Let* G *be a connected transitive graph such that for each* v ∈ V *there is a unique vertex* v' ∈ V *at distance* 3 *from* v. *Suppose* G *has degree* k, *where* n = 2k + 2. *Then* G *is a switching graph.*

Proof: Since Aut(G) is transitive, the set of $\frac{n}{2}$ pairs {v, v'} form a block system for Aut(G). Since the two vertices in one block are at distance 3 from each other, no vertex is adjacent to both vertices of a block. However the number of blocks is k + 1, so every vertex is adjacent to exactly one element of the blocks it does not itself lie in. Therefore G ≅ Sw(H), where H is the subgraph of G induced by any set of vertices containing exactly one element of each block. □

4.23 THEOREM *Let* G *be a transitive graph with* κ = k *and* n = 2k + 2. *Then either* G *has diameter* 2 *or* G *is a switching graph.*

Proof: Every vertex adjacent to a given vertex v is adjacent to at least one vertex at distance 2 from v, since otherwise κ < k. Therefore, if the diameter of G is greater than 2, there is a unique vertex v' at distance 3 from v. Thus G is a switching graph, by Theorem 4·22. □

4·24 THEOREM *Let* G *be a connected non-complete transitive graph with* $n \geq 7$ *and odd. Let* $\mathcal{D}(G)$ *be the set of elements of* Aut(G) *of the form* (a b)(c d). *If* $\mathcal{D}(G) \neq \emptyset$, *then* G *is a* NTLP.

Proof: Since Aut(G) is transitive, every $v \in V$ is contained in $\mathrm{supp}(\gamma)$ for some $\gamma \in \mathcal{D}(G)$. Thus $|\mathcal{D}(G)| \geq 2$. Since n is odd and $|\mathrm{supp}(\gamma)|$ is even for all $\gamma \in \mathcal{D}(G)$, we can find distinct $\gamma, \delta \in \mathcal{D}(G)$ such that $\mathrm{supp}(\gamma) \cap \mathrm{supp}(\delta) \neq \emptyset$.

There are essentially seven different ways in which γ and δ can overlap. In the first six cases, we can identify an intransitive subgroup Λ having exactly one non-trivial orbit. Therefore G is a NTPL in these cases, by Theorem 4·5. Let γ = (a b)(c d)

(i) If δ = (d e)(f g), take $\Lambda = \langle \gamma\delta\gamma\delta \rangle$.

(ii) If δ = (c d)(e f), take $\Lambda = \langle (c\ d) \rangle$.

$\Lambda \subseteq$ Aut(G) because {c, d} is ER.

(iii) If δ = (a c)(e f), take $\Lambda = \langle \gamma, \delta\gamma\delta \rangle$.

(iv) If δ = (a b)(c e), take $\Lambda = \langle \gamma\delta \rangle$.

(v) If δ = (a c)(d e), take $\Lambda = \langle \gamma, \delta \rangle$.

(vi) If δ = (a c)(b d), take $\Lambda = \langle \gamma, \delta \rangle$.

If none of the cases above occurs, the only type of overlap is as for γ = (a b)(c d) and δ = (a e)(c f); call this type (vii). Now define a relation $\sim$ on V.

(a) $x \sim x$ for all $x \in V$

(b) If $x \neq y \in V$, $x \sim y$ if and only if there are automorphisms γ = (x a)(y b) and δ = (x c)(y d) such that $a \neq c$ and $b \neq d$.

Clearly $\sim$ is symmetric. Now suppose that $s \neq y$, $x \sim y$ and there exists $\alpha = (x\ e)(z\ f) \in \mathcal{D}(G)$, where $\gamma \neq \alpha \neq \delta$ and $f \neq y$. Since only overlaps of type (vii) are allowed, $e \in \{a, b, c, d, y\}$, and either $z = y$ or $\{z, f\} = \{v, d\}$. However, in the latter case, α and $\delta\gamma\delta = (a\ c)(b\ d)$ have an overlap of type (ii). Therefore $z = y$.

We conclude that ~ is an equivalence relation with classes of size 2, contradicting the assumption that n is odd. □

4·25 THEOREM *Let* G *be a connected non-complete transitive graph with* n *odd and* $n \geq 7$. *If* G *has an automorphism of the form* $(a\ b\ c)(d\ e\ f)$, *then* G *is a* NTLP.

Proof: If $(a\ b\ c)(d\ e\ f) \in \text{Aut}(G)$ we find, by considering the edges between $\{a, b, c\}$ and $\{d, e, f\}$ that $(a\ b)(d\ e) \in \text{Aut}(G)$. The result now follows from Theorem 4·26. □

Theorems 4·24 and 4·25 undoubtedly have a common generalization, but we have made no serious attempt to find it.

4·26 Let G be a transitive graph and let $\Gamma = \text{Aut}(G)$. Let $\{(1)\} < \Lambda \leq \Gamma$. Then Λ has a unique representation

$$\Lambda = \Lambda^{(1)} \oplus \Lambda^{(2)} \oplus \cdots \oplus \Lambda^{(r)},$$

where the supports of the $\Lambda^{(i)}$ are non-empty and disjoint, and r is maximum. The subgroups $\Lambda^{(i)}$ are called the *fragments* of Λ.

Define a graph $H = H(G, \Lambda)$ as follows. V(H) is the set of non-trivial orbits of Λ. Two distinct vertices of H are adjacent if and only if the corresponding orbits are non-trivially joined in G.

4·27 LEMMA *If* Φ *is a fragment of* Λ *and* $\gamma \in \Gamma$, *then* Φ^{γ} *is a fragment of* Λ^{γ}.

Proof: obvious. □

4·28 LEMMA *Let* $1 \neq P \in Syl_p(\Gamma_1)$ *for some prime* p. *Then the supports of the fragments of* P *are the components of* H(G, P).

Proof: Let Φ be a fragment of P. Any orbit of P in supp(Φ) is trivially joined to each orbit not in supp(Φ), and so supp(Φ) is a union of components $V_1, V_2, \cdots, V_r$ of H(G, P). Suppose $r \geq 2$, let $\pi \in \Pi(V)$ have non-trivial cells $V_1, V_2, \cdots, V_r$ and fix any point not in supp(Φ). Then $\Gamma_\pi = \bigoplus_{i=1}^{r} \Gamma_\pi|_{C_i}$, and $\Phi \in Syl_p(\Gamma_\pi)$, so that $\Phi = \Phi^{(1)} \oplus \Phi^{(2)} \oplus \cdots \oplus \Phi^{(r)}$, where $\Phi^{(i)} \in Syl_p(\Gamma_\pi|_{C_i})$ by Lemma 1·9, contradicting the assumption that Φ is a fragment of P. □

4·29 LEMMA *Let* $1 \neq P \in Syl_p(\Gamma_1)$ *for some prime* p. *Let* Φ *be a fragment of* P *and let* $\gamma \in \Gamma$. *Then if* $\Phi^\gamma \leq P$ *and* supp(Φ) *is a union or orbits of* P, Φ^γ *is a fragment of* P.

Proof: Since supp(Φ^γ) is a component of H(G, P^γ) and the non-trivial orbits of Φ^γ are orbits of P, supp(Φ^γ) = supp(Φ') for some fragment Φ' of P. But then $\Phi^\gamma \leq \Phi'$, since $\Phi^\gamma \leq P$ and so $\Phi^\gamma = \Phi'$, since both Φ^γ and Φ' are in $Syl_p(\Gamma_{\theta(\Phi')})$. □

4·30 THEOREM *Let* $P \in Syl_p(\Gamma_1)$ *have fragments* $\Phi^{(1)}, \Phi^{(2)}, \cdots, \Phi^{(r)}$. *Suppose that some* $\Phi^{(i)}$ *is uniquely identified amongst* $\{\Phi^{(1)}, \Phi^{(2)}, \cdots, \Phi^{(r)}\}$ *by the sizes of its orbits and that, for every* $\gamma \in \Gamma$, $\Phi^{(i)\gamma} \leq P$ *only if the non-trivial orbits of* $\Phi^{(i)\gamma}$ *are orbits of* P. *Then* $|supp(\Phi^{(i)})| \geq \frac{1}{2}n$.

Proof: By Lemma 4·29, $\Phi^{(i)}$ is weakly closed in P with respect to Γ. The theorem now follows from Theorem 1·20. □

As an example of the use of Theorem 4·30, the automorphism group of a transitive graph G with 15 vertices cannot have a Sylow 2-subgroup of the form $\langle(2\ 3)(4\ 5)(6\ 7), (8\ 9)(10\ 11)(12\ 13)(14\ 15)\rangle$, since the fragment $\langle(2\ 3)(4\ 5)(6\ 7)\rangle$ has a support which is too small.

4·31 Eigenvalue Techniques

The results described in Sections 4·31 - 4·39 are special cases of more general results developed by Godsil and McKay [13].

Let $G \in G(V)$ and let $\pi = (V_1, V_2, \cdots, V_m) \in \underset{\sim}{\Pi}(V)$ be equitable. The m×n matrix $T = T(\pi)$ is defined by

$$T_{ij} = \begin{cases} k_i^{-\frac{1}{2}} & \text{if } j \in V_i \\ 0 & \text{otherwise,} \end{cases}$$

where $k_i = |V_i|$. Using T we define the *quotient matrix* of G by π to be $Q = Q(G, \pi) = TAT^T$, where A is the adjacency matrix of G. Note that Q is symmetric.

4·32 LEMMA *For* $1 \leq i, j \leq m$, *let* e_{ij} *be the number of vertices in* V_j *to which each vertex in* V_i *is adjacent in* G. *Then* $TA = QT$ *and* $Q_{ij} = (k_j/k_i)^{\frac{1}{2}} e_{ji}$, *for* $1 \leq i, j \leq m$.

Proof: For $1 \leq i, j \leq m$,

$$Q_{ij} = (TAT^T)_{ij} = \sum_{r=1}^{n} k_i^{-\frac{1}{2}} d(v, V_i) T_{jv}$$

$$= \sum_{r \in V_j} (k_i k_j)^{-\frac{1}{2}} e_{ji}$$

$$= (k_j/k_i)^{\frac{1}{2}} e_{ji}.$$

The equation TA = QT can now be verified directly. □

For any square matrix M, let σM denote the set of (distinct) eigenvalues of M. If $\lambda \in \sigma M$, define $\mu_M(\lambda)$ to be the multiplicity of λ. If $\lambda \notin \sigma M$ define $\mu_M(\lambda) = 0$.

4·33 LEMMA *For any m-vector* $\underset{\sim}{x}$ *and scalar* λ, $Q\underset{\sim}{x} = \lambda\underset{\sim}{x}$ *if and only if* $A(T^T\underset{\sim}{x}) = \lambda(T^T\underset{\sim}{x})$.

If $Q\underset{\sim}{x} = \lambda\underset{\sim}{x}$ then $T^TQ\underset{\sim}{x} = \lambda T^T\underset{\sim}{x}$ and so $AT^T\underset{\sim}{x} = \lambda T^T\underset{\sim}{x}$, by Lemma 4·32. If $AT^T\underset{\sim}{x} = \lambda T^T\underset{\sim}{x}$, then $TAT^T\underset{\sim}{x} = \lambda TT^T\underset{\sim}{x}$ and so $Q\underset{\sim}{x} = \lambda\underset{\sim}{x}$, since $TT^T = I$. □

4·34 COROLLARY *The characteristic polynomial of* Q *divides that of* A.

Proof: Suppose that $\lambda \in \sigma Q$. Then $\lambda \in \sigma A$ by the Lemma. Let $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots, \underset{\sim}{x}_r\}$ be a full set of orthonormal eigenvectors of Q for λ. Then for $1 \le i,j \le r$ we find $(T^T\underset{\sim}{x}_i)^T(T^T\underset{\sim}{x}_j) = \underset{\sim}{x}_i{}^T TT^T \underset{\sim}{x}_j = \underset{\sim}{x}_i{}^T\underset{\sim}{x}_j$, since $TT^T = I$. Therefore $\{T^Tx_1, T^Tx_2, \cdots, T^Tx_r\}$ is a set of orthornormal eigenvectors of A for λ. Therefore $\mu_Q(\lambda) \le \mu_A(\lambda)$. □

4·35 LEMMA $Q^r = TA^rT^T$ *for* $r = 0, 1, 2, \cdots$.

Proof: This is an easy consequence of the fact that $T^TT = I$. □

From now on we will assume that $V_1 = \{w\}$, for some $w \in V$. The next lemma follows immediately from Lemma 4·35.

4·36 LEMMA $(A^r)_{ww} = (Q^r)_{11}$, *for* $r = 0, 1, 2, \cdots$. □

We next recall two standard matrix theory results. Their proofs may be found in Lancaster [20], for example.

4·37 LEMMA *Let* M *be any real symmetric matrix, and let* $r \in \{0, 1, 2, \cdots\}$. *For* $\lambda \in \sigma M$ *let* $\{\underset{\sim}{x}_1(\lambda), \underset{\sim}{x}_2(\lambda), \cdots, \underset{\sim}{x}_{s}(\lambda)\}$ *be a full set of orthonormal eigenvectors of* M *for* λ, *where* $s_\lambda = \mu_M(\lambda)$.

Then (a) $\mathrm{tr}M^r = \sum_{\lambda\in\sigma M} s_\lambda \lambda^r$

(b) $M^r = \sum_{\lambda\in\sigma M} \lambda^r \sum_{i=1}^{s_\lambda} \underset{\sim}{x}_i(\lambda)\underset{\sim}{x}_i(\lambda)^T$. □

4·38 THEOREM *Let* G *be a transitive graph, and let* $\pi = (V_1, V_2, \cdots, V_m)$ *be an equitable partition, such that* $V_1 = \{w\}$. *Let* $Q = Q(G, \pi)$. *For any real number* λ, *define* $\rho(Q, \lambda)$ *as follows.*

(i) *If* $\lambda \notin \sigma Q$, *define* $\rho(Q, \lambda) = 0$.

(ii) *If* $\lambda \in \sigma Q$, *let* $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots, \underset{\sim}{x}_s\}$ *be a full set of orthonormal eigenvectors of* Q *for* λ. *Then define* $\rho(Q, \lambda) = \sum_{i=1}^{s} (\underset{\sim}{x}_i)_1^2$, *where* $(\underset{\sim}{x}_i)_1^2$ *denotes the square of the first entry of* $\underset{\sim}{x}_i$. *Then for any real number* λ, $\mu_A(\lambda) = n\rho(Q, \lambda)$.

Proof: For $r \in \{0, 1, 2, \cdots\}$,

$$\operatorname{tr} A^r = n(Q^r)_{11}, \text{ by Lemma 4·36,}$$

$$= n \sum_{\lambda \in \sigma Q} \lambda^r \rho(Q, \lambda), \text{ by Lemma 4·37(b),}$$

$$= n \sum_{\lambda \in \sigma A} \lambda^r \rho(Q, \lambda), \text{ by Corollary 4·32.} \qquad (1)$$

Alternatively,

$$\operatorname{tr} A^r = \sum_{\lambda \in \sigma A} \mu_A(\lambda)\lambda^r, \text{ by Lemma 4·37(a).} \qquad (2)$$

Since the elements of A are distinct (by definition), the claimed result follows on comparing (1) with (2). □

4·39 COROLLARY *Under the conditions of the Theorem,* $\sigma A = \sigma Q$. □

The usefulness of Theorem 4·38 is that it provides a necessary condition on a matrix Q in order that it be a quotient matrix of some transitive graph G. If the computed values $n\rho(Q, \lambda)$, for $\lambda \in \sigma Q$, are not positive integers, then G does not exist. We will find in Chapter 5 that this condition is very strong.

Theorem 4·38 can in fact be proved under the weaker assumption that G is *walk-regular,* and a generalized version holds for any graph G at all. (See Godsil and McKay [13] for further details.) We also note that the case of Theorem 4·38 for $\mu_Q(\lambda) = 1$ has recently been proved independently by Rees [39], who has used it in the search for symmetric graphs of degree three.

4·40 THEOREM [36] *Let* G *be a transitive graph with degree* k *and adjacency matrix* A. *If* λ *is a simple eigenvalue of* A, *then* $\lambda \in \{-k, -k+2, \cdots, k-2, k\}$.

Proof: Let $\underset{\sim}{x}$ be an eigenvector of A corresponding to λ. Since G is transitive, the entries of $\underset{\sim}{x}$ have equal absolute value. The theorem now follows on considering the first row of the equation $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$. □

4.41 THEOREM *Let* G *be a transitive graph with adjacency matrix* A. *Let* s *be the number of simple eigenvalues of* A. *Then* n *is even if* $s \geq 2$ *and divisible by* 4 *if* $s \geq 3$.

Proof: Since G is regular, $\underset{\sim}{c}$ is an eigenvector of A corresponding to the eigenvalue k, where k is the degree of G and $\underset{\sim}{c}$ is the n-vector with each entry 1. Suppose that λ is a simple eigenvalue of A other than k, and let $\underset{\sim}{y}$ be a corresponding eigenvector. Since the entries of $\underset{\sim}{y}$ have equal absolute value, and $\underset{\sim}{y}$ is orthogonal to $\underset{\sim}{c}$, n must be even.

Suppose that $\underset{\sim}{z}$ is one eigenvector corresponding to a simple eigenvalue other than k or λ. Then, as before, the entries of $\underset{\sim}{z}$ have equal absolute value. The mutual orthogonality of $\underset{\sim}{c}$, $\underset{\sim}{y}$ and $\underset{\sim}{z}$ now implies that n is divisible by 4. □

CHAPTER FIVE

CONSTRUCTION OF TRANSITIVE GRAPHS

In this chapter we describe the construction of all the transitive graphs with 19 or fewer vertices. This can be seen as an application of Algorithm 2·31, although we actually used an ancestor of that algorithm.

Since many of the steps of the construction required extensive computation, it is necessary to present them in the order in which they were actually performed. Not to do so would mean that we could not present intermediate results. The outcome of this is that the order in some places does not appear particularly sensible, since a few of the techniques used for eliminating subcases were not invented until after the optimum point for their application had passed.

The transitive graphs up to order 11 and some of those of order 12 were found previously by Yap [46]. To the best of our knowledge, ours is the first attempt to exhaustively catalogue the transitive graphs of any higher order, except when the order is prime (see Turner [9]).

5·1 An overview

Throughout this chapter, G is a transitive graph with $V(G) = V = \{1, 2, \cdots, n\}$, degree k and automorphism group Γ.

Define $\mathcal{G}$ to be a set containing one graph isomorphic to each transitive graph G which satisfies the following conditions.

(i) $n \in \{8, 9, 10, 12, 14, 15, 16, 18\}$

(ii) $3 \le k \le (n-1)/2$

(iii) G is not a NTLP.

(iv) G is not a switching graph.

(v) Γ is not regular.

(vi) G has connectivity k.

In Sections 5·2 - 5·3 we will identify all those transitive graphs of order 19 or less which are not isomorphic to a member of $\mathcal{G}$. In Sections 5·4 - 5·24 we will seek a collection $\underset{\sim}{Q}$ of 962131 matrices such that for each $G \in \mathcal{G}$ and some $\Lambda \in J(\Gamma)$, $Q(G, \theta(\Lambda)) \in \underset{\sim}{Q}$ (for some labelling of G). In Sections 5·25 - 5·30 we will use a battery of tests to identify a subset $\underset{\sim}{Q}^*$ of 709 elements of $\underset{\sim}{Q}$ with the same property as $\underset{\sim}{Q}$. In Sections 5·31 - 5·36 we will use $\underset{\sim}{Q}^*$ to construct $\mathcal{G}$.

5·2 Identification of transitive graphs not in $\mathcal{G}$.

A basic source of data was the catalogue of 9-vertex graphs produced by Baker, Dewdney and Szilard [2]. A direct search produced a list of all transitive graphs with nine or fewer vertices. The results coincided with the list of Yap [46]. By Theorem 4·5, the transitive NTLPs with $n \le 18$ are all lexicographic products of these graphs. The transitive switching graphs with $n \le 18$ were found with the help of Theorem 1·3. The transitive strongly regular graphs were extracted from Weisfeiler [43].

In order to construct those graphs with regular automorphism group a list of all the groups of order up to 19 was prepared, with help from C. Godsil. This list appears in Appendix 1. A complete list of all the Cayley graphs for each group was computed and those with regular groups selected.

Suppose that G is a transitive graph with a prime number of vertices, p. Since $p \mid |\Gamma|$, Γ contains an element of order p, which clearly must be a single p-cycle. Therefore G is a Cayley graph of the cyclic group Z_p.

Finally, we can investigate the transitive graphs with connectivity less than their degree.

5·3 THEOREM *Let* G *be a transitive graph, not a* NTPL, *with* n *vertices, degree* k *and connectivity* κ. *If* $n \leq 19$, $3 \leq k \leq (n-1)/2$ *and* $\kappa < k$, *then* G *is isomorphic to the graph drawn in Figure* 5·1.

Proof: Let a be the size of the atomic parts of G. The conditions on n and k ensure that $k \leq 8$ and, since κ is non-prime (Theorem 4·7(3)), the only possibilities are $\kappa = 4$ and $\kappa = 6$. By Theorem 4·9, $k \leq \kappa + a - 2$ and by Theorem 4·7(2) $a \mid \kappa$. If $a = 2$, then $k \leq \kappa$, so $a = 3$ and therefore $\kappa = 6$. This gives $k \leq 7$ and so $k = 7$. By the conditions on n and k we have $n = 18$, $k = 7$, $a = 3$ and $\kappa = 6$.

Since the atomic parts are connected and transitive, they must be triangles. Call them $A_1, A_2, \cdots, A_6$. Let H be the graph with vertices $A_1, A_2, \cdots, A_6$ and with A_i adjacent to A_j if and only if the corresponding atomic parts are joined in G by at least one edge. By Theorem 4·7(3), H is regular with degree 2 and since G is connected, H is connected. Therefore H is a hexagon. Assume for convenience that $A_1, A_2, \cdots, A_6$ are the vertices of the hexagon in cyclic order.

Now consider a vertex v in A_1. Since $k = 7$, v must be adjacent to every vertex in A_6 and to two vertices in A_2 (without loss of generality). Since the total number of edges leaving A_1 is odd, and A_2 and A_6 are blocks, the set-wise stabiliser Λ of A_1 in Γ

fixes A_2 and A_6 set-wise. Since also Λ acts transitively on A_1, every vertex in A_1 is adjacent to all of A_6 and to two vertices in A_2.

Considering the other atomic parts in similar fashion, we conclude that the pairs A_1A_6, A_2A_3 and A_4A_5 are joined by every possible edge and that the pairs A_1A_2, A_3A_4 and A_5A_6 are each joined in an equitable fashion by six edges. It is easy to see that this can be done in essentially only one way, yielding the graph in Figure 5·1. □

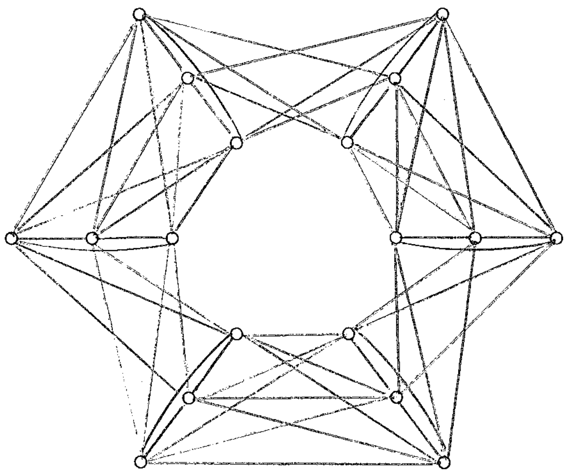


Figure 5·1

5·4 Numerical partitions

A *numerical partition* of n is a sequence σ of the form $(n; n_1^{m_1}, n_2^{m_2}, \cdots, n_r^{m_r})$ such that $1 \le n_1 < n_2 < \cdots < n_r$, $m_i > 0$ for $1 \le i \le r$ and $\sum_{i=}^{r} m_i n_i = n$. Superscripts equal to one are usually omitted. Define $r_{n_i}(\sigma) = m_i$ for $1 \le i \le r$ and $r_j(\sigma) = 0$ if $j \notin \{n_1, n_2, \cdots, n_r\}$. Also define $R(\sigma) = \{j \mid j \ge 2,\ r_j(\sigma) \ne 0\}$.

A partition $\pi \in \Pi^*(V)$ has an *associated numerical partition* $\sigma(\pi) = (n;\ n_1^{m_1}, \cdots, n_r^{m_r})$, where m_i is the number of cells of π of size n_i, for $1 \le i \le r$. If Λ is a permutation group of degree n, then $\sigma(\theta(\Lambda))$ will be abbreviated to $\sigma(\Lambda)$.

The first step in the construction of $\underset{\sim}{Q}$ will be to find a set Σ of numerical partitions with the following property. For every $G \in \boldsymbol{G}$ there is some $\Lambda \in J(\Gamma)$ such that $\sigma(\Lambda) \in \Sigma$. Such a set Σ will be called *sufficient*.

The first theorem identifies a number of types of numerical partition which can be eliminated from any sufficient set without destroying the sufficiency.

5·5 THEOREM *Let* $G \in \boldsymbol{G}$, $\Lambda \in J(\Gamma)$, *and* $\sigma = \sigma(\Lambda)$. *For each* i, *define* $r_i = r_i(\sigma)$. *Also define* $R = R(\sigma)$, $T = \sum_{i \in R} r_i$ *and* $m = \max\{r_i \mid i \in R\}$. *Then none of the following conditions are satisfied.*

(R1) $r_1 \ge 2$ *and* $m = 1$.

(R2) $t = 1$.

(R3) $r_1 = 1$ *and* $t = 2$.

(R4) *For some* $i \ge 2$, $r_i = 1$ *and* $(i, j) = 1$ *for all* $i \ne j \in R$.

(R5) *For some prime* p, $r_1 = p$ *and* $m < p$.

(R6) $\max R > 10$.

Proof: By Theorem 4·5, we know that G cannot have any non-trivial ER subsets, since G is not a NTLP. If R1 is satisfied, $\text{fix}(\Lambda)$ is ER, since $N_\Gamma(\Lambda)$ fixes each of the non-trivial orbits of Λ. If R2 is satisfied, the non-trivial orbit of Λ is ER. If R3 is satisfied, G is strongly regular. If R4 is satisfied, the orbit of size i is ER, since the coprimality condition ensures that it is trivially joined to every other orbit.

Suppose R5 is satisfied. Then by Theorem 1·16, a Sylow p-subgroup P of $N_\Gamma(\Lambda)$ acts transitively on fix(P). Also, P permutes the non-trivial orbits of Λ, by Theorem 1·15(a), and so P fixes each non-trivial orbit of $\langle \Lambda \rangle$ set-wise, since $m < p$. Therefore fix(Λ) is ER.

Suppose R6 is satisfied. Let $\ell \geq 11$ be the length of the longest orbit of Λ. Since $n \leq 18$, condition R4 is satisfied if $\ell = 11, 13, 15, 16$ or 17. If $\ell = 14$, either R1 or R4 is satisfied. So suppose $\ell = 12$. The only possibilities for σ which do not satisfy any of the conditions R1 - R5 are $(18;\ 1^2,\ 2^2,\ 12)$ and $(18;\ 1,\ 2,\ 3,\ 12)$.

Suppose $\sigma = (18;\ 1^2,\ 2^2,\ 12)$. Since the neighbourhood of any fixed point is a union of orbits, the degree of G is at most 5. Now, if any point in a 2-orbit is adjacent to a point in the 12-orbit, it is adjacent to at least six such points. Therefore the 12-orbit is ER.

Suppose $\sigma = (18;\ 1,\ 2,\ 3,\ 12)$. Then the degree of G is at most 5, as before. Therefore the 2-orbit is not joined at all to the 12-orbit, as before, and is trivially joined to the 3-orbit, since $(2, 3) = 1$. Therefore the 2-orbit is ER. □

5·6 THEOREM *Let* Σ^1 *be the set of all numerical partitions* σ *of* n, *where* $n \in \{8, 9, 10, 12, 14, 15, 16, 18\}$, $1 \leq r_1(\sigma) < n$, $r_1(\sigma) | n$ *and which satisfy none of the conditions* R1 - R6. *Then* Σ^1 *is sufficient.*

Proof: Let $G \in \mathcal{G}$. Then by Theorems 1·15, 1·16 and 5·5, $\Gamma_1 \in J(\Gamma)$ and $\sigma(\Gamma_1) \in \Sigma^1$. □

Altogether, Σ^1 contains 154 numerical partitions, as detailed below.

$\underline{n}$	$\underline{\text{partitions}}$
8	3
9	3
10	4
12	13
14	14
15	23
16	39
18	55

Since the total number of numerical partitions of elements of $\{8, 9, 10, 12, 14, 15, 16, 18\}$ is 1098, we have made considerable progress. However, if the computation of $\underset{\sim}{Q}$ is to be made feasible, the size of Σ^1 must be reduced much further.

Let $J^*(\Gamma)$ denote the set of subgroups $\Lambda \in J(\Gamma)$ which satisfy the additional property that $\text{fix}(\Lambda)$ is a block for Γ. Recall from Chapter 1 that $\Gamma_1 \in J^*(\Gamma)$ and $\langle \text{Syl}_p(\Gamma_1)\rangle \in J^*(\Gamma)$ if $p \mid |\Gamma_1|$.

5·7 LEMMA *Let* $G \in \boldsymbol{G}$, *and let* $1 \neq P \in \text{Syl}_p(\Gamma_1)$ *for some prime* p. *If* $\text{fix}(P') = \text{fix}(P)$ *for every* $P' \in \text{Syl}_p(\Gamma_1)$, $P \in J^*(\Gamma)$.

Proof: We already know that $P \in J(\Gamma)$. Furthermore, $\text{fix}(P) = \text{fix}(\langle \text{Syl}_p(\Gamma_1)\rangle)$ and so $P \in J^*(\Gamma)$ since $\langle \text{Syl}_p(\Gamma_1)\rangle \in J^*(\Gamma)$. □

5·8 LEMMA *Let* $G \in \boldsymbol{G}$, *and let* $p \mid |\Gamma_1|$ *be prime. Suppose that* Γ_1 *has a non-trivial orbit* W *of length* ℓ. *Then the orbits of* $\langle \text{Syl}_p(\Gamma_1)\rangle$ *on* W *are of equal size* r, *where* $r = 1$ *or* $r \geq p$. *Also, if* $p \mid \ell$ *then* $r > 1$.

Proof: The equality of the orbit sizes follows from Theorem 1·15(e), since $\langle \text{Syl}_p(\Gamma_1)\rangle \leq \Gamma_1$. Now suppose that $r > 1$. Then any $x \in W$ is moved by some $P \in \text{Syl}_p(\Gamma_1)$, so that $r \geq p$. The final assertion follows from Theorem 1·15(c). □

Let σ be a numerical partition. For prime p, we will say that σ satisfies *condition A_p* if

(i) for every $\ell \in R(\sigma)$, either $\ell < p$ or $p|\ell$,

(ii) for some $\ell \in R(\sigma)$, ℓ is not a power of p, and

(iii) for some $\ell \in R(\sigma)$, $p|\ell$.

Similarly, σ satisfies *condition B_p* if

(iv) for some $\ell \in R(\sigma)$, $1 < \ell < p$, and

(v) for some $\ell \in R(\sigma)$, $p|\ell$.

5·9 LEMMA *Let $G \in \mathcal{G}$, $p \geq 2$ be prime and $P \in Syl_p(\Gamma_1)$. Then if $\sigma(\Gamma_1)$ satisfies condition A_p, $P \in J^*(\Gamma)$ and $P \neq \Gamma_1$.*

Proof: $P \neq 1$ by (iii) and $P \neq \Gamma_1$ by (ii). Furthermore, any $P' \in Syl_p(\Gamma_1)$ fixes each orbit of size less than P and moves each point in each orbit of size divisible by p. Therefore $P \in J^*(\Gamma)$ by Lemma 5·7. □

5·10 LEMMA *Let $G \in \mathcal{G}$, $p \geq 3$ be prime and $\Lambda = \langle Syl_p(\Gamma_1)\rangle$. If $\sigma(\Gamma_1)$ satisfies condition B_p then $\Lambda \in J^*(\Gamma)$ and $\Lambda \neq \Gamma_1$.*

Proof: $|\Lambda| > 1$ by (v) and $\Lambda \neq \Gamma_1$ by (iv). Therefore $\Lambda \in J^*(\Gamma)$. □

5·11 THEOREM *Let Σ^2 be the set of numerical partitions formed from Σ^1 by deleting any member satisfying A_2, A_3, A_5, B_3 or B_5. Then Σ^2 is sufficient.*

Proof: Notice firstly that, due to the definition of Σ^1, $\sigma(\Lambda) \in \Sigma^1$ for any $\Lambda \in J^*(\Gamma)$, for any $G \in \mathcal{G}$.

(a) Suppose that $\sigma = \sigma(\Gamma_1)$ and that σ satisfies A_p, for some $p \in \{2, 3, 5\}$. Let $P \in Syl_p(\Gamma_1)$. Then $\sigma(P) \in \Sigma^1$, by Lemma 5·9. Furthermore the orbit lengths of $\sigma(P)$ are all powers of P, so that $\sigma(P)$ satisfies no A_q or B_q. Therefore $\sigma(P) \in \Sigma^2$.

(b) Suppose that $\sigma = \sigma(\Gamma_1)$ and that σ satisfies B_3. Let $\Lambda = \langle Syl_3(\Gamma_1)\rangle$ and $\sigma' = \sigma(\Lambda)$. Then $\sigma' \in \Sigma^1$ by Lemma 5·10, and so $\sigma' \in \Sigma^2$ unless σ' satisfies A_2, A_3, A_5, B_3 or B_5. By its definition, Λ has at least three fixed points and no 2-orbits.

(i) Suppose σ' satisfies A . Then the possible non-trivial orbit lengths of Λ are 4, 6, 8 and 10, with at least one orbit of size 6 or 10. The only such partitions in Σ^1 are $(16;\ 1^4,\ 6^2)$ and $(18;\ 1^6,\ 6^2)$. Both of these satisfy A_3 and so are treated below.

(ii) Suppose σ' satisfies A_3. Then since $Syl_3(\Lambda) = Syl_3(\Gamma_1)$ all the elements of $Syl_3(\Gamma_1)$ have the same fixed points, and so $\sigma(P) \in \Sigma^2$ for $P \in Syl_3(\Gamma_1)$, as in case (a).

(iii) Suppose σ' satisfies A_5 or B_5. Then we find that Σ^1 does not contain any such partitions.

(iv) Suppose σ' satisfies B_3. Then Λ has a 2-orbit, which is impossible.

(c) Suppose that $\sigma = \sigma(\Gamma_1)$ and that σ satisfies B_5. Let $\Lambda = \langle Syl_5(\Gamma_1)\rangle$ and $\sigma' = \sigma(\Lambda)$. Then Λ has at least 3 fixed points and no orbits of size 2, 3 or 4. The only such partitions in Σ^1 which satisfy A_2, A_3, A_5, B_3 or B_5 are $(16;\ 1^4,\ 6^2)$ and $(18;\ 1^6,\ 6^2)$. In either case Lemma 5·8 and condition R6 prove that the two 6-orbits are orbits of Γ_1. However, σ satisfies B_5 and so Γ_1 has an orbit of length 5 or 10. For $\sigma' = (16;\ 1^4,\ 6^2)$ this is impossible, and for $\sigma' = (18;\ 1^6,\ 6^2)$ we get $\sigma = (18;\ 1,\ 5,\ 6^2)$, which violates R4. Therefore $\sigma' \in \Sigma^2$. □

The definition of Σ^2 ensures that, for any $G \in \mathcal{G}$, there is some $\Lambda \in J^*(\Gamma)$ such that $\sigma(\Lambda) \in \Sigma^2$. Moreover, we can assume that $\Lambda = \Gamma_1$ unless $\sigma(\Gamma_1)$ satisfies A_2, A_3, A_5, B_3 or B_5. In the latter case, we can assume that either $\Lambda \in Syl_p(\Gamma_1)$ or $\Lambda = \langle Syl_p(\Gamma_1)\rangle$ for some $p \in \{2, 3, 5\}$. □

5·12 THEOREM *Form the set* Σ^3 *from* Σ^2 *by removing the numerical partitions* $(14; 1, 3, 4, 6)$, $(15; 1, 3^2, 4^2)$, $(16; 1^2, 3^2, 4^2)$, $(18; 1, 3, 6, 8)$, $(18; 1, 3, 4^2, 6)$, $(18; 1, 3^3, 4^2)$ *and* $(18; 1, 3^2, 4, 6)$. *Then* Σ^3 *is sufficient.*

Proof: In each case there are less than three fixed points, and the orbit sizes are not powers of the same prime. Therefore, by the preceding remarks, we can assume that we are dealing with $\sigma(\Gamma_1)$ in each case.

Suppose $\sigma(\Gamma_1) = (18; 1, 3, 6, 8)$. Then the degree k of G is either 3, 6 or 8. Let $\theta(\Gamma_1) = \{V_1, V_2, V_3, V_4\}$, where $|V_1| = 1$, $|V_2| = 3$, $|V_3| = 6$ and $|V_4| = 8$, and let each vertex in V_i be adjacent to e_{ij} vertices of V_j, for $1 \le i,j \le 4$. Now $e_{34} = 0$, 4 or 8 and $e_{24} = 0$ or 8. Hence $k \neq 3$, or else V_4 is ER. Suppose instead that $k = 6$. Then $e_{24} = 0$, which implies that $e_{34} = 4$ since otherwise V_4 would be ER. Similarly, since V_2 cannot be ER, $e_{23} > 0$. The only other possibility is $e_{23} = 2$ since otherwise the vertices in V_3 would have degree greater than 6. (They are all adjacent to V_1 as well.) But then we must have $e_{22} = 3$, which is impossible. Therefore $k = 8$. However we must then have $e_{24} = 0$ (or V_2 is ER), and so $e_{22} = 2$ and $e_{23} = 6$, making V_2 ER anyhow. Therefore $(18; 1, 3, 6, 8)$ cannot occur at all.

For each of the other numerical partitions, note that $Syl_3(\Gamma_1) \neq \{1\}$. Let $\sigma' = \sigma(P)$ for some $P \in Syl_3(\Gamma_1)$, and let f be the number of fixed points of $\Lambda = \langle Syl_3(\Gamma_1)\rangle$. Recall from Lemma 5·8 that Λ either fixes a 4-orbit point-wise or is transitive on it, and from Corollary 1·18 that $f|n$. In each case we will show that $\sigma' \in \Sigma^3$.

Say $\sigma(\Gamma_1) = (14; 1, 3, 4, 6)$. Since $f|14$, Λ is transitive on the 4-orbit. Therefore $\sigma' = (14; 1^2, 3^4) \in \Sigma^3$.

Say $\sigma(\Gamma_1) = (15; 1, 3^2, 4^2)$. Since $f|15$, Λ cannot fix both 4-orbits. If it fixes exactly one, $\sigma(\Lambda) = (15; 1^5, 3^2, 4)$, which violates R4. Hence $\sigma' = (15; 1^3, 3^4) \in \Sigma^3$.

Say $\sigma(\Gamma_1) = (16; 1^2, 3^2, 4^2)$. Since $f|16$, Λ is transitive on both 4-orbits. Therefore $\sigma' = (16; 1^4, 3^4) \in \Sigma^3$.

Say $\sigma(\Gamma_1) = (18; 1, 3, 4^2, 6)$ or $(18; 1, 3^3, 4^2)$. Since $f|18$, Λ cannot fix exactly one 4-orbit. Therefore $\sigma' = (18; 1^3, 3^5)$ or $(18; 1^9, 3^3)$, both of which are in Σ^3. If $\sigma(\Gamma_1) = (18; 1^2, 3^2, 4, 6)$ then $\sigma' = (18; 1^3, 3^5)$ or $(18; 1^6, 3^4)$, both of which are in Σ^3. □

The reason we went to the trouble of eliminating the seven numerical partitions in Theorem 5·12 is that in each of the remaining partitions, the cell sizes are all powers of the same prime. Theorem 1·15(c) and the fact that $Syl_p(\langle Syl_p(\Gamma_1)\rangle) = Syl_p(\Gamma_1)$, immediately imply the following theorem.

5·13 THEOREM *For any* $G \in \mathcal{G}$, *there is* $p \in \{2, 3, 5, 7\}$ *such that* $\sigma(P) \in \Sigma^3$ *for* $P \in Syl_p(\Gamma_1)$. □

The only numerical partition in Σ^3 which actually involves $p = 7$ is $(16; 1^2, 7^2)$. We will eliminate this partition and a few other potentially troublesome partitions in the next theorem.

