Subgraphs of random k-edge-coloured k-regular graphs

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Abstract

Let $G = G(n)$ be a randomly chosen k-edge-coloured k-regular graph with $2n$ vertices, where $k = k(n)$. Equivalently, G is the union of a random set of k disjoint perfect matchings. Let $h = h(n)$ be a graph with $m = m(n)$ edges such that $m^2 + mk = o(n)$. Using a switching argument, we find an asymptotic estimate of the expected number of subgraphs of G isomorphic to h. Isomorphisms may or may not respect the edge colouring, and other generalisations are also presented. Special attention is paid to matchings and cycles.

The results in this paper are essential to a forthcoming paper of McLeod in which an asymptotic estimate for the number of k-edge-coloured k-regular graphs for $k = o(n^{5/6})$ is found.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If $A \subseteq E(G)$, then $G[A]$ is the subgraph of G induced by A. In a *partial k-colouring* of G, each edge of G is either left *uncoloured* or assigned one colour from a fixed sequence of k colours, such that two adjacent edges cannot be assigned the same colour (although we do allow the possibility that both are uncoloured). Let $E_q(G)$ denote the set of edges in G coloured with the q-th colour, for $1 \leq q \leq k$, and let $E_0(G)$ denote the set of uncoloured edges in G. If g is a partially k-coloured graph on $V(G)$, then we say that g *occurs* in G if $E_i(g) \subseteq E_i(G)$ for $0 \leq i \leq k$. If a partial k-colouring assigns a colour to every edge of G, it is simply called a *k-colouring*. Throughout this paper the terms k-colouring and k-edge-colouring are used synonymously, as no other types of colouring are considered.

The results presented in this paper arose out of a need to know the expected number of particular small subgraphs in random k-coloured k-regular graphs (see [11]). The following two theorems, proved in Section 2, provide asymptotic results for this problem. Here, and throughout the paper, our asymptotics are with respect to $n \to \infty$, with other variables being functions of n. Also, whenever random objects are mentioned it will be assumed that the underlying distribution is the discrete uniform distribution.

Before presenting these theorems, we need to define two types of graph isomorphism. Let G_1 and G_2 be partially k-coloured graphs. The first type of isomorphism, which we shall call a *colour-preserving isomorphism*, is a bijection $f : V(G_1) \to V(G_2)$ such that $f(E_i(G_1)) = E_i(G_2)$ for $0 \leq i \leq k$. If there is such an isomorphism from G_1 to G_2, we write $G_1 \cong G_2$. The colour-preserving automorphism group of G_1 is denoted by $\mathrm{Aut}(G_1)$. The second type of isomorphism does not require edge colour to be preserved but does maintain the distinction between coloured and uncoloured edges. We call this a *colour-blind isomorphism*. We define it as a bijection $f_* : V(G_1) \to V(G_2)$ such that $f_*(E_0(G_1)) = E_0(G_2)$ and $f_*\big(\bigcup_{i=1}^k E_i(G_1)\big) = \bigcup_{i=1}^k E_i(G_2)$. If there is such an isomorphism from G_1 to G_2, we write $G_1 \cong_* G_2$. The colour-blind automorphism group of G_1 is denoted by $\mathrm{Aut}_*(G_1)$.

Let $(x)_y = x(x-1)\cdots(x-y+1)$ denote the falling factorial.

Theorem 1.1. *Let G be a randomly chosen k-coloured k-regular graph with $2n$ vertices. Let $h = h(n)$ be a k-coloured graph with m edges, v vertices (none of them isolated) and at most ℓ edges of each colour, where $m(\ell + k) = o(n)$. Then the expected number of subgraphs in G isomorphic to h by a colour-preserving isomorphism is*

$$\frac{(2n)_v}{(2n)^m|\mathrm{Aut}(h)|}\left(1 + O\left(\frac{m(\ell+k)}{n}\right)\right).$$

Theorem 1.2. *Let G be a randomly chosen k-coloured k-regular graph with $2n$ vertices. Let $h = h(n)$ be a k-coloured graph with m edges and v vertices (none of them isolated). Suppose that in each of the $\psi_h(k)$ possible colourings of h there are at most L edges of any one colour. If $m(L + k) = o(n)$ then the expected number of subgraphs in G isomorphic to h by a colour-blind isomorphism is*

$$\frac{(2n)_v\,\psi_h(k)}{(2n)^m|\mathrm{Aut}_*(h)|}\left(1 + O\left(\frac{m(L+k)}{n}\right)\right).$$

The main results of Section 2 are Theorems 2.8 and 2.9, which generalise Theorems 1.1 and 1.2 by allowing h to have uncoloured edges as well.

The results of Section 2 are then applied to particular types of subgraphs; perfect matchings in Section 3 and cycles in Section 4. In these sections we obtain a series of asymptotic results, including the distribution of the number of cycles in a random k-coloured k-regular graph. In Section 4 we also produce a more accurate version of the results in Section 2 for the case where the subgraph is a 3-cycle.

All of our subgraph counts are obtained using switching arguments. Previous examples of this technique can be found in [5, 7, 8], though we use an innovative arrangement of the calculations, as we will explain later.

Before proceeding, we require a few more definitions. A *matching* in a graph G is a set of edges that are pairwise vertex-disjoint. A matching is said to be *perfect* if every vertex of G is an endpoint of exactly one edge in the matching. Given any two matchings of G, we say that they are *disjoint* if they are edge-disjoint. In this paper we restrict our attention to the set of perfect matchings $\mathcal{M}(n)$ in the complete graph K_{2n} with vertex set $V(K_{2n}) = \{1, \ldots, 2n\}$, for $n \geq 1$. Simple counting reveals that $|\mathcal{M}(n)| = (2n)!/(2^n n!)$.

2 Switchings and subgraphs

In this section we estimate the expected number of subgraphs of a random partially k-coloured complete graph G which are isomorphic to a specified graph. We begin by determining the probability that a subgraph with a specified edge colouring appears in a specified location in G.

Let $(m_1, \ldots, m_k)$ be a sequence of k disjoint perfect matchings in K_{2n}. Each such sequence corresponds to a partially k-coloured complete graph $G = G(m_1, \ldots, m_k)$ with $V(G) = V(K_{2n})$, $E_i(G) = m_i$ for $1 \le i \le k$, and $E_0(G) = E(K_{2n}) - (m_1 \cup \cdots \cup m_k)$. Define $\mathcal{G}(k)$ to be the set of all such partially k-coloured complete graphs, and note that for all $G \in \mathcal{G}(k)$, $|E_i(G)| = n$ for $1 \le i \le k$.

Let g be a partially k-coloured graph on $V(K_{2n})$. Let $k_0(g)$ be the maximum number of uncoloured edges in g which share a common endpoint, and let $\ell_i(g) = |E_i(g)|$ for $1 \le i \le k$. Define $\mathcal{G}(k, g)$ to be the set of all graphs in $\mathcal{G}(k)$ in which g occurs, that is

$$\mathcal{G}(k, g) = \{G \in \mathcal{G}(k) \mid E_i(g) \subseteq E_i(G) \text{ for } 0 \le i \le k\}.$$

Suppose x and y are vertices of g such that $xy \notin E(g)$. (The notation xy refers to the unordered pair $\{x, y\}$, and similarly elsewhere.) Let $\mathcal{H}$ be the set of all partially coloured graphs which are identical to g except that they contain the edge xy. Let g_0 be the graph in $\mathcal{H}$ in which xy is an uncoloured edge. Define α, $0 \le \alpha \le k$, to be the number of different colours of those edges in g incident with vertices x or y. Without loss of generality, assume that this set of α colours comprises the $(k-\alpha+1)$-th to k-th colours of the k colours used to partially k-colour g. Let g_i be the graph in $\mathcal{H}$ in which edge xy is coloured with the i-th colour, for $1 \le i \le k-\alpha$. Hence $\mathcal{G}(k, g) = \bigcup_{i=0}^{k-\alpha} \mathcal{G}(k, g_i)$.