5·14 THEOREM *Form* Σ^4 *from* Σ^3 *by deleting the numerical partitions* $(8; 1^2, 3^2)$, $(10; 1^2, 4^2)$, $(12; 1^2, 5^2)$, $(16; 1^2, 7^2)$, $(18; 1^2, 8^2)$, $(16; 1^8, 2^2, 4)$, $(14; 1^2, 2^2, 8)$, $(16; 1^4, 2^2, 8)$ *and* $(18; 1^6, 2^2, 8)$.

Then Σ^4 *is sufficient.*

Proof: In each case we can assume that $\sigma = \sigma(P)$ for $P \in Syl_p(\Gamma_1)$, where $p \in \{2, 3, 5, 7\}$.

Suppose firstly that σ is of the form $(2r + 2; 1^2, r^2)$. Since $G \in \mathcal{G}$, G has degree r and the two fixed points are not adjacent. If they are adjacent to the same r-orbit, fix(P) is ER. On the other hand, if the fixed points are adjacent to different r-orbits, G has diameter greater than two, and so is a switching graph, by Theorem 4•23.

Suppose $\sigma(P) = (16; 1^8, 2^2, 4)$. From the remark preceding Theorem 5•12, we can assume that fix(P) is a block of Γ. (This may not be true for some of the partitions involved in the proof of Theorem 5•12.) Therefore there is an element $\gamma \in \Gamma$ such that $supp(P^\gamma) \cap supp(P) = \emptyset$. Let $P' = \langle P^\gamma, P \rangle$. Then $\sigma(P') = (16; 2^4, 4^2)$. Let v be a vertex in one of the 2-orbits. Then P'_v is a 2-group fixing a vertex, but strictly larger than P, contradicting the assumption that $P \in Syl_2(\Gamma_1)$.

Suppose now that $\sigma(P) = (12 + 2r; 1^{2r}, 2^2, 8)$, for $r \in \{1, 2, 3\}$. Since $N_\Gamma(P)$ acts transitively on fix(P), and there are no non-trivial ER subsets, half of the fixed points are adjacent to one 2-orbit and half to the other, and each point in a 2-orbit is adjacent to 4 points in the 8-orbit. Therefore a point in a 2-orbit has degree at least 4 + r. Also, a fixed point has degree at most 2r + 1. Therefore $r \geq 3$. For the case $(18; 1^6, 2^2, 8)$ we infer from the foregoing that each fixed point is adjacent to the other five fixed points and to one of the 2-orbits. However, this implies that fix(P) can be partitioned into two ER subsets. □

Our final attack on the number of numerical partitions $\sigma \in \Sigma$ is aimed at those with $r_1(\sigma) = 1$. The reason is that for these partitions the property that $N_\Gamma(\Lambda)$ acts transitively on $\mathrm{fix}(\Lambda)$ is trivial, and so of no use in reducing the number of quotient matrices.

5·15 LEMMA *Let* $G \in \mathcal{G}$ *and* $P \in \mathrm{Syl}_p(\Gamma_1)$ *for some prime* p. *Suppose that* $|\mathrm{fix}(P)| = 1$ *and that* P *has an orbit* W *of length* p. *Let* $w \in W$. *Then, if* $|P_w| > 1$, $P_w \in J(\Gamma)$.

Proof: Since $|\mathrm{fix}(P)| = 1$, $P \in \mathrm{Syl}_p(\Gamma)$. Furthermore $P_w = P \cap P'$, where $P' \in \mathrm{Syl}_p(\Gamma_w)$, and is of the largest size possible for any intersection of two distinct Sylow p-subgroups of Γ, since $[P : P_w] = p$. Let $\gamma \in \Gamma$ be such that $P_w^{\ \gamma} \leq \Gamma_1$. Since $P_w^{\ \gamma}$ is a p-group, $P_w^{\ \gamma\delta} \leq P$ for some $\delta \in \Gamma_1$. By Lemma 1·10, $P_w^{\ \gamma\delta} = P_w^{\ \beta}$ for some $\beta \in N_\Gamma(P)$. But $|\mathrm{fix}(P)| = 1$ and so $\beta \in \Gamma_1$. Hence $P_w^{\ \gamma} = P_w^{\ \alpha}$ where $\alpha = \beta\delta^{-1} \in \Gamma_1$. Therefore $P_w \in J(\Gamma)$ by Theorem 1·16. □

5·16 THEOREM *Form the set* Σ *of numerical partitions by deleting* $(9;\ 1,\ 2^2,\ 4)$, $(15;\ 1,\ 2,\ 4,\ 8)$, $(15;\ 1,\ 2,\ 4^3)$, $(15;\ 1,\ 2^3,\ 8)$, $(15;\ 1,\ 2^3,\ 4^2)$, $(15;\ 1,\ 2^5,\ 4)$ *and* $(16;\ 1,\ 3^2,\ 9)$ *from* Σ^4. *Then* Σ *is sufficient.*

Proof: In each case we can assume (as shown earlier) that the partition σ to be deleted is $\sigma(P)$ for some $P \in \mathrm{Syl}_p(\Gamma_1)$, where $p = 2$ or 3. In each case let $\sigma' = \sigma(P_w)$, where w is a vertex in an orbit of length p. Since there is at least one orbit of length greater than p, $|P_w| > 1$.

For each σ we consider all the possible values of σ'. Apparent possibilities not mentioned violate either R1 or R5, and so cannot actually occur.

If $\sigma = (9;\ 1,\ 2^2,\ 4)$ then $\sigma' = (9;\ 1^3,\ 2^3)$, which is in Σ. If $\sigma = (15;\ 1,\ 2,\ 4,\ 8)$ then $\sigma' = (15;\ 1^3,\ 2^6)$, $(15;\ 1^3,\ 2^4,\ 4)$ or $(15;\ 1^3,\ 4^3)$, all of which are in Σ. If $\sigma = (15;\ 1,\ 2,\ 4^3)$ then $\sigma' = (15;\ 1^3,\ 2^6)$, $(15;\ 1^3,\ 2^4,\ 4)$ or $(15;\ 1^3,\ 4^3)$, all of which are in Σ. If $\sigma = (15;\ 1,\ 2^3,\ 8)$ then $\sigma' = (15;\ 1^3,\ 2^6)$ or $(15;\ 1^5,\ 2^5)$, both of which are in Σ. If $\sigma = (15;\ 1,\ 2^3,\ 4^2)$ then $\sigma' = (15;\ 1^3,\ 2^6)$, $(15;\ 1^3,\ 2^4,\ 4)$ or $(15;\ 1^5,\ 2^5)$, all of which are in Σ. If $\sigma = (15;\ 1,\ 2^5,\ 4)$ then $\sigma' = (15;\ 1^3,\ 2^6)$, $(15;\ 1^3,\ 2^4,\ 4)$, $(15;\ 1^5,\ 2^5)$ or $(15;\ 1^9,\ 2^3)$. The first three are in Σ. For the case $\sigma' = (15;\ 1^9,\ 2^3)$ see below. If $\sigma = (16;\ 1,\ 3^3,\ 9)$ then $\sigma' = (16;\ 1^4,\ 3^4)$, which is in Σ.

Suppose that $\sigma(P) = (15;\ 1,\ 2^5,\ 4)$, where the 2-orbits of P are $V_1, V_2, \cdots, V_5$, and suppose that $\sigma(P_w) = (15;\ 1^9,\ 2^3)$ for any $w \in V_1 \cup V_2 \cup \cdots \cup V_5$. In each case two of the 2-orbits of P_w are in the 4-orbit of P, since $[P : P_w] = 2$. Without loss of generality then, fixing V_1 leaves V_2 unfixed, and fixing V_2 leaves either V_1 or V_3 unfixed. But then fixing V_4 leaves either V_1 and V_2 or V_2 and V_3 unfixed, contrary to hypothesis. Therefore we can find $w \in V_1 \cup V_2 \cup \cdots \cup V_5$ such that $\sigma(P_w) \neq (15;\ 1^9,\ 2^3)$. □

The set Σ comprises the 57 numerical partitions given in Table 5·1.

5·17 Neighbourhood partitions

Let $G \in \mathcal{G}$ and $\Lambda \in J(\Gamma)$. Since $\Lambda \leq \Gamma_1$, $\theta(\Lambda)$ induces a partition θ' on $N(1, G)$. The associated numerical partition $\sigma = \sigma(\theta')$ is called the *neighbourhood partition* corresponding to the pair (G, Λ). In other words, σ specifies the sizes of the orbits of Λ to which a fixed point is adjacent. Since $\Lambda \in J(\Gamma)$, σ is independent of the choice of fixed point.

$(8;\ 1^2,\ 2^3)$
$(8;\ 1^4,\ 2^2)$

$(9;\ 1,\ 2^4)$
$(9;\ 1^3,\ 2^3)$

$(10;\ 1^2,\ 2^2,\ 4)$
$(10;\ 1^2,\ 2^4)$
$(10;\ 1,\ 3^3)$

$(12;\ 1^2,\ 2,\ 4^2)$
$(12;\ 1^2,\ 2^3,\ 4)$
$(12;\ 1^2,\ 2^5)$
$(12;\ 1^4,\ 4^2)$
$(12;\ 1^4,\ 2^2,\ 4)$
$(12;\ 1^4,\ 2^4)$
$(12;\ 1^6,\ 2^3)$
$(12;\ 1^3,\ 3^3)$
$(12;\ 1^6,\ 3^2)$

$(14;\ 1^2,\ 4^3)$
$(14;\ 1^2,\ 2^2,\ 4^2)$
$(14;\ 1^2,\ 2^4,\ 4)$
$(14;\ 1^2,\ 2^6)$
$(14;\ 1^2,\ 3^4)$

$(15;\ 1,\ 2^7)$
$(15;\ 1^3,\ 4^3)$
$(15;\ 1^3,\ 2^4,\ 4)$
$(15;\ 1^2,\ 2^6)$
$(15;\ 1^5,\ 2^5)$
$(15;\ 1^3,\ 3^4)$

$(16;\ 1^2,\ 2,\ 4^3)$
$(16;\ 1^2,\ 2^3,\ 8)$
$(16;\ 1^2,\ 2^3,\ 4^2)$
$(16;\ 1^2,\ 2^5,\ 4)$
$(16;\ 1^2,\ 2^7)$
$(16;\ 1^4,\ 4^3)$
$(16;\ 1^4,\ 2^2,\ 4^2)$
$(16;\ 1^4,\ 2^4,\ 4)$
$(16;\ 1^4,\ 2^6)$
$(16;\ 1^8,\ 4^2)$
$(16;\ 1^8,\ 2^4)$
$(16;\ 1,\ 3^5)$
$(16;\ 1^4,\ 3^4)$
$(16;\ 1,\ 5^3)$

$(18;\ 1^2,\ 4^2,\ 8)$
$(18;\ 1^2,\ 4^4)$
$(18;\ 1^2,\ 2^2,\ 4,\ 8)$
$(18;\ 1^2,\ 2^2,\ 4^3)$
$(18;\ 1^2,\ 2^4,\ 8)$
$(18;\ 1^2,\ 2^4,\ 4^2)$
$(18;\ 1^2,\ 2^6,\ 4)$
$(18;\ 1^2,\ 2^8)$
$(18;\ 1^6,\ 4^3)$
$(18;\ 1^6,\ 2^2,\ 4^2)$
$(18;\ 1^6,\ 2^4,\ 4)$
$(18;\ 1^6,\ 2^6)$
$(18;\ 1^3,\ 3^5)$
$(18;\ 1^6,\ 3^4)$
$(18;\ 1^9,\ 3^3)$
$(18;\ 1^3,\ 5^3)$

Table 5·1

5·18 LEMMA *Let* $G \in \mathcal{G}$, $\Lambda \in J(\Gamma)$ *and* $\sigma_1 = \sigma(\Lambda)$. *Then the corresponding neighbourhood partition* σ_2 *satisfies the following conditions, where* k *is the degree of* G.

(a) σ_2 *is a numerical partition of* k.

(b) *For all* i, $r_i(\sigma_2) \leq r_i(\sigma_1)$.

(c) $r_1(\sigma_2) < r_1(\sigma_1)$.

(d) $r_1(\sigma_2) < k$.

(e) $r_1(\sigma_1)r_1(\sigma_2)$ *is even.*

(f) *If* $r_1(\sigma) \geq 2$, *there is some* $i \geq 2$ *such that* $0 < r_i(\sigma_2) < r_i(\sigma_1)$.

Proof: Conditions (a), (b) and (c) are obvious. Condition (d) is necessary to prevent G from being disconnected. Condition (e) follows from the fact that the subgraph fix(Λ) is regular. Finally, if condition (f) was not satisfied, fix(Λ) would be ER. □

Let T be the set of all partition pairs (σ_1, σ_2) such that $\sigma_1 \in \Sigma$ and σ_2 satisfies conditions (a) - (f) of Lemma 5·18. Then for each $G \in \mathcal{G}$, there is some $\Lambda \in J(\Gamma)$ such that $(\sigma_1, \sigma_2) \in T$, where $\sigma_1 = \sigma(\Lambda)$ and σ_2 is the corresponding neighbourhood partition. We can further assume that $\Lambda \in \mathrm{Syl}_p(\Gamma_1)$ for some $p \in \{2, 3, 5\}$, except possibly when σ_1 is one of the partitions $(9; 1^3, 2^3)$, $(15; 1^3, 2^6)$, $(15; 1^3, 4^3)$, $(15; 1^5, 2^5)$ and $(16; 1^4, 3^4)$.

5·19 Complete-join matrices

Let G be any graph and let $\pi = (V_1, V_2, \cdots, V_m)$ be a partition of V. Define $K(G, \pi)$ to be the graph whose vertices are $V_1, V_2, \cdots, V_m$ and where V_i is adjacent to V_j if either $i \neq j$ and V_i is completely joined to V_j in G or $i = j$ and the subgraph V_i is complete. Thus $K(G, \pi)$ may have loops on some vertices.

We will also regard the vertices of $K(G, \pi)$ to be labelled with the size of the corresponding cell of π, and will refer to this label as the *size* of the vertex. The set of size-preserving automorphisms of $K(G, \pi)$ will be denoted by $\mathrm{Auts}(K(G, \pi))$. Subgraphs of $K(G, \pi)$ will be considered to inherit their vertex sizes from $K(G, \pi)$.

If $\Lambda \le \Gamma$, $K(G, \theta(\Lambda))$ will be abbreviated to $K(G, \Lambda)$. Note that if $\Lambda \in J(\Gamma)$, $K(G, \Lambda)$ determines both $\sigma(\Lambda)$ and the corresponding neighbourhood partition.

The next major step in the construction of $\underset{\sim}{Q}$ will be to find a family $\underset{\sim}{K}$ of graphs such that, for any $G \in \boldsymbol{G}$, there is some $\Lambda \in J(\Gamma)$ such that $K(G,\Lambda) \in \underset{\sim}{K}$.

Let $f \ge 1$, $r \ge 1$ and $s \ge 0$. Define $\boldsymbol{F}(f, r, s)$ to be the set of all $f{\times}r$ 0-1 matrices F with the following properties.

(a) Each row of F has exactly s ones.

(b) The columns of F are in lexicographic order.

(c) Let H be the graph with adjacency matrix $\left[\begin{array}{c|c} 0 & F \\ \hline F^T & 0 \end{array}\right]$. Then the group of automorphisms of H which fix the partition $\{1, 2, \cdots, f | f+1, f+2, \cdots, f+r\}$ acts transitively on $\{1, 2, \cdots, f\}$.

In Table 5·2 we give the size of $\boldsymbol{F}(f, r, s)$ for various f, r and s. Obviously, $|\boldsymbol{F}(f, r, 0)| = 1$ for any f, r.

The reason for our interest in $\boldsymbol{F}(f, r, s)$ is revealed in the following theorem.

5·20 THEOREM *Let* $G \in \boldsymbol{G}$, *and* $\Lambda \in J(\Gamma)$. *Let* $\sigma_1 = \sigma(\Lambda)$ *and let* σ_2 *be the corresponding neighbourhood partition. Let* $1 = \ell_1 < \ell_2 < \cdots < \ell_t$ *be the different orbit sizes of* Λ. *Then for some ordering of the orbits of* Λ *in non-decreasing order of size and some matrix* M, *the adjacency matrix of* $K(G, \Lambda)$ *is*

r	s	f = 1	2	3	4	5	6	8	9
1	1	1	1	1	1		1		
2	1	1	2	1	4		11	36	
	2	1	1	1	1		1		
3	1	1	2	2	4		26		281
	2	1	2	2	4		26		281
	3	1	1	1	1		1		
4	1	1	2	2	5		26	141	
	2	1	3	3	10		81	386	
	3	1	2	2	5		26	141	
	4	1	1	1			1		
5	1	1	2	2		2			
	2	1	3	3		13			
	3	1	3	3		13			
	4	1	2	2					
6	1	1	2	2	5		27		
	2	1	3	4	14		226		
	3	1	4	4	22		436		
	4	1	3	4			226		
7	1	1	2						
	2	1	3						
	3	1	4						
	4	1	4						
8	1	1	2						
	2	1	3						
	3	1	4						
	4	1	5						

Table 5·2

$$\left[\begin{array}{c|c|c|c|c} T & F_2 & F_3 & \cdots & F_t \\ \hline F_2^T & & & & \\ \cline{1-1} F_3^T & & & & \\ \cline{1-1} \vdots & & M & & \\ \cline{1-1} F_t^T & & & & \end{array}\right],$$

where T *is the adjacency matrix of a transitive graph of order* $r_1(\sigma_1)$ *and degree* $r_1(\sigma_2)$, *and* $F_i \in \boldsymbol{F}(r_1(\sigma_1), r_{\ell_i}(\sigma_1), r_{\ell_i}(\sigma_2))$, *for* $2 \le i \le t$.

Moreover, let H *be the graph whose adjacency matrix* A(H) *is formed from* $K(G, \theta(\Lambda))$ *by setting* $M = 0$. *Let* m *be the order of* H.

(i) *The maximum degree of* H *is less than the degree of* G.

(ii) Auts(H) *acts transitively on the set* $\{1, 2, \cdots, r_1(\sigma_1)\}$, *and*

(iii) Auts(H) *does not have a subgroup which fixes* $\{r_1(\sigma_1), r_1(\sigma_1) + 1, \cdots, m\}$ *point-wise and has exactly one non-trivial orbit.*

Proof: T is the subgraph induced by fix(Λ) and so is transitive since $\Lambda \in J(\Gamma)$. Each F_i depicts the way in which each fixed point is joined to the orbits of size ℓ_i. Condition (b) (of the definition of $\boldsymbol{F}(f, r, s)$ above) can be satisfied by simply permuting the columns of each F_i. Condition (c) follows from condition (ii) above, which follows from the observation that Auts(H) contains the representation of $N_\Gamma(\Lambda)$ on the orbits of Λ. Condition (iii) is necessary to prevent G from containing a non-trivial ER subset. Finally, if any vertex of H has degree equal to the degree of G, the corresponding orbit of Λ is non-trivial and ER.

5·21 LEMMA *Assume the notation of Theorem 5·20. If* H *and* H* *correspond to the same partition pair* (σ_1, σ_2) *and are isomorphic via a mapping preserving the vertex sizes, they correspond to the same family of graphs in* $\mathbf{G}$.

Proof: obvious. □

5·22 Construction of $\underset{\sim}{K}$

Let K^1 be the set of the 650 graphs H, as defined in Theorem 5·19, which correspond to some $(\sigma_1, \sigma_2) \in T$ and satisfy all the requirements of Theorem 5·18. Only one member of each isomorphism class (defined Lemma 5·21) is included. The following table gives the size of K^1 for each order n and degree k.

n \ k	3	4	5	6	7	8
8	2					
9		3				
10	3	2				
12	7	13	17			
14	4	7	6	10		
15		11		19		
16	13	24	45	63	73	
18	10	30	46	57	89	96

Let $G \in \mathbf{G}$. Then there is some $\Lambda \in J(\Gamma)$ such that the graph H corresponding to $K \doteq K(G, \Lambda)$ is in K^1. Let F be the set of vertices of K (or H) of size one, and let N be the set of vertices of K (or H) of size greater than one which are adjacent in H to at least one vertex of size one. The corresponding subsets of G will be denoted by F and N* respectively.

Suppose now that $|N^*| < k$. Since $k < \frac{n}{2}$ and $|F| \leq \frac{n}{2}$, N* is a cutset of G of size less than k. Since this is not possible for

$G \in \mathcal{G}$, the corresponding H can be eliminated. Altogether 43 graphs are thus eliminated from K^1 giving a new set K^2 containing 607 graphs.

The next step is to determine the possible induced subgraphs N of K. This computation is quite complicated and so will only be described in broad outline.

For each $v \in F$ let N_v be the subgraph of K induced by those vertices in N which are adjacent to v. Since Auts(K) acts transitively on F, the N_v are all isomorphic. Each possible subgraph N_v can be determined, and then the possible imbeddings of these subgraphs in N can be enumerated by a backtrack procedure, subject to the requirements of correct overlap and to degree restrictions (non-trivial ER orbits are avoided). The resulting graphs H' (presumably subgraphs of K containing all the edges within $F \cup N$) can be eliminated if Auts(H') does not act transitively on F. The set K^3 of all generated H' has 946 members distributed as below. Many members of K^2 yielded no members of K^3.

n \ k	3	4	5	6	7	8
8	2					
9		4				
10	3	3				
12	7	14	18			
14	4	9	8	20		
15		13		24		
16	13	26	44	74	132	
18	10	34	48	79	134	223

The next step is to determine which vertices of each $H' \in K^3$ could have loops in some $K \in \underset{\sim}{K}$ which has H' as a subgraph. This produces a set K^4 of 8088 graphs distributed as below. At this stage we can say that for each $K \in \underset{\sim}{K}$, the subgraph of K containing all loops and all edges in $F \cup N$ is in K^4.

n \ k	3	4	5	6	7	8
8	5					
9		16				
10	11	12				
12	20	54	64			
14	21	63	72	163		
15		75		178		
16	53	174	256	534	810	
18	51	248	352	728	1083	3045

Because of the large size of K^4 an effort will be made to reduce it before proceding further. The following techniques can be applied.

(a) Let $G \in \mathcal{G}$ correspond to some $H'' \in K^4$. Then G has nk/2 edges altogether. Let ℓ be the number of edges of G represented by edges of H", and let $p \in \{2, 3, 5\}$ be the prime dividing the vertex sizes of H". Then $p|(nk/2 - \ell)$, since the remaining edges of G are between non-trivial orbits of Λ and in non-complete subgraphs within the orbits of Λ. This requirement eliminates 3040 cases.

(b) Let $H'' \in K^4$ and let J be the subgraph of H" induced by these vertices adjacent to a given vertex $v \in F$. If $\overline{J}$ is disconnected, where G is any graph in $\mathcal{G}$ which corresponds to H". Therefore $\overline{G}$ is a NTLP, by Corollary 4·17, and so G is a NTLP, contrary to the assumption that $\ddot{G} \in \mathcal{G}$. This requirement eliminates 258 graphs H".

Let K^5 be the set of graphs in K^4 which have not been eliminated in (a) or (b) above. Then K^5 contains 4790 graphs, distributed as below.

n \ k	3	4	5	6	7	8
8	3					
9		7				
10	9	7				
12	12	32	40			
14	15	38	48	109		
15		35		85		
16	34	107	170	340	497	
18	34	141	230	452	640	1705

A complex breadth-first process has been used to fill in any extra edges necessary to make up each $H'' \in K^5$ to the possible graphs $K(G, \Lambda)$ from which it could be derived.

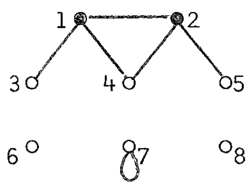
The possible sites for a new edge e of H" were broken into a number of classes which are necessarily invariant under $\mathrm{Auts}(K(G, \Lambda))$. For example, the sizes of the end-vertices and whether one or both of these vertices was in N were used. The resulting classes were then arranged in a convenient order. The program was designed to insert the edges only in order of class and, once the program had decided that no more edges of a given class were appropriate, it tested whether or not the size-preserving automorphism group was still transitive on F. The answer was almost always "yes", testifying to the success of the following theorem, which was used repeatedly. A method of isomorph-rejection was also used, ensuring that no subcase was ever considered more than once.

5·23 THEOREM *Let* Λ *and* $\Phi \le \Lambda$ *be permutation groups acting on a set* X. *Suppose* Φ *and* Λ *have a common orbit* W *and let* $w \in W$. *If* $Y \subseteq X$ *is fixed set-wise by* Λ_w, *and* $\gamma \in \Lambda$, *then* $Y^\gamma = Y^\phi$ *for some* $\phi \in \Phi$.

Proof: Since W is a common orbit of Λ and Φ, there is an element $\phi \in \Phi$ such that $w^\phi = w^\gamma$. Then $\gamma \in \Lambda_w \phi$. Since Y is fixed by Λ_w, we must have $Y^\gamma = Y^\phi$. □

Let $K \in \underset{\sim}{K}$ and let L be a spanning subgraph of K for which we know that Auts(K) ≤ Auts(L) and that Auts(L) acts transitively on F. Suppose that $\{x, y\} \subseteq V(K)$ is fixed by $\text{Auts}(L)_v$, where $v \in F$. Then Theorem 5·23 tells us that the orbit of {x, y} under Auts(K) is the same as the orbit of {x, y} under Auts(L).

EXAMPLE: K^5 contains the following graph L.



Here F = {1, 2}, vertices 3 - 8 have size 2 and $\text{Auts}(L) = \langle (1\ 2)(3\ 5), (6\ 8) \rangle$. The pair {3, 7} is fixed by $\text{Auts}(L)_1$, and so its orbit under Auts(K) is the same as under Auts(L), namely {{3, 7}, {5, 7}}. Therefore if we insert one of these edges we must insert the other one also.

The computation just described produced a set K^6 of 223159 graphs and required about 3½ hours of computer time on a Cyber 73 computer. K^6 satisfies the requirements for the set $\underset{\sim}{K}$, but we will first attempt to reduce its size (with only slight success).

(a) For each $K \in K^6$, each 2-orbit was examined to determine whether its degree in K was impossibly low. For example, if $\sigma = (18;\ 1^2,\ 2^8)$ and $k = 8$, no 2-orbit can have degree 0 in K, since its vertices cannot possibly be given degree more than seven by adding non-complete joins between orbits. This process eliminates 18555 cases.

(b) Let $K \in K^6$, and let c be the size of a component of the subgraph $N(1, \bar{K})$. Then by Theorem 4·20 applied to $\bar{G}$, either $n \leq 9$ or $2c > n - k - 1$. This test eliminates only 1556 cases.

(c) Suppose $n = 2k + 2$ and $v, w \in F$. Then if v and w are not adjacent, $N(v, K) \cap N(w, K) \neq \emptyset$, since otherwise G would be a switching graph, by Theorem 4·23. This test eliminates 3447 cases.

Let $\underset{\sim}{K}$ denote the set of all elements of K^6 not eliminated by the tests above. Then $\underset{\sim}{K}$ has 199601 elements distributed as below.

n	graphs
8	2
9	13
10	14
12	140
14	976
15	4452
16	12355
18	181649

5·24 Construction of $\underset{\sim}{Q}$

We are now in a position to construct a family of quotient matrices satisfying the requirements for $\underset{\sim}{Q}$.

For each $K \in \underset{\sim}{K}$, a simple backtrack program has been used to list all feasible ways of joining each orbit (vertex of K) to the other orbits or to itself. Having done this, another backtrack scheme produced 962131 possible quotient matrices. This scheme made considerable use of Theorem 5·23, and also used its knowledge of Auts(K) to eliminate isomorphs.

This set of possible quotient matrices is much too large, so we will expend considerable effort in reducing its size.

5·25 Necessary conditions on $Q \in \underset{\sim}{Q}$

We have computed a set $\underset{\sim}{Q}$ of 962131 matrices with the following property. For each $G \in \boldsymbol{\mathcal{G}}$ there is some $\Lambda \in J(\Gamma)$ such that $Q(G, \theta(\Lambda)) \in \underset{\sim}{Q}$. For the remainder of the chapter, $F = \mathrm{fix}(\Lambda)$.

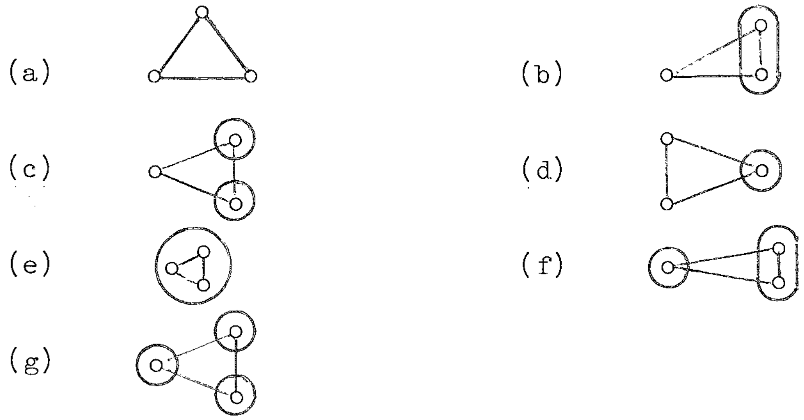
Quotient matrices are somewhat awkward for exact computation, since their entries are sometimes irrational. Consequently we have devised a somewhat different representation. Let $G \in \boldsymbol{\mathcal{G}}$, $\Lambda \in J(\Gamma)$, $\theta(\Lambda) = (V_1, V_2, \cdots, V_m)$ and $Q = Q(G, \theta(\Lambda))$. Define the symmetric m×m matrix $R = R(G, \Lambda)$ by

$$R_{ij} = Q_{ij}(\min\{|V_i|, |V_j|\}/\max\{|V_i|, |V_j|\})^{\frac{1}{2}}, \text{ for } 1 \le i,j \le m.$$

If $|V_i| \le |V_j|$, then R_{ij} is the number of vertices in V_i adjacent to each vertex in V_j. The quotient matrix Q is represented in the computer by R.

We now describe in detail a battery of tests which can be applied to each element of $\underset{\sim}{Q}$. These tests are so successful that all but 709 matrices can be eliminated. In other words, 961422 of them proved to be not equal to $Q(G, \theta(\Lambda))$ for any $G \in \boldsymbol{\mathcal{G}}$ and $\Lambda \in J(\Gamma)$.

5·26 The tests described in this section are only employed if there are no orbits of size greater than 4. Since G is transitive, each vertex lies on the same number of triangles. Call this number t. A triangle of G can appear in Q in seven different ways as indicated below, where the open circles represent non-trivial orbits of Λ.



The exact number of triangles of types (a) - (e) can be calculated from Q. For types (f) and (g) a table (165 entries) can be constructed by hand giving upper and lower bounds on the number of triangles, for each possible combination of the appropriate entries of R. For example, if $|V_i| = |V_j| = |V_\ell| = 4$, $R_{ij} = R_{i\ell} = 3$ and $R_{j\ell} = 2$, there are either 16 or 20 triangles of type (g). This table, plus the calculations for types (a) - (e), can be used to calculate the exact number t of triangles on a vertex $v \in F$ and bounds on the number of triangles on every other vertex of G.

NC1 : t is independent of the choice of $v \in F$.

NC2 : The upper and lower bounds for each $v \notin F$ include t.

NC3 : nt is divisible by 3.

The justification for NC3 is that nt/3 is the number of triangles in G.

Tests NC1 - NC3 are remarkably successful, eliminating all but 62818 elements of $\underset{\sim}{Q}$.

5·27 Consider the graph $H = H(G, \Lambda)$ defined in Section 4·26. Obviously, H can be determined from $Q(G, \theta(\Lambda))$. The nature of the components of H will be indicated by a symbol such as $H \sim (24, 222)$, which indicates that there are two components, one corresponding to a

2-orbit and a 4-orbit, and the other to three 2-orbits.

NC4 : If H has a component of type 22, 33 or 24, n is even.

The cases 22 and 33 are justified by Theorems 4·24 and 4·25. For the case 24, an examination of the possible ways of joining the two orbits, and the possible contents of the orbits, reveals that Γ has an element of the form (a b)(c d), whose support is the 4-orbit. Therefore, n is even by Theorem 4·25.

Recall from the remark following Lemma 5·18 that we can assume $\Lambda \in Syl_p(\Gamma_1)$, for some $p \in \{2, 3, 5\}$, unless $\sigma(\Lambda)$ is one of $(9; 1^3, 2^3)$, $(15; 1^3, 2^6)$, $(15; 1^3, 2^4, 4)$, $(15; 1^3, 4^3)$, $(15; 1^5, 2^5)$ and $(16; 1^4, 3^4)$. Many possible component types for H can be immediately eliminated by the use of Theorem 4·30.

NC5 : The following possibilities for H are impossible.

n = 12, H ~ (22, 222)

n = 14, H ~ (22, 2222), (222, 24)

n = 15, H ~ (222, 2222)

n = 16, H ~ (22, 22, 222), (22, 22, 24), (22, 222), (22, 2222), (22, 22222), (222, 2222), (222, 24), (2222, 24), (33, 333), (224, 24) (24, 44)

n = 18, H ~ (22, 22, 2222), (22, 22, 224), (22, 22, 44), (22, 222, 222), (22, 222, 24), (22, 2222), (22, 222222), (22, 224), (22, 44), (222, 22222), (222, 24), (2222, 224), (2222, 44), (22222, 24), (224, 44), (33, 333), (2224, 24), (24, 244)

5·28 Let J be the graph whose adjacency matrix is obtained from Q by changing every non-zero entry to one. If $\bar{N}(1, J)$ is disconnected, then so is $\bar{N}(1, G)$. Consequently, by Corollary 4·17 we have

NC6 : $\bar{N}(1, J)$ is connected.

5·29 For each orbit V_i of Λ, we can divide the vertices of G into classes according to their distance from V_i. This division can be determined from Q. Suppose that for each d, there are $\Delta(d)$ vertices at distance d or more from vertex 1.

NC7 : For each orbit V_i and each d, there are at most $\Delta(d)$ vertices at distance d or more from V_i.

NC8 : For any V_i, let d_i be the maximum distance of any vertex of G from V_i. Then if $0 < d < d_i$, there are at least k vertices of G at distance d from C_i

Condition NC8 follows from the assumption that G has connectivity k.

Tests NC4 - NC8 are not very successful, eliminating only 4365 cases, leaving 58454 cases remaining.

5·30 By far the most powerful necessary conditions which we have applied to $Q \in \underset{\sim}{Q}$ are based on the eigenvalue techniques described in Sections 4·31 - 4·40.

These computations were different from any of the earlier computations in that they necessitated the use of floating-point arithmetic, with its associated rounding error problems. The eigenvalues and eigenvectors of each Q were computed using adapted versions of routines from the IMSL library. These use Householder's method for reduction to tridiagonal form and the QR method for completing the diagonalisation. Since Q is real and symmetric, high accuracy eigenvalues can be expected (see Wilkinson [45]). The eigenvector problem is not so well conditioned, especially when there are eigenvalues which are close together. However, if we have a collection of eigenvalues, each significantly different in value from any eigenvalue not in the collection, the space spanned by all the corresponding eigenvectors will be found quite accurately (see [45]).

Since our largest Q has order 12, and the computations used approximately 14 decimal digit accuracy, the eigenvalues computed generally had errors of 10^{-11} or less. However we assumed only that their errors were less than 10^{-3}. We also checked that the computed set of eigenvectors was orthonormal to high accuracy (inner products less than 10^{-9} from their proper value). The latter check never failed.

Suppose that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$ are the computed eigenvalues, and $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots, \underset{\sim}{x}_m$ are the corresponding computed (orthonormal) eigenvectors. Let $\lambda_i \le \lambda_{i+1} \le \cdots \le \lambda_{i+r-1}$ be a contiguous subsequence of $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$ such that $\lambda_{j+1} - \lambda_j \le 10^{-3}$ for $i \le j \le i + r - 2$, $\lambda_i - \lambda_{i-1} > 10^{-3}$ (or $i = 1$) and $\lambda_{i+r} - \lambda_{i+r-1} > 10^{-3}$ (or $i + r - 1 = m$). It is possible that these r computed eigenvalues actually represent more than one eigenvalue of Q. However, the validity as opposed to the strength) of the tests described below will not suffer if we assume that we are considering an eigenvalue of Q with multiplicity r. In fact this assumption reduces the probability of a matrix Q being rejected accidentally due to errors in the computed eigenvectors (for the reasons noted above).

Now let $v \in F$ and compute $\rho(v) = \sum_{j=i}^{i+r-1} (\underset{\sim}{x}_j)_v^2$, where $(\underset{\sim}{x}_j)_v^2$ denotes the square of the v-th entry of $\underset{\sim}{x}_j$. From Corollary 4.34, Theorem 4.38 and Theorem 4.40 we obtain the following conditions.

NC9 : $\rho(v)$ is independent of the choice of $v \in F$ (verified within 10^{-3}).

NC10 : $n\rho(1)$ is an integer (verified within 10^{-2}).

NC11 : The integer nearest to $n\rho(1)$ is at least r.

NC12 : If $r = 1$ then λ_i is an integer (verified within 10^{-2}).

The wide deviation from integer allowed in NC10 and NC12 was designed to eliminate any chance of a matrix Q being rejected solely because of rounding error. It is highly likely that a number of matrices were passed when they should have been failed, but this is of minor importance.

Tests NC9 - NC12 eliminate all but 709 of the 58454 matrices to which they have been applied. The remaining matrices are distributed as below.

n	matrices
8	1
9	5
10	5
12	45
14	17
15	38
16	263
18	335

5·31 Generation of $\mathcal{G}$

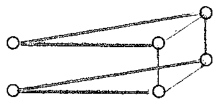
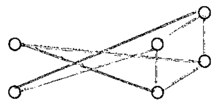
In the previous section we have constructed a family $\underset{\sim}{Q}^*$ of 709 matrices, such that for any $G \in \mathcal{G}$ there is some $\Lambda \in J(\Gamma)$ such that $Q(G, \theta(\Lambda)) \in \underset{\sim}{Q}^*$. The problem now is to use $\underset{\sim}{Q}^*$ to construct $\mathcal{G}$.

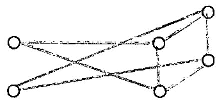
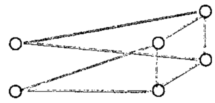
The quotient matrices Q for which Λ has an orbit of size 5 or more will be investigated by hand (see Section 5·32). There are only 7 of these. The remaining cases have been processed by a program which we now describe.

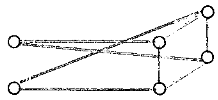
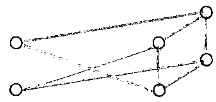
The edges of G which correspond to completely joined orbits are determined trivially. It is also possible to fill in the contents of each orbit, since for each order up to four there is only one transitive graph of each degree. The main difficulty is in making the non-complete joins between orbits.

For each of the pairs $(m_1, m_2) = (2, 2), (2, 4), (4, 2), (4, 4)$ and $(3, 3)$ a table T_1 was constructed (by machine) giving every possible equitable means of joining an orbit of length m_1 to an orbit of length m_2. This was indexed by m_1, m_2 and the number of edges between the orbits. A second table T_2 contained one member of each equivalence class in T_1, where two members of T_1 are equivalent if one can be obtained from the other by performing *allowable* permutations of the second orbit. For $m_2 = 2$ or 3, any permutation is allowable, but for $m_2 = 4$ only the eight automorphisms of a square are allowable, since a 4-orbit may contain a square or its complement.

For example, for $m_1 = 2$, $m_2 = 4$ and 4 edges between the orbits, T_1 contains six entries as below. For convenience we have drawn a square in each 4-orbit. T_2 contains just the first and fifth entries of T_1.

(a)  (b) 

(c)  (d) 

(e)  (f) 

Now define the graph $H = H(G, \Lambda)$ as in Section 4·26, but with loops omitted. As we noted earlier, H can be determined from Q. Each edge of H thus indicates a non-complete join in G between orbits of Λ. The edges of H were then labelled with *weights*, roughly indicating the complexity of the necessary join in G. More precisely, the higher an edge weight the greater the advantage in using T_2 rather than T_1 as the source of possible joins. The method of Prim [38] was then used to find a spanning forest of maximum total weight. The program MAXSPF in Nijenhuis and Wilf [35] was adapted for this purpose. Suppose that H has s vertices, t edges and c components. Prim's method produces a sequence $e_1, e_2, \cdots, e_{s-c}$ of edges of H with the property that at least one end-vertex of each edge is not an end-vertex of any earlier edge in the sequence.

The remaining $t - s + c$ edges of H were arranged in a sequence $e_{s-c+1}, e_{s-c+2}, \cdots, e_t$. The order was fairly arbitrary, except that edges which completed a neighbourhood (i.e. after insertion of the appropriate edges in G, the neighbourhood of some vertex of G would be known for the first time) were given precedence.

The complete list of possible transitive graphs G for each quotient matrix Q was then constructed using a backtrack program which inserted non-complete joins in G in the order $e_1, e_2, \cdots, e_t$. For $1 \le i \le s-c$ at least one of the orbits involved was only trivially joined to other orbits (before the join represented by e_i was inserted) and so could be subject to any allowable permutation without changing the graph at that stage. Therefore the list of possible joins could be drawn from T_2 rather than T_1. Joins corresponding to later edges of H were drawn from T_1. The only other non-trivial means of shortening the search was to keep track of the neighbourhoods of the vertices of G. Each time a new neighbourhood was completed, it was examined to see if its vertices had the same degree sequence as those of any earlier neighbourhoods.

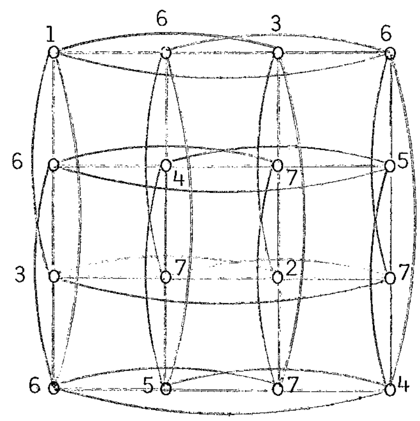
Any completed graphs generated by the program were tested to see if they were transitive, and if so whether they were isomorphic to any earlier generated graph (for the same quotient matrix). The program which tested the transitivity kept an eye open for transpositions in the automorphism group (none were ever found) but otherwise no effort was made to identify transitive graphs not in $\mathcal{G}$.

Despite the elaborate preparations, it was expected that this computation would occupy the computer for several hours at least. In fact it was all over in 12 minutes. More than half that time was taken up by the ten largest cases, each of which had two or more 4-orbits.

For the 702 quotient matrices processed, a total of 8584 graphs were produced. Of these, 7863 were intransitive and 127 were isomorphs, so that 594 transitive graphs were produced altogether. Out of the 702 quotient matrices, 120 produced no transitive graph, 584 produced one each, and 5 produced two each. In Figure 5·2 we give an example of a quotient matrix (actually $R(G, \Lambda)$ - see

$$R(G, \Lambda) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\sigma(\Lambda) = (16;\ 1^2,\ 2^3,\ 4^2)$$



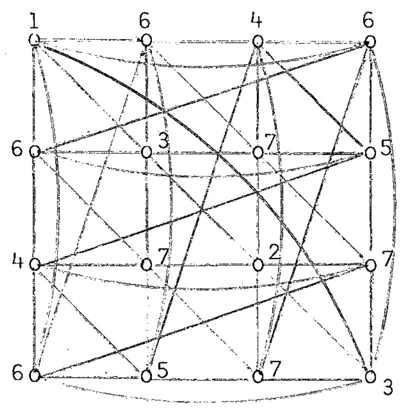


Figure 5·2

Section 5·25) having two transitive realisations. The two graphs drawn are strongly regular and so not actually in $\mathcal{G}$, but this was not noticed at the time. The labelling on each determines the orbit of Λ to which each vertex belongs.

5·32 Special Cases

We will now consider the seven quotient matrices in $\underset{\sim}{Q}^*$ which have not been processed by the program described in the previous section. In this section "x ~ y" means "x is adjacent to y in G" and "without loss of generality" is assumed at each step. Also, N(x, G) will be abbreviated to N(x) and $\bar{N}(x, G)$ to $\bar{N}(x)$.

(a) $\sigma(\Lambda) = (16;\ 1,\ 5^3)$,

$$R(G, \Lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 2 \\ 0 & 2 & 0 & 3 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

Any realisation is clearly strongly regular, and so not in $\mathcal{G}$.

(b) $\sigma(\Lambda) = (16;\ 1,\ 5^3)$,

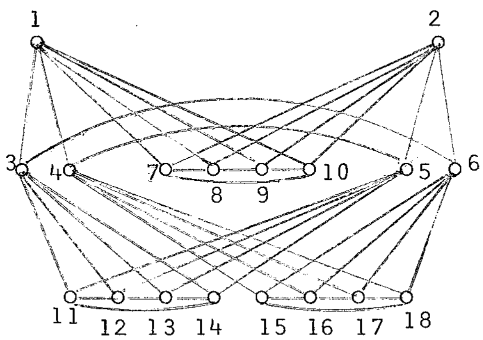
$$R(G, \Lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix}$$

Any realisation is strongly regular and so not in $\mathcal{G}$.

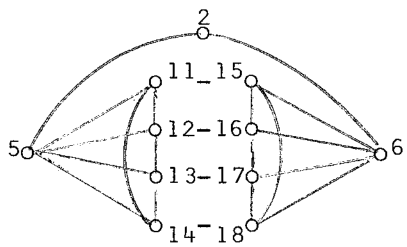
(c) $\sigma(\Lambda) = (18;\ 1^2,\ 2^2,\ 4,\ 8)$,

$$R(G,\ \Lambda) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}$$

Let $\theta(\Lambda) = (1|2|3,\ 4|5,\ 6|7,\ 8,\ 9,\ 10|11,\ 12,\ \cdots,\ 18)$. $N(1)$ is clearly isomorphic to $C_4 \cup \bar{K}_2$. In order that $N(3)$, $N(4)$, $N(5)$ and $N(6)$ be isomorphic to $N(1)$, we must have the situation below.



Now $\bar{N}(1)$ has the form

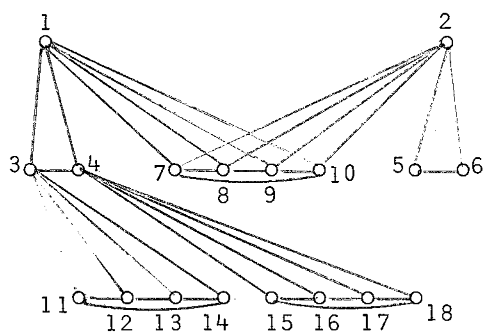


where each of {11, 12, 13, 14} is adjacent to one of {15, 16, 17, 18}, and vice-versa. Without loss of generality, $11 \sim 7$ and $11 \sim 15$. In order that $\bar{N}(11) \cong \bar{N}(1)$ we must have $13 \sim 9$ and $13 \sim 17$. The graph we have at the moment has automorphisms (8 10), (12 14) and (16 18). Therefore we can say $12 \sim 8$ and $12 \sim 16$ which forces $14 \sim 10$ and $14 \sim 18$. Considering $\bar{N}(11)$ and $\bar{N}(8)$ we must have $8 \sim 18$ and $10 \sim 16$. Similarly $7 \sim 17$ and $9 \sim 16$. The resulting graph is indeed transitive.

(d) $\sigma(\Lambda) = (18;\ 1\ ,\ 2\ ,\ 4,\ 8)$,

$$R(G,\ \Lambda) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}$$

Let $\theta(\Lambda) = (1|2|3,\ 4|5,\ 6|7,\ 8,\ 9,\ 10|11,\ 12,\ \cdots,\ 18)$. Obviously $N(1)$ is isomorphic to $C_4 \cup K_2$. Since $N(3) \cong N(4) \cong N(1)$ we have the following situation.



By considering vertex 1 we see that each vertex has two adjacent vertices at distance 3. Considering vertex 3 similarly, we have $5 \sim 11,\ 12,\ 13,\ 14$ and $6 \sim 15,\ 16,\ 17,\ 18$. To get $N(11) \cong N(1)$ we must have a triangle equivalent to $7 \sim 15 \sim 11 \sim 7$. Considering the vertices at distance 3 from 11 and 15 and then $N(9)$, we have another triangle $9 \sim 13 \sim 17 \sim 9$. Similarly we have $8 \sim 12 \sim 16 \sim 8$ and $10 \sim 16 \sim 18 \sim 10$. The resulting graph is transitive and can be identified as the cartesian product of a triangle and an octahedron.

(e) $\sigma(\Lambda) = (18;\ 1^2,\ 2^2,\ 4,\ 8)$,

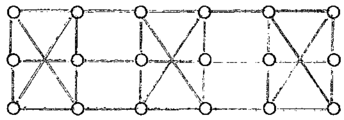
$$R(G,\ \Lambda) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{bmatrix}$$

This case can be handled in a way similar to case (d). However it is clear that the neighbourhood of vertex 1 is isomorphic to $K_5 \cup K_2$, and it is very easy to see that the only 18-vertex graph with each vertex having this neighbourhood is $K_3 \times K_6$. This graph is transitive, of course, and does have a quotient matrix of the form above (take Λ to be the stabiliser of two vertices in the same 6-clique).

(f) $\sigma(\Lambda) = (18;\ 1^2,\ 2^2,\ 4,\ 8)$

$$R(G,\ \Lambda) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{bmatrix}$$

Let $\theta(\Lambda) = (1|2|3,\ 4|5,\ 6|7,\ 8,\ 9,\ 10|11,\ 12,\ \cdots,\ 18)$. Obviously $N(1)$ is isomorphic to $K_5 \cup \bar{K}_2$. Since G is transitive it must contain 3 disjoint 6-cliques, forming a block-system for Λ, with 3 blocks of size 6. Now consider the spanning subgraph E of G containing just those edges not in any 6-clique. Then E is a transitive graph with degree two, and so is isomorphic to either $6C_3$, $3C_6$, $2C_9$ or C_{18}. In the present case, E contains a hexagon (say $1 \sim 3 \sim 5 \sim 2 \sim 6 \sim 4 \sim 1$) and so is isomorphic to $3C_6$. We can also see that this hexagon (and thus the other two) has two vertices in each 6-clique. The only possibility is the graph drawn schematically below, where it is to be understood that any two vertices in the same row are adjacent.



(g) $\sigma(\Lambda) = (18;\ 1^3,\ 5^3)$

$$R(G, \Lambda) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 4 & 1 & 1 \\ 0 & 1 & 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 4 \end{bmatrix}$$

The neighbourhood of each vertex is clearly isomorphic to $K_5 \cup K_2$, and so this is the same graph as in case (e), namely $K_3 \times K_6$.

5·33 Consolidation

A file has been constructed containing all the transitive graphs of order at most 19 which are not in $\mathcal{G}$ (see Section 5·1). This file has been merged with the collection of 594 graphs described in Section 5·31 and the extra handful of graphs constructed in the previous section. Isomorphic copies of the same graph have been eliminated. For convenience, every transitive graph up to order 9 has been retained, while those of order greater than 9 with $k > (n-1)/2$ have been deleted. The result is a set of 546 non-isomorphic transitive graphs. As this number indicates, many of the elements of $\mathcal{G}$ have been constructed via more than one quotient matrix. The total number of transitive graphs of order ≤ 19 and degree ≤ 8 can be found in Table 5·3. The same information restricted to connected graphs is given in Table 5·4. The entries for the remaining degrees are easily deduced, since the number of transitive graphs of order n and degree k is equal to the number with order n and degree $n - k - 1$, and all transitive graphs with $k \geq (n - 1)/2$ are connected.

Where there is any overlap, our results have been compared with those of Yap [46], who considered transitive graphs of order up to 12 (except for those of degree 5), Rees [39] who constructed the symmetric graphs of degree 3 up to 40 vertices, and Hall [18] who constructed all the graphs up to order 11 with isomorphic neighbourhoods. The only discrepancy found is with Yap's catalogue, which omits two transitive graphs on 12 vertices.

Another check on our results has been carried out by generating all the Cayley graphs of every group of order 19 or less, and verifying that each is present in the catalogue. Since the bulk of transitive graphs seem to be Cayley graphs (all but 9 of our 546 are) this is probably quite a good check. A further test has been to generate transitive graphs from those in the catalogue using unary and binary operations, and to check that the resulting graphs are present. An example is the D(G) construction defined in Theorem 1·22.

The complete list of 546 transitive graphs is given in Appendix 2 together with many of the properties of each graph, and representations of each graph as Cayley graphs or products of other graphs.

order	degree									total
	0	1	2	3	4	5	6	7	8	
1	1									1
2	1	1								2
3	1	0	1							2
4	1	1	1	1						4
5	1	0	1	0	1					3
6	1	1	2	2	1	1				8
7	1	0	1	0	1	0	1			4
8	1	1	2	3	3	2	1	1		14
9	1	0	2	0	3	0	2	0	1	9
10	1	1	2	3	4	4	3	2	1	22
11	1	0	1	0	2	0	2	0	1	8
12	1	1	4	7	11	13	13	11	7	74
13	1	0	1	0	3	0	4	0	3	14
14	1	1	2	3	6	6	9	9	6	56
15	1	0	3	0	8	0	12	0	12	48
16	1	1	3	7	16	27	40	48	48	286
17	1	0	1	0	4	0	7	0	10	36
18	1	1	4	7	16	24	38	45	54	380
19	1	0	1	0	4	0	10	0	14	60

Table 5·3

order	degree 0	1	2	3	4	5	6	7	8	total
1	1									1
2	0	1								1
3	0	0	1							1
4	0	0	1	1						2
5	0	0	1	0	1					2
6	0	0	1	2	1	1				5
7	0	0	1	0	1	0	1			3
8	0	0	1	2	3	2	1	1		10
9	0	0	1	0	3	0	2	0	1	7
10	0	0	1	3	3	4	3	2	1	18
11	0	0	1	0	2	0	2	0	1	7
12	0	0	1	4	10	12	13	11	7	64
13	0	0	1	0	3	0	4	0	3	13
14	0	0	1	3	5	6	8	9	6	51
15	0	0	1	0	7	0	12	0	12	44
16	0	0	1	4	13	25	39	47	48	272
17	0	0	1	0	4	0	7	0	10	35
18	0	0	1	5	12	23	36	45	53	365
19	0	0	1	0	4	0	10	0	14	59

Table 5.4

The group information given for each graph was computed using the algorithm described in McKay [26]. Those graphs in the catalogue which are planar were identified in the list of Imrich [10] (we did not ourselves test any graphs for planarity). Those graphs with primitive automorphism groups were found with the help of the simple lemma below. The algorithms used for the other given properties of each graph do not deserve special mention, except that for chromatic number we used Miller's method [34].

5·34 *Let* Γ *be a transitive group acting on* V. *For each pair* $e = \{x, y\}$, *where* $x \neq y \in V$, *define the graph* G_e *with vertex set* V *and edge set* $\{x^\gamma y^\gamma \mid \gamma \in \Gamma\}$. *Then* Γ *is primitive if and only if each* G_e *is connected.*

Proof: If B is a non-trivial block of Γ, and $x \neq y \in B$, then G_e is disconnected obviously. Conversely, if G_e is disconnected, then Γ is imprimitive, since $\Lambda \leq \mathrm{Aut}(G_e)$ and $\mathrm{Aut}(G_e)$ is imprimitive. □

5·35 Concluding remarks

The computation described in this chapter occupied a Cyber 73 computer for a total of about 14 hours, of which probably 80% was taken up with the 18-vertex transitive graphs. With the experience thus gained, the computer time required could be considerably reduced, but it would seem unlikely that the transitive graphs on 20 vertices could be found in less than 20 hours, using the same techniques.

On the other hand Cayley graphs are easy to generate. Those up to order 19 were generated and sorted according to isomorphism type in less than 15 minutes. There is no reason why all the Cayley graphs cannot be found for most groups for which the number of different Cayley graphs is not too large (less than a million say). The major problem is in the construction of those transitive graphs which are not Cayley graphs. As we have already stated, there are only 9 such graphs in our catalogue (not counting complements). A few of these are well-known (Petersen's graph, its linegraph, and the linegraph of K_6) but the others appear here for the first time. If we had some way of constructing these graphs separately, our labour would be very greatly reduced. One approach would be to use Theorem 1·23, but we would need to know the possible orders of minimum transitive subgroups of Aut(G). As C. Godsil has shown [12], these can have orders much larger than n, in general.

An alternative approach to the generation of transitive graphs would be to generate those with primitive automorphism groups separately. This could be done using similar methods to the ones used here, using the primitivity assumption to limit the number of subcases. For example, Γ_1 and $\langle Syl_p(\Gamma_1)\rangle$ either have 1 or n fixed points. The graphs with imprimitive groups could be constructed via a block system (each block contains the same transitive graph and the action of the group on the blocks is transitive).

Yet another technique is based on the following lemma. The proof is elementary but we will omit it anyway.

5·36 LEMMA *Let* Γ *be a transitive group with degree* $n = pm$, *where* p *is a prime and* $1 \leq m \leq p$. *Then* Γ *has an element of order* p *without fixed points.* □

By a stroke of luck, the requirements of the lemma are met by n = 20, 21 and 22. Of these, n = 20 should be the hardest case (since m is the largest) but initial investigations suggest that the generation of the transitive graphs for $20 \leq n \leq 22$ by this method is probably a practical proposition. We hope to be able to give more details, and the results of the computation, in a future paper.

APPENDIX ONE

GROUPS OF ORDERS 5 TO 19

In this Appendix we present a list of the groups of order n, for $5 \le n \le 19$, together with various items of data on each. This information is required for Appendix 2, in which we will list the Cayley graphs for each group.

On the first line of the description for each group, we give the identification number of the group, a common name (if any) and an abstract presentation. The second line has the following information.

inv = number of elements of order two

max = maximum order of an element

cntr = size of centre

comm = size of commutator subgroup

The next few lines contain generators for a regular permutation representation of the group. The last line gives a list of elements of the group. The first *inv* elements are the elements of order two, if any. The remaining elements constitute one member of each pair $\{\gamma, \gamma^{-1}\}$, where γ is an element of order greater than two. Each element is given as a word of minimum length in the generators and their inverses.

Group 5-1 $Z_5 = \langle \alpha \mid \alpha^5 = 1 \rangle$

inv = 0 *max* = 5 *cntr* = 5 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5)$

elements: , α α^2

Group 6-1 $Z_6 = \langle \alpha \mid \alpha^6 = 1 \rangle$

inv = 1 *max* = 6 *cntr* = 6 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6)$

elements: α^3, α α^2

Group 6-2 $D_6 = \langle \alpha, \beta \mid \alpha^2 = \beta^3 = (\alpha\beta)^2 = 1 \rangle$

inv = 3 *max* = 3 *cntr* = 1 *comm* = 3

$\alpha = (1\ 2)(3\ 6)(4\ 5)$

$\beta = (1\ 3\ 4)(2\ 5\ 6)$

elements: α $\alpha\beta$ $\alpha\beta^{-1}$, β

Group 7-1 $Z_7 = \langle \alpha \mid \alpha^7 = 1 \rangle$

inv = 0 *max* = 7 *cntr* = 7 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7)$

elements: , α α^2 α^3

Group 8-1 $Z_8 = \langle \alpha \mid \alpha^8 = 1 \rangle$

inv = 1 *max* = 8 *cntr* = 8 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$

elements: α^4, α α^2 α^3

Group 8-2 $Z_2 \otimes Z_4 = \langle \alpha, \beta \mid \alpha^4 = \beta^2 = 1,\ \alpha\beta = \beta\alpha \rangle$

inv = 3 *max* = 4 *cntr* = 8 *comm* = 1

$\alpha = (1\ 2\ 5\ 3)(4\ 6\ 8\ 7)$

$\beta = (1\ 4)(2\ 6)(3\ 7)(5\ 8)$

elements: β α^2 $\alpha^2\beta$, α $\alpha\beta$

Group 8-3 $Z_2{}^3 = \langle \alpha,\beta,\gamma \mid \alpha^2 = \beta^2 = \gamma^2 = 1,\ \alpha\beta = \beta\alpha,\ \alpha\gamma = \gamma\alpha,\ \beta\gamma = \gamma\beta \rangle$

inv = 7 *max* = 2 *cntr* = 8 *comm* = 1

$\alpha = (1\ 2)(3\ 5)(4\ 6)(7\ 8)$

$\beta = (1\ 3)(2\ 5)(4\ 7)(6\ 8)$

$\gamma = (1\ 4)(2\ 6)(3\ 7)(5\ 8)$

elements: $\alpha\ \beta\ \gamma\ \alpha\beta\ \alpha\gamma\ \beta\gamma\ \alpha\beta\gamma$

Group 8-4 $D_8 = \langle \alpha,\beta \mid \alpha^4 = \beta^2 = (\alpha\beta)^2 = 1 \rangle$

inv = 5 *max* = 4 *cntr* = 2 *comm* = 2

$\alpha = (1\ 2\ 5\ 3)(4\ 7\ 8\ 6)$

$\beta = (1\ 4)(2\ 6)(3\ 7)(5\ 8)$

elements: $\beta\ \alpha^2\ \alpha\beta\ \beta\alpha\ \alpha^2\beta,\ \alpha$

Group 8-5 $Q = \langle \alpha,\beta \mid \alpha^4 = 1,\ \alpha^2 = \beta^2,\ \alpha\beta\alpha = \beta \rangle$

inv = 1 *max* = 4 *cntr* = 2 *comm* = 2

$\alpha = (1\ 2\ 6\ 3)(4\ 8\ 5\ 7)$

$\beta = (1\ 4\ 6\ 5)(2\ 7\ 3\ 8)$

elements: $\alpha^2,\ \alpha\ \beta\ \alpha\beta$

Group 9-1 $Z_9 = \langle \alpha \mid \alpha^9 = 1 \rangle$

inv = 0 *max* = 9 *cntr* = 9 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$

elements: $,\ \alpha\ \alpha^2\ \alpha^3\ \alpha^4$

Group 9-2 $Z_3{}^2 = \langle \alpha,\beta \mid \alpha^3 = \beta^3 = 1,\ \alpha\beta = \beta\alpha \rangle$

inv = 0 *max* = 3 *cntr* = 9 *comm* = 1

$\alpha = (1\ 2\ 3)(4\ 6\ 8)(5\ 7\ 9)$

$\beta = (1\ 4\ 5)(2\ 6\ 7)(3\ 8\ 9)$

elements: $,\ \alpha\ \beta\ \alpha\beta\ \alpha\beta^{-1}$

Group 10-1 $Z_{10} = \langle\alpha|\alpha^{10} = 1\rangle$

inv = 1 *max* = 10 *cntr* = 10 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10)$

elements: α^5, α α^2 α^3 α^4

Group 10-2 $D_{10} = \langle\alpha,\beta|\alpha^5 = \beta^2 = (\alpha\beta)^2 = 1\rangle$

inv = 5 *max* = 5 *cntr* = 1 *comm* = 5

$\alpha = (1\ 2\ 5\ 7\ 3)(4\ 8\ 10\ 9\ 6)$

$\beta = (1\ 4)(2\ 6)(3\ 8)(5\ 9)(7\ 10)$

elements: β $\alpha\beta$ $\beta\alpha$ $\alpha^2\beta$ $\beta\alpha^2$, α α^2

Group 11-1 $Z_{11} = \langle\alpha|\alpha^{11} = 1\rangle$

inv = 0 *max* = 11 *cntr* = 11 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)$

elements: , α α^2 α^3 α^4 α^5

Group 12-1 $Z_{12} = \langle\alpha|\alpha^{12} = 1\rangle$

inv = 1 *max* = 12 *cntr* = 12 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12)$

elements: α^6, α α^2 α^3 α^4 α^5

Group 12-2 $Z_2 \otimes Z_6 = \langle\alpha,\beta|\alpha^6 = \beta^2 = 1,\ \alpha\beta = \beta\alpha\rangle$

inv = 3 *max* = 6 *cntr* = 12 *comm* = 1

$\alpha = (1\ 2\ 5\ 9\ 7\ 3)(4\ 6\ 10\ 12\ 11\ 8)$

$\beta = (1\ 4)(2\ 6)(3\ 8)(5\ 10)(7\ 11)(9\ 12)$

elements: β α^3 $\alpha^3\beta$, α α^2 $\alpha\beta$ $\alpha^2\beta$

Group 12-3 $D_{12} = \langle\alpha,\beta|\alpha^6 = \beta^2 = (\alpha\beta)^2 = 1\rangle$

inv = 7 *max* = 6 *cntr* = 2 *comm* = 3

$\alpha = (1\ 2\ 5\ 9\ 7\ 3)(4\ 8\ 11\ 12\ 10\ 6)$

$\beta = (1\ 4)(2\ 6)(3\ 8)(5\ 10)(7\ 11)(9\ 12)$

elements: β $\alpha\beta$ $\beta\alpha$ α^3 $\alpha^2\beta$ $\beta\alpha^2$ $\alpha^3\beta$, α α^2

Group 12-4 $A_4 = \langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = 1,\ \alpha\beta = \beta\alpha,\ \beta\gamma = \gamma\alpha\beta,\ \alpha\gamma = \gamma\beta \rangle$

inv = 3 *max* = 3 *cntr* = 1 *comm* = 4

$\alpha = (1\ 2)(3\ 6)(4\ 11)(5\ 10)(7\ 9)(8\ 12)$

$\beta = (1\ 3)(2\ 6)(4\ 7)(5\ 12)(8\ 10)(9\ 11)$

$\gamma = (1\ 4\ 5)(2\ 7\ 8)(3\ 9\ 10)(6\ 11\ 12)$

elements: $\alpha\ \beta\ \alpha\beta,\ \gamma\ \alpha\gamma\ \alpha\gamma^{-1}\ \beta\gamma$

Group 12-5 $\langle \alpha, \beta \mid \alpha^6 = 1,\ \alpha^3 = \beta^2,\ \alpha\beta = \beta\alpha^{-1} \rangle$

inv = 1 *max* = 6 *cntr* = 2 *comm* = 3

$\alpha = (1\ 2\ 6\ 12\ 9\ 3)(4\ 10\ 8\ 5\ 11\ 7)$

$\beta = (1\ 4\ 12\ 5)(2\ 7\ 9\ 8)(3\ 10\ 6\ 11)$

elements: $\beta^2,\ \alpha\ \beta\ \alpha^2\ \alpha\beta\ \beta\alpha$

Group 13-1 $Z_{13} = \langle \alpha \mid \alpha^{13} = 1 \rangle$

inv = 0 *max* = 13 *cntr* = 13 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13)$

elements: $,\ \alpha\ \alpha^2\ \alpha^3\ \alpha^4\ \alpha^5\ \alpha^6$

Group 14-1 $Z_{14} = \langle \alpha \mid \alpha^{14} = 1 \rangle$

inv = 1 *max* = 14 *cntr* = 14 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14)$

elements: $\alpha^7,\ \alpha\ \alpha^2\ \alpha^3\ \alpha^4\ \alpha^5\ \alpha^6$

Group 14-2 $D_{14} = \langle \alpha, \beta \mid \alpha^7 = \beta^2 = (\alpha\beta)^2 = 1 \rangle$

inv = 7 *max* = 7 *cntr* = 1 *comm* = 7

$\alpha = (1\ 2\ 5\ 9\ 11\ 7\ 3)(4\ 8\ 12\ 14\ 13\ 10\ 6)$

$\beta = (1\ 4)(2\ 6)(3\ 8)(5\ 10)(7\ 12)(9\ 13)(11\ 14)$

elements: $\beta\ \alpha\beta\ \beta\alpha\ \alpha^2\beta\ \beta\alpha^2\ \alpha^3\beta\ \beta\alpha^3,\ \alpha\ \alpha^2\ \alpha^3$

Group 15-1 $Z_{15} = \langle \alpha \mid \alpha^{15} = 1 \rangle$

inv = 0 *max* = 15 *cntr* = 15 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15)$

elements: $,\ \alpha\ \alpha^2\ \alpha^3\ \alpha^4\ \alpha^5\ \alpha^6\ \alpha^7$

Group 16-1 $\quad Z_{16} = \langle\alpha|\alpha^{16} = 1\rangle$

inv = 1 *max* = 16 *cntr* = 16 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16)$

elements: α^8, $\alpha\ \alpha^2\ \alpha^3\ \alpha^4\ \alpha^5\ \alpha^6\ \alpha^7$

Group 16-2 $\quad Z_2 \otimes Z_8 = \langle\alpha,\beta|\alpha^8 = \beta^2 = 1,\ \alpha\beta = \beta\alpha\rangle$

inv = 3 *max* = 8 *cntr* = 16 *comm* = 1

$\alpha = (1\ 2\ 5\ 9\ 13\ 11\ 7\ 3)(4\ 6\ 10\ 14\ 16\ 15\ 12\ 8)$

$\beta = (1\ 4)(2\ 6)(3\ 8)(5\ 10)(7\ 12)(9\ 14)(11\ 15)(13\ 16)$

elements: $\beta\ \alpha^4\ \alpha^4\beta$, $\alpha\ \alpha^2\ \alpha\beta\ \alpha^3\ \alpha^2\beta\ \alpha^3\beta$

Group 16-3 $\quad {Z_4}^2 = \langle\alpha,\beta|\alpha^4 = \beta^4 = 1,\ \alpha\beta = \beta\alpha\rangle$

inv = 3 *max* = 4 *cntr* = 16 *comm* = 1

$\alpha = (1\ 2\ 6\ 3)(4\ 7\ 12\ 9)(5\ 8\ 13\ 10)(11\ 14\ 16\ 15)$

$\beta = (1\ 4\ 11\ 5)(2\ 7\ 14\ 8)(3\ 9\ 15\ 10)(6\ 12\ 16\ 13)$

elements: $\alpha^2\ \beta^2\ \alpha^2\beta^2$, $\alpha\ \beta\ \alpha\beta\ \alpha\beta^{-1}\ \alpha^2\beta\ \alpha\beta^2$

Group 16-4 $\quad {Z_2}^2 \otimes Z_4 = \langle\alpha,\beta,\gamma|\alpha^4 = \beta^2 = \gamma^2 = 1,\ \alpha\beta = \beta\alpha,\ \beta\gamma = \gamma\beta,\ \alpha\gamma = \gamma\alpha\rangle$

inv = 7 *max* = 4 *cntr* = 16 *comm* = 1

$\alpha = (1\ 2\ 6\ 3)(4\ 7\ 12\ 9)(5\ 8\ 13\ 10)(11\ 14\ 16\ 15)$

$\beta = (1\ 4)(2\ 7)(3\ 9)(5\ 11)(6\ 12)(8\ 14)(10\ 15)(13\ 16)$

$\gamma = (1\ 5)(2\ 8)(3\ 10)(4\ 11)(6\ 13)(7\ 14)(9\ 15)(12\ 16)$

elements: $\beta\ \gamma\ \alpha^2\ \beta\gamma\ \alpha^2\beta\ \alpha^2\gamma\ \alpha^2\beta\gamma$, $\alpha\ \alpha\beta\ \alpha\gamma\ \alpha\beta\gamma$

Group 16-5 $\quad {Z_2}^4 = \langle\alpha,\beta,\gamma,\delta|\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 1,\ \alpha\beta = \beta\alpha,\ \alpha\gamma = \gamma\alpha,$
$\alpha\delta = \delta\alpha,\ \beta\gamma = \gamma\beta,\ \beta\delta = \delta\beta,\ \gamma\delta = \delta\gamma\rangle$

inv = 15 *max* = 2 *cntr* = 16 *comm* = 1

$\alpha = (1\ 2)(3\ 6)(4\ 7)(5\ 8)(9\ 12)(10\ 13)(11\ 14)(15\ 16)$

$\beta = (1\ 3)(2\ 6)(4\ 9)(5\ 10)(7\ 12)(8\ 13)(11\ 15)(14\ 16)$

$\gamma = (1\ 4)(2\ 7)(3\ 9)(5\ 11)(6\ 12)(8\ 14)(10\ 15)(13\ 16)$

$\delta = (1\ 5)(2\ 8)(3\ 10)(4\ 11)(6\ 13)(7\ 14)(9\ 15)(12\ 16)$

elements: $\alpha\ \beta\ \gamma\ \delta\ \alpha\beta\ \alpha\gamma\ \alpha\delta\ \beta\gamma\ \beta\delta\ \gamma\delta\ \alpha\beta\gamma\ \alpha\beta\delta\ \alpha\gamma\delta\ \beta\gamma\delta\ \alpha\beta\gamma\delta$

Group 16-6 $Z_2 \otimes D_8 = \langle \alpha,\beta,\gamma | \alpha^4 = \beta^2 = \gamma^2 = (\alpha\beta)^2 = 1,\ \beta\gamma = \gamma\beta,\ \alpha\gamma = \gamma\alpha \rangle$

inv = 11 *max* = 4 *cntr* = 4 *comm* = 2

α = (1 2 6 3)(4 9 12 7)(5 8 13 10)(11 15 16 14)

β = (1 4)(2 7)(3 9)(5 11)(6 12)(8 14)(10 15)(13 16)

γ = (1 5)(2 8)(3 10)(4 11)(6 13)(7 14)(9 15)(12 16)

elements: $\beta\ \gamma\ \alpha^2\ \alpha\beta\ \beta\alpha\ \beta\gamma\ \alpha^2\beta\ \alpha^2\gamma\ \alpha\beta\gamma\ \beta\alpha\gamma\ \alpha^2\beta\gamma,\ \alpha\ \alpha\gamma$

Group 16-7 $Z_2 \otimes Q = \langle \alpha,\beta,\gamma | \alpha^4 = \beta^2 = 1,\ \alpha^2 = \gamma^2,\ \alpha\gamma = \gamma\alpha^{-1},\ \gamma\beta = \beta\gamma,\ \alpha\beta = \beta\alpha \rangle$

inv = 3 *max* = 4 *cntr* = 4 *comm* = 2

α = (1 2 7 3)(4 8 14 11)(5 10 6 9)(12 16 13 15)

β = (1 4)(2 8)(3 11)(5 12)(6 13)(7 14)(9 15)(10 16)

γ = (1 5 7 6)(2 9 3 10)(4 12 14 13)(8 15 11 16)

elements: $\beta\ \alpha^2\ \alpha^2\beta,\ \alpha\ \gamma\ \alpha\beta\ \alpha\gamma\ \beta\gamma\ \alpha\beta\gamma$

Group 16-8 $\langle \alpha,\beta,\gamma | \alpha^4 = \beta^2 = (\beta\gamma)^2 = 1,\ \alpha^2 = \gamma^2,\ \alpha\gamma = \gamma\alpha^{-1},\ \alpha\beta = \beta\alpha \rangle$

inv = 7 *max* = 4 *cntr* = 4 *comm* = 2

α = (1 2 7 3)(4 8 14 11)(5 10 6 9)(12 16 13 15)

β = (1 4)(2 8)(3 11)(5 13)(6 12)(7 14)(9 16)(10 15)

γ = (1 5 7 6)(2 9 3 10)(4 12 14 13)(8 15 11 16)

elements: $\beta\ \alpha^2\ \beta\gamma\ \beta\gamma^{-1}\ \alpha^2\beta\ \alpha\beta\gamma\ \alpha\beta\gamma^{-1},\ \alpha\ \gamma\ \alpha\beta\ \alpha\gamma$

Group 16-9 $\langle \alpha,\beta,\gamma | \alpha^4 = \beta^2 = 1,\ \alpha\beta = \beta\alpha,\ \beta = \gamma^2,\ \alpha\gamma = \gamma\alpha^{-1}\beta \rangle$

inv = 7 *max* = 4 *cntr* = 4 *comm* = 2

α = (1 2 7 3)(4 8 14 11)(5 13 15 10)(6 12 16 9)

β = (1 4)(2 8)(3 11)(5 6)(7 14)(9 10)(12 13)(15 16)

γ = (1 5 4 6)(2 9 8 10)(3 12 11 13)(7 15 14 16)

elements: $\beta\ \alpha^2\ \alpha\gamma\ \alpha\gamma^{-1}\ \gamma\alpha\ \alpha^{-1}\gamma\ \alpha^2\beta,\ \alpha\ \gamma\ \alpha\beta\ \alpha^2\gamma$

Group 16-10 $\langle \alpha, \beta \mid \alpha^4 = \beta^4 = 1,\ \alpha\beta = \beta\alpha^{-1} \rangle$

inv = 3 *max* = 4 *cntr* = 4 *comm* = 2

$\alpha = (1\ 2\ 6\ 3)(4\ 9\ 12\ 7)(5\ 10\ 13\ 8)(11\ 14\ 16\ 15)$

$\beta = (1\ 4\ 11\ 5)(2\ 7\ 14\ 8)(3\ 9\ 15\ 10)(6\ 12\ 16\ 13)$

elements: $\alpha^2\ \beta^2\ \alpha^2\beta^2,\ \alpha\ \beta\ \alpha\beta\ \beta\alpha\ \alpha^2\beta\ \alpha\beta^2$

Group 16-11 $\langle \alpha, \beta \mid \alpha^8 = \beta^2 = 1,\ \beta\alpha\beta = \alpha^5 \rangle$

inv = 3 *max* = 8 *cntr* = 4 *comm* = 2

$\alpha = (1\ 2\ 5\ 11\ 16\ 15\ 7\ 3)(4\ 9\ 12\ 8\ 14\ 16\ 13\ 10)$

$\beta = (1\ 4)(2\ 6)(3\ 8)(5\ 12)(7\ 13)(9\ 15)(10\ 11)(14\ 16)$

elements: $\beta\ \alpha\beta\alpha^{-1}\ \alpha^4,\ \alpha\ \alpha^2\ \alpha\beta\ \beta\alpha\ \alpha^3\ \alpha^2\beta$

Group 16-12 $D_{16} = \langle \alpha, \beta \mid \alpha^8 = \beta^2 = (\alpha\beta)^2 = 1 \rangle$

inv = 9 *max* = 8 *cntr* = 2 *comm* = 4

$\alpha = (1\ 2\ 5\ 9\ 13\ 11\ 7\ 3)(4\ 8\ 12\ 15\ 16\ 14\ 10\ 6)$

$\beta = (1\ 4)(2\ 6)(3\ 8)(5\ 10)(7\ 12)(9\ 14)(11\ 15)(13\ 16)$

elements: $\beta\ \alpha\beta\ \beta\alpha\ \alpha^2\beta\ \beta\alpha^2\ \alpha^4\ \alpha^3\beta\ \beta\alpha^3\ \alpha^4\beta,\ \alpha\ \alpha^2\ \alpha^3$

Group 16-13 $\langle \alpha, \beta \mid \alpha^8 = \beta^2 = 1,\ \beta\alpha\beta = \alpha^3 \rangle$

inv = 5 *max* = 8 *cntr* = 2 *comm* = 4

$\alpha = (1\ 2\ 5\ 11\ 16\ 15\ 7\ 3)(4\ 9\ 14\ 6\ 13\ 8\ 12\ 10)$

$\beta = (1\ 4)(2\ 6)(3\ 8)(5\ 12)(7\ 14)(9\ 11)(10\ 15)(13\ 16)$

elements: $\beta\ \alpha^2\beta\ \alpha\beta\alpha\ \alpha\beta\alpha^{-1}\ \alpha^4,\ \alpha\ \alpha^2\ \alpha\beta\ \beta\alpha\ \alpha^3$

Group 16-14 $\langle \alpha, \beta \mid \alpha^8 = 1,\ \alpha^4 = \beta^2,\ \alpha\beta = \beta\alpha^{-1} \rangle$

inv = 1 *max* = 8 *cntr* = 2 *comm* = 4

$\alpha = (1\ 2\ 6\ 13\ 12\ 16\ 9\ 3)(4\ 10\ 15\ 8\ 5\ 11\ 14\ 7)$

$\beta = (1\ 4\ 12\ 5)(2\ 7\ 16\ 8)(3\ 10\ 13\ 11)(6\ 14\ 9\ 15)$

elements: $\beta^2,\ \alpha\ \beta\ \alpha^2\ \alpha\beta\ \beta\alpha\ \alpha^3\ \alpha^2\beta$

Group 17-1 $Z_{17} = \langle \alpha | \alpha^{17} = 1 \rangle$

inv = 0 *max* = 17 *cntr* = 17 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17)$

elements: , $\alpha\ \alpha^2\ \alpha^3\ \alpha^4\ \alpha^5\ \alpha^6\ \alpha^7\ \alpha^8$

Group 18-1 $Z_{18} = \langle \alpha | \alpha^{18} = 1 \rangle$

inv = 1 *max* = 18 *cntr* = 18 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18)$

elements: α^9, $\alpha\ \alpha^2\ \alpha^3\ \alpha^4\ \alpha^5\ \alpha^6\ \alpha^7\ \alpha^8$

Group 18-2 $Z_3 \otimes Z_6 = \langle \alpha,\beta | \alpha^6 = \beta^3,\ \alpha\beta = \beta\alpha \rangle$

inv = 1 *max* = 6 *cntr* = 18 *comm* = 1

$\alpha = (1\ 2\ 6\ 12\ 9\ 3)(4\ 7\ 13\ 17\ 15\ 10)(5\ 8\ 14\ 18\ 16\ 11)$

$\beta = (1\ 4\ 5)(2\ 7\ 8)(3\ 10\ 11)(6\ 13\ 14)(9\ 15\ 16)(12\ 17\ 18)$

elements: α^3, $\alpha\ \beta\ \alpha^2\ \alpha\beta\ \alpha\beta^{-1}\ \alpha^2\beta\ \alpha^2\beta^{-1}\ \alpha^3\beta$

Group 18-3 $Z_3 \otimes D_6 = \langle \alpha,\beta | \alpha^9 = \beta^2 = 1,\ \alpha\beta = \beta\alpha^2 \rangle$

inv = 3 *max* = 6 *cntr* = 3 *comm* = 3

$\alpha = (1\ 2\ 3)(3\ 8\ 6)(5\ 9\ 7)(10\ 12\ 14)(11\ 13\ 15)(16\ 18\ 17)$

$\beta = (1\ 4\ 10\ 16\ 11\ 5)(2\ 6\ 12\ 17\ 13\ 7)(3\ 8\ 14\ 18\ 15\ 9)$

elements: $\beta^3\ \alpha\beta^3\ \beta\alpha\beta^2$, $\alpha\ \beta\ \alpha\beta\ \beta\alpha\ \beta^2\ \alpha\beta^2\ \alpha\beta^{-2}$

Group 18-4 $D_{18} = \langle \alpha,\beta | \alpha^9 = \beta^2 = (\alpha\beta)^2 = 1 \rangle$

inv = 9 *max* = 9 *cntr* = 1 *comm* = 9

$\alpha = (1\ 2\ 5\ 9\ 13\ 15\ 11\ 7\ 3)(4\ 8\ 12\ 16\ 18\ 17\ 14\ 10\ 6)$

$\beta = (1\ 4)(2\ 6)(3\ 8)(5\ 10)(7\ 12)(9\ 14)(11\ 15)(13\ 17)(15\ 18)$

elements: $\beta\ \alpha\beta\ \beta\alpha\ \alpha^2\beta\ \beta\alpha^2\ \alpha^3\beta\ \beta\alpha^3\ \alpha^4\beta\ \beta\alpha^4$, $\alpha\ \alpha^2\ \alpha^3\ \alpha^4$

Group 18-5 $\quad Z_3 \text{ wr } Z_2 = \langle \alpha,\beta,\gamma \mid \alpha^3 = \beta^3 = \gamma^2 = (\alpha\gamma)^2 = (\beta\gamma)^2 = 1, \alpha\beta = \beta\gamma \rangle$

inv = 9 *max* = 3 *cntr* = 1 *comm* = 9

$\alpha = (1\ 2\ 3)(4\ 7\ 10)(5\ 8\ 11)(6\ 12\ 9)(13\ 17\ 15)(14\ 18\ 16)$

$\beta = (1\ 4\ 5)(2\ 7\ 8)(3\ 10\ 11)(6\ 14\ 13)(9\ 16\ 15)(12\ 18\ 17)$

$\gamma = (1\ 6)(2\ 9)(3\ 12)(4\ 13)(5\ 14)(7\ 15)(8\ 16)(10\ 17)(11\ 18)$

elements: $\gamma\ \alpha\gamma\ \beta\gamma\ \gamma\alpha\ \gamma\beta\ \alpha\beta\gamma\ \alpha\gamma\beta\ \beta\gamma\alpha\ \gamma\alpha\beta,\ \alpha\ \beta\ \alpha\beta\ \alpha\beta^{-1}$

Group 19-1 $\quad Z_{19} = \langle \alpha \mid \alpha^{19} = 1 \rangle$

inv = 0 *max* = 19 *cntr* = 19 *comm* = 1

$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19)$

elements: $,\ \alpha\ \alpha^2\ \alpha^3\ \alpha^4\ \alpha^5\ \alpha^6\ \alpha^7\ \alpha^8\ \alpha^9$

APPENDIX TWO

TRANSITIVE GRAPHS OF ORDER 2 TO 19

In this appendix we give a complete list of the transitive graphs of order n for $2 \le n \le 9$ and the transitive graphs of order n and degree k, for $10 \le n \le 19$ and $k \le (n - 1)/2$. This catalogue has been published in McKay [32]. Graph theoretic concepts not defined in this thesis have been defined in Behzad and Chartrand [4].