Lemma 2.1. *Let h be a partially k-coloured graph on $V(K_{2n})$ in which no colour is used on more than ℓ edges. For $k_0 = k_0(h)$, if $\ell + k + k_0 < n$ then $\mathcal{G}(k, h) \neq \emptyset$.*

Proof. By Corollary 5.3 of Plummer [12], in any graph on $V(K_{2n})$ in which every vertex has degree at least $n+\ell$, any matching with at most ℓ edges can be extended to a perfect matching. For $i = 1, 2, \ldots, k$ in turn, we can use Plummer's corollary to extend the matching $E_i(h)$ to a perfect matching m_i on $V(K_{2n})$. The graph to which Plummer's corollary is applied at the i-th stage has all the edges of K_{2n} except for the edges of h (other than $E_i(h)$) and the edges of the previously constructed perfect matchings $m_1, \ldots, m_{i-1}$. This graph has no vertex with more than $k+k_0 < n-\ell$ edges that were either in h initially or were subsequently assigned to one of the perfect matchings. This means that incident with each vertex there are at least $2n-1-(n-\ell-1) = n + \ell$ edges that we are free to use, so the hypotheses of Plummer's corollary are satisfied. The resulting perfect matchings $m_1, \ldots, m_k$ are such that $G(m_1, \ldots, m_k) \in \mathcal{G}(k, h)$. ∎

In the remainder of the paper we will be considering asymptotic results under hypotheses that imply $\ell + k + k_0 = o(n)$. Under this condition, Lemma 2.1 implies that sets we encounter such as $\mathcal{G}(k, g)$ and $\mathcal{G}(k, g_i)$ are non-empty for sufficiently large n. We will often make use of this fact implicitly.

We now apply a switching argument which will enable us to determine the ratio $\mathcal{G}(k, g_i)/\mathcal{G}(k, g_0)$, for $1 \le i \le k-\alpha$. Once we have this, it is straightforward to calculate the ratio $\mathcal{G}(k, g)/\mathcal{G}(k)$, the probability that g occurs in a random $G \in \mathcal{G}(k)$.

Given the graph g_q (where $1 \le q \le k-\alpha$), with its distinguished edge xy coloured using the q-th colour, and any $G \in \mathcal{G}(k, g_q)$, we define $\mathrm{Sw}(G, g_q)$ to be the set of all 4-tuples (u, v, w, z) such that u, v, w, x, y, z are distinct vertices of G and the following conditions are met:

$$\begin{array}{rlrl} A_1: & xy \in E_q(G); & A_7: & xy \in E_q(g_q); \\ A_2: & uv \in E_q(G); & A_8: & uv \notin E(g_q); \\ A_3: & wz \in E_q(G); & A_9: & wz \notin E(g_q); \\ A_4: & ux \in E_0(G); & A_{10}: & ux \notin E(g_q); \\ A_5: & vw \in E_0(G); & A_{11}: & vw \notin E(g_q); \\ A_6: & yz \in E_0(G); & A_{12}: & yz \notin E(g_q). \end{array}$$

If $\sigma = (u, v, w, z) \in \mathrm{Sw}(G, g_q)$, then the operation *switching down* $\mathrm{sw}(\sigma)$ creates graph G' from G by changing the edge set $E_q(G)$ to $E_q(G')$ and the edge set $E_0(G)$ to $E_0(G')$, where $E_q(G') = E_q(G) \cup \{ux, vw, yz\} - \{xy, uv, wz\}$ and $E_0(G') = E_0(G) \cup \{xy, uv, wz\} - \{ux, vw, yz\}$. Note that since the edges in $E_q(G')$ are a perfect matching and the switching operation does not involve any edge in g_q other than xy, graph g_0 occurs in graph G' and $G' \in \mathcal{G}(k, g_0)$. A pictorial representation of $\mathrm{sw}(\sigma)$ appears in Figure 1.

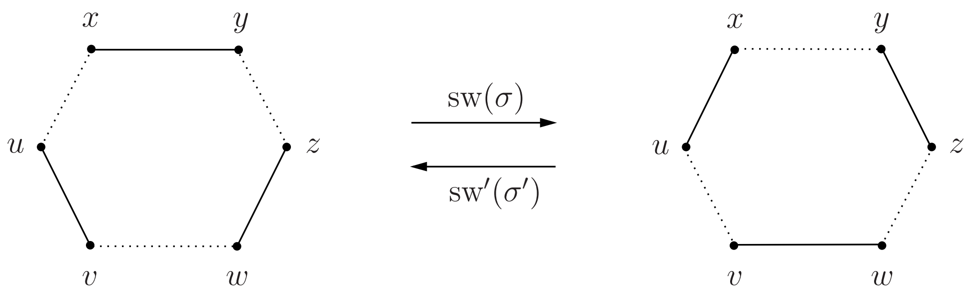

Figure 1: Switching down $\mathrm{sw}(\sigma)$ and switching up $\mathrm{sw}'(\sigma')$.

Given graph g_0 (with a distinguished uncoloured edge xy) and any $G' \in \mathcal{G}(k, g_0)$, define $\mathrm{Sw}'(G', g_0, q)$ to be the set of all 4-tuples (u, v, w, z) such that u, v, w, x, y, z are distinct vertices of G' and the following conditions are met:

$$\begin{array}{rlrl} B_1: & ux \in E_q(G'); & B_7: & ux \notin E(g_0); \\ B_2: & yz \in E_q(G'); & B_8: & yz \notin E(g_0); \\ B_3: & vw \in E_q(G'); & B_9: & vw \notin E(g_0); \\ B_4: & xy \in E_0(G'); & B_{10}: & xy \in E_0(g_0); \\ B_5: & uv \in E_0(G'); & B_{11}: & uv \notin E(g_0); \\ B_6: & wz \in E_0(G'); & B_{12}: & wz \notin E(g_0). \end{array}$$

If $\sigma' = (u, v, w, z) \in \mathrm{Sw}'(G', g_0, q)$, then the operation *switching up* $\mathrm{sw}'(\sigma')$ creates graph G from G' by changing the edge set $E_q(G')$ to $E_q(G)$ and the edge set $E_0(G')$

to $E_0(G)$ where $E_q(G) = E_q(G') \cup \{xy, uv, wz\} - \{ux, vw, yz\}$ and $E_0(G) = E_0(G') \cup \{ux, vw, yz\} - \{xy, uv, wz\}$. Note that since the edges in $E_q(G)$ are a perfect matching and the switching operation does not involve any edge in g_0 other than xy, graph g_q occurs in graph G and $G \in \mathcal{G}(k, g_q)$. Figure 1 also depicts the operation $\mathrm{sw}'(\sigma')$.