Throughout our description of the data given for each graph in the catalogue we will call the graph G and assume that $V(G) = V = \{1, 2, \cdots, n\}$. The degree of G will be denoted by k and the automorphism group of G by Γ. Also define $\alpha(G) = \theta(\Gamma_1)$ and let $\partial(G)$ be the partition of V such that vertices v and w are in the same cell if and only if $\partial(1, v) = \partial(1, w)$.

(a) *Set Notation:* A set of positive integers can be written as an octal integer by putting bit i equal to 1 if and only if i is in the set. The bits are numbered from 1 starting at the right hand (low order) end. For example, 251 (octal) is 10101001 (binary) and so represents the set $\{1, 4, 6, 8\}$.

(b) *First Line of Data:* The first item in this line is the *name* of G, for example L20 or P16. The letter indicates the order of G (A for 1, B for 2, etc.), and the numbers are allotted sequentially within each order. Care must be taken to avoid confusing names like K3 with the commonly accepted notations for special graphs, for example K_3, C_5, $K_{3,4}$. The latter notations will be used in this description of the catalogue, but never in the catalogue itself.

We now describe the other pieces of information which may occur on the first line.

(i) DEG: degree of G.

(ii) F: flags associated with G. Each flag is a single letter whose presence indicates a special property. If no flags apply, the F is omitted. The flags used are listed below.

X = disconnected.

N = not a Cayley graph.

T = distance transitive ($\partial(G) = \alpha(G)$).

R = distance regular ($\partial(G)$ is equitable) but not distance transitive (only case is P84).

V = Γ acts primitively on V.

I = Γ satisfies this condition: For any $v, w \in V$ there is $\gamma \in \Gamma$ such that $v^{\gamma} = w$ and $w^{\gamma} = v$.

A = antipodal (if $\partial(u, v) = \partial(u, w) = \Delta$ then $\partial(v, w) = \Delta$, where Δ is the diameter of G).

S = self-complementary ($\bar{G} \cong G$).

P = planar.

(iii) AUT: order of Γ_1.

(iv) P: partitions $\partial(G)$ and $\alpha(G)$. Each digit or letter gives the size of one cell of a partition π of V. Letters are used for cell sizes over 9; A for 10, B for 11, etc.

Case 1: If n = 2 of G is not a GRR, then π is $\alpha(G)$. The cells of $\alpha(G)$ are grouped by commas into the cells of $\partial(G)$. For example, P = (1, 4, 24, 1) indicates that $\alpha(G)$ has one 4-cell at distance 1 from vertex 1, a 2-cell and a 4-cell at distance 2, and a single 1-cell at distance 3. If G is disconnected, only vertices in the component containing vertex 1 are included; the presence of additional components is indicated by a "+" sign.

Case 2: If $n \neq 2$ and G is a GRR, then π is $\partial(G)$. To avoid confusion with Case 1, the cells are separated by slashes. For example, P = (1/6/8/1) indicates 6 vertices at distance 1 from vertex 1, 8 vertices at distance 2, and 1 vertex at distance 3.

(v) GIR: girth of G, unless G is acyclic.

(vi) CN: chromatic numbers of G and $\bar{G}$, respectively.

(vii) T: arc-transitivity of G, unless Γ is not transitive on 1-arcs, or k = 0, or k = 2.

(viii) Any other text on the first line indicates a common name for G, for example "PETERSEN GRAPH".

(c) *Adjacency Matrix* (omitted if d = 0)

$$A = a_2\ a_3\ a_4\ a_5,\ a_6\ \cdots\ a_n.$$

Each a_i is an octal representation (see part (a)) of the set of vertices preceding vertex i which are adjacent to i. Note that a_1 is omitted. The labelling of the vertices of G is consistent with the partition P described above. For example, if P = (1, 4, 24, 1), $\alpha(G)$ is {1|2, 3, 4, 5|6, 7|8, 9, 10, 11|12} and $\partial(G)$ is {1|2, 3, 4, 5|6, 7, 8, 9, 10, 11|12}.

Example: If A = 1 1 6, we have 2 adjacent to 1, 3 adjacent to 1 and 4 adjacent to 2 and 3.

(d) *Eigenvalues of Adjacency Matrix* (omitted if G is disconnected).

$$E = m_1\lambda_1\ m_2\lambda_2\ \cdots.$$

Each field gives one eigenvalue of the adjacency matrix of G. If the eigenvalue has multiplicity other than one, this multiplicity is written immediately before the eigenvalue, using an intervening "+" for nonnegative eigenvalues. If the eigenvalues for G are $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, those for $\bar{G}$ are
$-\lambda_{n-1} - 1 \le -\lambda_{n-2} - 1 \le \cdots \le -\lambda_1 - 1 \le n - k - 1$.

Example: E = -4 3-·4391 2+0 1·3417 5

The eigenvalues are -0·4391 (3 times), 0 (twice) and -4, 1·3417, 5 (once each).

(e) *Independent Sets and Cliques*

(omitted if G is disconnected).

$$K = (\alpha_3\ \alpha_4\ \cdots, \beta_3\ \beta_4\ \cdots).$$

α_i is the number of independent sets of size i in G, i.e. cliques of size i in $\bar{G}$, which include vertex 1.

β_i is the number of cliques of size i in G which include vertex 1.

Those numbers before the comma are α's; those after the comma are β's. The total number of independent sets or cliques of size i in G is $n\alpha_i/i$ or $n\beta_i/i$, respectively.

Example: K = (, 4 1). G has no independent sets of size 3 or greater. Vertex 1 is contained in 4 triangles and 1 clique of size 4.

(f) *Representations of* G.

The data provided about G contain a number of descriptors expressing G as a product etc. In explaining each descriptor type, H and J stand for the names of transitive graphs in the catalogue. As before, n and k are the order and degree of G, respectively. The variable i indicates a positive integer.

(i) -H: complement of H, unless G is self-complementary.

(ii) i[H]: G is the disjoint union of i copies of H ($i > 1$), unless $k \leq 1$.

(iii) L(H): linegraph of H, unless $k \leq 2$.

(iv) -L(H): complement of L(H), unless $k \leq 2$ or G is complete.

(v) SW(H): switching graph of H.

(vi) SW(H+): switching graph of the disjoint union $H \cup K_1$, unless H is complete or empty. The only example is L37. All switching graphs in the catalogue are either type (v) or type (vi).

(vii) D(H): H plus diagonals (see Section 1.22), provided H has diameter at least 3 and is connected.

(viii) -D(H): complement of D(H). This notation is omitted if H is bipartite and has diameter 3. In that case -D(H) is the disjoint union of two cliques. H is connected with diameter ≥ 3.

(ix) Wi(H): generalized linegraph of subdivision graph ($1 \leq i \leq 9$). Form a multigraph from H by replacing each edge by i parallel edges. Then subdivide each edge with a new vertex and take the linegraph of the result. Omitted if H has degree ≤ 1, or $k = 2$.
Every linegraph in the catalogue is of type (iii) or type (ix) except these:

$L(K_{1,m}) = K_m \quad (2 \leq m \leq 19)$,

$L(K_{3,m}) = K_3 \times K_m \quad (4 \leq m \leq 6)$.

(x) -Wi(H): complement of Wi(H) ($1 \leq i \leq 9$), unless H has degree ≤ 1, or Wi(H) has degree 2.

(xi) H[J]: lexicographic product of H around J, unless $k \le 1$. If H is empty (i.e. $\bar{H}$ is complete), the notation (ii) is used instead.

(xii) H × J: cartesian product of H and J, unless either H or J is empty.

(xiii) -H × J: complement of H × J, unless G is complete.

(xiv) H * J: tensor product of H and J, unless $k \le 1$.

(xv) -H * J: complement of H * J, unless G is either empty or complete.

(xvi) i/m: G is the Cayley graph $C(\Lambda, \Omega)$, where Λ is the ith group of order n, and the connection set Ω is specified by the octal number m (see (a)). The groups and their elements are numbered in the order they are listed in Appendix 1; an element and its inverse have the same ordinal. Ω is not canonical in any sense.
Example: If $n = 16$, the notation 3/123 represents $C(\Lambda, \Omega)$, where Λ is group 16-3 and Ω is $\{\alpha^2, \beta^2, \beta^{\pm 1}, (\alpha\beta^{-1})^{\pm 1}\}$.
Cayley graph representation is only given if $2 \le k \le (n - 1)/2$.

TRANSITIVE GRAPHS ON 2 VERTICES

B1 DEG=0 F=XTVIAP AUT=1 P=(1,+) CN=1,2
-B2 SW(A1)

B2 DEG=1 F=TVIAP AUT=1 P=(1,1) CN=2,1 T=1
A=1 E=-1 1 K=(,) -B1

TRANSITIVE GRAPHS ON 3 VERTICES

C1 DEG=0 F=XTVIAP AUT=2 P=(1,+) CN=1,3
-C2 -L(C2)

C2 DEG=2 F=TVIAP AUT=2 P=(1,2) GIR=3 CN=3,1 TRIANGLE
A=1 3 E=2-1 2 K=(,1) -C1

TRANSITIVE GRAPHS ON 4 VERTICES

D1 DEG=0 F=XTVIAP AUT=6 P=(1,+) CN=1,4
-D4

D2 DEG=1 F=XTIP AUT=2 P=(1,1,+) CN=2,2 T=1
A=1 0 4 -D3 -L(D3) SW(B1) SW(B2) -B2XB2

D3 DEG=2 F=TIAP AUT=2 P=(1,2,1) GIR=4 CN=2,2 SQUARE
A=1 1 6 E=-2 2+0 2 K=(,) -D2 B2[B1] B2XB2 -B1XB2 -B2*B2

D4 DEG=3 F=TVIAP AUT=6 P=(1,3) GIR=3 CN=4,1 T=2 TETRAHEDRON
A=1 3 7 E=3-1 3 K=(,3 1) -D1 B2[B2]

TRANSITIVE GRAPHS ON 5 VERTICES

E1 DEG=0 F=XTVIAP AUT=24 P=(1,+) CN=1,5
-E3

E2 DEG=2 F=TVISP AUT=2 P=(1,2,2) GIR=5 CN=3,3 PENTAGON
A=1 1 4 12 E=2-1.61803 2+.61803 2 K=(,) -L(E2) 1/1

E3 DEG=4 F=TVIA AUT=24 P=(1,4) GIR=3 CN=5,1 T=2
A=1 3 7 17 E=4-1 4 K=(,6 4 1) -E1

TRANSITIVE GRAPHS ON 6 VERTICES

F1 DEG=0 F=XTVIAP AUT=120 P=(1,+) CN=1,6
-F8

F2 DEG=1 F=XTIP AUT=8 P=(1,1,+) CN=2,3 T=1
A=1 0 4 0,20 -F7 -L(D4)

F3 DEG=2 F=XTIP AUT=12 P=(1,2,+) GIR=3 CN=3,2
A=1 3 0 10,30 2[C2] -F5 SW(C2) 1/4 2/10

F4 DEG=2 F=TIAP AUT=2 P=(1,2,2,1) GIR=6 CN=2,3 HEXAGON
A=1 1 4 2,30 E=-2 2-1 2+1 2 K=(1,) -F6 SW(C1) -B2XC2 B2*C2 1/2 2/6

TRANSITIVE GRAPHS ON 6 VERTICES (CONTD)

F5 DEG=3 F=TIA AUT=12 P=(1,3,2) GIR=4 CN=2,3 T=3
A=1 1 1 16,16 E=-3 4+0 3 K=(1,) -F3 -L(F3) D(F4) B2[C1] -B1XC2

F6 DEG=3 F=IP AUT=2 P=(1,12,2) GIR=3 CN=3,2 PRISM
A=1 1 5 12,26 E=2-2 2+0 1 3 K=(,1) -F4 -L(F4) W3(B2) -W1(C2) B2XC2 -B2*C2

F7 DEG=4 F=TIAP AUT=8 P=(1,4,1) GIR=3 CN=3,2 T=1 OCTAHEDRON
A=1 1 7 7,36 E=2-2 3+0 4 K=(,4) -F2 L(D4) -W1(F2) C2[B1] -B2XC1

F8 DEG=5 F=TVIA AUT=120 P=(1,5) GIR=3 CN=6,1 T=2
A=1 3 7 17,37 E=5-1 5 K=(,10 10 5 1) -F1 B2[C2] C2[B2]

TRANSITIVE GRAPHS ON 7 VERTICES

G1 DEG=0 F=XTVIAP AUT=720 P=(1,+) CN=1,7
-G4

G2 DEG=2 F=TVIP AUT=2 P=(1,2,2,2) GIR=7 CN=3,4 HEPTAGON
A=1 1 4 2,20 50 E=2-1.80194 2-.44504 2+1.24698 2 K=(3,) -G3 -D(G2) 1/1

G3 DEG=4 F=VI AUT=2 P=(1,22,2) GIR=3 CN=4,3
A=1 3 5 3,34 72 E=2-2.24698 2-.55496 2+.80194 4 K=(,3) -G2 -L(G2) D(G2)

G4 DEG=6 F=TVIA AUT=720 P=(1,6) GIR=3 CN=7,1 T=2
A=1 3 7 17,37 77 E=6-1 6 K=(,15 20 15 6 1) -G1

TRANSITIVE GRAPHS ON 8 VERTICES

H1 DEG=0 F=XTVIAP AUT=5040 P=(1,+) CN=1,8
-H14

H2 DEG=1 F=XTIP AUT=48 P=(1,1,+) CN=2,4 T=1
A=1 0 4 0,20 0 100 -H13

H3 DEG=2 F=XTIP AUT=16 P=(1,2,1,+) GIR=4 CN=2,4
A=1 1 6 0,20 20 140 2[D3] -H11 D2[B1] B2XD2 B2*D3 1/4 2/5 3/104 4/22 5/2

H4 DEG=2 F=TIAP AUT=2 P=(1,2,2,2,1) GIR=8 CN=2,4 OCTAGON
A=1 1 4 2,20 10 140 E=-2 2-1.41421 2+0 2+1.41421 2 K=(6 1,) -H12 1/10 4/11

H5 DEG=3 F=XTIP AUT=144 P=(1,3,+) GIR=3 CN=4,2 T=2
A=1 3 7 0,20 60 160 2[D4] -H8 SW(D2) SW(D4) D2[B2] 1/5 2/22 3/70 4/42 5/3

H6 DEG=3 F=I AUT=2 P=(1,12,22) GIR=4 CN=3,4
A=1 1 1 10,24 52 26 E=2-2.41421 -1 2+.41421 2+1 3 K=(3,) -H10 D(H4) 1/11 4/26

H7 DEG=3 F=TIAP AUT=6 P=(1,3,3,1) GIR=4 CN=2,4 T=2 CUBE
A=1 1 1 14,12 6 160 E=-3 3-1 3+1 3 K=(3 1,) -H9 SW(D1) SW(D3) -W4(B2) B2XD3 -B2XD4 B2*D4 2/11 3/45 4/15

H8 DEG=4 F=TIA AUT=144 P=(1,4,3) GIR=4 CN=2,4 T=3
A=1 1 1 1,36 36 36 E=-4 6+0 4 K=(3 1,) -H5 D(H7) B2[D1] D3[B1] -B1XD4

TRANSITIVE GRAPHS ON 8 VERTICES (CONTD)

H9 DEG=4 F=I AUT=6 P=(1,13,3) GIR=3 CN=4,2
A=1 1 5 15,12 62 146 E=3-2 3+0 2 4 K=(,3 1) -H7 W4(B2) B2XD4 -B2XD3 -B2*D4

H10 DEG=4 F=IP AUT=2 P=(1,22,12) GIR=3 CN=4,3 ANTIPRISM
A=1 1 5 13,6 54 162 E=2-2 2-1.41421 0 2+1.41421 4 K=(,3) -H6 -D(H4)

H11 DEG=5 F=I AUT=16 P=(1,14,2) GIR=3 CN=4,2
A=1 3 3 7,13 74 174 E=-3 4-1 2+1 5 K=(,6 2) -H3 -L(H3) -W2(D2) B2[D2] D3[B2]
-B1XD3 -B2XD2 -B2*D3

H12 DEG=5 F=I AUT=2 P=(1,122,2) GIR=3 CN=4,2
A=1 1 5 13,27 56 136 E=2-2.41421 2-1 2+.41421 1 5 K=(,6 1) -H4 -L(H4)
-W1(D3)

H13 DEG=6 F=TIA AUT=48 P=(1,6,1) GIR=3 CN=4,2 T=1
A=1 1 7 7,37 37 176 E=3-2 4+0 6 K=(,12 8) -H2 -W1(H2) B2[D3] D4[B1] -B1XD2
-B2XD1 -B2*D2

H14 DEG=7 F=TVIA AUT=5040 P=(1,7) GIR=3 CN=8,1 T=2
A=1 3 7 17,37 77 177 E=7-1 7 K=(,21 35 35 21 7 1) -H1 B2[D4] D4[B2]

TRANSITIVE GRAPHS ON 9 VERTICES

I1 DEG=0 F=XTVIAP AUT=40320 P=(1,+) CN=1,9
-I9

I2 DEG=2 F=XTIP AUT=144 P=(1,2,+) GIR=3 CN=3,3
A=1 3 0 10,0 30 40 240 3[C2] -I7 1/4 2/4

I3 DEG=2 F=TIP AUT=2 P=(1,2,2,2,2) GIR=9 CN=3,5 NONAGON
A=1 1 4 2,20 10 100 240 E=2-1.87939 2-1 2+.34730 2+1.53209 2 K=(10 4,) -I8
1/10

I4 DEG=4 F=TVIS AUT=8 P=(1,4,4) GIR=3 CN=3,3 T=1
A=1 3 1 11,24 12 154 162 E=4-2 4+1 4 K=(2,2) L(F5) -L(F5) C2XC2 -C2XC2 C2*C2
-C2*C2 2/12

I5 DEG=4 F=I AUT=2 P=(1,22,22) GIR=3 CN=3,3
A=1 3 1 1,34 32 124 252 E=2-2.87939 2-.65270 2+.53209 2+1 4 K=(3,1) -I6
-D(I3) 1/14

I6 DEG=4 F=I AUT=2 P=(1,22,22) GIR=3 CN=3,3
A=1 1 3 15,24 12 144 342 E=2-2 2-1.53209 2-.34730 2+1.87939 4 K=(1,3) -I5
D(I3) 1/11

I7 DEG=6 F=TIA AUT=144 P=(1,6,2) GIR=3 CN=3,3 T=1
A=1 1 1 17,17 17 176 176 E=2-3 6+0 6 K=(1,9) -I2 -L(I2) C2[C1] -C1XC2

I8 DEG=6 F=I AUT=2 P=(1,222,2) GIR=3 CN=5,3
A=1 3 5 13,27 17 174 372 E=2-2.53209 2-1.34730 2+0 2+.87939 6 K=(,10 4) -I3
-L(I3)

I9 DEG=8 F=TVIA AUT=40320 P=(1,8) GIR=3 CN=9,1 T=2
A=1 3 7 17,37 77 177 377 E=8-1 8 K=(,28 56 70 56 28 8 1) -I1 C2[C2]

TRANSITIVE GRAPHS ON 10 VERTICES

J1 DEG=0 F=XTVIAP AUT=362880 P=(1,+) CN=1,10

J2 DEG=1 F=XTIP AUT=384 P=(1,1,+) CN=2,5 T=1
A=1 0 4 0,20 0 100 0 400

J3 DEG=2 F=XTIP AUT=20 P=(1,2,2,+) GIR=5 CN=3,6
A=1 1 4 12,0 40 0 300 240 2[E2] 1/4 2/40

J4 DEG=2 F=TIAP AUT=2 P=(1,2,2,2,2,1) GIR=10 CN=2,5 POLYGON
A=1 1 4 2,20 10 100 40 600 E=-2 2-1.61803 2-.61803 2+.61803 2+1.61803 2
K=(15 10 1,) B2*E2 1/10 2/24

J5 DEG=3 F=I AUT=2 P=(1,12,22,2) GIR=4 CN=2,5
A=1 1 1 12,6 4 10 320 340 E=-3 2-1.61803 2-.61803 2+.61803 2+1.61803 3
K=(9 4 1,) D(J4) 1/3 2/7

J6 DEG=3 F=IP AUT=2 P=(1,12,22,2) GIR=4 CN=3,5 PRISM
A=1 1 1 12,6 10 104 240 520 E=2-2.61803 2-.61803 2-.38197 1 2+1.61803 3
K=(9 4,) B2XE2 1/21 2/41

J7 DEG=3 F=NTVI AUT=12 P=(1,3,6) GIR=5 CN=3,5 T=3 PETERSEN GRAPH
A=1 1 1 10,22 10 102 144 224 E=4-2 5+1 3 K=(9 2,) -L(E3)

J8 DEG=4 F=XTI AUT=2880 P=(1,4,+) GIR=3 CN=5,2 T=2
A=1 3 7 17,0 40 140 340 740 2[E3] SW(E3) 1/24 2/140

J9 DEG=4 F=I AUT=32 P=(1,4,14) GIR=4 CN=3,5 T=1
A=1 1 1 1,36 30 106 106 630 E=2-3.23607 5+0 2+1.23607 4 K=(6 2,) E2[B1] 1/14
2/130

J10 DEG=4 F=TIA AUT=24 P=(1,4,4,1) GIR=4 CN=2,5 T=2
A=1 1 1 1,34 32 26 16 740 E=-4 4-1 4+1 4 K=(6 4 1,) SW(E1) -W5(B2) -B2XE3
B2*E3 1/12 2/33

J11 DEG=4 F=IAP AUT=2 P=(1,22,22,1) GIR=3 CN=4,4 ANTIPRISM
A=1 1 3 15,24 12 44 302 740 E=2-2.23607 4-1 0 2+2.23607 4 K=(3,3) SW(E2)
-D(J11) -D(J6) 1/6 2/43

TRANSITIVE GRAPHS ON 11 VERTICES

K1 DEG=0 F=XTVIAP AUT=3628800 P=(1,+) CN=1,11

K2 DEG=2 F=TVIP AUT=2 P=(1,2,2,2,2,2) GIR=11 CN=3,6 POLYGON
A=1 1 4 2,20 10 100 40 400,1200
E=2-1.91899 2-1.30972 2-.28463 2+.83083 2+1.68251 2 K=(21 20 5,) 1/10

K3 DEG=4 F=VI AUT=2 P=(1,22,22,2) GIR=3 CN=4,4
A=1 1 3 15,24 12 102 44 640,1700
E=2-2.20362 2-1.59435 2-.47889 2-.23648 2+2.51334 4 K=(6,3) D(K2) -D(K3) 1/24

K4 DEG=4 F=VI AUT=2 P=(1,22,222) GIR=4 CN=3,6
A=1 1 1 1,34 32 104 242 424,1212
E=2-3.22871 2-1.08816 2+.37279 2+.54620 2+1.39788 4 K=(9 4,) 1/5

TRANSITIVE GRAPHS ON 12 VERTICES

L1 DEG=0 F=XTVIAP AUT=39916800 P=(1,+) CN=1,12

L2 DEG=1 F=XTIP AUT=3840 P=(1,1,+) CN=2,6 T=1
A=1 0 4 0,20 0 100 0 400,0 2000

L3 DEG=2 F=XTIP AUT=2592 P=(1,2,+) GIR=3 CN=3,4
A=1 3 0 10,0 30 40 0 240,400 2400 2[F3] 4[C2] 1/20 2/20 3/400 4/40 5/10

L4 DEG=2 F=XTIP AUT=256 P=(1,2,1,+) GIR=4 CN=2,6
A=1 1 6 0,20 0 20 240 100,100 3000 3[D3] F2[B1] B2XF2 1/10 2/5 3/110 4/5 5/4

L5 DEG=2 F=XTIP AUT=24 P=(1,2,2,1,+) GIR=6 CN=2,6
A=1 1 4 2,30 0 100 0 400,1200 500 2[F4] B2*F3 B2*F4 C2*D2 1/4 2/10 3/102 5/2

L6 DEG=2 F=TIAP AUT=2 P=(1,2,2,2,2,2,1) GIR=12 CN=2,6 POLYGON
A=1 1 4 2,20 10 100 40 400,200 3000 E=-2 2-1.73205 2-1 2+0 2+1 2+1.73205 2
K=(28 35 15 1,) 1/2 3/120

L7 DEG=3 F=XTIP AUT=6912 P=(1,3,+) GIR=3 CN=4,3 T=2
A=1 3 7 0,20 0 60 260 100,1100 3100 3[D4] F2[B2] 1/11 2/7 3/34 4/7 5/21

L8 DEG=3 F=XTI AUT=864 P=(1,3,2,+) GIR=4 CN=2,6 T=3
A=1 1 1 16,16 0 100 100 100,1600 1600 2[F5] D2[C1] B2*F5 1/5 2/12 3/106 5/3

L9 DEG=3 F=XIP AUT=24 P=(1,12,2,+) GIR=3 CN=3,4
A=1 1 5 12,26 0 100 100 200,1500 1600 2[F6] W3(D2) B2XF3 C2XD2 1/21 2/21 3/401
5/11

L10 DEG=3 F=P AUT=2 P=(1,12,22,22) GIR=3 CN=3,4
A=1 1 5 10,4 2 102 240 120,440 3020 E=3-2 3-1 2+0 3+2 3 K=(18 10,1) W1(D4)
4/11

L11 DEG=3 AUT=4 P=(1,12,122,12) GIR=4 CN=2,6
A=1 1 1 14,10 4 2 2 620,540 340 E=-3 2-1.73205 3-1 3+1 2+1.73205 3
K=(19 15 5 1,) 3/124

L12 DEG=3 F=I AUT=2 P=(1,12,22,22) GIR=4 CN=3,6
A=1 1 1 12,6 10 4 200 500,1240 520 E=2-2.73205 3-1 2+0 2+.73205 2+2 3
K=(19 16 5,) D(L6) 1/41 3/32

L13 DEG=3 F=IAP AUT=2 P=(1,12,22,12,1) GIR=4 CN=2,6 PRISM
A=1 1 1 6,12 10 4 300 220,140 3400 E=-3 2-2 -1 4+0 1 2+2 3 K=(19 16 5 1,)
B2XF4 B2*F6 2/14 3/122

L14 DEG=4 F=XTIP AUT=384 P=(1,4,1,+) GIR=3 CN=3,4 T=1
A=1 1 7 7,36 0 100 100 700,700 3600 2[F7] L(H5) -D(L26) F3[B1] 1/24 2/120
3/600 5/12

L15 DEG=4 AUT=4 P=(1,112,122,2) GIR=3 CN=4,3
A=1 1 5 15,6 20 110 42 442,1300 2700 E=4-2 2-.73205 3+0 2+2.73205 4 K=(10,3 1)
W2(C2) 3/56

L16 DEG=4 AUT=2 P=(1,22,1222) GIR=3 CN=3,4
A=1 1 1 11,6 24 12 60 450,702 1304 E=3-2.56155 3-1 2+1 3+1.56155 4 K=(12 4,1)
4/103

TRANSITIVE GRAPHS ON 12 VERTICES (CONTD)

L17 DEG=4 F=I AUT=2 P=(1,22,122,2) GIR=4 CN=2,6
A=1 1 1 1,6 34 32 22 14,1540 1640 E=-4 2-1.73205 2-1 2+0 2+1 2+1.73205 4
K=(13 10 5 1,) 1/50 3/47

L18 DEG=4 F=I AUT=4 P=(1,22,14,2) GIR=3 CN=3,4
A=1 1 1 11,6 24 22 114 212,1440 2340 E=2-3 4-1 0 2+1 2+2 4 K=(12 6,1)
-D(L21) B2XF6 C2XD3 1/30 2/25 3/501 5/14

L19 DEG=4 F=I AUT=12 P=(1,13,23,2) GIR=4 CN=2,6
A=1 1 1 1,34 34 12 22 6,1700 1640 E=-4 -2 4-1 4+1 2 4 K=(13 10 5 1,) B2XF5
2/16 3/214

L20 DEG=4 F=IAP AUT=4 P=(1,4,24,1) GIR=3 CN=3,4 T=1 CUBOCTAHEDRON
A=1 1 5 3,30 6 50 304 60,1102 3600 E=5-2 3+0 3+2 4 K=(11 3,2) L(H7) -D(L10)
4/50

L21 DEG=4 F=IP AUT=2 P=(1,22,22,12) GIR=3 CN=3,4 ANTIPRISM
A=1 1 3 15,12 24 104 42 600,1440 3300 E=4-2 2-.73205 3+0 2+2.73205 4
K=(10 1,3) -D(L12) 1/44 3/205

L22 DEG=4 F=IA AUT=2 P=(1,22,222,1) GIR=3 CN=3,4
A=1 1 1 11,24 12 4 202 454,322 740 E=2-2.73205 2-2 3+0 2+.73205 2+2 4
K=(12 5,1) 1/22 3/403

L23 DEG=4 F=I AUT=64 P=(1,4,14,2) GIR=4 CN=2,6 T=1
A=1 1 1 1,36 30 30 6 6,1700 1700 E=-4 2-2 6+0 2+2 4 K=(13 11 5 1,) F4[B1]
B2*F7 C2*D3 1/42 2/110 3/221 5/24

L24 DEG=4 F=I AUT=4 P=(1,22,124) GIR=4 CN=3,6
A=1 1 1 1,6 60 50 224 222,1114 512 E=2-3 2-2 0 6+1 4 K=(13 6,) D(L13) 1/14
2/43 3/132 5/42

L25 DEG=5 F=XTI AUT=86400 P=(1,5,+) GIR=3 CN=6,2 T=2
A=1 3 7 17,37 0 100 300 700,1700 3700 2[F8] SW(F3) SW(F8) D2[C2] F3[B2] 1/25
2/121 3/610 5/13

L26 DEG=5 F=I AUT=64 P=(1,14,4,2) GIR=3 CN=4,3
A=1 3 7 3,23 60 160 14 414,1700 3700 E=-3 8-1 2+3 5 K=(4,6 2) SW(F2) SW(F4)
-D(L18) -D(L35) F4[B2] 1/43 2/114 3/245 5/61

L27 DEG=5 AUT=1 P=(1/5/6) GIR=3 CN=3,4
A=1 1 1 15,11 50 66 306 412,1160 3106 E=-3 2-2.73205 2-1 2+0 2+.73205 2+2 5
K=(7 1,3) 3/225

L28 DEG=5 F=I AUT=2 P=(1,122,222) GIR=3 CN=3,4
A=1 1 5 1,1 50 124 252 526,272 166 E=2-3.73205 2-1 2-.26795 5+1 5 K=(9 4,1)
D(L22) 1/61 3/413

L29 DEG=5 F=I AUT=2 P=(1,122,222) GIR=3 CN=4,3
A=1 3 7 1,1 22 42 170 264,1350 724 E=-3 2-2.73205 2-1 2+0 2+.73205 2+2 5
K=(7,3 1) 1/13 3/174

L30 DEG=5 F=I AUT=12 P=(1,23,6) GIR=3 CN=4,3
A=1 3 1 11,31 44 12 314 222,1524 1342 E=6-2 3+1 2+2 5 K=(6,4 1) C2XD4 -C2*D4
1/31 2/27 3/434 4/17 5/31

TRANSITIVE GRAPHS ON 12 VERTICES (CONTD)

L31 DEG=5 AUT=1 P=(1/5/6) GIR=3 CN=3,4
A=1 1 1 15,15 74 42 210 702,622 3406 E=2-2.73205 2-2 -1 2+0 2+.73205 1 3 5
K=(6 1,4) 3/503

L32 DEG=5 F=I AUT=4 P=(1,122,24) GIR=3 CN=4,3
A=1 1 1 3,23 16 16 250 144,1630 1524 E=2-3 2-2 4+0 1 2+2 5 K=(7,3 1) 1/15
2/17 3/311 5/23

L33 DEG=5 F=I AUT=4 P=(1,14,24) GIR=3 CN=4,4
A=1 1 1 11,5 50 124 262 162,1216 516 E=3-3 2-1 6+1 5 K=(8 2,2) -L(F7) D(L20)
4/111

L34 DEG=5 F=TIA AUT=120 P=(1,5,5,1) GIR=4 CN=2,6 T=2
A=1 1 1 1,1 74 72 66 56,36 3700 E=-5 5-1 5+1 5 K=(10 10 5 1,) SW(F1) SW(F5)
-W6(B2) -B2XF8 B2*F8 2/111 3/163

L35 DEG=5 F=IA AUT=8 P=(1,14,14,1) GIR=3 CN=3,4
A=1 1 1 15,15 74 42 206 212,1422 3700 E=2-3 5-1 3+1 3 5 K=(6 2,4) SW(F6)
SW(F7) -D(L15) B2XF7 2/124 3/416

L36 DEG=5 F=I AUT=2 P=(1,122,222) GIR=3 CN=4,4
A=1 1 1 5,31 50 124 216 116,642 3122 E=2-3 2-1.73205 2-1 3+1 2+1.73205 5
K=(7,3) 1/7 3/350

L37 DEG=5 F=TIAP AUT=10 P=(1,5,5,1) GIR=3 CN=4,4 T=1 ICOSAHEDRON
A=1 3 5 3,31 50 114 22 560,606 3700 E=3-2.23607 5-1 3+2.23607 5 K=(5,5)
SW(E2+) -D(L37) 4/121

TRANSITIVE GRAPHS ON 13 VERTICES

M1 DEG=0 F=XTVIAP AUT=479001600 P=(1,+) CN=1,13

M2 DEG=2 F=TVIP AUT=2 P=(1,2,2,2,2,2,2) GIR=13 CN=3,7 POLYGON
A=1 1 4 2,20 10 100 40 400,200 2000 5000
E=2-1.94188 2-1.49702 2-.70921 2+.24107 2+1.13613 2+1.77091 2 K=(36 56 35 6,)
1/1

M3 DEG=4 F=VI AUT=4 P=(1,4,44) GIR=4 CN=4,7 T=1
A=1 1 1 1,20 10 142 144 54,1122 224 4412 E=4-2.65109 4+.27389 4+1.37720 4
K=(18 12,) 1/6

M4 DEG=4 F=VI AUT=2 P=(1,22,22,22) GIR=3 CN=4,5
A=1 1 3 15,24 12 44 102 400,1200 3500 3240
E=2-2.20623 2-1.70081 2-1.25595 2-.17097 2+.42692 2+2.90704 4 K=(15 4,3)
D(M2) 1/44

M5 DEG=4 F=VI AUT=2 P=(1,22,222,2) GIR=4 CN=3,7
A=1 1 1 1,34 32 4 202 414,222 2500 5240
E=2-3.43891 2-.80575 2-.46814 2-.36089 2+1.06170 2+2.01199 4 K=(18 16 5,)
-D(M4) 1/5

M6 DEG=6 F=TVIS AUT=6 P=(1,6,6) GIR=3 CN=5,5 T=1
A=1 3 1 15,11 43 124 312 432,654 3046 5360 E=6-2.30278 6+1.30278 6 K=(6,6)
1/15

M7 DEG=6 F=VIS AUT=2 P=(1,222,222) GIR=3 CN=5,5
A=1 3 3 15,5 3 132 74 244,1502 3350 3560
E=2-3.19783 2-1.96516 2-1.07010 2+.07010 2+.96516 2+2.19783 6 K=(6,6) 1/64

M8 DEG=6 F=VI AUT=2 P=(1,222,222) GIR=3 CN=4,5
A=1 3 5 3,1 1 174 172 164,1152 2524 5252
E=2-4.14811 2-.88018 2-.56468 2+.51496 2+.66799 2+1.41002 6 K=(9 4,3) -M9
D(M5) 1/16

M9 DEG=6 F=VI AUT=2 P=(1,222,222) GIR=3 CN=5,4
A=1 1 3 5,33 75 124 52 412,1224 3604 7602
E=2-2.41002 2-1.66799 2-1.51496 2-.43532 2-.11982 2+3.14811 6 K=(3,9 4) -M8
-D(M5) 1/52

TRANSITIVE GRAPHS ON 14 VERTICES

N1 DEG=0 F=XTVIAP AUT=6227020800 P=(1,+) CN=1,14

N2 DEG=1 F=XTIP AUT=46080 P=(1,1,+) CN=2,7 T=1
A=1 0 4 0,20 0 100 0 400,0 2000 0 10000

N3 DEG=2 F=XTIP AUT=28 P=(1,2,2,2,+) GIR=7 CN=3,8
A=1 1 4 2,20 50 0 200 0,0 2400 1200 3000 2[G2] 1/20 2/200

N4 DEG=2 F=TIAP AUT=2 P=(1,2,2,2,2,2,2,1) GIR=14 CN=2,7 POLYGON
A=1 1 4 2,20 10 100 40 400,200 2000 1000 14000
E=-2 2-1.80194 2-1.24698 2-.44504 2+.44504 2+1.24698 2+1.80194 2
K=(45 84 70 21 1,) B2*G2 1/2 2/140

N5 DEG=3 F=I AUT=2 P=(1,12,22,22,2) GIR=4 CN=2,7
A=1 1 1 12,6 10 4 200 100,240 120 3400 5400
E=-3 2-2.24698 2-.80194 2-.55496 2+.55496 2+.80194 2+2.24698 3
K=(33 44 25 6 1,) D(N4) 1/11 2/7

N6 DEG=3 F=IP AUT=2 P=(1,12,22,22,2) GIR=4 CN=3,7 PRISM
A=1 1 1 12,6 10 4 200 500,240 120 5000 12400
E=2-2.80194 2-1.44504 2-.80194 2+.24698 2+.55496 1 2+2.24698 3 K=(33 44 25 6,)
B2XG2 1/5 2/201

N7 DEG=3 F=TI AUT=24 P=(1,3,6,4) GIR=6 CN=2,7 T=4 HEAWOOD GRAPH
A=1 1 1 10,2 2 4 4 10,1240 1500 460 320 E=-3 6-1.41421 6+1.41421 3
K=(33 42 20 6 1,) 2/144

N8 DEG=4 F=XI AUT=28 P=(1,22,2,+) GIR=3 CN=4,6
A=1 3 5 3,34 72 0 200 200,400 3400 3600 7200 2[G3] 1/104 2/1200

N9 DEG=4 F=IAP AUT=2 P=(1,22,22,22,1) GIR=3 CN=4,5 ANTIPRISM
A=1 1 3 15,24 12 44 102 500,240 1400 6200 17000
E=2-2.24698 2-1.69202 2-1.35690 2-.55496 0 2+.80194 2+3.04892 4 K=(21 10,3)
1/60 2/504

N10 DEG=4 F=I AUT=2 P=(1,22,222,12) GIR=4 CN=2,7
A=1 1 1 1,32 34 14 22 2,4 3600 3300 3440
E=-4 2-2.24698 2-.80194 2-.55496 2+.55496 2+.80194 2+2.24698 4
K=(24 26 15 6 1,) B2*G3 1/12 2/164

TRANSITIVE GRAPHS ON 14 VERTICES (CONTD)

N11 DEG=4 F=IA AUT=2 P=(1,22,2222,1) GIR=4 CN=3,7
A=1 1 1 1,24 12 4 2 414,1222 450 4320 740
E=2-3.04892 2-2.24698 2-.55496 0 2+.80194 2+1.35690 2+1.69202 4 K=(24 22 5,)
1/22 2/214

N12 DEG=4 F=I AUT=128 P=(1,4,14,4) GIR=4 CN=3,7 T=1
A=1 1 1 1,36 30 6 6 30,600 3100 4600 13100 E=2-3.60388 2-.89008 7+0 2+2.49396 4
K=(24 28 15 3,) -D(N14) G2[B1] 1/30 2/1005

N13 DEG=4 F=TI AUT=24 P=(1,4,6,3) GIR=4 CN=2,7 T=2 DUAL OF HEAWOOD
A=1 1 1 1,30 24 14 12 6,22 2700 3240 1540 E=-4 6-1.41421 6+1.41421 4
K=(24 24 15 6 1,) 2/154

N14 DEG=5 F=I AUT=128 P=(1,14,4,4) GIR=3 CN=5,4
A=1 3 7 3,23 14 60 114 260,1200 2500 7200 16500
E=2-2.60388 7-1 2+.10992 2+3.49396 5 K=(12,6 2) -D(N12) G2[B2] 1/31 2/207

N15 DEG=5 F=I AUT=2 P=(1,122,2222) GIR=3 CN=4,5
A=1 1 5 5,11 50 24 242 122,1006 2412 3340 4720
E=2-2.69202 2-2.35690 2-1.24698 -1 2+.44504 2+1.80194 2+2.04892 5 K=(15 4,3)
D(N6) D(N9) 1/61 2/1114

N16 DEG=5 F=A AUT=1 P=(1/5/7/1) GIR=3 CN=4,5
A=1 1 1 11,15 60 6 202 530,406 710 2066 7300
E=2-3.21615 2-1.85926 -1 2-.38772 2-.16723 2+.96917 2+2.66119 5 K=(15 8,3)
2/226

N17 DEG=5 F=I AUT=2 P=(1,122,222,2) GIR=3 CN=4,5
A=1 1 5 11,5 70 164 12 406,1042 422 7200 16500
E=2-3.24698 2-1.55496 2-1.24698 2-.19806 2+.44504 2+1.80194 3 5 K=(15 8,3)
B2XG3 1/105 2/1201

N18 DEG=5 F=I AUT=2 P=(1,122,2222) GIR=4 CN=3,7
A=1 1 1 1,1 66 72 52 26,1110 2604 5050 12424
E=2-4.04892 2-1.24698 -1 2+.35690 2+.44504 2+.69202 2+1.80194 5 K=(18 16 5,)
D(N11) D(N5) 1/23 2/541

N19 DEG=5 F=I AUT=2 P=(1,122,222,2) GIR=4 CN=2,7
A=1 1 1 1,1 72 66 54 34,26 52 7500 7600
E=-5 2-1.80194 2-1.24698 2-.44504 2+.44504 2+1.24698 2+1.80194 5
K=(18 20 15 6 1,) 1/13 2/172

N20 DEG=6 F=XTI AUT=3628800 P=(1,6,+) GIR=3 CN=7,2 T=2
A=1 3 7 17,37 77 0 200 600,1600 3600 7600 17600 2[G4] SW(G4) 1/124 2/1600

N21 DEG=6 AUT=1 P=(1/6/7) GIR=3 CN=5,4
A=1 1 1 11,5 75 70 46 422,630 1456 3302 13206
E=2-3.21615 -2 2-1.85926 2-.38772 2-.16723 2+.96917 2+2.66119 6 K=(9,6 2)
2/233

N22 DEG=6 F=I AUT=2 P=(1,222,1222) GIR=3 CN=4,5
A=1 3 1 1,5 3 170 164 552,1224 2612 4134 12072
E=2-4.04892 2-1.80194 2-.44504 2+.35690 2+.69202 2+1.24698 2 6 K=(12 6,3)
1/122 2/1206

N23 DEG=6 F=I AUT=2 P=(1,222,1222) GIR=3 CN=4,5
A=1 3 5 3,1 1 170 164 152,1304 2642 5134 2472
E=2-4.04892 -2 2-1.24698 2+.35690 2+.44504 2+.69202 2+1.80194 6 K=(12 6,3)
D(N16) 1/46 2/1017

N24 DEG=6 F=TIA AUT=720 P=(1,6,6,1) GIR=4 CN=2,7 T=2
A=1 1 1 1,1 1 174 172 166,156 136 76 17600 E=-6 6-1 6+1 6 K=(15 20 15 6 1,)
SW(G1) -W7(B2) -B2XG4 B2*G4 1/52 2/173

N25 DEG=6 F=I AUT=2 P=(1,222,1222) GIR=3 CN=4,5
A=1 3 1 11,15 23 36 214 222,544 3142 5450 13520
E=2-2.69202 2-2.35690 2-1.80194 2-.44504 2+1.24698 2 2+2.04892 6 K=(9 2,6)
1/106 2/1203

N26 DEG=6 F=I AUT=2 P=(1,222,1222) GIR=3 CN=4,5
A=1 1 3 5,15 23 36 214 222,544 3142 5450 13520
E=2-2.69202 2-2.35690 -2 2-1.24698 2+.44504 2+1.80194 2+2.04892 6 K=(9 2,6)
1/54 2/570

N27 DEG=6 F=IA AUT=2 P=(1,222,222,1) GIR=3 CN=4,5
A=1 3 3 15,5 3 132 74 144,1142 2310 5460 17600
E=2-3.49396 6-1 2-.10992 2 2+2.60388 6 K=(9 4,6) SW(G3) -D(N28) 1/160 2/1214

N28 DEG=6 F=IA AUT=2 P=(1,222,222,1) GIR=3 CN=5,4
A=1 1 3 5,33 75 124 52 412,224 3204 7402 17600
E=2-2.60388 -2 6-1 2+.10992 2+3.49396 6 K=(6,9 4) SW(G2) -D(N17) -D(N27) 1/16
2/1055

TRANSITIVE GRAPHS ON 15 VERTICES

01 DEG=0 F=XTVIAP P=(1,+) CN=1,15

02 DEG=2 F=XTIP AUT=62208 P=(1,2,+) GIR=3 CN=3,5
A=1 3 0 10,0 30 40 0 240,400 0 2400 4000 24000 5[C2] 1/20

03 DEG=2 F=XTIP AUT=400 P=(1,2,2,+) GIR=5 CN=3,9
A=1 1 4 12,0 40 0 0 500,440 200 0 14000 10200 3[E2] 1/40

04 DEG=2 F=TIP AUT=2 P=(1,2,2,2,2,2,2,2) GIR=15 CN=3,8 POLYGON
A=1 1 4 2,20 10 100 40 400,200 2000 1000 10000 24000
E=2-1.95630 2-1.61803 2-1 2-.20906 2+.61803 2+1.33826 2+1.82709 2
K=(55 120 126 56 7,) 1/2

05 DEG=4 F=XTI AUT=691200 P=(1,4,+) GIR=3 CN=5,3 T=2
A=1 3 7 17,0 40 0 140 540,1540 200 4200 14200 34200 3[E3] 1/44

06 DEG=4 F=I AUT=4 P=(1,22,24,4) GIR=3 CN=3,5
A=1 1 1 11,4 42 24 22 214,412 500 1040 14240 16100
E=4-2.61803 4-.38197 2+.38197 2+1 2+2.61803 4 K=(30 32 10,1) C2XE2 1/60

07 DEG=4 F=NTIA AUT=8 P=(1,4,8,2) GIR=3 CN=4,6 T=1
A=1 3 1 11,20 4 110 144 2,210 1060 3002 5300 12440 E=5-2 4-1 5+2 4
K=(29 24 2,2) L(J7)

08 DEG=4 F=I AUT=2 P=(1,22,22,22,2) GIR=3 CN=3,5
A=1 1 3 15,24 12 44 102 400,200 1500 2240 7000 33000
E=2-2.16535 2-2 4-1 2-.12920 2+1.12920 2+3.16535 4 K=(28 20 1,3) D(04) 1/12

09 DEG=4 F=I AUT=2 P=(1,22,2222,2) GIR=4 CN=3,8
A=1 1 1 1,24 12 4 2 414,1222 410 4220 10540 4340
E=2-3.23607 2-1.82709 2-1.33826 2+.20906 2+1 2+1.23607 2+1.95630 4
K=(31 36 15 2,) 1/104

010 DEG=4 F=I AUT=2 P=(1,22,222,22) GIR=3 CN=3,5
A=1 1 1 11,24 12 54 122 2,4 3040 3100 11400 26200
E=2-2.95630 2-2 2-1.20906 2+.33826 2+.38197 2+.82709 2+2.61803 4 K=(30 32 11,1)
1/22

011 DEG=4 F=I AUT=2 P=(1,22,222,22) GIR=4 CN=3,8
A=1 1 1 1,34 32 4 202 14,22 2400 5200 12500 5240
E=2-3.57433 4-1 2-.27977 2+.40898 2+1 2+2.44512 4 K=(31 40 25 6,) 1/140

012 DEG=4 F=I AUT=4 P=(1,4,224,2) GIR=4 CN=3,8 T=1
A=1 1 1 1,24 12 30 6 120,50 1042 2104 6600 11600
E=2-3.23607 2-2 4-.61803 2+1.23607 4+1.61803 4 K=(31 36 16 2,) C2*E2 1/11

013 DEG=6 F=I AUT=2 P=(1,222,2222) GIR=3 CN=5,3
A=1 3 7 17,1 1 50 320 344,542 2510 1260 16504 15242
E=2-2.95630 2-2.61803 2-1.20906 2-.38197 2+.33826 2+.82709 2+3 6 K=(13,6 4 1)
1Fj44

014 DEG=6 F=I AUT=2 P=(1,222,2222) GIR=3 CN=3,5
A=1 3 5 3,1 41 134 72 104,42 3464 3312 3260 23510
E=2-3.16535 2-3 2-1.12920 2+.12920 4+1 2+2.16535 6 K=(15 8 1,4) 1/121

015 DEG=6 F=I AUT=2 P=(1,222,2222) GIR=3 CN=4,5
A=1 3 1 1,5 3 72 334 64,1112 2224 1412 16160 15150
E=2-3.78339 2-2.61803 2-.38197 2+0 2+.48883 2+1.54732 2+1.74724 6 K=(16 8,3)
1/16

016 DEG=6 F=I AUT=4 P=(1,24,224) GIR=3 CN=3,5
A=1 3 5 13,5 43 146 36 30,140 3300 3420 17410 17240
E=2-3 4-1.61803 2-1.23607 4+.61803 2+3.23607 6 K=(12 4 1,7) 1/122

017 DEG=6 F=I AUT=2 P=(1,222,222,2) GIR=3 CN=5,4
A=1 1 3 5,33 75 124 52 204,1402 2412 1224 17200 37400
E=2-2.61803 2-1.74724 2-1.54732 2-.48883 2-.38197 2+0 2+3.78339 6 K=(10,9 4)
-D(010) -D(011) 1/52

018 DEG=6 F=I AUT=2 P=(1,222,2222) GIR=3 CN=4,5
A=1 1 3 5,23 55 164 152 204,402 3220 7410 13012 27024
E=2-2.82709 2-2.33826 2-1.23607 2-.79094 2+0 2+.95630 2+3.23607 6 K=(12 4,7)
-D(017) 1/124

019 DEG=6 F=I AUT=2 P=(1,222,2222) GIR=3 CN=3,5
A=1 1 1 11,23 55 134 72 42,104 3404 7202 13214 7422
E=2-3 2-2.61803 2-.82709 2-.38197 2-.33826 2+1.20906 2+2.95630 6 K=(13 4 1,6)
D(08) 1/13

TRANSITIVE GRAPHS ON 15 VERTICES (CONTD)

O20 DEG=6 F=I AUT=48 P=(1,24,8) GIR=3 CN=5,3
A=1 3 1 11,31 71 104 12 614,422 3224 2442 15244 12702 E=8-2 4+1 2+3 6
K=(12,7 4 1) C2XE3 -C2*E3 1/64

O21 DEG=6 F=NTVI AUT=48 P=(1,6,8) GIR=3 CN=4,5 T=1
A=1 3 1 1,21 11 124 142 654,54 2342 2524 15032 2632 E=5-3 9+1 6 K=(16 8 2,3)
-L(F8) D(O7)

O22 DEG=6 F=I AUT=2 P=(1,222,2222) GIR=3 CN=3,5
A=1 3 1 1,1 1 174 172 424,1212 2124 5052 12164 5152
E=2-4.57433 2-1.27977 2-.59102 2+0 4+1 2+1.44512 6 K=(18 16 5,1) 1/62

O23 DEG=6 F=I AUT=5184 P=(1,6,26) GIR=4 CN=3,8 T=1
A=1 1 1 1,1 1 176 176 160,1016 1016 1016 16160 16160
E=2-4.85410 10+0 2+1.85410 6 K=(19 20 10 2,) D(O12) D(O9) E2[C1] 1/51

O24 DEG=6 F=I AUT=4 P=(1,24,224) GIR=3 CN=4,6
A=1 1 5 13,5 43 2 204 740,630 1262 1514 6464 12312
E=2-2.61803 4-2.23607 2-.38197 2+0 4+2.23607 6 K=(13 4,6) -D(O6) 1/15

TRANSITIVE GRAPHS ON 16 VERTICES

P1 DEG=0 F=XTVIAP P=(1,+) CN=1,16

P2 DEG=1 F=XTIP AUT=645120 P=(1,1,+) CN=2,8 T=1
A=1 0 4 0,20 0 100 0 400,0 2000 0 10000 0,40000

P3 DEG=2 F=XTIP AUT=6144 P=(1,2,1,+) GIR=4 CN=2,8
A=1 1 6 0,20 0 20 240 100,0 100 5000 2000 2000,60000 2[H3] 4[D3] H2[B1] B2XH2
D2XD2 B2*H3 D2*D3 1/20 2/3 3/3 4/30 5/24000 6/1200 7/3 8/42 9/30 10/3 11/3
12/240 13/5 14/4

P4 DEG=2 F=XTIP AUT=32 P=(1,2,2,2,1,+) GIR=8 CN=2,8
A=1 1 4 2,20 10 140 0 400,0 0 2000 5000 2400,14000 2[H4] B2*H4 1/100 2/400
6/1001 8/24 11/200 12/102 13/1000 14/100

P5 DEG=2 F=TIAP AUT=2 P=(1,2,2,2,2,2,2,2,1) GIR=16 CN=2,8 POLYGON
A=1 1 4 2,20 10 100 40 400,200 2000 1000 10000 4000,60000
E=-2 2-1.84776 2-1.41421 2-.76537 2+0 2+.76537 2+1.41421 2+1.84776 2
K=(66 165 210 126 28 1,) 1/10 12/210

P6 DEG=3 F=XTIP AUT=497664 P=(1,3,+) GIR=3 CN=4,4 T=2
A=1 3 7 0,20 0 60 260 100,0 1100 5100 2000 22000,62000 2[H5] 4[D4] H2[B2] 1/21
2/22 3/22 4/1004 5/60001 6/10004 7/22 8/1002 9/1002 10/22 11/24 12/2040 13/120
14/5

P7 DEG=3 F=XTIP AUT=288 P=(1,3,3,1,+) GIR=4 CN=2,8 T=2
A=1 1 1 14,12 6 160 0 400,0 2000 5000 2400 25000,12400 2[H7] B2XH3 D2XD3 B2*H5
B2*H7 D2*D4 2/201 3/12 4/402 5/441 6/1024 7/11 8/404 9/300 10/12 11/402 12/106
13/16

P8 DEG=3 F=XI AUT=32 P=(1,12,22,+) GIR=4 CN=3,8
A=1 1 1 12,6 50 124 0 400,0 400 1000 16000 12400,7000 2[H6] 1/101 2/42 6/2024
8/43 11/14 12/4040 13/60 14/101

P9 DEG=3 F=I AUT=2 P=(1,12,22,22,22) GIR=4 CN=3,8
A=1 1 1 12,6 10 4 240 120,200 100 5000 2400 24000,52000
E=2-2.84776 2-1.76537 -1 2-.41421 2-.23463 2+.84776 2+1 2+2.41421 3
K=(51 96 85 36 7,) D(P5) 1/11 12/54

P10 DEG=3 F=IAP AUT=2 P=(1,12,22,22,12,1) GIR=4 CN=2,8 PRISM
A=1 1 1 12,6 10 4 200 100,240 120 1400 5000 2400,70000
E=-3 2-2.41421 3-1 2-.41421 2+.41421 3+1 2+2.41421 3 K=(51 96 85 36 7 1,)
B2XH4 B2*H6 2/11 6/1042 12/301

P11 DEG=3 AUT=8 P=(1,12,122,122,2) GIR=4 CN=2,8
A=1 1 1 14,2 2 10 4 160,100 40 600 600 26000,16000
E=-3 2-2.23607 5-1 5+1 2+2.23607 3 K=(51 95 80 33 7 1,) 6/2011 9/440 12/114
13/404

P12 DEG=3 F=IA AUT=6 P=(1,3,6,23,1) GIR=6 CN=2,8 T=2
A=1 1 1 10,10 4 2 4 2,620 1140 300 440 1020,70000
E=-3 4-1.73205 3-1 3+1 4+1.73205 3 K=(51 94 75 27 7 1,) 8/105 11/102 12/412
13/1001

P13 DEG=4 F=XTI AUT=165888 P=(1,4,3,+) GIR=4 CN=2,8 T=3
A=1 1 1 1,36 36 36 0 400,400 400 400 17000 17000,17000 2[H8] D2[D1] H3[B1]
B2*H8 D3*D3 1/104 2/205 3/140 4/2042 5/40034 6/11400 7/140 8/2140 9/2102 10/140
11/140 12/5000 13/500 14/30

P14 DEG=4 F=XI AUT=288 P=(1,13,3,+) GIR=3 CN=4,4
A=1 1 5 15,22 46 152 0 400,400 1000 6400 5000 32400,27000 2[H9] W4(D2) B2XH5
D2XD4 2/23 3/46 4/224 5/20520 6/306 7/106 8/222 9/161 10/106 11/25 12/451
13/121

P15 DEG=4 F=XIP AUT=32 P=(1,22,12,+) GIR=3 CN=4,6
A=1 1 3 15,6 52 164 0 400,0 3000 2400 17000 12400,45400 2[H10] 1/120 2/420
6/11100 8/2041 11/220 12/2011 13/103 14/12

P16 DEG=4 F=IP AUT=2 P=(1,22,22,22,12) GIR=3 CN=4,6 ANTIPRISM
A=1 1 3 15,24 12 44 102 400,200 1500 2240 3000 32000,65000
E=2-2.17958 2-2 2-1.41421 2-.64885 2-.43355 0 2+1.41421 2+3.26197 4
K=(36 35 5,3) 1/6 12/4404

P17 DEG=4 F=I AUT=2 P=(1,22,1222,22) GIR=4 CN=3,8
A=1 1 1 1,30 24 12 4 402,1014 422 4240 2140 21200,50500
E=2-3.41421 2-1.84776 2-.76537 2-.58579 0 2+.76537 2+1.84776 2+2 4
K=(39 56 35 9,) 1/60 12/2003

P18 DEG=4 F=I AUT=2 P=(1,22,2222,12) GIR=4 CN=3,8
A=1 1 1 1,2 4 24 12 122,1054 50 120 14600 4300,50440
E=2-3.26197 2-2 2-1.41421 0 2+.43355 2+.64885 2+1.41421 2+2.17958 4
K=(39 55 30 6,) 1/14 12/4003

P19 DEG=4 F=IA AUT=2 P=(1,22,222,22,1) GIR=4 CN=2,8
A=1 1 1 1,32 34 2 4 14,22 1600 2600 2240 1500,74000
E=-4 2-2.61313 2-1.08239 6+0 2+1.08239 2+2.61313 4 K=(39 59 45 21 7 1,) 1/42
12/512

P20 DEG=4 F=N AUT=2 P=(1,112,111222,11) GIR=4 CN=3,8
A=1 1 1 1,30 2 4 24 14,1002 402 2320 4310 1540,6240
E=2-3.23607 -2 4-1.41421 0 2+1.23607 4+1.41421 2 4 K=(39 54 30 6,)

TRANSITIVE GRAPHS ON 16 VERTICES (CONTD)

P21 DEG=4 F=I AUT=2 P=(1,112,1222,22) GIR=4 CN=3,8
A=1 1 1 1,6 22 12 24 14,220 2110 1240 540 25000,52400
E=2-3.41421 -2 2-1.41421 2-.58579 3+0 2+1.41421 3+2 4 K=(39 56 35 9,) B2XH6
2/103 6/17 12/65

P22 DEG=4 AUT=8 P=(1,112,122,22,2) GIR=3 CN=4,4
A=1 1 5 15,6 42 142 20 410,1400 3400 200 10100 34000,72000
E=5-2 2-1.23607 5+0 2 2+3.23607 4 K=(36 34,3 1) W2(D3) 6/1250 9/2041 12/544
13/221

P23 DEG=4 F=I AUT=256 P=(1,4,14,4,2) GIR=4 CN=2,8 T=1
A=1 1 1 1,36 30 6 6 30,600 1100 1100 600 36000,36000
E=-4 2-2.82843 10+0 2+2.82843 4 K=(39 61 50 24 7 1,) H4[B1] 1/202 2/410
6/3011 9/230 10/240 12/134 13/412 14/44

P24 DEG=4 AUT=2 P=(1,112,11222,111) GIR=4 CN=3,8
A=1 1 1 1,6 30 24 14 402,202 2060 1050 700 34000,43100
E=2-3.23607 3-2 5+0 2+1.23607 3+2 4 K=(39 55 30 6,) 6/1054 9/2042

P25 DEG=4 F=I AUT=4 P=(1,22,124,22) GIR=4 CN=2,8
A=1 1 1 1,30 4 2 24 12,14 22 2640 5140 6300,1700
E=-4 2-2 4-1.41421 2+0 4+1.41421 2+2 4 K=(39 56 40 21 7 1,) D(P10) B2*H10
2/105 6/4401 8/441 11/103 12/4202 13/206 14/22

P26 DEG=4 AUT=1 P=(1/4/7/4) GIR=4 CN=2,8
A=1 1 1 1,30 6 14 2 22,10 24 6440 7100 3600,740
E=-4 -2 2-1.84776 2-1.41421 2-.76537 2+.76537 2+1.41421 2+1.84776 2 4
K=(39 56 40 21 7 1,) 12/35

P27 DEG=4 F=TIA AUT=24 P=(1,4,6,4,1) GIR=4 CN=2,8 T=2 4-CUBE
A=1 1 1 1,14 22 24 12 30,6 1500 1240 2440 2300,74000 E=-4 4-2 6+0 4+2 4
K=(39 57 40 21 7 1,) B2XH7 D3XD3 B2*H9 3/210 4/2050 5/20620 6/1103 9/214
10/110 11/240 13/1014

P28 DEG=4 F=I AUT=6 P=(1,13,36,2) GIR=4 CN=4,8
A=1 1 1 1,6 22 12 110 60,620 1104 50 10204 16000,61400
E=4-2.73205 -2 3+0 4+.73205 3+2 4 K=(39 54 25,) D(P12) 8/72 11/105 12/542
13/61

P29 DEG=5 F=XI AUT=2048 P=(1,14,2,+) GIR=3 CN=4,4
A=1 3 7 3,23 74 174 0 400,400 1400 2400 17400 17000,57000 2[H11] D2[D2] H3[B2]
1/105 2/442 3/407 4/2300 5/52402 6/3444 7/407 8/2402 9/2401 10/407 11/407
12/346 13/1060 14/31

P30 DEG=5 F=XI AUT=32 P=(1,122,2,+) GIR=3 CN=4,4
A=1 3 7 11,25 72 166 0 400,400 1400 3000 16400 17400,37000 2[H12] 1/121 2/122
6/5044 8/432 11/224 12/2340 13/134 14/13

P31 DEG=5 F=I AUT=256 P=(1,14,4,4,2) GIR=3 CN=4,4
A=1 3 7 3,23 14 60 114 260,1200 3200 500 10500 36000,76000
E=-3 2-1.82843 8-1 2+1 2+3.82843 5 K=(24 8,6 2) H4[B2] 1/203 2/414 6/3122
9/1046 10/242 12/174 13/225 14/45

P32 DEG=5 AUT=1 P=(1/5/8/2) GIR=3 CN=4,4
A=1 1 5 15,1 50 2 344 206,1010 2022 560 1242 26100,56200
E=-3 2-2.84776 2-1.76537 -1 2-.41421 2-.23463 2+.84776 2+2.41421 3 5
K=(27 20,3 1) 12/73

P33 DEG=5 AUT=1 P=(1/5/8/2) GIR=3 CN=4,4
A=1 1 5 15,1 50 2 344 206,1010 2022 560 1242 36000,46300
E=2-3 2-2.41421 3-1 2-.41421 2+.41421 1 2+2.41421 3 5 K=(27 20,3 1) 6/730

P34 DEG=5 F=A AUT=1 P=(1/5/9/1) GIR=3 CN=4,6
A=1 1 1 15,5 40 42 202 610,544 120 4012 14406 1330,17100
E=-3 2-2.41421 2-2.23607 2-1 2-.41421 2+.41421 2+2.23607 2+2.41421 5
K=(27 19,3) 12/4124

P35 DEG=5 AUT=1 P=(1/5/8/2) GIR=3 CN=4,6
A=1 1 5 5,11 60 22 202 524,10 2406 1150 6042 22300,52600
E=2-3 2-2.23607 5-1 3+1 2+2.23607 3 5 K=(27 20,3) 6/12110

P36 DEG=5 F=I AUT=2 P=(1,122,2222,2) GIR=3 CN=4,6
A=1 1 5 5,11 50 24 220 540,1022 442 4012 12006 23300,14700
E=2-3.17958 2-1.64885 2-1.43355 2-1 2-.41421 1 2+2.26197 2+2.41421 5
K=(27 20 5,3) 1/111 12/1064

P37 DEG=5 AUT=8 P=(1,122,1222,12) GIR=4 CN=2,8
A=1 1 1 1,1 74 70 64 42,22 16 16 17100 13600,7600
E=-5 2-2.23607 5-1 5+1 2+2.23607 5 K=(30 40 35 21 7 1,) 6/2431 9/2064 12/334
13/416

P38 DEG=5 AUT=1 P=(1/5/A) GIR=3 CN=4,6
A=1 3 1 11,21 14 12 6 104,1220 2242 440 13420 10710,15140
E=-3 2-2.23607 4-1.73205 2-1 4+1.73205 2+2.23607 5 K=(27 19,3) 13/241

P39 DEG=5 F=I AUT=6 P=(1,23,16,3) GIR=4 CN=2,8
A=1 1 1 1,1 6 64 62 54,32 52 34 7300 14700,13500
E=-5 4-1.73205 3-1 3+1 4+1.73205 5 K=(30 39 35 21 7 1,) 8/1130 11/441 12/711
13/1007

P40 DEG=5 F=I AUT=2 P=(1,122,1222,12) GIR=4 CN=2,8
A=1 1 1 1,1 14 32 46 66,72 44 30 14700 16300,15500
E=-5 2-2.41421 3-1 2-.41421 2+.41421 3+1 2+2.41421 5 K=(30 41 35 21 7 1,)
B2*H12 2/214 6/11240 12/436

P41 DEG=5 F=I AUT=2 P=(1,122,2222,2) GIR=3 CN=4,4
A=1 1 1 3,23 12 6 10 404,1130 644 4224 2150 21600,51500
E=2-2.84776 2-2.41421 2-1.76537 2-.23463 2+.41421 2+.84776 1 2+3 5
K=(27 20,3 1) 1/23 12/2150

P42 DEG=5 F=I AUT=2 P=(1,122,2222,2) GIR=4 CN=3,8
A=1 1 1 1,1 72 66 32 46,1010 2404 4030 12044 25200,52500
E=2-4.26197 2-1 2-.56645 2-.41421 2-.35115 1 2+1.17958 2+2.41421 5
K=(30 40 25 6,) 1/45 12/4043

P43 DEG=5 AUT=1 P=(1/5/7/3) GIR=3 CN=4,6
A=1 1 1 15,11 10 6 202 406,50 1066 2160 12500 21300,42700
E=2-3.14626 -3 4-1 2-.31784 2+.31784 2+1 2+3.14626 5 K=(27 24 5,3) 12/1310

P44 DEG=5 F=I AUT=12 P=(1,23,16,3) GIR=3 CN=4,4
A=1 1 1 11,31 6 44 42 214,412 1224 2422 3100 34100,60700 E=3-3 6-1 4+1 2+3 5
K=(27 21,3 1) B2XH9 D3XD4 3/242 4/1045 5/2341 6/2413 9/1062 10/52

P45 DEG=5 F=A AUT=2 P=(1,1112,111222,1) GIR=3 CN=4,4
A=1 1 1 11,31 14 12 6 242,1222 2104 5104 10440 24420,60700
E=2-3 2-2.23607 5-1 3+1 2+2.23607 3 5 K=(27 19,3 1) 6/643 9/1122

P46 DEG=5 F=A AUT=4 P=(1,122,1224,1) GIR=3 CN=4,6
A=1 1 5 3,3 14 2 202 120,140 2250 2444 5230 11424,3700
E=-3 4-2.23607 4-1 2+1 4+2.23607 5 K=(27 20,3) 9/620 11/111

P47 DEG=5 F=I AUT=4 P=(1,122,224,2) GIR=3 CN=4,4
A=1 1 1 3,23 6 12 110 604,250 124 2230 4144 31400,47400
E=-3 4-2.41421 2-1 4+.41421 2+1 2+3 5 K=(27 20,3 1) 2/602 6/6014 8/1112
11/207 12/4070 13/72 14/23

P48 DEG=5 F=I AUT=144 P=(1,14,34,3) GIR=4 CN=2,8
A=1 1 1 1,1 74 74 74 12,22 42 6 17200 17400,17100 E=-5 -3 6-1 6+1 3 5
K=(30 42 35 21 7 1,) B2XH8 B2*H11 2/111 4/1142 5/74400 6/2642 7/304 8/2114
9/2404 11/141 12/631 13/601

P49 DEG=5 F=I AUT=2 P=(1,122,1222,12) GIR=3 CN=4,6
A=1 1 1 11,25 14 12 6 442,1222 130 4144 700 31400,66200
E=2-3 2-2.41421 3-1 2-.41421 2+.41421 1 2+2.41421 3 5 K=(27 21,3) B2XH10
2/424 6/5240 12/2112

P50 DEG=5 F=IA AUT=16 P=(1,14,144,1) GIR=4 CN=3,8
A=1 1 1 1,1 74 30 244 30,1044 2422 1242 2412 1206,74100
E=2-3.82843 -3 4-1 6+1 2+1.82843 5 K=(30 34 15 3,) 2/52 6/1154 9/1144 10/304
12/4700

P51 DEG=5 F=I AUT=2 P=(1,122,22222) GIR=4 CN=4,8
A=1 1 1 1,1 70 64 210 104,1212 506 4050 12024 21442,11422
E=2-3.61313 -3 2-2.08239 2+.08239 6+1 2+1.61313 5 K=(30 32 10,) D(P19) 1/13
12/57

P52 DEG=5 F=N AUT=2 P=(1,1112,1111222) GIR=3 CN=4,4
A=1 1 1 5,25 6 2 310 214,1010 3010 4640 12620 4122,42142
E=4-2.41421 2-2.23607 -1 4+.41421 1 2+2.23607 3 5 K=(27 18,3 1)

P53 DEG=5 F=A AUT=8 P=(1,122,12222,1) GIR=4 CN=3,8
A=1 1 1 1,1 74 74 74 402,202 12 6 17040 17020,17100
E=2-4.23607 5-1 2+.23607 5+1 3 5 K=(30 40 25 6,) D(P11) 6/645 9/321 12/2405
13/501

P54 DEG=5 F=A AUT=2 P=(1,122,12222,1) GIR=3 CN=4,6
A=1 1 1 5,31 14 12 6 402,1202 1440 6220 2124 21150,14700
E=2-3 4-1.73205 3-1 1 4+1.73205 3 5 K=(27 19,3) 8/464 11/221 12/2103 13/141

P55 DEG=5 F=TVI AUT=120 P=(1,5,A) GIR=4 CN=4,8 T=2 -CLEBSCH GRAPH
A=1 1 1 1,1 14 142 44 522,224 160 6412 3050 630,64006 E=5-3 10+1 5
K=(30 30 5,) D(P27) 3/604 4/1051 5/2170 6/11220 9/1114 10/54 11/54 13/1023

P56 DEG=6 F=XTI AUT=18432 P=(1,6,1,+) GIR=3 CN=4,4 T=1
A=1 1 7 7,37 37 176 0 400,400 3400 3400 17400 17400,77000 2[H13] -D(P109)
D2[D3] H5[B1] 1/124 2/460 3/143 4/3024 5/71003 6/14006 7/460 8/2143 9/2403
10/143 11/160 12/7000 13/700 14/214

P57 DEG=6 F=I AUT=2 P=(1,1122,12222) GIR=3 CN=4,4
A=1 1 5 15,1 1 6 322 652,1104 444 4162 2152 26320,16250
E=-4 2-3.41421 2-1.41421 2-.58579 3+0 2+1.41421 3+2 6 K=(21 14,3 1) 2/243
6/6214 12/4071

P58 DEG=6 F=I AUT=6 P=(1,123,36) GIR=3 CN=4,4
A=1 3 7 1,1 1 42 22 102,1550 1464 4730 3324 744,41270
E=-4 4-2.73205 3+0 4+.73205 3+2 6 K=(21 12,3 1) D(P28) 8/1047 11/445 12/751
13/75

P59 DEG=6 AUT=4 P=(1,1122,1224) GIR=3 CN=4,6
A=1 1 1 11,5 5 36 26 416,342 342 5120 4510 33060,32450
E=4-3.23607 -2 4+0 4+1.23607 2+2 6 K=(21 14,3) D(P46) 9/1504 11/305

P60 DEG=6 F=A AUT=1 P=(1/6/8/1) GIR=3 CN=4,4
A=1 3 5 15,21 41 150 6 354,1500 2102 4422 10432 16240,66600
E=2-3.23607 3-2 2-1.23607 4+0 2+1.23607 2+3.23607 6 K=(18 9,6 1) 9/3022

P61 DEG=6 AUT=8 P=(1,1122,1222,2) GIR=3 CN=4,4
A=1 1 5 15,5 45 6 160 550,302 2242 232 10232 27400,57400
E=-4 4-2 2-1.23607 5+0 2 2+3.23607 6 K=(18 8,6 2) 6/1163 9/2035 12/372 13/227

P62 DEG=6 F=A AUT=1 P=(1/6/8/1) GIR=3 CN=4,4
A=1 1 5 1,35 61 134 2 422,734 410 5006 15042 5502,74600
E=2-3.41421 -2 2-1.84776 2-.76537 2-.58579 0 2+.76537 2+1.84776 4 6
K=(18 9,6 2) 12/2107

P63 DEG=6 F=A AUT=1 P=(1/6/8/1) GIR=3 CN=4,4
A=1 1 5 1,35 61 134 2 422,734 410 5102 15042 5406,74600
E=2-3.41421 2-2 2-1.41421 2-.58579 3+0 2+1.41421 2 4 6 K=(18 9,6 2) 6/365

P64 DEG=6 F=A AUT=1 P=(1/6/8/1) GIR=3 CN=4,4
A=1 1 1 5,35 31 154 2 422,734 410 5006 4502 35042,35600
E=2-3.23607 4-2 5+0 2+1.23607 2 4 6 K=(18 9,6 1) 6/5064

P65 DEG=6 F=I AUT=2 P=(1,222,12222) GIR=3 CN=4,6
A=1 1 5 3,13 25 6 130 470,304 242 3222 4614 26500,57040
E=2-2.61313 5-2 2-1.08239 2+1.08239 2+2 2+2.61313 6 K=(18 6,6) 1/62 12/2215

P66 DEG=6 F=I AUT=2 P=(1,222,1222,2) GIR=3 CN=4,6
A=1 1 1 11,23 55 6 134 72,242 2304 3022 4414 34600,73200
E=2-3.41421 2-2.17958 2-.64885 2-.58579 2-.43355 2+0 2 2+3.26197 6 K=(18 10,6)
-D(P77) 1/224 12/6402

P67 DEG=6 F=A AUT=1 P=(1/6/8/1) GIR=3 CN=4,6
A=1 1 5 1,1 51 74 62 12,1624 2050 5106 1526 4162,55600
E=2-4.14626 -2 2-1.31784 2-.68216 4+0 2+2 2+2.14626 6 K=(21 18 5,3) 12/1245

P68 DEG=6 F=I AUT=2 P=(1,222,12222) GIR=3 CN=4,6
A=1 1 5 13,1 1 146 14 422,1250 2720 5110 12460 11146,20546
E=2-4 2-1.84776 2-1.41421 2-.76537 2+.76537 2+1.41421 2+1.84776 2 6 K=(21 15,3)
1/144 12/5220

P69 DEG=6 F=I AUT=2 P=(1,222,1222,2) GIR=4 CN=2,8
A=1 1 1 1,1 1 36 174 172,162 154 116 66 36600,37200
E=-6 2-1.84776 2-1.41421 2-.76537 2+0 2+.76537 2+1.41421 2+1.84776 6
K=(24 35 35 21 7 1,) 1/250 12/536