Conditions A_1–A_{12} are inverse to B_1–B_{12} in the sense that, for any $G \in \mathcal{G}(k, g_q)$, $G' \in \mathcal{G}(k, g_0)$ and $\sigma = (u, v, w, v)$, we have that $\sigma \in \mathrm{Sw}(G, g_q)$ and $\mathrm{sw}(\sigma)$ produces G' from G if and only if $\sigma \in \mathrm{Sw}'(G', g_0, q)$ and $\mathrm{sw}'(\sigma)$ produces G from G'. This implies that

$$\sum_{G \in \mathcal{G}(k, g_q)} |\mathrm{Sw}(G, g_q)| = \sum_{G' \in \mathcal{G}(k, g_0)} |\mathrm{Sw}'(G', g_0, q)|. \tag{1}$$

Lemma 2.2. *Let* $G \in \mathcal{G}(k, g_q)$, $\ell = \ell_q(g_q)$ *and* $k_0 = k_0(g_q)$. *If* $\ell + k + k_0 < n$ *then*

$$4(n-\ell-k-k_0)(n-\ell-\tfrac{1}{2}k-\tfrac{1}{2}k_0) \le |\mathrm{Sw}(G, g_q)| \le 4(n-\ell)^2. \tag{2}$$

Proof. We need to bound the number of 4-tuples (u, v, w, z) such that conditions A_1–A_{12} are satisfied. The definition of g_q ensures that A_1 and A_7 hold already. We can choose vertices u and v in at least $2n-2\ell-(k-1)-k_0$ ways, since of the $2n$ choices of u and v which satisfy condition A_2, at most 2ℓ violate condition A_8, at most $k-1$ violate condition A_4, and at most k_0 violate condition A_{10}. We can then choose the vertices w and z in at least $2(n-1)-2\ell-2(k-1)-2k_0$ ways, applying a similar argument for conditions $A_3, A_5, A_6, A_9, A_{11}$ and A_{12}. This gives us a lower bound of $2(n-\ell-k-k_0)(2n-2\ell-k-k_0)$ on the number of 4-tuples, yielding the left inequality of (2). Note that conditions $A_1, A_2, A_3, A_7, A_8, A_9$ force $\{x, y\} \cap \{u, v, w, z\} = \emptyset$, and by using $n-1$ in place of n when choosing w, z we take into account the constraint $\{u, v\} \cap \{w, z\} = \emptyset$.

For the upper bound, note that vertices u and v can be chosen in at most $2(n-\ell)$ ways, and similarly for w and z. ∎

Lemma 2.3. *Let* $G' \in \mathcal{G}(k, g_0)$, $\ell = \ell_q(g_0)$ *and* $k_0 = k_0(g_0)$. *If* $\ell + k + k_0 + 1 < n$ *then*

$$2(n-\ell-k-k_0-1) \le |\mathrm{Sw}'(G', g_0, q)| \le 2(n-\ell-2). \tag{3}$$

Proof. The proof is similar to that of Lemma 2.2.

We need to bound the number of 4-tuples (u, v, w, z) such that conditions B_1–B_{12} are satisfied. The definition of g_0 ensures that B_4 and B_{10} hold already. We can choose the vertices u and z in exactly 1 way such that conditions B_1, B_2, B_7 and B_8 are satisfied (since uncoloured edge xy is given and no edge of $E_q(g_0)$ is incident with x or y). Vertices v and w can be chosen in at least $2(n-2)-2\ell-2(k-1)-2k_0$ ways, since of the $2(n-2)$ choices of v and w which satisfy condition B_3 and $\{v, w\} \cap \{u, x, y, z\} = \emptyset$, at most 2ℓ violate condition B_9, at most $k-1$ violate condition B_5 (likewise for B_6), and at most k_0 violate condition B_{11} (likewise for B_{12}). This gives us a lower bound of $2(n-\ell-k-k_0-1)$ on the number of 4-tuples, satisfying the left inequality of (3).

For the upper bound, note that there are at most $2(n-\ell-2)$ ways of choosing the vertices v and w. ∎

Now that we have calculated bounds on the number of ways of performing the switching down and switching up operations on graphs $G \in \mathcal{G}(k, g_q)$ and $G' \in \mathcal{G}(k, g_0)$ respectively, we can determine the relative sizes of the sets $\mathcal{G}(k, g_q)$ and $\mathcal{G}(k, g_0)$.

Lemma 2.4. *Let $k_0 = k_0(g_0)$. If $\ell_q(g_0) \le \beta n$ for some constant $\beta < 1$, and $k + k_0 = o(n)$, then*

$$\frac{|\mathcal{G}(k, g_q)|}{|\mathcal{G}(k, g_0)|} = \frac{1}{2(n - \ell_q(g_0))}\left(1 + O\left(\frac{k + k_0}{n}\right)\right),$$

uniformly over q.

Proof. From Lemmas 2.2 and 2.3 we have

$$\begin{aligned}\frac{|\mathcal{G}(k, g_0)|}{|\mathcal{G}(k, g_q)|}\frac{n{-}\ell_q(g_0){-}k{-}k_0(g_0){-}1}{2(n - \ell_q(g_q))^2} &\le \frac{\sum_{G' \in \mathcal{G}(k, g_0)} |\mathrm{Sw}'(G', g_0, q)|}{\sum_{G \in \mathcal{G}(k, g_q)} |\mathrm{Sw}(G, g_q)|} \\ &\le \frac{|\mathcal{G}(k, g_0)|}{|\mathcal{G}(k, g_q)|}\frac{n - \ell_q(g_0) - 2}{(n{-}\ell_q(g_q){-}k{-}k_0(g_q))(2n{-}2\ell_q(g_q){-}k{-}k_0(g_q))},\end{aligned}$$

since $\mathcal{G}(k, g_q)$ and $\mathcal{G}(k, g_0)$ are non-empty sets. It follows from (1) that

$$\frac{n{-}\ell_q(g_0){-}k{-}k_0(g_0){-}1}{2(n - \ell_q(g_q))^2} \le \frac{|\mathcal{G}(k, g_q)|}{|\mathcal{G}(k, g_0)|} \le \frac{n - \ell_q(g_0) - 2}{(n{-}\ell_q(g_q){-}k{-}k_0(g_q))(2n{-}2\ell_q(g_q){-}k{-}k_0(g_q))}.$$

This simplifies to yield the required result since $\ell_q(g_0) + 1 = \ell_q(g_q) \le \beta n$, $k_0(g_q) \le k_0$ and $k + k_0 = o(n)$. ∎

The next two lemmas follow directly from Lemma 2.4 recalling that $\sum_{i=0}^{k-\alpha} |\mathcal{G}(k, g_i)| = |\mathcal{G}(k, g)|$.

Lemma 2.5. *Let $k_0 = k_0(g_0)$. If $\ell_i(g_0) \le \beta n$ for some constant $\beta < 1$ and all i, and $k + k_0 = o(n)$, then*

$$\frac{|\mathcal{G}(k, g_q)|}{|\mathcal{G}(k, g)|} = \frac{1}{2(n - \ell_q(g_0))}\left(1 + O\left(\frac{k + k_0}{n}\right)\right)$$

uniformly over q.