P70 DEG=6 F=I AUT=256 P=(1,24,144) GIR=4 CN=3,8
A=1 1 1 1,1 1 176 120 450,450 3120 3126 4456 3126,4456
E=2-4.82843 -2 8+0 2+.82843 2+2 6 K=(24 28 15 3,) D(P20) D(P23) D(P50) D(P53)
H6[B1] 1/70 2/506 6/3324 9/516 10/305 12/4740 13/312 14/34

P71 DEG=6 AUT=2 P=(1,1122,12222) GIR=3 CN=4,4
A=1 1 5 15,21 51 6 164 554,1002 2402 4212 12222 26300,56240
E=4-2.73205 2-2 3+0 4+.73205 2 4 6 K=(18 8,6 1) 8/2072 11/125 12/2644 13/1124

P72 DEG=6 F=I AUT=2 P=(1,1122,1222,2) GIR=3 CN=4,4
A=1 1 5 15,21 51 6 164 554,222 2212 4102 12042 35200,72600
E=2-3.41421 2-2 2-1.41421 2-.58579 3+0 2+1.41421 2 4 6 K=(18 10,6 1) B2XH12
2/63 6/4644 12/2047

P73 DEG=6 F=IA AUT=2 P=(1,222,2222,1) GIR=3 CN=4,6
A=1 3 3 5,1 1 132 74 144,142 2110 5060 12314 5462,17600
E=2-4.02734 3-2 4+0 2+.33182 2+1.19891 2+2.49661 6 K=(21 17 5,3) 1/214
12/1522

P74 DEG=6 F=I AUT=4 P=(1,24,1224) GIR=3 CN=4,6
A=1 1 5 5,13 23 170 340 230,1002 404 6502 7024 26442,17014
E=2-2.82843 3-2 4-1.41421 4+1.41421 2+2.82843 6 K=(18 7,6) 1/150 12/1603
13/245 14/320

P75 DEG=6 F=I AUT=16 P=(1,114,144) GIR=3 CN=4,4
A=1 1 5 5,25 15 6 60 510,460 3110 5242 2702 15222,22612
E=2-2.82843 5-2 4+0 2+2 2+2.82843 6 K=(18 2,6 2) 2/415 6/1352 9/2161 10/123
12/1234

P76 DEG=6 F=A AUT=4 P=(1,114,11114,1) GIR=3 CN=4,4
A=1 1 5 5,25 15 170 2 770,4 2442 6412 2502 22422,76200
E=-4 3-2 4-1.41421 2+0 4+1.41421 4 6 K=(18 9,6 2) 6/2362 8/255

P77 DEG=6 F=I AUT=2 P=(1,222,222,12) GIR=3 CN=4,4
A=1 1 3 5,33 75 24 12 452,324 1402 2204 14600 36200,75400
E=2-2.49661 3-2 2-1.19891 2-.33182 4+0 2+4.02734 6 K=(15 1,9 4) 1/244 12/1017

P78 DEG=6 AUT=1 P=(1/6/9) GIR=3 CN=4,6
A=1 1 1 5,1 61 30 306 116,650 2024 4152 5252 3422,54244
E=-4 2-3.23607 2-2 5+0 2+1.23607 3+2 6 K=(21 14,3) 6/6610

P79 DEG=6 F=A AUT=1 P=(1/6/8/1) GIR=3 CN=4,4
A=1 1 1 11,37 5 134 32 442,314 3002 4104 11422 34500,57200
E=2-3.41421 -2 2-1.41421 2-1.23607 2-.58579 2+0 2+1.41421 2+3.23607 6
K=(18 8,6 1) 12/1642

P80 DEG=6 AUT=1 P=(1/6/9) GIR=3 CN=4,6
A=1 1 1 11,1 61 24 256 16,14 2304 6062 5072 3702,13340
E=-4 2-3.26197 -2 2-1.41421 0 2+.43355 2+.64885 2+1.41421 2+2.17958 6
K=(21 14,3) D(P34) 12/1303

P81 DEG=6 F=TVI AUT=72 P=(1,6,9) GIR=3 CN=4,4 T=1
A=1 3 7 1,21 61 104 22 430,1624 442 5050 16244 4702,51310 E=9-2 6+2 6
K=(18 6,6 2) L(H8) D4XD4 -D4*D4 3/423 4/217 5/1274 6/3066 9/433 10/423 11/243
13/1403

P82 DEG=6 F=I AUT=768 P=(1,6,16,2) GIR=4 CN=2,8 T=1
A=1 1 1 1,1 1 176 170 170,146 146 36 36 37400,37400 E=-6 3-2 8+0 3+2 6
K=(24 36 35 21 7 1,) H7[B1] B2*H13 D3*D4 2/640 3/540 4/2131 5/74110 6/5501
7/310 8/1414 9/3200 10/540 11/213 12/707 13/417 14/142

P83 DEG=6 F=A AUT=1 P=(1/6/8/1) GIR=3 CN=4,4
A=1 1 1 5,35 51 130 6 402,1310 2640 1026 11036 5502,56600
E=-4 2-2.17958 -2 2-1.41421 2-.64885 2-.43355 0 2+1.41421 2+3.26197 6
K=(18 8,6 2) 12/1107

P84 DEG=6 F=RI AUT=12 P=(1,6,36) GIR=3 CN=4,6 T=1 SHRIKHANDE GRAPH
A=1 3 5 3,11 61 30 104 42,614 3222 1250 11406 7540,30720 E=9-2 6+2 6
K=(18 4,6) 3/310 6/4550 11/640 13/1411

P85 DEG=6 F=I AUT=2 P=(1,222,12222) GIR=3 CN=4,6
A=1 1 1 11,21 11 140 56 526,264 2312 4044 12102 30624,31212
E=2-3.41421 2-3.26197 2-.58579 2+0 2+.43355 2+.64885 2 2+2.17958 6 K=(21 14,3)
1/122 12/3022

P86 DEG=6 AUT=8 P=(1,1122,12222) GIR=3 CN=4,4
A=1 1 5 15,5 45 6 170 570,1202 2602 222 10212 36100,76040
E=2-3.23607 4-2 5+0 2+1.23607 2 4 6 K=(18 8,6 2) D(P22) 6/2607 9/507 12/2072
13/324

P87 DEG=6 F=I AUT=16 P=(1,24,124,2) GIR=4 CN=2,8
A=1 1 1 1,1 1 170 174 172,126 56 146 36 36600,37200
E=-6 -2 4-1.41421 4+0 4+1.41421 2 6 K=(24 35 35 21 7 1,) 2/305 6/1171 8/2224
11/310 12/635 13/611 14/304

P88 DEG=6 AUT=1 P=(1/6/7/2) GIR=3 CN=4,4
A=1 1 7 1,15 31 160 6 412,1022 3120 1446 364 26600,56600
E=-4 4-2 2-1.23607 5+0 2 2+3.23607 6 K=(18 9,6 1) 6/5062

P89 DEG=6 F=I AUT=16 P=(1,114,124,2) GIR=3 CN=4,4
A=1 1 5 5,15 25 6 170 570,242 2212 302 10222 37000,76400 E=-4 5-2 6+0 2+2 4 6
K=(18 10,6 2) B2XH11 2/446 4/256 5/4545 6/317 7/446 8/1123 9/2442 11/146
12/746 13/1061

P90 DEG=6 F=I AUT=4 P=(1,222,1224) GIR=3 CN=4,6
A=1 1 1 11,11 21 146 146 146,320 2250 5024 4422 32414,33012
E=2-4 -2 4-1.41421 2+0 4+1.41421 2+2 6 K=(21 15,3) D(P54) 2/125 6/10151 8/545
11/223 12/2407 13/540 14/52

P91 DEG=6 F=IA AUT=4 P=(1,24,1124,1) GIR=3 CN=4,6
A=1 1 3 5,23 15 170 6 320,250 2442 1504 12412 5424,77000
E=2-2.82843 5-2 4+0 2+2 2+2.82843 6 K=(18 8,6) 2/610 6/11150 9/1430 10/310
12/3300

P92 DEG=6 AUT=1 P=(1/6/9) GIR=3 CN=4,4
A=1 1 5 1,31 17 164 16 502,1040 2002 6264 5412 7420,66220
E=4-2.73205 -2 2-1.23607 2+0 4+.73205 2+3.23607 6 K=(18 8,6 1) 13/261

P93 DEG=6 F=N AUT=2 P=(1,11112,111222) GIR=3 CN=4,6
A=1 3 5 11,1 1 30 206 762,510 450 6122 6062 14344,22344
E=-4 2-3.23607 4-1.41421 0 2+1.23607 4+1.41421 2 6 K=(21 13,3)

P94 DEG=6 F=I AUT=12 P=(1,123,36) GIR=3 CN=4,4
A=1 3 7 1,1 1 162 162 162,450 230 6224 6444 31110,47104 E=2-4 3-2 6+0 4+2 6
K=(21 15,3 1) D(P45) 3/605 4/2151 5/54064 6/2417 9/3101 10/605

P95 DEG=6 AUT=2 P=(1,11112,111222) GIR=3 CN=4,4
A=1 3 7 1,1 1 30 224 762,550 550 4304 2244 34122,32062
E=-4 2-3.23607 2-2 5+0 2+1.23607 3+2 6 K=(21 13,3 1) 6/751 9/364

P96 DEG=7 F=XTI AUT=203212800 P=(1,7,+) GIR=3 CN=8,2 T=2
A=1 3 7 17,37 77 177 0 400,1400 3400 7400 17400 37400,77400 2[H14] SW(H14)
SW(H5) D2[D4] H5[B2] 1/125 2/132 3/147 4/2446 5/62311 6/14206 7/612 8/2602
9/2503 10/147 11/164 12/7040 13/720 14/215

P97 DEG=7 F=I AUT=4 P=(1,124,224) GIR=3 CN=4,4
A=1 3 7 11,5 51 25 60 700,1540 3620 5152 14546 2632,43226 E=4-3 5-1 4+1 2+3 7
K=(12 2,9 1) 3/311 6/4463 11/324 13/163

P98 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 1 5 11,5 75 45 330 426,1442 3102 7430 4252 26222,70246
E=2-3.14626 2-3 3-1 2-.31784 2+.31784 2+1 2+3.14626 7 K=(12 1,9 4) 12/1722

P99 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 3 7 11,5 61 45 324 430,1426 3140 5252 7102 24720,70252
E=3-3 2-2.23607 4-1 3+1 2+2.23607 3 7 K=(12 3,9 2) 6/13121

P100 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 1 7 1,15 61 123 124 606,1434 1246 2032 11512 16260,74540
E=2-3 2-2.23607 4-1.73205 -1 4+1.73205 2+2.23607 7 K=(12 2,9 2) 13/1413

P101 DEG=7 F=I AUT=2 P=(1,1222,2222) GIR=3 CN=4,4
A=1 1 5 7,33 11 105 350 324,1102 2602 4056 12036 17540,27620
E=2-3.17958 2-2.41421 2-1.64885 2-1.43355 2+.41421 2+1 2+2.26197 3 7
K=(12 4,9 1) 1/131 12/3043

P102 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 1 3 5,25 17 105 214 450,1542 3604 4260 13162 14232,51432
E=2-3.17958 -3 2-1.64885 2-1.43355 -1 2-.41421 1 2+2.26197 2+2.41421 7
K=(12 2,9 3) 12/4646

P103 DEG=7 F=I AUT=2 P=(1,1222,11222) GIR=3 CN=4,4
A=1 3 3 7,13 11 105 74 474,1110 3204 2342 14322 16540,66620
E=3-3 2-2.41421 2-1 2-.41421 2+.41421 1 2+2.41421 3 7 K=(12 4,9 2) 2/261
6/14013 12/2323

P104 DEG=7 AUT=2 P=(1,1222,11222) GIR=3 CN=4,4
A=1 3 3 7,13 5 111 74 474,142 2222 7104 17210 12740,64720
E=3-3 4-1.73205 2-1 1 4+1.73205 3 7 K=(12 2,9 2) 8/1245 11/622 12/2722 13/156

P105 DEG=7 F=I AUT=2 P=(1,1222,11222) GIR=3 CN=4,4
A=1 3 3 13,7 1 1 360 714,1502 1602 2354 4334 26270,16164
E=-5 -3 2-2.41421 2-1 2-.41421 2+.41421 3+1 2+2.41421 7 K=(15 8,6 2) D(P33)
D(P83) 2/541 6/3711 12/1706

P106 DEG=7 AUT=2 P=(1,1222,11222) GIR=3 CN=4,6
A=1 3 3 3,3 21 41 360 714,1406 1412 2354 4334 26264,16170
E=-5 -3 4-1.73205 2-1 3+1 4+1.73205 7 K=(15 8,6) 8/2524 11/341 12/5424 13/642

P107 DEG=7 F=I AUT=8 P=(1,124,44) GIR=3 CN=4,4
A=1 3 7 11,5 31 45 252 126,1126 652 6300 15220 32540,71460 E=4-3 5-1 4+1 2+3 7
K=(12 4,9 3) 3/607 4/1076 5/3217 6/1725 9/317 10/427 11/247 13/463

P108 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 3 1 7,31 51 61 300 504,1414 3432 6132 646 23246,31710
E=3-3 2-2.41421 2-1 2-.41421 2+.41421 1 2+2.41421 3 7 K=(12 3,9 3) 6/12701

P109 DEG=7 F=I AUT=768 P=(1,16,6,2) GIR=3 CN=4,4
A=1 3 7 3,23 3 103 360 760,74 314 2074 4314 37400,77400 E=-5 11-1 3+3 7
K=(12 8,9 3) SW(H11) SW(H2) SW(H7) -D(P132) -D(P44) H7[B2] 2/447 3/324 4/2067
5/72510 6/10307 7/512 8/742 9/3201 10/541 11/147 12/5070 13/1260 14/123

P110 DEG=7 F=N AUT=2 P=(1,111112,11222) GIR=3 CN=4,4
A=1 1 7 15,23 5 105 56 466,230 2130 5302 13302 26540,56640
E=-3 4-2.41421 2-2.23607 4+.41421 1 2+2.23607 3 7 K=(12 2,9 2)

P111 DEG=7 F=I AUT=256 P=(1,124,44) GIR=3 CN=6,4
A=1 3 7 3,3 23 43 240 520,1520 3640 3254 4534 13254,24534
E=2-3.82843 9-1 2+1.82843 2+3 7 K=(12,9 3) D(P31) H6[B2] 1/223 2/57 6/2627
9/333 10/307 12/1741 13/525 14/251

P112 DEG=7 F=IA AUT=4 P=(1,124,124,1) GIR=3 CN=4,4
A=1 3 7 11,5 51 25 360 640,1520 2152 5146 2232 21226,77400
E=2-3.82843 9-1 2+1.82843 2+3 7 K=(12 6,9 1) SW(H12) SW(H6) -D(P138) -D(P49)
2/522 6/10564 9/2245 10/131 12/3340

P113 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 3 5 1,7 71 121 234 116,434 1152 3022 14640 33602,36260
E=3-3 2-2.23607 4-1 3+1 2+2.23607 3 7 K=(12 3,9 2) 6/11215

P114 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 1 1 15,15 75 75 374 202,1010 3402 3022 17006 17102,76042
E=2-2.84776 2-2.41421 2-1.76537 -1 2-.23463 2+.41421 2+.84776 1 5 7
K=(9 2,12 8) 12/2266

P115 DEG=7 F=I AUT=2 P=(1,1222,2222) GIR=3 CN=4,6
A=1 1 5 11,5 1 1 250 524,1352 726 5252 12526 1372,766
E=2-5.02734 3-1 2-.66818 2+.19891 4+1 2+1.49661 7 K=(18 16 5,3) D(P18) D(P36)
D(P42) D(P67) D(P73) 1/47 12/4057

P116 DEG=7 AUT=4 P=(1,1114,11114) GIR=3 CN=4,4
A=1 1 1 15,35 55 135 374 10,1402 1006 7102 17022 17042,67202
E=-3 4-2.41421 3-1 4+.41421 2+1 5 7 K=(9 2,12 8) -D(P88) 6/3626 8/3013

P117 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 1 1 1,15 61 45 314 336,1322 2016 4462 12414 24472,34322
E=2-4.26197 -3 -1 2-.56645 2-.41421 2-.35115 1 2+1.17958 2+2.41421 7
K=(15 8,6 1) D(P79) 12/1354

P118 DEG=7 AUT=2 P=(1,111112,11222) GIR=3 CN=4,4
A=1 3 1 17,31 5 105 56 456,1340 3340 4222 12122 34610,72510
E=3-3 2-2.23607 4-1 3+1 2+2.23607 3 7 K=(12 2,9 3) 6/557 9/565

P119 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,6
A=1 1 5 1,1 65 45 232 122,502 3654 3454 372 13232,23066
E=2-4.23607 2-2.41421 -1 2-.41421 2+.23607 2+.41421 2+1 2+2.41421 7 K=(15 8,6)
12/3403

P120 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 1 1 5,25 21 105 344 116,106 3430 6252 772 7252,13432
E=2-4.23607 -3 4-1 2+.23607 5+1 3 7 K=(15 8,6 1) D(P64) 6/5245

P121 DEG=7 F=I AUT=4 P=(1,1222,224) GIR=3 CN=4,4
A=1 1 5 7,33 3 103 36 456,360 2360 4610 15104 32510,73204
E=2-3 4-2.41421 -1 4+.41421 2+1 2+3 7 K=(12 4,9 2) 2/622 6/10706 8/1246
11/624 12/2370 13/334 14/53

P122 DEG=7 F=I AUT=2 P=(1,1222,2222) GIR=3 CN=6,4
A=1 1 5 11,5 55 35 12 406,1230 2544 3262 4562 25642,13522
E=2-3.49661 2-2.19891 2-1.33182 3-1 4+1 2+3.02734 7 K=(12,9 4) 1/311 12/1552

P123 DEG=7 F=I AUT=2 P=(1,1222,2222) GIR=3 CN=4,4
A=1 1 1 13,27 51 25 310 704,1152 626 5410 13404 32132,34246
E=2-3.61313 2-2.08239 5-1 2+.08239 2+1.61313 2+3 7 K=(12 4,9 1) D(P9) 1/33
12/2255

P124 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 1 1 15,25 1 121 344 156,1252 2422 4116 4716 14342,53430
E=-5 -3 2-2.23607 4-1 5+1 2+2.23607 7 K=(15 8,6 1) 6/5252

P125 DEG=7 F=I AUT=2 P=(1,1222,2222) GIR=3 CN=4,4
A=1 3 3 11,25 7 13 262 562,314 2314 5044 12430 34620,73140
E=2-3 2-2.84776 2-1.76537 2-.41421 2-.23463 2+.84776 2+2.41421 3 7 K=(12 4,9 2)
1/305 12/5260

P126 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,6
A=1 3 5 1,21 1 125 340 526,474 434 5172 7202 22352,24552
E=2-4.23607 4-1.73205 -1 2+.23607 2+1 4+1.73205 7 K=(15 8,6) 13/1304

P127 DEG=7 F=I AUT=2 P=(1,1222,2222) GIR=3 CN=4,4
A=1 3 3 1,1 13 7 22 42,770 1364 1670 11564 5654,3534
E=-5 2-2.84776 2-1.76537 2-.41421 2-.23463 2+.84776 2+1 2+2.41421 7
K=(15 8,6 2) D(P32) 1/53 12/676

P128 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 3 1 11,15 21 127 350 36,444 3122 6120 13016 16642,31550
E=2-3 2-2.41421 2-2.23607 -1 2-.41421 2+.41421 2+2.23607 2+2.41421 7
K=(12 2,9 2) 12/1632

P129 DEG=7 F=I AUT=4 P=(1,124,1124) GIR=3 CN=4,6
A=1 1 1 11,5 51 25 360 414,652 526 5312 3306 25072,13066
E=2-3.82843 2-3 3-1 6+1 2+1.82843 7 K=(15 7,6) D(P91) 2/431 6/13013 9/2330
10/644 12/4625

P130 DEG=7 F=TIA AUT=5040 P=(1,7,7,1) GIR=4 CN=2,8 T=2
A=1 1 1 1,1 1 1 374 372,366 356 336 276 176,77400 E=-7 7-1 7+1 7
K=(21 35 35 21 7 1,) SW(H1) SW(H8) -W8(B2) -B2XH14 B2*H14 2/644 4/1231 5/2754
6/6262 7/644 8/2454 9/1254 11/541 12/736 13/607

P131 DEG=7 F=I AUT=2 P=(1,1222,2222) GIR=3 CN=4,4
A=1 3 7 1,1 11 105 262 162,352 2326 5450 13424 21270,50564
E=2-4.26197 2-2.41421 2-.56645 2-.35115 2+.41421 2+1 2+1.17958 3 7 K=(15 8,6 1)
D(P41) 1/321 12/6043

P132 DEG=7 F=IA AUT=48 P=(1,16,16,1) GIR=3 CN=4,4
A=1 1 1 15,15 75 75 374 202,1006 1012 7022 7042 36102,77400 E=3-3 7-1 4+1 5 7
K=(9 3,12 8) SW(H13) SW(H3) SW(H9) -D(P61) -D(P89) B2XH13 2/464 4/1454 5/65405
6/7604 7/461 8/1231 9/2443 11/231 12/2531 13/701

P133 DEG=7 F=I AUT=16 P=(1,124,224) GIR=3 CN=4,4
A=1 1 1 3,3 43 23 6 12,770 3364 1654 11534 1714,41474
E=-5 4-2.41421 -1 4+.41421 4+1 3 7 K=(15 8,6 2) D(P47) 2/217 6/5207 8/2512
11/154 12/5142 13/1074 14/305

P134 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,6
A=1 3 5 5,33 41 111 220 466,1700 2426 2510 17022 34710,10356
E=2-3.14626 2-3 3-1 2-.31784 2+.31784 2+1 2+3.14626 7 K=(12 5,9) 12/3026

P135 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 1 5 1,35 1 105 350 136,1262 242 4132 16506 17424,4732
E=2-4.23607 2-2.23607 3-1 2+.23607 4+1 2+2.23607 7 K=(15 8,6 1) D(P60) 9/712

P136 DEG=7 AUT=8 P=(1,1222,11222) GIR=3 CN=4,4
A=1 1 1 3,3 43 23 360 774,64 70 6314 6314 26416,16416
E=2-4.23607 -3 4-1 2+.23607 5+1 3 7 K=(15 7,6 2) D(P24) D(P62) D(P63) 6/2566
9/2136 12/2613 13/515

P137 DEG=7 AUT=8 P=(1,1222,11222) GIR=3 CN=4,6
A=1 3 3 1,1 43 23 314 774,406 412 6360 6360 26074,16074
E=-5 -3 2-2.23607 4-1 5+1 2+2.23607 7 K=(15 9,6) 6/14450 9/1604 12/5203
13/1450

P138 DEG=7 F=IA AUT=4 P=(1,124,124,1) GIR=3 CN=4,4
A=1 1 1 13,7 73 67 14 132,246 1510 2604 15430 16444,77400
E=2-3 2-1.82843 7-1 2+1 2+3.82843 7 K=(9 1,12 6) SW(H10) SW(H4) -D(P112)
-D(P21) -D(P43) -D(P72) 2/614 6/6512 9/3015 10/132 12/3112

P139 DEG=7 F=I AUT=4 P=(1,124,224) GIR=3 CN=6,4
A=1 1 1 13,27 33 147 132 246,1010 2404 6510 7204 36430,37044
E=4-2.41421 2-1.82843 3-1 4+.41421 2+3.82843 7 K=(9,12 6) D(P16) -D(P17)
-D(P66) 1/151 12/4174 13/1432 14/47

P140 DEG=7 F=I AUT=16 P=(1,124,1124) GIR=3 CN=4,4
A=1 1 1 3,3 43 23 360 774,406 412 6254 16134 6254,46134 E=-5 2-3 5-1 6+1 3 7
K=(15 9,6 2) D(P76) 3/614 4/616 5/3750 6/3534 7/621 8/634 9/1075 10/622

P141 DEG=7 AUT=1 P=(1/7/8) GIR=3 CN=4,4
A=1 1 1 5,31 21 121 344 356,1152 2422 4216 12424 24616,34152
E=-5 -3 2-2.41421 2-1 2-.41421 2+.41421 3+1 2+2.41421 7 K=(15 9,6 1) D(P35)
6/5072

P142 DEG=7 AUT=2 P=(1,1222,11222) GIR=3 CN=4,4
A=1 1 1 11,25 47 33 60 414,1112 3206 5524 3650 12532,24646
E=2-3 4-2.23607 3-1 2+1 4+2.23607 7 K=(12 3,9 2) 9/3007

P143 DEG=7 AUT=2 P=(1,1222,11222) GIR=3 CN=4,6
A=1 1 1 11,25 27 53 60 414,1212 3106 3544 5630 12532,24646
E=2-3 4-2.23607 3-1 2+1 4+2.23607 7 K=(12 1,9) 11/641

TRANSITIVE GRAPHS ON 17 VERTICES

Q1 DEG=0 F=XTVIAP P=(1,+) CN=1,17

Q2 DEG=2 F=TVIP AUT=2 P=(1,2,2,2,2,2,2,2,2) GIR=17 CN=3,9 POLYGON
A=1 1 4 2,20 10 100 40 400,200 2000 1000 10000 4000,40000 120000
E=2-1.9659 2-1.7004 2-1.2053 2-.5473 2+.1845 2+.8915 2+1.4780 2+1.8649 2
K=(78 220 330 252 84 8,) 1/100

Q3 DEG=4 F=VI AUT=4 P=(1,4,44,4) GIR=4 CN=3,9 T=1
A=1 1 1 1,2 4 120 50 214,70 122 406 4040 22200,30100 141400
E=4-2.90570 4-.48793 4+.34415 4+2.04948 4 K=(48 80 55 12,) 1/11

Q4 DEG=4 F=VI AUT=2 P=(1,22,22,22,22) GIR=3 CN=4,6
A=1 1 3 15,24 12 44 102 500,240 1400 2200 12000 5000,70000 164000
E=2-2.2478 2-1.7814 2-1.7526 2-.8090 2-.3138 2-.1010 2+1.6626 2+3.3430 4
K=(45 56 15,3) D(Q2) 1/210

Q5 DEG=4 F=VI AUT=2 P=(1,22,2222,22) GIR=4 CN=3,9
A=1 1 1 1,2 4 24 12 122,1054 50 120 14400 14200,40440 120300
E=2-3.1712 2-2.5133 2-1.0207 2-.2224 2+.1645 2+1.0760 2+1.3176 2+2.3695 4
K=(48 80 55 12,) 1/14

Q6 DEG=4 F=VI AUT=2 P=(1,22,222,222) GIR=4 CN=3,9
A=1 1 1 1,34 32 4 2 14,22 400 4200 12500 5240,12400 105200
E=2-3.66638 2-1.51590 2-1.07447 2-.36279 2+.27275 2+.65968 2+.93069 2+2.75642 4
K=(48 84 75 36 7,) 1/300

Q7 DEG=6 F=VI AUT=2 P=(1,222,22222) GIR=3 CN=5,6
A=1 3 3 5,1 1 70 130 212,1424 1344 2542 10304 4442,61120 162050
E=2-3.4818 2-2.7749 2-2.0632 2-.2746 2+.7905 2+1.1152 2+1.5512 2+2.1377 6
K=(27 20,3) 1/130

Q8 DEG=6 F=VI AUT=2 P=(1,222,222,22) GIR=3 CN=5,5
A=1 1 3 5,33 75 24 12 452,324 1402 2204 14200 34400,65400 172200
E=2-2.32874 2-2.27974 2-1.80144 2-.76974 2-.62442 2+.11235 2+.45729 2+4.23444 6
K=(21 4,9 4) -D(Q6) 1/61

Q9 DEG=6 F=VI AUT=2 P=(1,222,22222) GIR=3 CN=5,6
A=1 1 5 3,25 13 142 344 510,260 3020 7010 10014 24022,72202 65404
E=2-2.7212 2-2.1884 2-1.6932 2-1.6218 2-1.0408 2+1.5022 2+1.8222 2+2.9410 6
K=(24 12,6) 1/141

Q10 DEG=6 F=VI AUT=2 P=(1,222,2222,2) GIR=3 CN=4,6
A=1 1 1 11,23 55 134 72 4,1002 2214 1422 12042 5104,73400 67200
E=2-3.4530 2-2.0142 2-1.3563 2-.8611 2-.3034 2+.0835 2+1.3770 2+3.5275 6
K=(24 16 5,6) 1/64

Q11 DEG=6 F=VI AUT=2 P=(1,222,2222,2) GIR=3 CN=4,6
A=1 3 3 5,1 1 132 74 462,314 2102 5044 4150 10160,65200 172400
E=2-4.2137 2-1.5681 2-1.3063 2-.8899 2-.0379 2+1.0560 2+1.1642 2+2.7956 6
K=(27 28 10,3) -D(Q8) 1/112

Q12 DEG=6 F=VI AUT=2 P=(1,222,22222) GIR=4 CN=3,9
A=1 1 1 1,1 1 172 174 64,1112 2204 5402 12024 25012,50164 24152
E=2-4.87165 2-1.42769 2-1.03525 2+.34905 2+.40355 2+.52869 2+.84421 2+2.20910 6
K=(30 40 25 6,) 1/51

TRANSITIVE GRAPHS ON 17 VERTICES (CONTD)

Q13 DEG=6 F=VI AUT=2 P=(1,222,22222) GIR=3 CN=4,6
A=1 1 1 11,21 11 56 326 40,100 3064 3112 12044 25102,23224 43412
E=2-3.71854 2-2.98668 2-.64833 2-.38282 2-.12926 2+.66906 2+1.64253 2+2.55403 6
K=(27 24 5,3) 1/106

Q14 DEG=8 F=VIS AUT=4 P=(1,44,44) GIR=3 CN=5,5
A=1 3 5 13,31 15 23 7 724,652 2552 1364 3242 32160,74510 165604
E=4-3.39363 4-.85622 4-.14378 4+2.39363 8 K=(12 4,12 4) D(Q3) -D(Q3) 1/226

Q15 DEG=8 F=VIS AUT=2 P=(1,2222,2222) GIR=3 CN=5,5
A=1 3 3 5,11 61 27 17 660,1710 2532 1274 16114 15062,61544 162342
E=2-3.5340 2-2.3488 2-2.0952 2-1.8536 2+.8536 2+1.0952 2+1.3488 2+2.5340 8
K=(12 4,12 4) 1/56

Q16 DEG=8 F=VI AUT=2 P=(1,2222,2222) GIR=3 CN=6,5
A=1 1 5 13,35 33 21 211 122,1054 2706 1646 11650 26720,33506 47246
E=2-3.9802 2-1.9750 2-1.6169 2-1.1717 2-.8507 2+1.0038 2+2.2685 2+2.3222 8
K=(12,12 4) -Q21 1/232

Q17 DEG=8 F=VI AUT=2 P=(1,2222,2222) GIR=3 CN=6,5
A=1 1 3 15,37 37 105 43 472,334 2402 5204 16700 35640,55412 136224
E=2-3.0067 2-2.7357 2-1.4373 2-1.2431 2-.8017 2+.2969 2+1.2405 2+3.6871 8
K=(9,15 8) -Q19 -D(Q5) 1/152

Q18 DEG=8 F=TVIS AUT=8 P=(1,8,8) GIR=3 CN=6,6 T=1
A=1 3 3 5,21 55 43 231 624,1170 2432 4066 11514 33252,7306 134740
E=8-2.56155 8+1.56155 8 K=(12,12) 1/213

Q19 DEG=8 F=VI AUT=2 P=(1,2222,2222) GIR=3 CN=5,6
A=1 3 3 5,23 15 5 3 214,1422 2674 5732 1562 22354,23750 43760
E=2-4.6871 2-2.2405 2-1.2969 2-.1983 2+.2431 2+.4373 2+1.7357 2+2.0067 8
K=(15 8,9) -Q17 D(Q5) 1/143

Q20 DEG=8 F=VIS AUT=2 P=(1,2222,2222) GIR=3 CN=5,5
A=1 3 5 13,21 11 27 17 714,662 2364 1552 13104 27042,70560 164350
E=2-4.02917 2-2.59037 2-.90996 2-.58521 2-.41479 2-.09004 2+1.59037 2+3.02917 8
K=(12 4,12 4) 1/305

Q21 DEG=8 F=VI AUT=2 P=(1,2222,2222) GIR=3 CN=5,6
A=1 1 3 15,15 63 105 43 572,374 324 4452 6406 31206,75610 76620
E=2-3.32223 2-3.26849 2-2.00383 2-.14928 2+.17175 2+.61690 2+.97501 2+2.98017 8
K=(12 4,12) -Q16 1/145

Q22 DEG=8 F=VI AUT=2 P=(1,2222,2222) GIR=3 CN=4,6
A=1 3 5 13,5 3 1 1 774,772 764 752 10724 24652,52524 125252
E=2-5.41898 2-1.12173 2-.67658 2-.53621 2+.50866 2+.58809 2+.82969 2+1.82706 8
K=(18 16 5,6) -Q23 D(Q10) D(Q11) -D(Q4) 1/216

Q23 DEG=8 F=VI AUT=2 P=(1,2222,2222) GIR=3 CN=6,4
A=1 1 3 5,13 65 173 375 524,252 2052 1124 16012 35024,77004 177002
E=2-2.8271 2-1.8297 2-1.5881 2-1.5087 2-.4638 2-.3234 2+.1217 2+4.4190 8
K=(6,18 16 5) -Q22 D(Q4) -D(Q10) -D(Q11) 1/126

TRANSITIVE GRAPHS ON 18 VERTICES

R1 DEG=0 F=XTVIAP P=(1,+) CN=1,18

R2 DEG=1 F=XTIP AUT=10321920 P=(1,1,+) CN=2,9 T=1
A=1 0 4 0,20 0 100 0 400,0 2000 0 10000 0,40000 0 200000

R3 DEG=2 F=XTIP AUT=1866240 P=(1,2,+) GIR=3 CN=3,6
A=1 3 0 10,0 30 40 0 240,400 0 2400 4000 0,24000 40000 240000 2[I2] 3[F3]
6[C2] 1/100 2/100 3/200 4/4000 5/1000

R4 DEG=2 F=XTIP AUT=576 P=(1,2,2,1,+) GIR=6 CN=2,9
A=1 1 4 2,30 0 100 0 400,0 1200 500 2000 0,40000 120000 42000 3[F4] B2*I2
C2*F2 1/10 2/20 3/6 4/14 5/12

R5 DEG=2 F=XTIP AUT=36 P=(1,2,2,2,2,+) GIR=9 CN=3,10
A=1 1 4 2,20 10 100 240 0,1000 0 0 0 12000,5000 30000 24000 2[I3] 1/20 4/10000

R6 DEG=2 F=TIAP AUT=2 P=(1,2,2,2,2,2,2,2,2,1) GIR=18 CN=2,9 POLYGON
A=1 1 4 2,20 10 100 40 400,200 2000 1000 10000 4000,40000 20000 300000
E=-2 2-1.87939 2-1.53209 2-1 2-.34730 2+.34730 2+1 2+1.53209 2+1.87939 2
K=(91 286 495 462 210 36 1,) B2*I3 1/200 4/401

R7 DEG=3 F=XTI AUT=124416 P=(1,3,2,+) GIR=4 CN=2,9 T=3
A=1 1 1 16,16 0 100 0 100,100 3200 3200 400 400,400 160000 160000 3[F5] F2[C1]
1/11 2/41 3/7 4/141 5/13

R8 DEG=3 F=XIP AUT=576 P=(1,12,2,+) GIR=3 CN=3,6
A=1 1 5 12,26 0 100 0 200,100 3200 3100 400 400,20000 140400 160000 3[F6]
W3(F2) B2XI2 C2XF2 1/101 2/11 3/201 4/4001 5/10040

R9 DEG=3 F=I AUT=2 P=(1,12,22,22,22,2) GIR=4 CN=2,9
A=1 1 1 12,6 10 4 240 120,200 100 5000 2400 2000,4000 150000 160000
E=-3 2-2.53209 2-1.34730 2-.87939 4+0 2+.87939 2+1.34730 2+2.53209 3
K=(73 180 225 146 49 8 1,) D(R6) 1/41 4/421

R10 DEG=3 F=IP AUT=2 P=(1,12,22,22,22,2) GIR=4 CN=3,9 PRISM
A=1 1 1 12,6 10 4 240 120,200 100 5000 2400 4000,42000 120000 250000
E=2-2.87939 2-2 2-.87939 2-.65270 2+0 2+.53209 1 2+1.34730 2+2.53209 3
K=(73 180 225 146 49 8,) B2XI3 1/5 4/10001

R11 DEG=3 F=TIA AUT=12 P=(1,3,6,6,2) GIR=6 CN=2,9 T=3 PAPPUS GRAPH
A=1 1 1 10,4 2 4 10 2,500 440 220 1040 1020,300 32000 144000
E=-3 6-1.73205 4+0 6+1.73205 3 K=(73 178 210 116 35 8 1,) 3/22 5/620

R12 DEG=3 AUT=4 P=(1,12,22,24,4) GIR=3 CN=3,6
A=1 1 5 2,22 10 4 40 20,100 200 2100 4200 11000,20400 45000 102400
E=4-2 4-1.30278 4+0 1 4+2.30278 3 K=(72 166 165 52,1) W1(F5) 3/401

R13 DEG=3 AUT=1 P=(1/3/6/6/2) GIR=6 CN=2,9
A=1 1 1 10,10 2 4 2 4,1100 200 420 120 640,1040 124000 16000
E=-3 2-1.9696 2-1.7321 2-1.2856 2-.6840 2+.6840 2+1.2856 2+1.7321 2+1.9696 3
K=(73 178 210 117 35 8 1,) 4/61

R14 DEG=4 F=XTIP AUT=36864 P=(1,4,1,+) GIR=3 CN=3,6 T=1
A=1 1 7 7,36 0 100 0 100,1300 1300 7200 400 400,60400 60400 360000 3[F7] L(L7)
I2[B1] 1/110 2/12 3/220 4/4014 5/10030

R15 DEG=4 F=XTI AUT=576 P=(1,4,4,+) GIR=3 CN=3,6 T=1
A=1 3 1 11,24 12 154 162 0,1000 0 7000 2000 21000,74000 26000 245000 2[I4]
L(L8) C2XF3 C2*F3 2/300 3/410 5/6000

R16 DEG=4 F=XI AUT=36 P=(1,22,22,+) GIR=3 CN=3,6
A=1 3 1 1,34 32 124 252 0,1000 0 2000 1000 26000,35000 36000 55000 2[I5] 1/120
4/5000

R17 DEG=4 F=XI AUT=36 P=(1,22,22,+) GIR=3 CN=3,6
A=1 1 3 15,12 24 144 342 0,1000 0 2000 14000 36000,25000 13000 305000 2[I6]
1/420 4/12000

R18 DEG=4 AUT=72 P=(1,13,233,23) GIR=4 CN=2,9
A=1 1 1 1,34 34 20 4 10,2 2 2 16100 16040,11600 3600 5600
E=-4 2-2.64575 6-1 6+1 2+2.64575 4 K=(58 114 115 69 28 8 1,) 3/106 4/622
5/720

R19 DEG=4 AUT=1 P=(1/4/8/5) GIR=4 CN=2,9
A=1 1 1 1,14 2 12 4 30,20 2 24 6440 5500,2700 12140 15200
E=-4 2-2.20893 2-1.62871 2-1.21157 2-1 2+1 2+1.21157 2+1.62871 2+2.20893 4
K=(58 112 105 62 28 8 1,) 4/621

R20 DEG=4 F=IAP AUT=2 P=(1,22,22,22,22,1) GIR=3 CN=3,6 ANTIPRISM
A=1 1 3 15,24 12 44 102 500,240 1400 2200 12000 5000,30000 144000 360000
E=2-2.22668 2-2 2-1.53209 2-1.18479 2-.34730 3+0 2+1.87939 2+3.41147 4
K=(55 84 35 1,3) 1/6 4/1003

R21 DEG=4 AUT=2 P=(1,22,2222,122) GIR=3 CN=3,6
A=1 3 1 1,4 42 20 10 12,24 1010 2020 740 4240,10500 106400 51200
E=2-3 2-2 4-1.30278 3+0 2+1 4+2.30278 4 K=(57 102 75 21,1) 3/1020

R22 DEG=4 F=I AUT=2 P=(1,22,2222,122) GIR=4 CN=2,9
A=1 1 1 1,24 12 14 22 20,10 2 4 3140 11600,6600 16040 15100
E=-4 2-2 2-1.87939 2-1.53209 2-.34730 2+.34730 2+1.53209 2+1.87939 2+2 4
K=(58 112 105 63 28 8 1,) B2*I6 1/240 4/630

R23 DEG=4 F=I AUT=2 P=(1,22,2222,122) GIR=4 CN=3,9
A=1 1 1 1,2 4 24 12 120,50 122 4054 3000 20440,20300 102400 241200
E=2-3 2-2.87939 2-.87939 2-.65270 0 2+.53209 2+1 2+1.34730 2+2.53209 4
K=(58 112 95 29,) 1/30 4/1041

R24 DEG=4 F=I AUT=2 P=(1,22,222,122,2) GIR=4 CN=2,9
A=1 1 1 1,32 34 4 2 14,22 600 2400 1200 2440,1300 74000 134000
E=-4 2-2.87939 2-1 2-.65270 2-.53209 2+.53209 2+.65270 2+1 2+2.87939 4
K=(58 116 120 71 28 8 1,) B2*I5 1/210 4/650

R25 DEG=4 F=I AUT=2 P=(1,22,2222,122) GIR=4 CN=3,9
A=1 1 1 1,24 12 4 2 14,22 2010 5020 740 10440,4300 102400 241200
E=2-3.41147 2-2 2-1.53209 2-.34730 3+0 2+1.18479 2+1.87939 2+2.22668 4
K=(58 112 100 43 7,) 1/22 4/1030

R26 DEG=4 F=IA AUT=2 P=(1,22,222,222,1) GIR=3 CN=3,6
A=1 1 1 11,24 12 4 2 54,122 240 500 12400 5200,40400 120200 314000
E=2-2.87939 2-2.53209 2-1.34730 2-.65270 0 2+.53209 2+.87939 2+1 2+3 4
K=(57 104 80 21,1) 1/102 4/4600

R27 DEG=4 F=I AUT=4 P=(1,22,24,14,2) GIR=3 CN=3,6
A=1 1 1 11,4 2 22 24 212,414 140 1100 2040 10300,20440 144000 234000
E=2-3 4-2 5+0 4+1 2+3 4 K=(57 104 80 22,1) C2XF4 2/6 3/50 5/1005

R28 DEG=4 F=I AUT=512 P=(1,4,14,4,4) GIR=4 CN=3,9 T=1
A=1 1 1 1,36 30 6 6 30,600 1100 600 1100 24000,52000 124000 252000
E=2-3.75877 2-2 9+0 2+.69459 2+3.06418 4 K=(58 118 130 82 28 4,) I3[B1] 1/402
4/2401

R29 DEG=4 F=I AUT=8 P=(1,4,44,14) GIR=4 CN=2,9 T=1
A=1 1 1 1,10 2 20 4 30,6 14 22 740 14500,14240 3300 3440 E=-4 4-2 4-1 4+1 4+2 4
K=(58 112 105 64 28 8 1,) B2*I4 C2*F4 2/420 3/120 5/740

R30 DEG=5 F=XTI AUT=124416000 P=(1,5,+) GIR=3 CN=6,3 T=2
A=1 3 7 17,37 0 100 0 300,1300 3300 7300 400 20400,60400 160400 360400 3[F8]
F2[C2] I2[B2] 1/111 2/13 3/304 4/4414 5/1520

R31 DEG=5 F=I AUT=512 P=(1,14,4,4,4) GIR=3 CN=5,5
A=1 3 7 3,23 14 60 114 260,1200 500 3200 4500 24000,52000 164000 352000
E=2-2.75877 11-1 2+1.69459 2+4.06418 5 K=(40 32,6 2) I3[B2] 1/61 4/2106

R32 DEG=5 AUT=2 P=(1,122,22222,2) GIR=3 CN=3,6
A=1 1 5 1,21 50 24 330 344,10 4 2042 14022 2402,45002 116200 66100
E=2-3 4-2.30278 2-1 2+0 4+1.30278 2+2 3 5 K=(44 58 25 2,2) 3/411

R33 DEG=5 AUT=72 P=(1,23,233,13) GIR=4 CN=2,9
A=1 1 1 1,1 74 72 60 50,30 6 6 6 34300,33400 27400 17400
E=-5 2-2.64575 6-1 6+1 2+2.64575 5 K=(46 78 80 57 28 8 1,) 3/124 4/263 5/760

R34 DEG=5 AUT=1 P=(1/5/A/2) GIR=3 CN=3,6
A=1 1 5 1,31 10 22 202 604,140 60 2006 16402 1214,44120 13100 304500
E=2-2.7321 2-2.5634 2-1.6223 2-1.1953 -1 2+.2465 2+.7321 2+2.3169 2+2.8177 5
K=(43 50 10 1,3) 4/2051

R35 DEG=5 F=I AUT=2 P=(1,122,2222,22) GIR=3 CN=3,6
A=1 1 1 11,25 10 104 12 6,330 344 1042 10422 1200,40500 155000 162400
E=2-3 2-2.53209 2-1.34730 2-1 2-.53209 2+.65270 2+.87939 2+2.87939 3 5
K=(43 52 15 2,3) B2XI6 1/405 4/3001

R36 DEG=5 F=I AUT=2 P=(1,122,2222,22) GIR=3 CN=3,6
A=1 1 1 1,21 50 124 254 134,42 2022 12 6 24200,52100 135000 72400
E=2-3.87939 2-1.87939 2-1.65270 2-.46791 2+0 2+.34730 2+1.53209 2+2 3 5
K=(45 68 45 12,1) B2XI5 1/501 4/14001

R37 DEG=5 AUT=1 P=(1/5/8/4) GIR=4 CN=2,9
A=1 1 1 1,1 54 12 6 34,70 22 44 62 15300,31600 26500 16600
E=-5 2-2.20893 2-1.62871 2-1.21157 2-1 2+1 2+1.21157 2+1.62871 2+2.20893 5
K=(46 76 75 56 28 8 1,) 4/334

R38 DEG=5 F=N AUT=8 P=(1,14,444) GIR=3 CN=4,6
A=1 1 1 11,5 60 124 150 614,202 2402 1002 10102 14020,52040 64004 322010
E=8-2.30278 8+1.30278 3 5 K=(44 56 20,2)

R39 DEG=5 F=I AUT=8 P=(1,14,44,4) GIR=3 CN=3,6
A=1 1 1 11,5 60 124 150 614,42 22 2006 4012 14200,22400 171000 146100
E=4-3 4-1 4+0 4+2 3 5 K=(44 60 30 4,2) B2XI4 C2XF6 2/301 3/211 5/3001

TRANSITIVE GRAPHS ON 18 VERTICES (CONTD)

R40 DEG=5 F=I AUT=2 P=(1,122,2222,22) GIR=4 CN=2,9
A=1 1 1 1,1 66 72 32 46,4 10 44 30 36400,37000 26600 17100
E=-5 2-2.87939 2-1 2-.65270 2-.53209 2+.53209 2+.65270 2+1 2+2.87939 5
K=(46 80 80 56 28 8 1,) D(R9) 1/203 4/703

R41 DEG=5 F=I AUT=2 P=(1,122,22222,2) GIR=3 CN=3,6
A=1 1 1 1,21 24 50 206 112,510 1204 4042 12022 230,144 165000 152400
E=2-3.53209 2-2.34730 2-1.87939 -1 2-.12061 2+.34730 2+1.53209 4+2 5
K=(45 64 35 6,1) D(R26) 1/301 4/4700

R42 DEG=5 F=I AUT=2 P=(1,122,2222,22) GIR=4 CN=3,9
A=1 1 1 1,1 72 66 46 32,10 2004 4030 2044 24400,53000 124600 53100
E=2-4.41147 5-1 2-.53209 2+.18479 2+.65270 2+1.22668 2+2.87939 5
K=(46 80 75 36 7,) 1/45 4/2130

R43 DEG=5 AUT=1 P=(1/5/7/5) GIR=3 CN=3,6
A=1 1 1 15,11 6 102 50 206,10 1066 2460 2400 22200,61100 120300 252400
E=2-3.1650 2-2.7321 -1 2-.5938 2-.4375 2-.3367 2+.7321 2+1.0313 2+3.5017 5
K=(43 56 25 1,3) 4/1121

R44 DEG=5 F=I AUT=6 P=(1,23,226,2) GIR=4 CN=3,9
A=1 1 1 1,1 70 70 104 202,1024 442 5014 422 21044,2412 124200 252100
E=2-4 4-1.87939 -1 4+.34730 4+1.53209 2+2 5 K=(46 72 50 14,) 1/15 4/2222

R45 DEG=5 AUT=1 P=(1/5/9/3) GIR=3 CN=3,6
A=1 3 5 11,21 40 6 340 202,1020 2460 14 3002 10110,41700 36000 270100
E=2-2.73205 4-1.79129 5-1 2+.73205 4+2.79129 5 K=(42 44 10 1,4) 3/1102

R46 DEG=5 AUT=1 P=(1/5/7/5) GIR=3 CN=3,6
A=1 1 1 15,15 2 10 74 42,220 1102 1106 200 26200,35100 22600 262100
E=2-2.9696 2-2.2856 2-1.6840 -1 2-.3160 2+.2679 2+.2856 2+.9696 2+3.7321 5
K=(42 48 20 3,4) 4/4620

R47 DEG=5 F=I AUT=12 P=(1,23,66) GIR=3 CN=3,6
A=1 3 1 1,1 40 120 310 20,1040 3010 2422 2214 21112,4242 14124 101444
E=6-2.73205 -1 6+.73205 4+2 5 K=(45 62 30 6,1) D(R11) 3/112 5/1026

R48 DEG=5 F=I AUT=8 P=(1,14,44,4) GIR=4 CN=2,9
A=1 1 1 1,1 62 16 16 62,44 50 24 30 35400,33200 26500 16300
E=-5 4-2 4-1 4+1 4+2 5 K=(46 76 75 56 28 8 1,) 2/23 3/47 5/325

R49 DEG=5 F=I AUT=2 P=(1,122,2222,22) GIR=4 CN=2,9
A=1 1 1 1,1 70 64 16 16,42 22 30 44 35200,32500 16600 27100
E=-5 2-2 2-1.87939 2-1.53209 2-.34730 2+.34730 2+1.53209 2+1.87939 2+2 5
K=(46 76 75 56 28 8 1,) 1/211 4/615

R50 DEG=5 F=I AUT=2 P=(1,122,2222,22) GIR=3 CN=4,6
A=1 1 1 11,25 10 4 244 130,6 12 2042 14022 25200,12500 103300 44700
E=2-3.22668 2-2.18479 5-1 2-.53209 2+.65270 2+2.41147 2+2.87939 5
K=(43 52 20,3) D(R10) D(R20) 1/7 4/1403

R51 DEG=5 F=A AUT=1 P=(1/5/B/1) GIR=4 CN=3,9
A=1 1 1 1,1 24 74 110 4,1602 1442 424 11002 20050,41022 120150 56100
E=2-3.8490 2-2.7321 -1 2-.9383 2+.0902 2+.7321 2+.8480 2+1.6329 2+2.2161 5
K=(46 72 50 13,) D(R13) 4/2602

R52 DEG=5 AUT=2 P=(1,122,1222,1112) GIR=3 CN=3,6
A=1 1 5 5,31 74 2 202 412,1206 40 4020 14000 23600,14100 130400 324200
E=4-2.73205 5-1 2+.26795 4+.73205 2+3.73205 5 K=(42 48 20 2,4) 3/244 5/1007

R53 DEG=5 F=I AUT=24 P=(1,23,26,4) GIR=3 CN=3,6
A=1 3 1 1,1 70 70 44 14,1012 22 4024 442 26100,51500 26200 211600
E=2-4 9-1 6+2 5 K=(45 68 50 14,1) C2XF5 2/501 3/207 5/2441

R54 DEG=6 F=XTI AUT=186624 P=(1,6,2,+) GIR=3 CN=3,6 T=1
A=1 1 1 17,17 17 176 176 0,1000 1000 1000 17000 17000,17000 176000 176000
2[I7] -D(R139) F3[C1] 1/424 2/214 3/1600 4/13000 5/15000

R55 DEG=6 F=XI AUT=36 P=(1,222,2,+) GIR=3 CN=5,6
A=1 3 5 13,27 17 174 372 0,1000 1000 3000 6000 36000,37000 77000 175000 2[I8]
1/504 4/15000

R56 DEG=6 AUT=72 P=(1,123,233,3) GIR=3 CN=6,3
A=1 1 5 15,35 75 12 206 100,1020 3040 602 10602 30602,27000 147000 317000
E=7-2 2-.64575 6+0 2+4.64575 6 K=(27,10 10 5 1) W3(C2) 3/246 4/4514 5/10071

R57 DEG=6 F=A AUT=1 P=(1/6/A/1) GIR=3 CN=4,6
A=1 1 1 1,31 21 44 116 402,1214 1624 2220 1016 1320,30152 50142 276000
E=2-4.08832 3-2 2-1.28142 2+0 2+.32052 2+.32955 2+1.97601 2+2.74366 6
K=(34 38 15,3) 4/1621

R58 DEG=6 AUT=1 P=(1/6/9/2) GIR=3 CN=5,5
A=1 1 1 5,11 75 100 2 402,650 640 5406 2270 1026,41036 54600 227200
E=2-3.50810 3-2 2-.86428 2-.67684 2-.25067 2+0 2+1.55887 2+3.74102 6
K=(31 24,6 2) 4/10407

R59 DEG=6 F=I AUT=2 P=(1,222,122222) GIR=3 CN=3,6
A=1 1 3 15,1 1 146 12 24,50 120 3504 15442 5310,42660 6504 207042
E=2-3.41147 2-3 2-1.87939 2-1 2+.34730 2+1.18479 2+1.53209 2 2+2.22668 6
K=(34 32 5 1,3) 1/64 4/3204

R60 DEG=6 F=I AUT=2 P=(1,222,122222) GIR=3 CN=3,6
A=1 1 3 15,1 1 146 50 120,12 24 7042 6504 5310,42660 13504 25442
E=2-3.41147 2-3 -2 2-1.53209 2-.34730 2+1 2+1.18479 2+1.87939 2+2.22668 6
K=(34 32 5 1,3) 1/442 4/10630

R61 DEG=6 F=A AUT=4 P=(1,24,2224,1) GIR=3 CN=5,6
A=1 3 5 15,3 43 30 140 660,710 6 4006 5440 12500,25220 52210 374000
E=9-2 4+0 4+3 6 K=(30 16,7 2) L(I4) 3/426

R62 DEG=6 AUT=2 P=(1,222,122222) GIR=3 CN=3,6
A=1 3 1 11,21 51 170 12 24,44 102 6440 7100 23414,15422 23204 214602
E=2-3 4-2.30278 4-1 4+1.30278 2 2+3 6 K=(32 26 5 1,5) 3/620

R63 DEG=6 F=I AUT=2 P=(1,222,12222,2) GIR=3 CN=3,6
A=1 3 5 13,1 1 36 50 520,544 1142 4104 12042 11110,20460 162600 155200
E=2-3.87939 2-1.87939 2-1.65270 2-1 2-.46791 2+.34730 2+1.53209 2 2+3 6
K=(33 32 15 3,4) 1/510 4/6404

R64 DEG=6 F=I AUT=2 P=(1,222,12222,2) GIR=3 CN=3,6
A=1 3 5 13,1 1 36 50 120,544 1142 4504 3042 11110,20460 162600 155200
E=2-3.87939 -2 2-1.65270 2-1.53209 2-.46791 2-.34730 2+1 2+1.87939 2+3 6
K=(33 32 15 3,4) 1/150 4/4017

R65 DEG=6 F=IA AUT=4 P=(1,24,2224,1) GIR=3 CN=3,6
A=1 1 1 1,21 11 124 52 146,36 740 630 12104 12042,45024 25012 360600
E=2-4 2-3 2-1 6+0 5+2 6 K=(35 42 20 3,2) C2*F6 2/124 3/1406 5/5005

R66 DEG=6 F=I AUT=2 P=(1,222,222,122) GIR=3 CN=5,5
A=1 1 3 5,33 75 52 124 212,424 2404 1202 14000 31000,66000 171200 166400
E=2-2.53209 3-2 2-1.34730 2-1.22668 2-.18479 2+0 2+.87939 2+4.41147 6
K=(28 10,9 4) 1/230 4/10074

R67 DEG=6 F=I AUT=2 P=(1,222,2222,12) GIR=3 CN=4,6
A=1 1 3 5,23 55 152 164 202,404 2024 1012 11210 6420,17000 172000 365000
E=2-2.87939 2-2.53209 -2 2-1.34730 2-.65270 2+0 2+.53209 2+.87939 2+4 6
K=(30 22 5,7) 1/142 4/4303

R68 DEG=6 F=IA AUT=2 P=(1,222,22222,1) GIR=3 CN=5,6
A=1 1 5 3,25 13 150 360 22,1014 1420 2210 13002 7004,50244 24502 374000
E=2-2.53209 2-2.41147 3-2 2-1.34730 2+0 2+.87939 2+2.18479 2+3.22668 6
K=(31 22,6) 1/610 4/10146

R69 DEG=6 F=I AUT=2 P=(1,222,12222,2) GIR=3 CN=3,6
A=1 1 5 13,5 43 170 14 22,44 102 3404 15402 5320,2650 126400 257000
E=2-3 2-2.22668 2-1.87939 2-1.18479 2-1 2+.34730 2+1.53209 2 2+3.41147 6
K=(31 24 5 1,6) 1/406 4/3210

R70 DEG=6 F=I AUT=2 P=(1,222,12222,2) GIR=3 CN=3,6
A=1 1 3 5,15 23 170 44 102,24 12 3404 15402 5310,2660 126400 257000
E=2-3 2-2.22668 -2 2-1.53209 2-1.18479 2-.34730 2+1 2+1.87939 2+3.41147 6
K=(31 24 5 1,6) 1/602 4/10154

R71 DEG=6 F=I AUT=2 P=(1,222,2222,12) GIR=3 CN=3,6
A=1 1 1 11,23 55 42 104 72,134 604 4602 2414 1222,74000 151200 326400
E=2-3.06418 2-3 2-1 2-.69459 6+0 2 2+3.75877 6 K=(31 26 5 1,6) 1/26 4/3005

R72 DEG=6 F=IA AUT=2 P=(1,222,22222,1) GIR=3 CN=3,6
A=1 1 1 1,21 51 72 134 42,104 1204 6402 13014 7022,2454 1322 374000
E=2-3.75877 2-3 -2 6+0 2+.69459 2+1 2+3.06418 6 K=(34 38 15 1,3) 1/640
4/10123

R73 DEG=6 F=A AUT=1 P=(1/6/A/1) GIR=3 CN=5,5
A=1 1 1 11,35 45 30 2 422,630 2204 5042 504 21412,60704 15102 236200
E=2-3.09096 3-2 2-1.86164 2-.66781 2-.09662 2+0 2+2.55623 2+3.16080 6
K=(31 18,6 2) 4/1254

R74 DEG=6 F=IA AUT=2 P=(1,222,22222,1) GIR=3 CN=4,6
A=1 3 1 11,11 21 24 12 622,614 2144 1142 11050 26120,10304 104442 360600
E=2-3.22668 2-2.53209 2-2.18479 2-1.34730 2+0 2+.87939 3+2 2+2.41147 6
K=(33 30 5,4) 1/106 4/6420

R75 DEG=6 F=IA AUT=2 P=(1,222,22222,1) GIR=3 CN=3,6
A=1 3 1 1,1 1 142 144 134,72 264 4512 2224 21412,50060 124110 303600
E=2-4.41147 2-2.53209 2-1.34730 2+0 2+.18479 2+.87939 2+1.22668 3+2 6
K=(36 46 25 6,1) 1/700 4/14021

R76 DEG=6 F=I AUT=8 P=(1,24,144,2) GIR=3 CN=3,6
A=1 3 1 1,1 1 170 104 422,412 3044 1164 2154 4072,532 143200 234600
E=2-4.75877 6-1 2-.30541 2+0 3+2 2+2.06418 6 K=(36 52 35 9,1) -D(R66) 1/160
4/6110

R77 DEG=6 AUT=2 P=(1,11112,122222) GIR=3 CN=3,6
A=1 3 3 15,1 1 36 60 120,1410 1410 4300 2240 22542,15142 32444 35104
E=2-3.64575 5-2 4+0 2+1 2+1.64575 2+3 6 K=(33 30 10 1,4) 3/33 5/2720

R78 DEG=6 F=I AUT=4 P=(1,24,224,12) GIR=3 CN=3,6
A=1 3 5 13,5 43 36 146 140,30 1500 2220 12210 5440,74000 163000 317000
E=2-3 5-2 6+0 2+1 2+4 6 K=(30 22 5 1,7) 2/640 3/70 5/4660

R79 DEG=6 AUT=1 P=(1/6/B) GIR=3 CN=3,6
A=1 1 1 1,31 21 120 12 426,504 214 2204 10152 4740,23212 17440 54026
E=2-3.6458 2-2.8794 -2 2-.8794 2-.6527 2+.5321 2+1.3473 2+1.6458 2+2.5321 6
K=(34 34 10 1,3) 4/2143

R80 DEG=6 AUT=1 P=(1/6/8/3) GIR=4 CN=2,9
A=1 1 1 1,1 1 164 142 72,134 66 154 16 132,76400 71600 27600
E=-6 2-1.9696 2-1.7321 2-1.2856 2-.6840 2+.6840 2+1.2856 2+1.7321 2+1.9696 6
K=(37 60 70 56 28 8 1,) 4/475

R81 DEG=6 F=I AUT=8 P=(1,24,1244) GIR=4 CN=3,9
A=1 1 1 1,1 1 170 204 202,1104 2442 2422 15014 11154,20562 5134 2472
E=2-4.75877 3-2 2-.30541 2+0 6+1 2+2.06418 6 K=(37 52 35 9,) D(R24) D(R51)
1/214 4/2132

R82 DEG=6 AUT=1 P=(1/6/B) GIR=3 CN=3,6
A=1 1 5 11,21 41 24 106 422,1640 2214 1032 1240 16500,20306 134002 260150
E=2-3.64575 3-2 4-1.30278 2+0 2+1.64575 4+2.30278 6 K=(33 30 10 1,4) 3/423

R83 DEG=6 F=A AUT=2 P=(1,222,22222,1) GIR=3 CN=3,6
A=1 1 1 11,1 41 120 50 246,506 664 712 1124 22052,1422 102214 374000
E=2-4 4-2.30278 4+0 4+1.30278 3+2 6 K=(35 40 15 2,2) 3/1030

R84 DEG=6 F=IA AUT=2 P=(1,222,22222,1) GIR=3 CN=3,6
A=1 1 1 11,21 11 56 326 40,1100 3064 3112 2044 21102,21224 42412 374000
E=2-4 2-2.53209 2-1.34730 2-.53209 2+0 2+.65270 2+.87939 2 2+2.87939 6
K=(34 38 15 2,3) 1/34 4/3014

R85 DEG=6 F=I AUT=2 P=(1,222,2222,12) GIR=4 CN=2,9
A=1 1 1 1,1 1 134 72 172,174 106 46 62 114,74600 76200 75400
E=-6 2-2.53209 2-1.34730 2-.87939 4+0 2+.87939 2+1.34730 2+2.53209 6
K=(37 62 70 56 28 8 1,) B2*I8 1/212 4/770

R86 DEG=6 F=I AUT=16 P=(1,24,18,2) GIR=3 CN=3,6
A=1 3 1 1,31 31 170 102 14,1412 422 4024 11044 26042,11504 46600 331200
E=4-3 6-1 2+0 3+2 2+3 6 K=(32 28 10 2,5) C2XF7 2/144 3/250 5/6014

R87 DEG=6 F=I AUT=16 P=(1,24,128) GIR=3 CN=3,6
A=1 1 1 1,31 31 170 204 202,502 1024 6412 2422 21014,45104 130442 47044
E=4-3 3-2 2+0 6+1 2+3 6 K=(33 28 10 2,4) D(R27) 2/540 3/226 5/1350

R88 DEG=6 F=I AUT=12 P=(1,6,26,3) GIR=4 CN=2,9 T=1
A=1 1 1 1,1 1 52 124 154,132 66 74 162 116,33600 56600 65600
E=-6 6-1.73205 4+0 6+1.73205 6 K=(37 60 70 56 28 8 1,) 3/123 5/672

R89 DEG=6 AUT=1 P=(1/6/B) GIR=3 CN=4,6
A=1 1 5 1,11 45 154 102 120,404 2060 5006 2622 15430,15220 102412 223042
E=2-3.20893 2-2.62871 2-2.21157 -2 2+.21157 2+.62871 2+1 2+1.20893 2+3 6
K=(33 28 5,4) 4/4407

TRANSITIVE GRAPHS ON 18 VERTICES (CONTD)

R90 DEG=6 F=I AUT=31104 P=(1,6,26,3) GIR=4 CN=2,9 T=1
A=1 1 1 1,1 1 176 176 160,160 160 16 16 16,77000 77000 77000
E=-6 2-3 12+0 2+3 6 K=(37 66 75 57 28 8 1,) F4[C1] B2*I7 C2*F5 1/242 2/62
3/125 4/636 5/752

R91 DEG=6 AUT=4 P=(1,222,1244) GIR=3 CN=3,6
A=1 3 1 11,21 11 36 204 202,1044 502 3104 4442 30520,31050 106450 47120
E=4-3 -2 4-1.30278 4+1 4+2.30278 6 K=(33 30 10 2,4) 3/460

R92 DEG=7 F=A AUT=1 P=(1/7/9/1) GIR=3 CN=3,6
A=1 1 1 15,1 15 1 374 56,214 2322 252 4204 33062,22342 31162 137400
E=2-4.8490 2-1.9383 2-.9098 2-.7321 2-.1520 2+.6329 1 2+1.2161 2+2.7321 7
K=(27 30 15 3,4) 4/14450

R93 DEG=7 AUT=72 P=(1,223,133,3) GIR=3 CN=6,3
A=1 3 5 13,7 47 147 36 300,1240 3140 430 10430 30430,67000 157000 337000
E=-3 6-2 2-.64575 6+0 2+4.64575 7 K=(18,13 10 5 1) 3/261 4/4551 5/10466

R94 DEG=7 AUT=1 P=(1/7/A) GIR=3 CN=5,5
A=1 1 5 15,5 45 1 270 206,1062 1430 4342 6502 10350,66406 3312 271220
E=2-4.08832 -3 2-2 2-1.28142 2+0 2+.32052 2+.32955 2+1.97601 2+2.74366 7
K=(25 16,6 2) D(R73) 4/2247

R95 DEG=7 AUT=2 P=(1,11122,112222) GIR=3 CN=3,6
A=1 1 5 5,5 31 51 6 412,254 134 5700 3700 26442,16422 61262 111162
E=2-3.64575 -3 4-2 4+0 2+1 2+1.64575 2+3 7 K=(24 14 5 1,7) 3/71 5/2271

R96 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=4,6
A=1 1 1 5,11 31 145 350 324,1032 446 4072 2066 15410,23404 164202 352102
E=2-3.87939 -3 2-1.65270 2-1.53209 2-.46791 2-.34730 2+1 2+1.87939 2+3 7
K=(24 16 5,7) 1/303 4/4172

R97 DEG=7 AUT=1 P=(1/7/A) GIR=3 CN=5,6
A=1 1 5 5,11 31 141 350 406,12 1134 3064 2462 17222,13102 122604 324242
E=2-3.20893 -3 2-2.62871 2-2.21157 2+.21157 2+.62871 2+1 2+1.20893 2+3 7
K=(24 12,7) 4/4217

R98 DEG=7 AUT=1 P=(1/7/A) GIR=3 CN=3,6
A=1 1 5 11,1 45 61 150 222,1444 1424 4072 2254 2126,31312 65502 56602
E=2-3.7321 2-2.8177 2-2.3169 2-.2679 2-.2465 1 2+1.1953 2+1.6223 2+2.5634 7
K=(25 18 5 1,6) 4/3444

R99 DEG=7 AUT=1 P=(1/7/A) GIR=3 CN=5,6
A=1 3 5 1,31 13 25 220 560,62 604 4310 16450 7046,57120 32602 201516
E=4-2.79129 4-2 2-.73205 1 4+1.79129 2+2.73205 7 K=(23 10,8) 3/622

R100 DEG=7 AUT=2 P=(1,11122,112222) GIR=3 CN=3,6
A=1 1 5 1,21 41 121 6 772,650 2530 4610 12510 24026,52046 120126 250246
E=4-3.73205 2-.73205 4-.26795 5+1 2+2.73205 7 K=(26 24 10 2,5) 3/254 5/14502

R101 DEG=7 F=I AUT=6 P=(1,223,226) GIR=3 CN=6,3
A=1 3 1 1,7 47 147 34 32,1024 2412 3120 4610 33060,24510 153220 134450
E=4-2.87939 2-2 4-.65270 4+.53209 1 2+4 7 K=(21,10 10 5 1) 1/115 4/14141

R102 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=5,5
A=1 1 1 5,11 65 171 250 124,1016 416 3042 4422 34102,72202 65450 113424
E=3-3 2-2.22668 2-1.53209 2-1.18479 2-.34730 2+1 2+1.87939 2+3.41147 7
K=(22 4,9 4) 1/451 4/2456

R103 DEG=7 AUT=4 P=(1,124,2224) GIR=3 CN=4,6
A=1 1 5 11,11 65 65 374 374,1002 2402 6010 6004 34102,72202 134042 272022
E=4-3.30278 4-1 4+.30278 4+1 5 7 K=(22 18 5,9) D(R12) 3/611

R104 DEG=7 F=I AUT=6 P=(1,223,226) GIR=3 CN=3,6
A=1 3 5 3,1 1 1 10 420,1762 1754 724 664 564,31312 51152 61252
E=2-5 4-1.53209 4-.34730 3+1 4+1.87939 7 K=(28 32 15 2,3) D(R44) D(R84) 1/35 4/11222

R105 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=5,5
A=1 1 5 7,13 23 43 216 516,1024 450 6240 16120 24120,52240 164610 153104
E=2-2.87939 2-2.41147 4-2 2-.65270 2+.53209 1 2+2.18479 2+3.22668 7
K=(22 8,9 2) 1/423 4/3442

R106 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=5,5
A=1 1 5 7,13 23 43 120 240,216 2116 5024 2450 31520,31640 63610 115504
E=-3 2-2.53209 2-2.41147 2-2 2-1.34730 2+0 2+.87939 2+2.18479 2+3.22668 7
K=(22 8,9 2) 1/443 4/10315

R107 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=6,3
A=1 3 7 17,37 1 1 102 202,1520 1640 2710 5304 25220,12540 165210 152504
E=-3 2-2.87939 2-2.53209 2-1.34730 2-.65270 2+0 2+.53209 2+.87939 2+4 7
K=(21,10 10 5 1) 1/311 4/4435

R108 DEG=7 AUT=1 P=(1/7/A) GIR=3 CN=3,6
A=1 1 1 5,21 15 51 54 22,1424 3050 1744 5216 32302,1562 44132 306602
E=2-3.6458 -3 2-2.8794 2-.8794 2-.6527 2+.5321 2+1.3473 2+1.6458 2+2.5321 7
K=(25 18 5 1,6) D(R34) 4/2155

R109 DEG=7 F=A AUT=1 P=(1/7/9/1) GIR=3 CN=5,5
A=1 1 1 5,11 55 135 264 202,1012 3022 3520 11424 152,44056 166042 77400
E=2-3.50810 -3 2-2 2-.86428 2-.67684 2-.25067 2+0 2+1.55887 2+3.74102 7
K=(22 10,9 4) 4/10247

R110 DEG=7 AUT=4 P=(1,1222,244) GIR=3 CN=5,5
A=1 3 7 1,1 63 163 16 16,620 1140 4540 3220 35050,16444 67030 132424
E=-3 8-2 4+0 4+3 7 K=(21 2,10 4) 3/1122

R111 DEG=7 F=I AUT=2 P=(1,1222,2222,2) GIR=3 CN=5,5
A=1 1 5 11,25 55 35 370 764,12 2006 4042 12022 26202,16102 175000 372400
E=2-3.53209 2-2.34730 2-1.53209 2-1 2-.34730 2-.12061 2+1 2+1.87939 5 7
K=(21 12,10 4) B2XI8 1/125 4/7004

R112 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=5,6
A=1 1 1 5,31 41 121 350 324,1216 516 3042 14422 25050,12424 160212 350106
E=2-4.06418 4-2 2-1.69459 7+1 2+2.75877 7 K=(25 16,6) 1/445 4/12250

R113 DEG=7 F=I AUT=144 P=(1,16,26,2) GIR=4 CN=2,9
A=1 1 1 1,1 1 1 374 374,172 346 316 326 272,76 177000 176400
E=-7 2-2 6-1 6+1 2+2 7 K=(31 56 70 56 28 8 1,) 1/243 2/423 3/161 4/676 5/774

R114 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=5,6
A=1 1 1 5,11 31 45 310 704,1050 2424 5016 2416 11162,20662 66222 116142
E=2-3.41147 3-3 2-1.53209 2-.34730 2+1 2+1.18479 2+1.87939 2+2.22668 7
K=(25 16,6) D(R68) 1/611 4/1623

R115 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=4,6
A=1 1 5 1,21 21 41 310 304,1252 2526 1152 10626 5450,3424 160146 150232
E=2-4.22668 2-3.18479 2-1.53209 2-.34730 5+1 2+1.41147 2+1.87939 7
K=(27 24 5,4) D(R74) 1/107 4/6052

R116 DEG=7 F=I AUT=4 P=(1,124,2224) GIR=3 CN=5,6
A=1 1 5 5,11 31 45 254 134,1240 520 6012 16006 25302,22462 55062 112702
E=5-3 2-2 2+0 6+1 2+3 7 K=(24 12,7) 2/47 3/74 5/2266

R117 DEG=7 F=A AUT=2 P=(1,1222,111222,1) GIR=3 CN=3,6
A=1 1 1 13,7 7 13 60 14,1314 3120 3240 11650 5524,4632 10546 374400
E=2-3.73205 4-2 4-.73205 2-.26795 1 4+2.73205 7 K=(23 14 5 1,8) -D(R52)
3/1422 5/14602

R118 DEG=7 F=I AUT=2 P=(1,1222,2222,2) GIR=3 CN=5,5
A=1 1 1 13,27 33 147 132 246,430 1044 6010 16004 25204,12510 175000 372400
E=2-2.87939 4-2 2-1.22668 2-.65270 2-.18479 2+.53209 1 2+4.41147 7
K=(19 4,12 6) 1/461 4/12106

R119 DEG=7 F=I AUT=2 P=(1,1222,2222,2) GIR=3 CN=5,5
A=1 1 1 7,13 73 67 152 226,510 1204 6010 16004 25024,12450 175000 372400
E=-3 2-2.53209 2-2 2-1.34730 2-1.22668 2-.18479 2+0 2+.87939 2+4.41147 7
K=(19 4,12 6) 1/245 4/2346

R120 DEG=7 AUT=1 P=(1/7/A) GIR=3 CN=5,6
A=1 3 1 5,15 21 105 324 122,6 1250 3450 6112 24640,62170 57022 13612
E=2-3.5634 2-2.6223 2-2.1953 2-.7535 2-.7321 1 2+1.3169 2+1.8177 2+2.7321 7
K=(24 12,7) 4/5045

R121 DEG=7 F=I AUT=16 P=(1,124,28) GIR=3 CN=6,3
A=1 1 1 3,23 63 163 16 16,1210 504 7030 4604 13110,64424 73050 126444
E=3-3 4-2 6+0 2+1 2+4 7 K=(21,10 10 5 1) 2/621 3/307 5/2613

R122 DEG=7 AUT=1 P=(1/7/A) GIR=3 CN=3,6
A=1 1 1 11,21 55 11 250 406,106 1102 7620 7620 366,26406 46056 111270
E=2-3.7321 2-3.5017 2-1.0313 2-.2679 2+.3367 2+.4375 2+.5938 1 2+3.1650 7
K=(25 20 5 1,6) 4/12212

R123 DEG=7 F=I AUT=8 P=(1,124,244) GIR=3 CN=5,5
A=1 1 5 3,3 43 23 6 412,1330 744 3270 4564 21030,50444 142504 345210
E=2-3.75877 6-2 2+.69459 5+1 2+3.06418 7 K=(24 16,7 2) 1/161 4/6460

R124 DEG=7 F=I AUT=8 P=(1,124,244) GIR=3 CN=5,5
A=1 1 1 3,3 43 23 16 16,1330 744 3270 4564 21210,50504 142444 345030
E=2-3.75877 3-3 6+0 2+.69459 2+1 2+3.06418 7 K=(25 16,6 2) 1/215 4/10654

R125 DEG=7 AUT=1 P=(1/7/A) GIR=3 CN=3,6
A=1 3 1 11,1 25 41 310 216,74 1624 6452 10322 21150,3542 62604 344062
E=2-3.73205 4-2.79129 2-.26795 5+1 4+1.79129 7 K=(26 20 5 1,5) 3/512

R126 DEG=7 F=I AUT=240 P=(1,25,A) GIR=3 CN=6,3
A=1 3 1 11,31 71 171 204 12,1414 1022 6424 5042 32444,25102 152504 125602
E=10-2 5+1 2+4 7 K=(20,11 10 5 1) C2XF8 -C2*F8 2/113 3/704 5/14441

R127 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=4,6
A=1 1 5 1,1 11 5 350 324,1052 2426 5050 2424 21272,10566 34342 32322
E=2-4.75877 -3 2-2 2-.30541 2+0 6+1 2+2.06418 7 K=(28 28 10,3) D(R57) D(R72)
1/47 4/1546

R128 DEG=7 AUT=4 P=(1,124,2224) GIR=3 CN=5,6
A=1 1 5 11,5 31 45 250 124,16 2016 5620 3540 25302,12702 113062 64462
E=5-3 4-1.30278 4+1 4+2.30278 7 K=(24 12,7 2) D(R61) 3/427

R129 DEG=7 F=I AUT=2 P=(1,1222,22222) GIR=3 CN=3,6
A=1 1 5 1,1 1 1 366 372,352 326 4450 13024 24250,52124 120252 250126
E=2-5.41147 2-1.53209 2-.81521 2-.34730 2+.22668 5+1 2+1.87939 7
K=(30 40 25 6,1) D(R41) D(R75) 1/701 4/6244

R130 DEG=7 F=I AUT=2 P=(1,1222,2222,2) GIR=4 CN=2,9
A=1 1 1 1,1 1 1 372 366,156 236 346 332 274,174 176400 177000
E=-7 2-1.87939 2-1.53209 2-1 2-.34730 2+.34730 2+1 2+1.53209 2+1.87939 7
K=(31 56 70 56 28 8 1,) 1/53 4/537

R131 DEG=7 AUT=1 P=(1/7/A) GIR=3 CN=3,6
A=1 1 5 7,31 41 101 34 602,1062 1510 306 12604 16042,72510 45216 25360
E=2-3.64575 -3 2-2 4-1.30278 2+0 2+1.64575 4+2.30278 7 K=(24 14 5 1,7) 3/1061

R132 DEG=7 F=I AUT=144 P=(1,16,26,2) GIR=3 CN=3,6
A=1 1 1 15,21 55 55 374 374,202 2006 2042 2012 34022,34102 177000 176400
E=2-4 2-2 6-1 6+1 5 7 K=(22 20 10 2,9) -D(R93) B2XI7 1/425 2/215 3/1411
4/13200 5/7100

R133 DEG=7 F=A AUT=1 P=(1/7/9/1) GIR=3 CN=5,5
A=1 3 1 5,21 51 117 130 6,1460 1210 2534 3062 24540,15016 64242 347400
E=2-3.09096 -3 2-2 2-1.86164 2-.66781 2-.09662 2+0 2+2.55623 2+3.16080 7
K=(22 10,9 2) 4/10354

R134 DEG=7 F=A AUT=1 P=(1/7/9/1) GIR=3 CN=4,6
A=1 3 1 15,15 5 101 350 252,6 74 2322 7022 16520,3342 70610 327400
E=2-4.1650 2-1.5938 2-1.4375 2-1.3367 2-.7321 2+.0313 1 2+2.5017 2+2.7321 7
K=(24 18 5,7) -D(R43) 4/5424