Proof. We begin by noting that

$$\begin{aligned}\frac{|\mathcal{G}(k, g)|}{|\mathcal{G}(k, g_q)|} &= \frac{|\mathcal{G}(k, g_0)|}{|\mathcal{G}(k, g_q)|} + \sum_{i=1}^{k-\alpha} \frac{|\mathcal{G}(k, g_i)|}{|\mathcal{G}(k, g_q)|} \\ &= \frac{|\mathcal{G}(k, g_0)|}{|\mathcal{G}(k, g_q)|} + \frac{|\mathcal{G}(k, g_0)|}{|\mathcal{G}(k, g_q)|} \sum_{i=1}^{k-\alpha} \frac{|\mathcal{G}(k, g_i)|}{|\mathcal{G}(k, g_0)|}.\end{aligned}$$

Applying Lemma 2.4 we obtain

$$\begin{aligned}\frac{|\mathcal{G}(k,g)|}{|\mathcal{G}(k,g_q)|} &= 2(n-\ell_q(g_0))\Big(1+O\Big(\frac{k+k_0}{n}\Big)\Big)+(k-\alpha)\Big(1+O\Big(\frac{k+k_0}{n}\Big)\Big)\\ &= 2(n-\ell_q(g_0))\Big(1+O\Big(\frac{k+k_0}{n}\Big)\Big).\end{aligned}$$

∎

Lemma 2.6. *Let $k_0 = k_0(g_0)$. If $\ell_i(g_0) \le \beta n$ for some constant $\beta < 1$ and all i, and $k + k_0 = o(n)$, then*

$$\frac{|\mathcal{G}(k,g_0)|}{|\mathcal{G}(k,g)|} = 1 - \tfrac{1}{2}\sum_{i=1}^{k-\alpha}\frac{1}{n-\ell_i(g_0)} + O\Big(\frac{k(k+k_0)}{n^2}\Big).$$

Proof. We can express the ratio as

$$\frac{|\mathcal{G}(k,g)|}{|\mathcal{G}(k,g_0)|} = 1 + \sum_{i=1}^{k-\alpha}\frac{|\mathcal{G}(k,g_i)|}{|\mathcal{G}(k,g_0)|}.$$

Since $k + k_0 = o(n)$ and $\ell_i(g_0) \le \beta n$ for all i, we can apply Lemma 2.4 and obtain

$$\frac{|\mathcal{G}(k,g)|}{|\mathcal{G}(k,g_0)|} = 1 + \tfrac{1}{2}\sum_{i=1}^{k-\alpha}\frac{1}{n-\ell_i(g_0)}\Big(1+O\Big(\frac{k+k_0}{n}\Big)\Big),$$

from which the lemma follows. ∎

We are now in a position to determine the probability that graph g occurs in a random graph in $\mathcal{G}(k)$.

Note that the graph classes whose sizes are compared in Lemmas 2.5 and 2.6 are not connected by a switching or even a sequence of switchings. This marks a break from the traditional method of analysis, which would employ chains of switchings to compare the parts of a partition of $\mathcal{G}(k)$ defined by all the possible colourings (and non-colourings) of the edges of g. Computing the size of any particular part would then require a sum over all parts, which can be a non-trivial task [5]. The new arrangement of the analysis avoids this sum and has the added advantage that we can choose to apply Lemmas 2.5 and 2.6 to the edges of g in any convenient order. For example, in the following, we treat the uncoloured edges of g first because the error terms behave better that way.

Theorem 2.7. *Let $g = g(n)$ be a partially k-coloured graph on $V(K_{2n})$ such that $|E(g)| = r + m$ and $|E_0(g)| = r$. Define $k_0 = k_0(g)$. If $m(k + k_0) = o(n)$ and $rk(k + k_0) = o(n^2)$, then the probability that g occurs in a random $G \in \mathcal{G}(k)$ is*

$$\frac{1}{2^m\prod_{i=1}^{k}(n)_{\ell_i(g)}}\Big(1-\frac{k}{2n}\Big)^r\Big(1+O\Big(\frac{m(k+k_0)}{n}+\frac{rk(k+k_0)}{n^2}\Big)\Big).$$

Proof. Let $E(g) = \{e_1, \ldots, e_{r+m}\}$ where $E_0(g) = \{e_1, \ldots, e_r\}$, and, for $0 \le j \le r+m$, define $g^{(j)}$ to be the subgraph $g[\{e_1, \ldots, e_j\}]$ with edge colours (or lack of colours) induced from g. It follows that $\mathcal{G}(k, g^{(0)}) = \mathcal{G}(k)$ and $g^{(r+m)} = g$.

The probability that g occurs in G is given by

$$\frac{|\mathcal{G}(k,g)|}{|\mathcal{G}(k)|} = \prod_{j=0}^{r+m-1} \frac{|\mathcal{G}(k, g^{(j+1)})|}{|\mathcal{G}(k, g^{(j)})|}. \tag{4}$$

Clearly $k_0(g^{(j)}) \le k_0(g)$ and $\ell_i(g^{(j)}) \le \ell_i(g)$ for all i, j, and the condition $m(k + k_0) = o(n)$ implies that $\ell_i(g) = o(n)$ for all i. Since the conditions for Lemmas 2.5 and 2.6 are satisfied, we can use these lemmas to estimate the ratios on the right side of (4). Note that $\alpha = 0$ and $\ell_i(g^{(j+1)}) = 0$ for $1 \le i \le k$ in each application of Lemma 2.6. ∎

Let h be a partially k-coloured graph with no isolated vertices, where $|V(h)| \le 2n$. We now determine the expected number of subgraphs in a randomly chosen graph $G \in \mathcal{G}(k)$ isomorphic to h under each of the two types of isomorphism defined in Section 1. Let the set of subgraphs of G for which there exists a colour-preserving isomorphism to h be given by $\mathcal{I}(h, G) = \{E \subseteq E(G) \mid G[E] \cong h\}$. Similarly, define the set of subgraphs of G for which there exists a colour-blind isomorphism to h by $\mathcal{I}_*(h, G) = \{E \subseteq E(G) \mid G[E] \cong_* h\}$.

Theorem 2.8. *Suppose that the partially k-coloured graph $h = h(n)$ has m coloured edges, r uncoloured edges and v vertices (none of them isolated). Define $k_0 = k_0(h)$. If $m(k + k_0) = o(n)$ and $rk(k + k_0) = o(n^2)$, then the expected value of $|\mathcal{I}(h, G)|$ for random $G \in \mathcal{G}(k)$ is*

$$\frac{(2n)_v}{2^m |\mathrm{Aut}(h)| \prod_{i=1}^{k} (n)_{\ell_i(h)}} \left(1 - \frac{k}{2n}\right)^r \left(1 + O\left(\frac{m(k+k_0)}{n} + \frac{rk(k+k_0)}{n^2}\right)\right).$$

Proof. The number of equivalence classes of injections from $V(h)$ into $V(G)$ is $(2n)_v / |\mathrm{Aut}(h)|$, where injections are considered equivalent if they induce the same colouring on the edges in the image. The probability that the image of one of these injections is a subgraph of a random graph in $\mathcal{G}(k)$ is given by Theorem 2.7, since the condition $m(k + k_0) = o(n)$ implies that $\ell_i(h) = o(n)$ for all i. The result follows immediately. ∎

To estimate the colour-blind count $|\mathcal{I}_*(h, G)|$, we just need to sum Theorem 2.8 over all possible colourings of the coloured edges of h. For a variable x, define

$$\psi_h(k, x) = \sum_{C \in C_k(h)} \big((x)_{\ell_1(C)} (x)_{\ell_2(C)} \cdots (x)_{\ell_k(C)}\big)^{-1},$$

where $C_k(h)$ is the set of all k-colourings of the coloured edges of h, and $\ell_i(C)$ is the number of times the i-th colour is used by colouring C. Also define $\psi_h(k) = |C_k(h)|$, which is of course just the edge-chromatic polynomial of the coloured part of h.

Theorem 2.9. *Suppose that the partially k-coloured graph h has m coloured edges, r uncoloured edges and v vertices (none of them isolated). Define $k_0 = k_0(h)$. If $m(k + k_0) = o(n)$ and $rk(k + k_0) = o(n^2)$, then the expected value of $|\mathcal{I}_*(h, G)|$ for random $G \in \mathcal{G}(k)$ is*

$$\frac{(2n)_v\, \psi_h(k, n)}{2^m |\mathrm{Aut}(h)|} \left(1 - \frac{k}{2n}\right)^r \left(1 + O\left(\frac{m(k + k_0)}{n} + \frac{rk(k + k_0)}{n^2}\right)\right).$$

Theorems 1.1 and 1.2, stated in Section 1, are special cases of Theorems 2.8 and 2.9 respectively. We prove them both now.

Proofs of Theorems 1.1 and 1.2. A k-coloured k-regular graph can be uniquely extended to $G \in \mathcal{G}(k)$ by joining each non-adjacent pair of vertices by an uncoloured edge. To obtain Theorem 1.1 from Theorem 2.8, note that $\prod_{i=1}^{k} (n)_{\ell_i(g)} = n^m \bigl(1 + O(m\ell/n)\bigr)$ if $m\ell = o(n)$. To obtain Theorem 1.2 from Theorem 2.9, apply the similar fact that $\psi_h(k, n) = \psi_h(k)n^{-m}\bigl(1 + O(mL/n)\bigr)$ if $mL = o(n)$. ∎

3 Perfect Matchings

In this section we present a series of results about perfect matchings which follow from the theorems in Section 2. Note that the first three theorems provide actual counts rather than expectations.

Let $\overline{G}$ denote the complement of graph G.

Theorem 3.1. *Let G be a graph on $2n$ vertices m edges and maximum degree Δ. If $\Delta m = o(n^2)$, then the number of perfect matchings in $\overline{G}$ is*

$$\frac{(2n)!}{2^n\, n!} \left(1 - \frac{1}{2n}\right)^m \left(1 + O\left(\frac{\Delta m}{n^2}\right)\right).$$

Proof. Let g be a copy of G with all edges uncoloured. The quantity we require is the number of elements of $\mathcal{G}(1)$ which contain g. This is equal to $|\mathcal{G}(1)|$ times the probability that a randomly chosen member of $\mathcal{G}(1)$ contains g. The first quantity is $|\mathcal{M}(n)| = (2n)!/(2^n n!)$ and the second is given by Theorem 2.7. ∎

In [4] Godsil proved that the number of perfect matchings in the complement of a graph G is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \alpha(G, x)\, dx,$$

where $\alpha(G, x)$ is the matchings polynomial of G. Hence Theorem 3.1 provides an asymptotic estimate of this integral in the case where G is moderately sparse. A much more accurate estimate for regular graphs will be given in [11].

Theorem 3.2. *Suppose $0 \le t < k$ and $k^2(t+1) = o(n)$. Then the number of k-regular graphs on $2n$ vertices with a distinguished set of t disjoint perfect matchings is*

$$\frac{(2nk-2nt)!\,(2n)!^{\,t}}{t!\,(nk-nt)!\,2^{nk}\bigl(k-t\bigr)!^{\,2n}\,n!^{\,t}}\exp\biggl(-\frac{k^2-t-1}{4}-\frac{k^3}{24n}+O\biggl(\frac{k^2(t+1)}{n}\biggr)\biggr).$$

Proof. Given the bounds on k and t, we have from [9] that the number of $(k-t)$-regular graphs is equal to

$$\frac{(2nk-2nt)!}{(nk-nt)!\,2^{nk-nt}(k-t)!^{\,2n}}\exp\biggl(-\frac{(k-t)^2-1}{4}-\frac{(k-t)^3}{24n}+O\biggl(\frac{(k-t)^2}{n}\biggr)\biggr). \tag{5}$$

Let G be a $(k-t)$-regular graph. Then Theorem 3.1 gives us the number of perfect matchings in $\overline{G}$. Since this quantity, within the variability allowed by the error term, is independent of the structure of G, the number of sets of t disjoint perfect matchings in $\overline{G}$ is given by

$$\begin{aligned}&\frac{1}{t!}\prod_{i=k-t}^{k-1}\biggl(\frac{(2n)!}{2^n\,n!}\Bigl(1-\frac{1}{2n}\Bigr)^{ni}\Bigl(1+O\Bigl(\frac{k^2}{n}\Bigr)\Bigr)\biggr)\\ &\qquad=\frac{1}{t!}\biggl(\frac{(2n)!}{2^n\,n!}\biggr)^{t}\Bigl(1-\frac{1}{2n}\Bigr)^{nt(2k-t-1)/2}\biggl(1+O\biggl(\frac{k^2t}{n}\biggr)\biggr).\end{aligned}$$

The formula in the theorem statement is obtained by multiplying by (5), taking an asymptotic expansion of $(1-1/(2n))^{nt(2k-t-1)/2}$, and applying the condition $k^2(t+1) = o(n)$. ∎

Note that Theorem 3.2 is not applicable when $k = t$. To deal with this case, we have the following theorem.

Theorem 3.3. *For $k = o(n^{1/3})$, the number of sets of k disjoint perfect matchings in K_{2n} is*

$$\frac{1}{k!}\biggl(\frac{(2n)!}{2^n\,n!}\biggr)^{k}\exp\biggl(-\frac{k(k-1)}{4}+O\biggl(\frac{k^3}{n}\biggr)\biggr). \tag{6}$$

Proof. The proof is identical to that of Theorem 3.2 except that the starting point is the number of 0-regular graphs, namely 1. ∎

This result is an improvement over Theorem 5.4 in [2], in which Bollobás obtained the same formula for fixed k. It may be possible to increase the range of k in (6) by using a modified switching argument similar to that used in Section 4. However in [11] an estimate for $k = o(n^{5/6})$ will be presented. While this stronger result is obtained using different methods to those employed here, it relies on the results given in Section 2 of this paper.

Bollobás and McKay [3] found the expectation of the number of perfect matchings in a random k-regular graph for $k = o(n^{1/3})$. The following theorem generalises this to sets of disjoint perfect matchings. Note that the expectation refers to the uniform distribution on k-regular graphs, not that on k-regular k-coloured graphs.