R135 DEG=7 F=I AUT=24 P=(1,34,226) GIR=3 CN=3,6
A=1 1 1 1,1 41 21 240 120,1716 1476 1254 1252 20534,10532 141246 30526
E=2-5 4-2 11+1 7 K=(29 32 15 2,2) D(R32) D(R65) D(R83) 2/125 3/1504 5/14520

R136 DEG=7 F=A AUT=1 P=(1/7/9/1) GIR=3 CN=3,6
A=1 3 5 13,11 11 161 350 26,340 3500 1024 14006 22540,77002 10276 77400
E=2-3.7321 2-2.2161 2-1.6329 2-.8480 2-.2679 2-.0902 2+.9383 1 2+3.8490 7
K=(22 16 5 1,9) -D(R46) 4/12302

R137 DEG=8 F=XTI P=(1,8,+) GIR=3 CN=9,2 T=2
A=1 3 7 17,37 77 177 377 0,1000 3000 7000 17000 37000,77000 177000 377000
2[I9] SW(I9) F3[C2] 1/524 2/314 3/1610 4/17000 5/17000

R138 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=5,5
A=1 1 3 5,35 41 43 141 204,624 3512 7112 12724 15224,63252 41532 201536
E=2-4.50810 2-1.86428 2-1.67684 2-1.25067 3+0 2+.55887 2+2 2+2.74102 8
K=(18 8,10 2) 4/14207

R139 DEG=8 F=I AUT=31104 P=(1,26,6,3) GIR=3 CN=6,3
A=1 3 7 17,7 37 47 247 640,1640 3640 130 10130 30130,77000 177000 377000
E=-4 14-1 2+5 8 K=(9,19 20 10 2) SW(I2) -D(R132) -D(R172) -D(R21) -D(R53)
F4[C2] 1/342 2/66 3/170 4/4363 5/4336

R140 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=5,5
A=1 1 5 15,15 51 65 221 724,156 406 7420 14302 15612,22512 175042 126162
E=-4 2-3.1650 2-2.7321 2-.5938 2-.4375 2-.3367 2+.7321 2+1.0313 2+3.5017 8
K=(16 4,12 6) 4/10732

R141 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,5
A=1 3 5 13,5 43 33 35 146,350 2560 626 10616 23300,55440 157120 167050
E=2-3.53209 2-2.34730 2-2 2-1.53209 2-.34730 2-.12061 2+1.87939 2+2 4 8
K=(15 6,13 4) -D(R119) 1/134 4/7220

R142 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=5,5
A=1 1 5 1,5 55 65 361 132,222 3154 2122 15414 2672,33052 52426 335202
E=2-3.74102 2-3 2-1.55887 2-1 0 2+.25067 2+.67684 2+.86428 2+3.50810 8
K=(16 4,12 6) 4/12213

R143 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=3,6
A=1 1 1 11,5 5 153 21 764,1512 1112 2224 14562 16456,30550 106076 114626
E=2-4.64575 4-2.30278 3+0 2+.64575 4+1.30278 2+2 8 K=(20 14 5 1,8) D(R45)
3/433

R144 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=6,3
A=1 1 1 5,25 15 65 275 312,222 2402 3514 15130 6072,46342 35152 167006
E=-4 2-2.9696 2-2.2856 2-1.6840 2-.3160 2+.2679 2+.2856 2+.9696 2+3.7321 8
K=(15,13 10 5 1) 4/4655

R145 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,6
A=1 3 5 3,3 5 103 45 630,1054 1122 4374 12572 6720,46650 103322 245454
E=2-4.41147 2-3 2-1 2-.87939 0 2+.18479 2+1.22668 2+1.34730 2+2.53209 8
K=(19 10,9) 1/614 4/11231

R146 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,6
A=1 3 5 13,11 21 61 111 36,254 2522 714 10662 15644,23702 127144 57142
E=2-3.53209 2-3.41147 2-2.34730 2-.12061 3+0 2+1.18479 2+2 2+2.22668 8
K=(18 6,10) 1/132 4/14231

R147 DEG=8 F=N AUT=2 P=(1,2222,12222) GIR=3 CN=5,6
A=1 3 5 3,21 51 111 261 146,454 322 1624 11612 17244,27502 116134 66072
E=4-3 4-2.30278 0 4+1.30278 4+2 8 K=(17 6,11 2)

R148 DEG=8 AUT=2 P=(1,2222,12222) GIR=3 CN=3,6
A=1 3 1 11,21 51 55 123 146,512 264 674 732 17054,27122 117404 267202
E=2-4 4-2.30278 2-1 3+0 4+1.30278 2+3 8 K=(17 12 5 1,11) 3/1070

R149 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,5
A=1 3 1 1,31 71 115 263 606,334 472 5042 3104 35014,73022 72254 134522
E=2-3.53209 2-2.34730 2-2.22668 2-1.18479 2-.12061 3+0 2+2 2+3.41147 8
K=(15 6,13 4) -D(R118) -D(R71) 1/550 4/6306

R150 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=6,3
A=1 1 5 15,35 3 175 7 612,1462 1300 4504 16142 7422,74124 53412 334150
E=2-3 4-2.30278 2-1.64575 2-1 0 4+1.30278 2+3.64575 8 K=(14,14 10 5 1) 3/1305

R151 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,5
A=1 3 7 7,11 61 23 215 146,1342 1544 632 10634 26300,56440 165050 353120
E=2-3.22668 2-2.87939 2-2.18479 2-1 2-.65270 2+.53209 2+1 2+2.41147 4 8
K=(15 6,13 4) 1/324 4/15003

R152 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=6,3
A=1 3 5 15,1 35 45 135 650,452 2006 5342 11250 35360,26122 106642 327022
E=2-3.8794 2-1.8794 2-1.6527 2-1.6458 2-.4679 0 2+.3473 2+1.5321 2+3.6458 8
K=(15,13 10 5 1) 4/5622

R153 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,5
A=1 3 1 1,5 43 107 47 630,764 2752 5024 13012 21134,51072 130530 270270
E=2-4.41147 2-2.87939 2-1 2-.65270 2+.18479 2+.53209 2+1 2+1.22668 4 8
K=(18 10,10 4) 1/522 4/15011

R154 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,5
A=1 1 5 3,35 33 61 111 36,410 2220 7226 7416 2744,44742 127302 57444
E=-4 2-3.22668 2-2.18479 4-1 2-.53209 2+.65270 2+2.41147 2+2.87939 8
K=(16 6,12 4) 1/72 4/2725

R155 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,6
A=1 1 5 3,35 33 61 311 36,1102 3044 5226 3416 22650,54720 66244 316502
E=2-3.22668 2-2.53209 2-2.18479 2-1.34730 2-1 0 2+.87939 2+2.41147 2+3 8
K=(15 2,13 4) -D(R23) 1/702 4/5243

R156 DEG=8 F=I AUT=4 P=(1,224,1224) GIR=3 CN=3,6
A=1 3 5 13,5 43 43 305 36,746 746 630 170 35220,75410 133120 273050
E=2-4 4-2 4-1 4+1 2+2 4 8 K=(16 10 5 1,12) -D(R78) 2/614 3/1413 5/7600

R157 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,5
A=1 3 7 7,1 1 25 13 170,764 752 1544 11342 25504,13242 164530 152270
E=2-4.41147 -4 4-1 2-.53209 2+.18479 2+.65270 2+1.22668 2+2.87939 8
K=(19 10,9 4) D(R109) 1/216 4/10764

R158 DEG=8 F=IA AUT=2 P=(1,2222,2222,1) GIR=3 CN=5,5
A=1 3 5 13,5 43 47 107 742,1744 530 270 15120 36050,50234 124432 377000
E=2-4.06418 2-1.69459 8-1 2+1 2+2.75877 4 8 K=(15 8,13 4) SW(I8) -D(R180)
1/544 4/7120

R159 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,6
A=1 1 5 13,11 61 13 25 170,1700 1640 4346 2546 23246,15506 126232 56434
E=-4 2-3.53209 2-2.34730 2-1.87939 2-.12061 2+.34730 2+1.53209 4+2 8
K=(18 6,10) 1/152 4/4467

R160 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=3,6
A=1 1 5 13,3 45 101 241 170,524 252 726 656 27224,57412 17264 27512
E=2-4 2-3.41147 2-1.87939 3+0 2+.34730 2+1.18479 2+1.53209 2+2.22668 8
K=(19 14 5 1,9) D(R69) 1/226 4/3162

R161 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=3,6
A=1 3 1 1,13 65 111 261 146,524 252 672 734 27224,17412 117024 267012
E=2-4 2-3 4-1 5+0 2+2 2+3 8 K=(17 10 5 1,11) 2/252 5/11350

R162 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=3,6
A=1 1 5 3,25 53 121 251 36,754 762 322 454 35024,73012 133404 275202
E=2-4 2-2.22668 2-1.87939 2-1.18479 3+0 2+.34730 2+1.53209 2+3.41147 8
K=(16 10 5 1,12) 1/644 4/12606

R163 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,5
A=1 1 5 3,27 57 5 203 740,232 2434 4114 12062 33224,35412 115660 63710
E=2-3.22668 2-3 2-2.18479 2-1 2-.87939 0 2+1.34730 2+2.41147 2+2.53209 8
K=(16 2,12 4) 1/234 4/3314

R164 DEG=8 AUT=2 P=(1,111122,12222) GIR=3 CN=6,3
A=1 3 5 13,11 51 151 351 36,1120 3060 7520 17260 2506,44246 164206 352406
E=4-3 2-1.64575 4-1 3+0 2+2 2+3.64575 8 K=(14,14 10 5 1) 3/315 5/3225

R165 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=3,6
A=1 3 5 13,5 43 43 305 170,434 232 746 746 27220,57410 117120 267050
E=2-4 2-2 2-1.87939 2-1.53209 2-.34730 2+.34730 2+1.53209 2+1.87939 4 8
K=(16 10 5 1,12) -D(R67) 1/464 4/13003

R166 DEG=8 AUT=2 P=(1,2222,12222) GIR=3 CN=5,6
A=1 3 5 13,5 43 13 225 146,636 636 5060 13110 34540,32340 151520 361250
E=4-3.30278 2-2 2-1 4+.30278 2+1 2+2 4 8 K=(16 8,12) 3/1230

R167 DEG=8 F=I AUT=6 P=(1,26,2223) GIR=3 CN=3,6
A=1 1 3 3,25 3 15 105 134,642 640 4130 5770 12770,17146 17416 17226
E=3-4 4-1.87939 4+.34730 4+1.53209 2+2 8 K=(19 14 5 1,9) D(R70) 1/642 4/10555

R168 DEG=8 AUT=2 P=(1,11222,111222) GIR=3 CN=6,3
A=1 1 5 15,21 51 35 235 602,1170 2144 5022 15012 26542,56342 31306 231446
E=-4 4-2.73205 4-1 2+.26795 4+.73205 2+3.73205 8 K=(15,13 10 5 1) 3/265
5/10770

R169 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=5,5
A=1 3 5 11,21 57 161 171 620,1006 2012 7240 17240 34016,44536 67006 213660
E=2-3 2-2.74366 2-1.97601 2-1 2-.32955 2-.32052 0 2+1.28142 2+4.08832 8
K=(13 2,15 8) 4/3126

R170 DEG=8 F=I AUT=2 P=(1,2222,12222) GIR=3 CN=5,6
A=1 3 3 5,23 55 5 3 630,572 374 3204 15402 21264,51512 132650 74720
E=2-4.41147 2-2.53209 2-1.34730 2-1 0 2+.18479 2+.87939 2+1.22668 2+3 8
K=(18 10,10) 1/306 4/6550

R171 DEG=8 F=I AUT=2048 P=(1,8,18) GIR=3 CN=3,6 T=1
A=1 1 7 1,7 21 101 221 776,550 226 6270 6270 6506,70550 70226 306506
E=4-4 9+0 4+2 8 K=(20 20 10 2,8) D(R86) I4[B1] C2*F7 2/624 3/313 5/5452

R172 DEG=8 F=IA AUT=144 P=(1,26,26,1) GIR=3 CN=3,6
A=1 1 1 1,31 41 131 131 774,772 606 4116 4126 4036,70446 70246 377000
E=2-5 8-1 6+1 4 8 K=(19 20 10 2,9) SW(I7) -D(R56) 1/434 2/134 3/1603 4/13014
5/7050

R173 DEG=8 F=TIA AUT=40320 P=(1,8,8,1) GIR=4 CN=2,9 T=2
A=1 1 1 1,1 1 1 1 774,772 766 756 736 676,576 376 377000 E=-8 8-1 8+1 8
K=(28 56 70 56 28 8 1,) SW(I1) -W9(B2) -B2XI9 B2*I9 1/252 2/462 3/163 4/776
5/377

R174 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=3,6
A=1 1 1 15,1 55 45 121 374,256 416 3402 16300 34472,34262 123446 14732
E=2-4.6458 2-2.5321 2-1.3473 2-.5321 0 2+.6458 2+.6527 2+.8794 2+2.8794 8
K=(19 14 5 1,9) D(R136) 4/3262

R175 DEG=8 F=I AUT=512 P=(1,44,144) GIR=3 CN=3,6
A=1 3 5 13,1 1 1 1 776,764 752 752 764 14624,62152 114624 262152
E=2-5.75877 2-1.30541 9+0 2+1.06418 2+2 8 K=(24 28 15 3,4) D(R63) D(R64)
D(R76) D(R92) -D(R31) I5[B1] 1/314 4/6245

R176 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=5,5
A=1 1 7 15,21 61 7 251 460,1216 206 7500 17114 4672,51720 16146 325142
E=2-3.16080 2-3 2-2.55623 2-1 0 2+.09662 2+.66781 2+1.86164 2+3.09096 8
K=(16 6,12 2) 4/3312

R177 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=5,5
A=1 3 5 1,21 7 55 221 550,1044 2152 2232 15330 5530,72426 6762 53446
E=2-4.09096 2-2.86164 2-1.66781 2-1.09662 3+0 2+1.55623 2+2 2+2.16080 8
K=(18 10,10 2) 4/5701

R178 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=5,5
A=1 1 5 15,27 1 105 163 330,642 36 7606 6014 33312,27500 124152 321360
E=-4 2-2.7321 2-2.5634 2-1.6223 2-1.1953 2+.2465 2+.7321 2+2.3169 2+2.8177 8
K=(16 4,12 4) 4/2754

R179 DEG=8 AUT=2 P=(1,2222,12222) GIR=3 CN=5,5
A=1 1 5 13,23 55 41 301 740,522 254 6246 6506 27224,17412 111434 261232
E=6-3 2-1 3+0 6+2 8 K=(17 4,11 2) 3/455

R180 DEG=8 F=IA AUT=2 P=(1,2222,2222,1) GIR=3 CN=5,5
A=1 1 3 15,37 37 105 43 472,334 2300 1440 14604 34602,55412 36224 377000
E=-4 2-2.75877 10-1 2+1.69459 2+4.06418 8 K=(13 4,15 8) SW(I3) -D(R111)
-D(R158) 1/650 4/10665

R181 DEG=8 F=I AUT=24 P=(1,26,234) GIR=3 CN=3,6
A=1 3 5 5,13 23 5 203 770,770 606 56 126 35540,75230 36540 236230
E=3-4 8-1 6+2 8 K=(18 10 5 1,10) 2/464 3/1160 5/1167

R182 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=6,5
A=1 3 7 15,31 15 105 61 660,1016 1304 3642 12012 24560,64702 166422 37340
E=-4 2-2.73205 4-1.79129 4-1 2+.73205 4+2.79129 8 K=(15,13 6) 3/523

R183 DEG=8 AUT=2 P=(1,11222,111222) GIR=3 CN=3,6
A=1 1 5 5,21 11 105 45 170,1602 1144 5622 5612 26532,16272 22566 12356
E=2-4.64575 2-3 2-1 5+0 2+.64575 4+2 8 K=(20 14 5 1,8) D(R117) 3/1423 5/5425

R184 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=4,6
A=1 1 1 5,15 15 1 205 760,732 3216 4462 7440 31036,30356 40762 230332
E=2-5.08832 2-2.28142 2-.67948 2-.67045 3+0 2+.97601 2+1.74366 2+2 8
K=(21 18 5,7) D(R134) 4/6164

R185 DEG=8 F=IA AUT=2 P=(1,2222,2222,1) GIR=3 CN=4,6
A=1 3 5 13,25 13 5 3 572,374 664 712 10704 24642,51520 126250 377000
E=2-4.75877 8-1 2-.30541 0 2+2.06418 2+3 8 K=(18 16 5,10) SW(I5) -D(R190)
-D(R35) 1/146 4/6303

R186 DEG=8 F=IA AUT=8 P=(1,44,44,1) GIR=3 CN=5,5
A=1 3 3 15,25 53 31 207 612,170 1624 2146 11502 34444,23310 146260 377000
E=4-3 8-1 0 4+3 8 K=(14 4,14 4) SW(I4) -D(R186) -D(R39) 2/74 3/1503 5/3252

R187 DEG=8 AUT=1 P=(1/8/9) GIR=3 CN=5,5
A=1 1 1 15,1 51 111 145 470,1126 416 4162 6264 35306,7662 11256 232612
E=-4 2-3.8490 2-2.7321 2-.9383 2+.0902 2+.7321 2+.8480 2+1.6329 2+2.2161 8
K=(19 8,9 2) D(R133) D(R58) 4/10273

R188 DEG=8 F=I AUT=512 P=(1,44,144) GIR=3 CN=3,6
A=1 1 1 1,7 71 107 271 776,530 246 530 246 36030,76006 76006 336030
E=2-4 2-3.06418 2-.69459 9+0 2+3.75877 8 K=(16 12 5 1,12) D(R28) I6[B1] 1/606
4/3360

R189 DEG=8 F=I AUT=12 P=(1,26,36) GIR=3 CN=5,6
A=1 3 5 13,5 43 5 203 36,606 146 5660 13550 36330,3360 105710 306470
E=-4 6-2.73205 6+.73205 4+2 8 K=(18 4,10) 3/136 5/10356

R190 DEG=8 F=IA AUT=2 P=(1,2222,2222,1) GIR=3 CN=6,4
A=1 1 3 5,13 65 173 375 524,252 2052 1124 6012 31024,75004 176002 377000
E=2-3 2-2.06418 8-1 0 2+.30541 2+4.75877 8 K=(10,18 16 5) SW(I6) -D(R185)
-D(R25) -D(R36) -D(R42) -D(R50) 1/36 4/12446

TRANSITIVE GRAPHS ON 19 VERTICES

S1 DEG=0 F=XTVIAP P=(1,+) CN=1,19

S2 DEG=2 F=TVIP AUT=2 P=(1,2,2,2,2,2,2,2,2,2) GIR=19 CN=3,10 POLYGON
A=1 1 4 2,20 10 100 40 400,200 2000 1000 10000 4000,40000 20000 200000 500000
E=2-1.973 2-1.759 2-1.355 2-.803 2-.165 2+.491 2+1.094 2+1.578 2+1.892 2
K=(105 364 715 792 462 120 9,) 1/2

S3 DEG=4 F=VI AUT=2 P=(1,22,22,22,22,2) GIR=3 CN=4,7
A=1 1 3 15,24 12 44 102 500,240 1400 2200 12000 5000,44000 30000 320000 740000
E=2-2.158 2-2.138 2-1.520 2-1.268 2-.665 2-.081 2+.291 2+2.069 2+3.470 4
K=(66 120 70 6,3) D(S2) 1/401

S4 DEG=4 F=VI AUT=2 P=(1,22,2222,222) GIR=4 CN=3,10
A=1 1 1 1,24 12 2 4 420,210 422 4214 2100 21040,42000 21000 300300 300440
E=2-3.114 2-2.776 2-1.482 2-.312 2+.133 2+.537 2+.929 2+1.413 2+2.672 4
K=(69 152 155 66 7,) 1/22

S5 DEG=4 F=VI AUT=2 P=(1,22,2222,222) GIR=4 CN=3,10
A=1 1 1 1,10 60 24 12 2,4 114 62 13000 7000,20440 40300 201400 502200
E=2-3.327 2-2.562 2-.969 2-.394 2-.261 2-.181 2+1.585 2+1.726 2+2.383 4
K=(69 152 160 78 14,) 1/402

S6 DEG=4 F=VI AUT=2 P=(1,22,222,222,2) GIR=4 CN=3,10
A=1 1 1 1,34 32 4 2 14,22 2400 1200 10400 24200,2500 1240 250000 524000
E=2-3.7317 2-1.9241 2-.8788 2-.8636 2+.2237 2+.3258 2+.7749 2+1.0882 2+2.9855 4
K=(69 156 185 126 49 8,) 1/30

S7 DEG=6 F=VI AUT=2 P=(1,222,222,222) GIR=3 CN=5,5
A=1 1 3 5,33 75 124 52 412,224 2204 1402 10000 24000,71400 66200 171000 666000
E=2-2.623 2-1.840 2-1.682 2-1.647 2-.830 2-.266 2+.059 2+1.266 2+4.564 6
K=(36 20,9 4) 1/601

S8 DEG=6 F=VI AUT=2 P=(1,222,22222,2) GIR=3 CN=4,7
A=1 1 5 3,25 13 102 44 550,1360 22 14 11410 6220,14404 114202 270200 564400
E=2-2.727 2-2.233 2-2.153 2-1.749 2-.984 2+.230 2+.923 2+2.217 2+3.477 6
K=(39 36 5,6) 1/242

S9 DEG=6 F=VI AUT=2 P=(1,222,2222,22) GIR=3 CN=4,7
A=1 1 1 11,23 55 134 72 214,422 2 4 6042 11104,55200 36400 256000 535000
E=2-3.492 2-2.323 2-1.468 2-1.064 2-.246 2-.174 2+.310 2+1.497 2+3.961 6
K=(39 40 15,6) D(S3) 1/520

S10 DEG=6 F=VI AUT=2 P=(1,222,22222,2) GIR=3 CN=4,7
A=1 1 1 1,21 51 134 72 4,1002 42 104 12014 25022,12254 5522 57400 37200
E=2-3.9169 2-2.9413 2-1.4357 2-.4258 2+.0965 2+.6237 2+.7815 2+.9132 2+3.3048 6
K=(42 56 30 6,3) 1/124

S11 DEG=6 F=VI AUT=2 P=(1,222,222222) GIR=4 CN=4,10
A=1 1 1 1,1 1 160 150 104,1042 2134 1072 5602 12604,2214 101422 220054 540122
E=2-4.535 2-2.836 2-1.535 2-.032 2+.610 2+.699 2+1.420 2+1.579 2+1.631 6
K=(45 68 45 12,) 1/510

S12 DEG=6 F=VI AUT=2 P=(1,222,22222,2) GIR=4 CN=3,10
A=1 1 1 1,1 1 174 172 74,132 2004 5002 10044 24102,50054 24122 252400 525200
E=2-5.086 2-1.198 2-.671 2-.478 2-.388 2-.346 2+1.028 2+1.318 2+2.820 6
K=(45 80 75 36 7,) -D(S7) 1/222

S13 DEG=6 F=VI AUT=2 P=(1,222,222222) GIR=3 CN=4,7
A=1 1 3 15,1 1 12 24 344,542 50 120 12110 25060,14304 114442 26604 51602
E=2-3.279 2-3.241 2-1.667 2-1.044 2-.884 2+1.227 2+1.869 2+1.904 2+2.115 6
K=(42 48 10,3) D(S6) 1/441

S14 DEG=6 F=VI AUT=6 P=(1,6,66) GIR=3 CN=5,7 T=1
A=1 1 7 5,21 43 14 102 340,424 60 4012 12402 35040,6420 114210 301700 63204
E=6-2.28514 6-1.22188 6+2.50702 6 K=(39 32,6) 1/301

S15 DEG=6 F=VI AUT=2 P=(1,222,22222,2) GIR=3 CN=4,7
A=1 3 5 3,1 1 130 70 144,142 414 4222 12304 5442,42120 21050 312400 705200
E=2-3.897 2-2.638 2-1.433 2-1.029 2-.580 2+.715 2+1.013 2+2.182 2+2.667 6
K=(42 52 20,3) 1/620

S16 DEG=6 F=VI AUT=2 P=(1,222,22222,2) GIR=3 CN=4,7
A=1 1 1 11,21 11 56 326 40,1100 1064 2112 1224 2412,41102 22044 336000 355000
E=2-4.131 2-2.071 2-2.020 2-.560 2+.125 2+.372 2+.410 2+1.711 2+3.163 6
K=(42 56 30 6,3) 1/203

S17 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=4,7
A=1 3 5 3,25 13 5 3 100,1040 1472 6334 3664 3712,42704 121642 113270 207530
E=2-4.700 2-2.339 2-1.258 2-1.059 2-.342 2+.065 2+.458 2+2.501 2+2.673 8
K=(24 20 5,9) 1/660

S18 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=5,7
A=1 1 3 5,15 23 101 241 634,632 2434 1232 14304 34442,41506 122246 252160 525150
E=2-3.992 2-2.552 2-2.319 2-2.237 2+.863 2+.908 2+1.504 2+1.809 2+2.017 8
K=(24 16,9) D(S8) 1/131

S19 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=4,7
A=1 1 5 13,11 61 101 241 416,226 1220 2410 15724 16652,15162 16154 112506 605246
E=2-4.296 2-2.823 2-1.601 2-1.229 2-.977 2+1.332 2+1.404 2+1.988 2+2.202 8
K=(24 16 5,9) 1/560

S20 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=5,5
A=1 1 7 17,1 1 25 13 712,664 2250 5520 6346 11546,52110 125060 310342 704544
E=2-4.595 2-2.643 2-1.700 2-.553 2-.104 2+.225 2+1.061 2+1.101 2+3.209 8
K=(24 16,9 4) 1/151

S21 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=5,6
A=1 1 7 7,25 13 15 23 570,1370 2250 1520 14604 34602,46044 131102 344502 730244
E=2-3.508 2-2.957 2-1.742 2-1.149 2-.836 2-.655 2+1.414 2+2.122 2+3.311 8
K=(21 8,12 4) -D(S4) 1/702

S22 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=5,5
A=1 3 7 7,13 65 101 41 334,472 1440 6300 15604 16602,54410 134220 342454 321322
E=2-3.406 2-2.803 2-2.788 2-.746 2-.089 2+.209 2+.550 2+1.312 2+3.760 8
K=(21 8,12 4) 1/47

S23 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=4,7
A=1 1 3 15,15 63 105 43 572,374 1002 2004 14604 34602,55410 136220 254324 134452
E=2-4.082 2-2.399 2-1.662 2-1.050 2-.573 2+.110 2+.143 2+1.718 2+3.796 8
K=(21 16 5,12) -D(S26) 1/612

S24 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=5,7
A=1 1 3 15,11 21 15 23 320,450 452 4324 7304 33442,1754 102762 57106 37046
E=2-4.044 2-3.640 2-1.387 2-.394 2-.128 2+.356 2+.534 2+1.704 2+2.998 8
K=(24 16,9) 1/36

S25 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=5,5
A=1 1 5 3,33 75 41 301 22,1014 1614 2622 7062 13114,5710 112660 165406 272206
E=2-3.441 2-3.001 2-2.005 2-1.044 2-.745 2-.339 2+.828 2+2.591 2+3.157 8
K=(21 4,12 4) 1/621

S26 DEG=8 F=VI AUT=2 P=(1,2222,2222,2) GIR=3 CN=7,4
A=1 1 3 5,13 65 173 375 524,252 2012 1024 15004 36002,46052 31124 375000 776000
E=2-3.037 2-1.914 2-1.634 2-1.529 2-.493 2-.431 2-.262 2+.245 2+5.055 8
K=(15,18 16 5) -D(S9) 1/161

S27 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=5,5
A=1 1 3 5,21 51 173 175 564,352 24 4012 12202 25404,76004 175002 255064 136112
E=2-3.195 2-2.450 2-2.025 2-1.191 2-.731 2-.707 2+.748 2+1.152 2+4.399 8
K=(18 4,15 8) -D(S5) 1/701

S28 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=5,5
A=1 3 3 5,11 61 27 17 412,1224 2460 1310 16270 15530,4544 110342 327042 353104
E=2-3.426 2-2.239 2-2.185 2-1.847 2-1.349 2-.069 2+1.950 2+2.360 2+2.805 8
K=(21 8,12 4) 1/123

S29 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=4,7
A=1 3 5 13,1 1 3 5 734,672 2364 1552 12104 25042,60760 60750 250524 524252
E=2-5.2514 2-2.1468 2-1.8321 2+.1451 2+.4232 2+.5142 2+.6938 2+.8476 2+2.6063 8
K=(27 28 10,6) D(S15) 1/614

S30 DEG=8 F=VI AUT=2 P=(1,2222,22222) GIR=3 CN=4,7
A=1 3 5 3,1 1 1 1 774,772 2124 5052 10724 24652,50524 124252 210764 104752
E=2-5.890 2-1.363 2-.945 2-.441 2-.180 2+.616 2+1.190 2+1.466 2+1.546 8
K=(30 40 25 6,3) D(S10) D(S12) D(S16) 1/74

<u>Additional Information</u>

(a) Two graphs are *cospectral* if their adjacency matrices have the same eigenvalues and multiplicities. We list here all families of cospectral graphs in the catalogue. The complements of each member of a family form another family.

12 vertices: L15 L21, L27 L29.

16 vertices: P33 P49, P35 P45, P61 P88, P63 P72,
P64 P86, P75 P91, P78 P95, P81 P84,
P97 P107, P98 P134, P99 P113 P118, P103 P108,
P105 P141, P111 P112, P120 P136, P124 P137,
P142 P143.

(b) The following graphs are the only ones in the catalogue which are not Cayley graphs:

J7, O7, O21, P20, P52, P93, P110, R38, R147.

(c) The switching classes of transitive graphs of even order are shown in Table 1. It is easy to show that G and H are switching equivalent if and only if $\bar{G}$ and $\bar{H}$ are. Thus each family in Table 1 provides another by complementing each member. However the following graphs are actually switching equivalent to their own complements:

B1, J3, J6, J7, R15, R32, R38, R39, R147, R148, R161, R179.

Table 1 does not include the following graphs, as they are unique in their switching classes: L10, L16, L37, P74 and P139. It may be worth noticing that each family of cospectral graphs is related also by switching. In fact, two switching equivalent regular graphs of the same degree are necessarily cospectral.

(d) The self-complementary transitive graphs in the catalogue are E2, I4, M6, M7, Q14, Q15, Q18 and Q20.

(e) The connected planar transitive graphs (excluding polygons) with $4 \le n \le 19$ are D4, F6, F7, H7, H10, J6, J11, L10, L13, L20, L21, L37, N6, N9, P10, P16, R10 and R20.

(f) The distance-regular connected graphs in the catalogue, excluding polygons and those with $k > (n - 1)/2$, are H7, I4, J7, J10, L34, L37, M6, N7, N13, N24, O7, O21, P27, P55, P81, P84, P130, Q18, R11 and R173. Of these, only P84 is not distance-transitive.

(g) Γ will act primitively on V if n is prime or if G is an empty graph. Excluding complements, the only other examples in the catalogue where this occurs are for I4, J7, O21, P55 and P81.

(h) The following are all those graphs in the catalogue whose arc-transitivity is at least one. We exclude disconnected graphs, polygons, and those whose complements are disjoint unions of complete graphs.

H7, I4, J7, J9, J10, $\overline{\text{J7}}$, L20, L23, L34, L37, $\overline{\text{L30}}$, M3, M6, N7, N12, N13, N24, O7, O12, O21, O23, $\overline{\text{O20}}$, $\overline{\text{O21}}$, P12, P23, P27, P55, P81, P82, P84, P130, $\overline{\text{P55}}$, $\overline{\text{P81}}$, Q3, Q18, R11, R28, R29, R88, R90, R171, R173, $\overline{\text{R126}}$, S14.

(i) The only connected graph in the catalogue which has no Hamiltonian cycle is Petersen's graph (J7), which has Hamiltonian paths and cycles of length 9.

B1	-B1 .								
D1	-D2 .								
F1	-F3 ,	F2	F4 .						
H1	-H5 ,	H2	-H3	H7 ,	H4	-H6 .			
J1	-J8 ,	J2	J10 ,	J3	-J3 ,	J4	J5 ,		
J6	-J6 ,	J7	-J7 ,	J9	-J11 .				
L1	-L25 ,	L2	-L14	L34 ,	L3	-L8 ,	L4	L23	-L35 ,
L5	-L9	L19 ,	L6	L17	-L31 ,	L7	L26 ,		
L11	-L22	L36 ,	L12	L27	L29 ,	L13	-L18	L32 ,	
L15	L21	-L28 ,	L20	-L33 ,	L24	-L30 .			
N1	-N20 ,	N2	N24 ,	N3	-N8 ,	N4	N19 ,	N5	N10 ,
N6	-N17 ,	N7	N13 ,	N9	-N22 ,	N11	-N25 ,	N12	-N27 ,
N14	N28 ,	N15	N26 ,	N16	N21 ,	N18	N23 .		
P1	-P96 ,	P2	-P56	P130,	P3	-P29	P82	-P132,	
P4	-P30	P87	-P116,	P5	P69	-P114,	P6	-P13	P109,
P7	-P14	P48	-P89	P140,	P8	-P15	-P76	P133,	
P9	-P62	P127,	P10	P40	-P63	-P72	P105	P141,	
P11	P37	-P64	-P86	P124	P137,				
P12	P39	-P71	P106,	P16	P83	-P131,			
P17	-P32	-P125,	P18	P80	-P101,	P19	-P123,		
P20	-P52	P93	-P110,	P21	-P33	-P49	P57	-P103	-P108,
P22	-P53	P61	P88	-P120	-P136,	P23	-P111	-P112,	
P24	-P35	-P45	P78	P95	-P99	-P113	-P118,		
P25	-P47	P90	-P121,	P26	-P41	P68 ,			
P27	-P44	P94	-P97	-P107,	P28	-P54	P58	-P104,	
P31	-P70	P138,	P34	P128,	P36	-P85	P102,		
P38	P100,	P42	-P66	P117,	P43	-P67	P98	P134,	
P46	-P59	P142	P143,	P50	-P75	-P91	P129,		
P51	-P65 ,	P55	-P81	-P84 ,	P60	-P135,	P73	-P122,	
P77	-P115,	P79	-P119,	P92	-P126 .				
R1	-R137,	R2	R173,	R3	-R54 ,	R4	R113,	R5	-R55 ,
R6	R130,	R7	R90 ,	R8	-R132,	R9	R85 ,	R10	-R111,
R11	R88 ,	R12	-R103,	R13	R80 ,	R14	-R172,	R15	-R15 ,
R16	-R17 ,	R18	R33 ,	R19	R37 ,	R20	-R153,	R21	-R166,
R22	R49 ,	R23	-R141,	R24	R40 ,	R25	-R151,	R26	-R165,
R27	-R156,	R28	-R158,	R29	R48 ,	R30	R139,	R31	R180,
R32	-R32 ,	R34	R178,	R35	-R36 ,	R38	-R38 ,	R39	-R39 ,
R41	R159,	R42	R157,	R43	R140,	R44	R167,	R45	R182,
R46	R144,	R47	R189,	R50	R154,	R51	R187,	R52	R168,
R53	R181,	R56	R93 ,	R57	R94 ,	R58	R109,	R59	-R74 ,
R60	R114,	R61	R110,	R62	-R83 ,	R63	-R84 ,	R64	R96 ,
R65	-R86 ,	R66	R119,	R67	R107,	R68	R106,	R69	-R75 ,
R70	R102,	R71	-R76 ,	R72	R124,	R73	R133,	R77	R95 ,
R78	R121,	R79	R108,	R81	R127,	R82	R131,	R87	R116,
R89	R97 ,	R91	R128,	R92	-R136,	R98	-R120,	R99	-R125,
R100	-R117,	R101	-R104,	R105	-R115,	R112	-R123,	R118	-R129,
R122	-R134,	R126	-R135,	R138	-R142,	R143	-R150,	R145	-R149,
R146	-R163,	R147	-R147,	R148	-R148,	R152	-R174,	R155	-R160,
R161	-R161,	R162	-R170,	R164	-R183,	R169	-R184,	R171	-R186,
R175	-R190,	R176	-R177,	R179	-R179,	R185	-R188 .		

TABLE 1 - Switching classes of transitive graphs (-X is the complement of X)

APPENDIX THREE

EXAMPLES OF ALGORITHM 2·31 OUTPUT

In this Appendix we give two examples of the automorphism group generators produced by Algorithm 2·31. In each case we will use the notation defined in Section 2·32.

Example 1

In our first example G is the 5-dimensional cube defined as follows.

$$V(G) = \{(i,j,k,\ell,m) \mid i,j,k,\ell,m \in \{0,1\}\}$$

$$E(G) = \{(i_1,j_1,k_1,\ell_1,m_1)(i_2,j_2,k_2,\ell_2,m_2) \mid (i_1-i_2)^2 + (j_1-j_2)^2 + (k_1-k_2)^2 + (\ell_1-\ell_2)^2 + (m_1-m_2)^2 = 1\}$$

The elements of V(G) are numbered 1, 2, ···, 32 in lexicographic order.

For this graph we find K = 5, $w_1 = 1$, $w_2 = 16$, $w_3 = 24$, $w_4 = 28$ and $w_5 = 30$. The output produced is as below. The execution time was 0·18 seconds.

(2 3)(6 7)(10 11)(14 15)(18 19)(22 23)(26 27)(30 31)

$|\Gamma^{(4)}| = 2$ $\quad|\theta(\Gamma^{(4)})| = 24$

(3 5)(4 6)(11 13)(12 14)(19 21)(20 22)(27 29)(28 30)

$|\Gamma^{(3)}| = 6$ $\quad|\theta(\Gamma^{(3)})| = 16$

(5 9)(6 10)(7 11)(8 12)(21 25)(22 26)(23 27)(24 28)

$|\Gamma^{(2)}| = 24$ $\quad|\theta(\Gamma^{(2)})| = 10$

(9 17)(10 18)(11 19)(12 20)(13 21)(14 22)(15 23)(16 24)

$|\Gamma^{(1)}| = 120$ $\quad|\theta(\Gamma^{(1)})| = 6$

(1 2)(3 4)(5 6)(7 8)(9 10)(11 12)(13 14)(15 16)(17 18)(19 20)(21 22)
(23 24)(25 26)(27 28)(29 30)(31 32)

$|\Gamma| = 3840$ $\quad|\theta(\Gamma)| = 1$

Example 2

In our first example $G = C_5[C_5]$ where each C_5 is labelled in cyclic order and the product is labelled as in the definition (Section 1·3). The elements V(G) will be called 1, 2, ···, 25 in lexicographic order.

For this graph we find K = 10, $w_1 = 1$, $w_2 = 3$, $w_3 = 11$, $w_4 = 13$, $w_5 = 16$, $w_6 = 18$, $w_7 = 21$, $w_8 = 23$, $w_9 = 6$ and $w_{10} = 8$. The output below was generated in 0·23 seconds.

(7 10)(8 9)

$|\Gamma^{(9)}| = 2$ $\quad |\theta(\Gamma^{(9)})| = 23$

(6 7 8 9 10)

$|\Gamma^{(8)}| = 10$ $\quad |\theta(\Gamma^{(8)})| = 21$

(22 25)(23 24)

$|\Gamma^{(7)}| = 20$ $\quad |\theta(\Gamma^{(7)})| = 19$

(21 22 23 24 25)

$|\Gamma^{(6)}| = 100$ $\quad |\theta(\Gamma^{(6)})| = 17$

(17 20)(18 19)

$|\Gamma^{(5)}| = 200$ $\quad |\theta(\Gamma^{(5)})| = 15$

(16 17 18 19 20)

$|\Gamma^{(4)}| = 1000$ $\quad |\theta(\Gamma^{(4)})| = 13$

(12 15)(13 14)

$|\Gamma^{(3)}| = 2000$ $\quad |\theta(\Gamma^{(3)})| = 11$

(11 12 13 14 15)

(6 21)(7 22)(8 23)(9 24)(10 25)(11 16)(12 17)(13 18)(14 19)(15 20)

$|\Gamma^{(2)}| = 20000$ $\quad |\theta(\Gamma^{(2)})| = 7$

(2 5)(3 4)

$|\Gamma^{(1)}| = 40000$ $\quad |\theta(\Gamma^{(1)})| = 5$

(1 2 3 4 5)

(1 6 11 16 21)(2 7 12 17 22)(3 8 13 18 23)(4 9 14 19 24)(5 10 15 20 25)

$|\Gamma| = 1000000$ $\quad$ $|\theta(\Gamma)| = 1$

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