Theorem 3.4. *Suppose* $0 \le t < k$ *and* $k^2(t+1) = o(n)$. *Then the expected number of sets of* t *disjoint perfect matchings in a random* k*-regular graph on* $2n$ *vertices is*

$$\frac{(2nk-2nt)!\,(nk)!\,(2n)!^t\,k!^{2n}}{t!\,(nk-nt)!\,(2nk)!\,(k-t)!^{2n}\,n!^t}\exp\biggl(\frac{t}{4}+O\biggl(\frac{k^2(t+1)}{n}\biggr)\biggr).$$

Proof. The theorem follows directly from Theorem 3.2 and the case $t = 0$ of (5). ∎

4 Cycles

In this section, we compare some results about cycles in random k-coloured k-regular graphs to those of random k-regular graphs. We first consider the expected number of cycles of length z in a randomly chosen graph. Recall that the chromatic polynomial of a z-cycle, k being the number of colours, is $\lambda(z,k) = (k-1)^z + (-1)^z(k-1)$.

Theorem 2.9 provides the expectation for $zk = o(n)$, but since the factor $\psi(k,n)$ is somewhat opaque, we will give the slightly weaker result implied by Theorem 1.2.

Theorem 4.1. *Let* G *be a random* k*-coloured* k*-regular graph on* $2n$ *vertices,* $k \ge 1$. *Then, if* $z \ge 3$ *and* $z(z+k) = o(n)$, *the expected number of* z*-cycles in* G *is*

$$\frac{\lambda(z,k)}{2z}\biggl(1+O\biggl(\frac{z(z+k)}{n}\biggr)\biggr).$$

Proof. In applying Theorem 1.2 to a z-cycle h, we have $m = z$, $L \le z/2$, $|\mathrm{Aut}_*(h)| = 2z$ and $\psi_h(k) = \lambda(z,k)$. ∎

It is interesting to compare Theorem 4.1 to the corresponding result for random k-regular graphs. This problem has been studied several times; see [13] for a survey. As yet only the case $k = o(n^{1/3})$ has been solved. In the following we give the result for $z = o(n^{1/2})$ for comparison with Theorem 4.1, even though the theory can handle arbitrary z.

Theorem 4.2. *Let* H_k *be a random* k*-regular graph for* $1 \le k = o(n^{1/3})$. *Then, if* $3 \le z = o(n^{1/2})$, *the expected number of* z*-cycles in* H_k *is*

$$\frac{(k-1)^z}{2z}\biggl(1+O\biggl(\frac{z^2+k^3}{n}\biggr)\biggr).$$

Proof. Theorem 4.6 in [6] says that the number of k-regular graphs on $2n$ vertices containing a specific z-cycle is

$$\frac{(2nk-2z)!}{2^{nk-z}(nk-z)!\,k!^{\,2n-z}(k-2)!^{\,z}}\exp\biggl(-\frac{k^2-1}{4}+O\Bigl(\frac{z+k^3}{n}\Bigr)\biggr).$$

Dividing by the total number of graphs, we find that the probability that H_k contains a specific z-cycle is

$$\frac{2^z k^z (k-1)^z (2nk-2z)!(nk)!}{(nk-z)!\,(2nk)!}\exp\biggl(O\Bigl(\frac{z+k^3}{n}\Bigr)\biggr).$$

Multiplying by the number of positions in H_k in which a z-cycle may occur, namely $(2n)_z/(2z)$, and applying Stirling's formula, gives the desired result. ∎

A natural next step is to determine the distribution of the z-cycles in a randomly chosen graph. In [10] it is shown that for $k = k(n) \ge 3$, $z = z(n) \ge 3$ and $(k-1)^{2z-1} = o(n)$ the distribution of z-cycles in a random k-regular graph approaches a Poisson distribution with mean $(k-1)^z/(2z)$ as n approaches infinity. For random k-coloured k-regular graphs, a similar result might be possible using the same technique, but we will leave that question for a later paper. For the present, we will be content with showing that the distribution of z-cycles in a random k-coloured k-regular graph is asymptotically Poisson in the case that both the degree and the cycle length are bounded. It will be clear that very slowly growing k or z could be handled by the same method.

Theorem 4.3. *Let $k \ge 1, z \ge 3, c \ge 0$ be integer constants. Then the probability that a random k-coloured k-regular graph of order $2n$ has exactly c cycles of length z is*

$$\frac{1}{c!}\biggl(\frac{\lambda(z,k)}{2z}\biggr)^{c}\exp\biggl(-\frac{\lambda(z,k)}{2z}+o(1)\biggr).$$

Proof. Let $X = X(n)$ be the number of z-cycles in a randomly chosen k-coloured k-regular graph with $2n$ vertices.

Consider any fixed $i \ge 0$. The ith factorial moment $\mathbb{E}_i(X)$ of X is the expected number of sequences of i distinct z-cycles in G. Write the number of such sequences as $Y_0 + Y_1$, where Y_0 counts sequences of i vertex-disjoint z-cycles, and Y_1 counts the remaining cases.

First we estimate $\mathbb{E}(Y_0)$. In the language of Theorem 1.2, we have $m = v = iz$, $L \le iz$, $|\mathrm{Aut}_*(h)| = (2z)^i i!$ and $\psi_h(k) = \lambda(z,k)^i$. Including a factor of $i!$ since we are counting ordered sequences, this gives us

$$\mathbb{E}(Y_0) = \biggl(\frac{\lambda(z,k)}{2z}\biggr)^{i}\bigl(1+O(n^{-1})\bigr). \tag{7}$$

Next we bound $\mathbb{E}(Y_1)$. The number of isomorphism types of subgraphs h induced by a sequence of z-cycles, not all disjoint, is bounded, but all of them have $m \le iz$, $v \le m-1$, $L \le iz$ and $|\mathrm{Aut}_*(h)| \ge 1$. Therefore, Theorem 1.2 gives that $\mathbb{E}(Y_1) = O(n^{-1})$.

Combining these two estimates, we have that

$$\mathbb{E}_i(X) \to \left(\frac{\lambda(z,k)}{2z}\right)^i$$

as $n \to \infty$ for any fixed i. This implies that X converges in distribution to the Poisson distribution with mean $\lambda(z,k)/(2z)$ (see for example [1, Thm 1.22]), which is equivalent to the theorem statement. ∎

We briefly mention another simple application. A *rainbow cycle* in an edge-coloured graph is a cycle such that every edge has a different colour.

Theorem 4.4. *Suppose $k \ge z \ge 3$ and $zk = o(n)$. Then the expected number of rainbow cycles of length z in a random k-coloured k-regular graph of order $2n$ is*

$$\frac{(k)_z}{2z}\left(1 + O\left(\frac{zk}{n}\right)\right).$$

Proof. There are $(k)_z/(2z)$ isomorphism types of rainbow z-cycle if k colours are available. The expected number of each type is $1 + O(zk/n)$, by Theorem 1.1. ∎

Theorem 1.2, or Theorem 4.4, tell us that the expected number of 3-cycles, or *triangles*, in a k-regular k-coloured graph is given by $\frac{1}{6}k(k-1)(k-2)(1+O(k/n))$ for $k = o(n)$. However, a more precise estimate is required for the calculations in [11], so we now focus in more detail on the case where g is a 3-cycle. We obtain these results by using a simplified version of the switching argument (taken from [7]) which requires only four graph edges (two coloured and two uncoloured) rather than six. Theorem 2.7 is applied to assist in calculating the number of ways of performing the switching. We can assume that $k \ge 3$, since otherwise triangles are impossible.

Let t be a graph with vertex set $V(t) = \{x, y, z\} \subseteq V(K_{2n})$ and exactly two edges xz and yz coloured with the $(k-1)$-th and k-th colour, respectively. As before, $\mathcal{G}(k,t)$ is the set of all graphs in $\mathcal{G}(k)$ in which t occurs. Let $\mathcal{T}$ be the set of all partially coloured graphs which are identical to t except that they contain the edge xy. Define t_0 as the graph in $\mathcal{T}$ in which xy is an uncoloured edge, and define t_q as the graph in $\mathcal{T}$ in which edge xy is coloured with the q-th colour, for $1 \le q \le k-2$. Hence $\mathcal{G}(k,t) = \bigcup_{q=0}^{k-2} \mathcal{G}(k, t_q)$.

Given graph t_q, with $1 \le q \le k-2$, and any $G \in \mathcal{G}(k, t_q)$, define $\mathrm{Sw}_\Delta(G, t_q)$ to be the set of all ordered pairs (u, v) such that u, v, x, y are distinct vertices of G and the following conditions are met:

$$\begin{array}{llll} C_1: & xy \in E_q(G); & C_5: & xy \in E_q(t_q); \\ C_2: & uv \in E_q(G); & C_6: & uv \notin E(t_q); \\ C_3: & ux \in E_0(G); & C_7: & ux \notin E(t_q); \\ C_4: & vy \in E_0(G); & C_8: & vy \notin E(t_q). \end{array}$$

If $\sigma = (u, v) \in \mathrm{Sw}_\Delta(G, t_q)$, then the operation $\mathrm{sw}(\sigma)$ creates from G the graph $G' \in \mathcal{G}(k, t_0)$ from G by changing the edge set $E_q(G)$ to $E_q(G')$ and the edge set $E_0(G)$ to $E_0(G')$ where $E_q(G') = E_q(G) \cup \{ux, vy\} - \{xy, uv\}$ and $E_0(G') = E_0(G) \cup \{xy, uv\} - \{ux, vy\}$. A pictorial representation of $\mathrm{sw}(\sigma)$ appears in Figure 2.

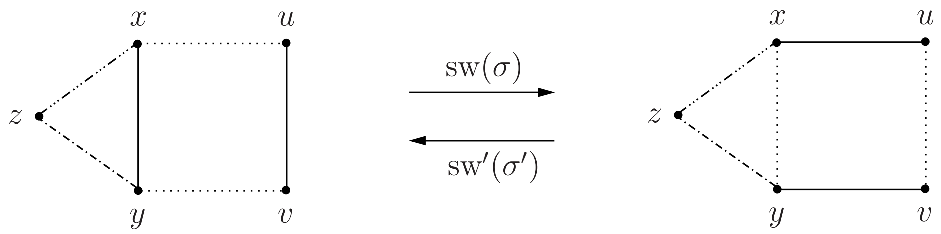

Figure 2: Switching operations $\mathrm{sw}(\sigma)$ and $\mathrm{sw}'(\sigma')$.

Given graph t_0 and any $G' \in \mathcal{G}(k, t_0)$, define $\mathrm{Sw}'_\Delta(G', t_0, q)$ to be the set of all ordered pairs (u, v) such that u, v, x, y are distinct vertices of G' and the following conditions are met:

$$\begin{array}{llll} D_1: & ux \in E_q(G'); & D_5: & ux \notin E(t_0); \\ D_2: & vy \in E_q(G'); & D_6: & vy \notin E(t_0); \\ D_3: & xy \in E_0(G'); & D_7: & xy \in E_0(t_0); \\ D_4: & uv \in E_0(G'); & D_8: & uv \notin E(t_0). \end{array}$$

If $\sigma' = (u, v) \in \mathrm{Sw}'_\Delta(G', t_0, q)$, then the operation $\mathrm{sw}'(\sigma')$ creates from G' the graph $G \in \mathcal{G}(k, t_q)$ from G' by changing the edge set $E_q(G')$ to $E_q(G)$ and the edge set $E_0(G')$ to $E_0(G)$ where $E_q(G) = E_q(G') \cup \{xy, uv\} - \{ux, vy\}$ and $E_0(G) = E_0(G') \cup \{ux, vy\} - \{xy, uv\}$. Figure 2 also depicts the operation $\mathrm{sw}'(\sigma')$.

Conditions C_1–C_8 are inverse to D_1–D_8 in the same sense as A_1–A_{12} were seen to be inverse to B_1–B_{12} in Section 2. This implies that

$$\sum_{G \in \mathcal{G}(k, t_q)} |\mathrm{Sw}_\Delta(G, t_q)| = \sum_{G' \in \mathcal{G}(k, t_0)} |\mathrm{Sw}'_\Delta(G', t_0, q)|. \tag{8}$$

Lemma 4.5. *If $G \in \mathcal{G}(k, t_q)$, then*

$$\mathbb{E}\big(|\mathrm{Sw}_\Delta(G, t_q)|\big) = (2n - 2k)\left(1 + O\left(\frac{k^2}{n^2}\right)\right), \tag{9}$$

for $3 \le k = o(n)$, uniformly over q.

Proof. Consider finding all pairs (u, v) such that conditions C_1–C_8 are satisfied. The definition of t_q ensures that C_1 and C_5 hold already. We can choose the vertices u and v in exactly $2n-4$ ways such that conditions C_2 and C_6–C_8 are satisfied. The probability that condition C_3 is also satisfied (conditional on the choice of u and v) is $(1-(k-2)/2n)(1+O(k^2/n^2))$ by Lemma 2.6, where we take g to be the graph consisting of t_q and the edge uv, and g_0 to be graph g with the addition of the uncoloured edge ux. By a similar argument we find that the probability that condition C_4 is also satisfied (conditional on the choice of u and v, and ux being uncoloured) is $(1-(k-2)/2n)$ $(1 + O(k^2/n^2))$. The result follows immediately. ∎

Lemma 4.6. *If $G' \in \mathcal{G}(k, t_0)$, then*

$$\mathbb{E}\big(|\mathrm{Sw}'_\Delta(G', t_0, q)|\big) = \left(1 - \frac{k-1}{2n}\right)\left(1 + O\left(\frac{k^2}{n^2}\right)\right), \tag{10}$$

for $3 \le k = o(n)$.

Proof. This is similar to the proof of Lemma 4.5. We consider finding all pairs (u, v) such that conditions D_1–D_8 are satisfied. Since uncoloured edge xy is given, we can choose the vertices u and v in exactly one way such that all conditions are satisfied except possibly for D_4. The probability that condition D_4 is also satisfied (conditional on ux and vy being coloured with the q-th colour) is $(1-(k-1)/2n)(1 + O(k^2/n^2))$ by Lemma 2.6, where we take g to be the graph consisting of t_0 and the edges ux and vy, and g_0 to be graph g with the addition of the uncoloured edge uv. The result follows immediately. ∎

We can now determine the relative sizes of the sets $\mathcal{G}(k, t_q)$ and $\mathcal{G}(k, t_0)$.

Lemma 4.7. *If $3 \le k = o(n)$, then*

$$\frac{|\mathcal{G}(k, t_q)|}{|\mathcal{G}(k, t_0)|} = \frac{1}{2n}\left(1 + \frac{k+1}{2n} + O\left(\frac{k^2}{n^2}\right)\right),$$

uniformly over q.

Proof. From Lemmas 4.5 and 4.6 we have

$$\begin{aligned}\sum_{G \in \mathcal{G}(k,t_q)} |\mathrm{Sw}_\Delta(G, t_q)| &= |\mathcal{G}(k, t_q)|\, \mathbb{E}\big(|\mathrm{Sw}_\Delta(G, t_q)|\big) \\ &= |\mathcal{G}(k, t_q)|\, (2n-2k)\left(1 + O\left(\frac{k^2}{n^2}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{G' \in \mathcal{G}(k,t_0)} |\mathrm{Sw}'_\Delta(G', t_0, q)| &= |\mathcal{G}(k, t_0)|\, \mathbb{E}\big(|\mathrm{Sw}'_\Delta(G', t_0, q)|\big) \\ &= |\mathcal{G}(k, t_0)|\left(1 - \frac{k-1}{2n}\right)\left(1 + O\left(\frac{k^2}{n^2}\right)\right).\end{aligned}$$

The result is obtained by substituting these equations into (8) and simplifying. ∎

The next two lemmas follow directly from Lemma 4.7, recalling that $\sum_{i=0}^{k-2} |\mathcal{G}(k,t_i)| = |\mathcal{G}(k,t)|$.

Lemma 4.8. *If* $3 \le k = o(n)$, *then*

$$\frac{|\mathcal{G}(k,t_q)|}{|\mathcal{G}(k,t)|} = \frac{1}{2n-3}\left(1 + O\left(\frac{k^2}{n^2}\right)\right)$$

uniformly over q.

Proof. We begin by noting that

$$\begin{aligned}\frac{|\mathcal{G}(k,t)|}{|\mathcal{G}(k,t_q)|} &= \frac{|\mathcal{G}(k,t_0)|}{|\mathcal{G}(k,t_q)|} + \sum_{i=1}^{k-2} \frac{|\mathcal{G}(k,t_i)|}{|\mathcal{G}(k,t_q)|} \\ &= \frac{|\mathcal{G}(k,t_0)|}{|\mathcal{G}(k,t_q)|} + \frac{|\mathcal{G}(k,t_0)|}{|\mathcal{G}(k,t_q)|} \sum_{i=1}^{k-2} \frac{|\mathcal{G}(k,t_i)|}{|\mathcal{G}(k,t_0)|}.\end{aligned}$$

Applying Lemma 4.7 we obtain

$$\begin{aligned}\frac{|\mathcal{G}(k,t)|}{|\mathcal{G}(k,t_q)|} &= (2n-k-1)\left(1 + O\left(\frac{k^2}{n^2}\right)\right) + (k-2)\left(1 + O\left(\frac{k^2}{n^2}\right)\right) \\ &= (2n-3)\left(1 + O\left(\frac{k^2}{n^2}\right)\right).\end{aligned}$$

∎

For completeness we include the following lemma.

Lemma 4.9. *If* $3 \le k = o(n)$, *then*

$$\frac{|\mathcal{G}(k,t_0)|}{|\mathcal{G}(k,t)|} = \left(1 - \frac{k-2}{2n} - \frac{3(k-2)}{4n^2}\right)\left(1 + O\left(\frac{k^3}{n^3}\right)\right).$$

Proof. We can express the ratio in the lemma statement as

$$\frac{|\mathcal{G}(k,t_0)|}{|\mathcal{G}(k,t)|} = \frac{|\mathcal{G}(k,t_0)|}{\sum_{i=0}^{k-2}|\mathcal{G}(k,t_i)|} = \frac{1}{1 + \sum_{i=1}^{k-2} \frac{|\mathcal{G}(k,t_i)|}{|\mathcal{G}(k,t_0)|}}.$$

Since $k = o(n)$ we can apply Lemma 4.7 and obtain

$$\frac{|\mathcal{G}(k,t_0)|}{|\mathcal{G}(k,t)|} = \frac{1}{1 + \frac{k-2}{2n}\left(1 + \frac{(k+1)}{2n} + O\left(\frac{k^2}{n^2}\right)\right)},$$

uniformly. Taking the Taylor series of the right side yields the required result. ∎

Theorem 4.10. *For* $3 \le k = o(n)$, *the probability that* t_q *occurs in a random* $G \in \mathcal{G}(k)$ *is*

$$\frac{1}{(2n-1)(2n-2)(2n-3)}\left(1 + O\left(\frac{k^2}{n^2}\right)\right),$$

uniformly over q.

Proof. The probability that t_q occurs in G is

$$\frac{|\mathcal{G}(k,t_q)|}{|\mathcal{G}(k)|} = \frac{|\mathcal{G}(k,t)|}{|\mathcal{G}(k)|}\frac{|\mathcal{G}(k,t_q)|}{|\mathcal{G}(k,t)|} = \frac{1}{2n{-}3}\left(1+O\left(\frac{k^2}{n^2}\right)\right)\frac{|\mathcal{G}(k,t)|}{|\mathcal{G}(k)|},$$

by Lemma 4.8.

To determine the probability that t occurs in graph G we start by noting that over all graphs in $\mathcal{G}(k)$ there are exactly $k!\binom{2n-1}{k}$ possible k-colourings of the edges incident to z. By symmetry, each k-colouring occurs in the same number of graphs. If we now consider only those graphs in $\mathcal{G}(k)$ in which the edges xz and yz are coloured with the $(k{-}1)$-th and k-th colour respectively, (as they are in t), then there are only $(k{-}2)!\binom{2n-3}{k-2}$ possible k-colourings of the edges incident with z. Hence, the proportion of graphs in $\mathcal{G}(k)$ in which t occurs is $1/((2n{-}1)(2n{-}2))$. The result follows immediately. ∎

For our next two lemmas we will again require the two types of graph isomorphism defined in Section 1, as well as the sets $\mathcal{I}(h,G)$ and $\mathcal{I}_*(h,G)$ defined in Section 2.

Theorem 4.11. *For random $G \in \mathcal{G}(k)$, where $3 \le k = o(n)$, the expected value of $|\mathcal{I}(t_q,G)|$ is*

$$1+\frac{3}{2n}+O\left(\frac{k^2}{n^2}\right).$$

Proof. The number of injections from $V(t_q)$ into $V(G)$ is $2n(2n{-}1)(2n{-}2)$. The probability that there exists a colour-preserving isomorphism from the image of one of these mappings to t_q is given by Theorem 4.10. The result follows immediately. ∎

Theorem 4.12. *For random $G \in \mathcal{G}(k)$, where $3 \le k = o(n)$, the expected number of triangles, regardless of edge colours, is*

$$k(k{-}1)(k{-}2)\left(\frac{1}{6}+\frac{1}{4n}+O\left(\frac{k^2}{n^2}\right)\right).$$

Proof. The number of injections from $V(t_q)$ into $V(G)$ is $2n(2n{-}1)(2n{-}2)$. The probability that there exists a colour-blind isomorphism from the image of one of these mappings to t_q is obtained by multiplying the formula in Theorem 4.10 by $k(k{-}1)(k{-}2)$, the number of ways of (properly) colouring t_q with k colours. However, there is a six-fold overcount because any subgraph which has a colour-blind isomorphism to t_q has six such isomorphisms. The result follows immediately. ∎

It is possible that the switching technique used in this section can also be used to improve upon the results in Section 2 for other small subgraphs.

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