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Hamiltonian Cycles in Cubic 3-Connected Bipartite Planar Graphs

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We show that 3-connected cubic bipartite planar graphs with fewer than 66 vertices are Hamiltonian.

1. INTRODUCTION

Unsolved problem 5 in [19] states what has become known as Barnette's conjecture. This is that every cubic 3-connected bipartite planar graph (C3CBP) is Hamiltonian. The constraints of the problem seem to set it somewhere between 4-connected planar graphs, 3-connected cubic planar graphs, and 3-connected cubic bipartite graphs. A famous result of Tutte [17] shows that the 4-connected planar graphs are Hamiltonian (see also Thomassen [15] for a more recent proof which also settles a conjecture of Plummer). Tutte [16] also showed that some 3-connected planar graphs are non-Hamiltonian. That the same is true for bipartite cubic 3-connected graphs is shown by a graph of Horton, see [2]. (A smaller example has now been found by Ellingham and Horton [5].)

This paper, while unable to settle the Barnette conjecture, aims to give evidence in its support. We are able to show that the conjecture is true for graphs of order up to and including 64. Some related results can be found in Goodey [7], Plummer and Pulleyblank [13], and Richmond and Wormald [14].

Recent work has been expended on trying to determine the order of the smallest non-Hamiltonian cubic 3-connected planar (C3CP) graph. Lederberg, Bosák, and Barnette (see [8]) have constructed non-Hamiltonian C3CP of order 38. Okamura [12] has shown that the smallest non-Hamiltonian C3CP has order at least 34. The reduction and cut techniques we use here are similar to those used by Okamura ([11, 12]), Barnette and Wegner [1], Butler ([3, 4]), and Goodey [6] in investigating C3CPs.

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One method we will use to find Hamiltonian cycles in C3CBPs is to separate at an edge cut, find Hamiltonian cucles in the parts, and then combine these cycles. A *k-face* is a face bounded by k edges. A *k-cut* is a set of k edges whose removal separates G into two parts, each with more than two vertices. A 4-cut is *essential* if neither part is a 4-face and it is *major* if neither part is a 4-face or one of the graphs R_1 or R_2 of Fig. 2. Since it is 3-connected, a C3CBP has no 2-cuts. A C3CB4 is a C3CBP with no 3-cuts or essential 4-cuts, so that any 4-cut has one part which is a 4-face. A C3CBP4* is a C3CBP with no 3-cuts or major 4-cuts, so that any 4-cut has one part which is a 4-face (R_0), R_1, or R_2. Figure 1 shows the three graphs with fewer than 16 vertices which are C3CBPs. Here C_3 is a C3CBP, C_2 is a C3CBP4*, and C_1 is a C3CBP4.

C3CBPs sometimes have many Hamiltonian cycles, allowing us to impose conditions on them. We say a C3CBP is H if it has a Hamiltonian cycle, H^+ (H^-) if it has a Hamiltonian cycle through (avoiding) any specified edge, and H^{+-} if any two edges can be specified, one in and one not in some Hamiltonian cycle. One further property, H^*, is a slight weakening of H^{+-}. H^* will be defined in Section 2.

With each of these properties we associate a number. Thus N is the largest number for which every C3CBP on at most N vertices is H. At the end of the paper we show that N is at least 64. We choose, however, to use N, N^+, N^-, N^{+-}, and N^* for the numbers associated with properties H, H^+, H^-, H^{+-}, and H^* so that such results as $N^+ \geqslant N^* + 8$ (Theorem 2) will remain relevant even after the computer results which provide the basic data are superseded.

2. Reductions

One of the basic tools used for finding Hamiltonian cycles in planar cubic graphs has been reductions. We will use the twelve basic reductions shown in Fig. 2, along with some variations of these. Each reduction R_i involves a subgraph of G (also called R_i) with certain edges shown bold.

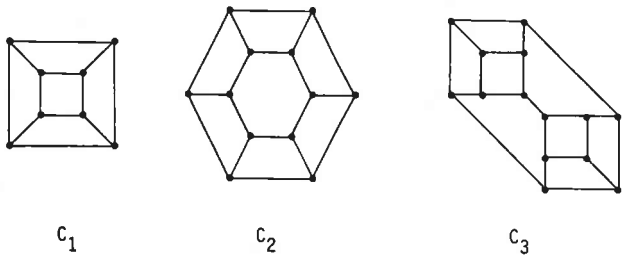

Figure 1

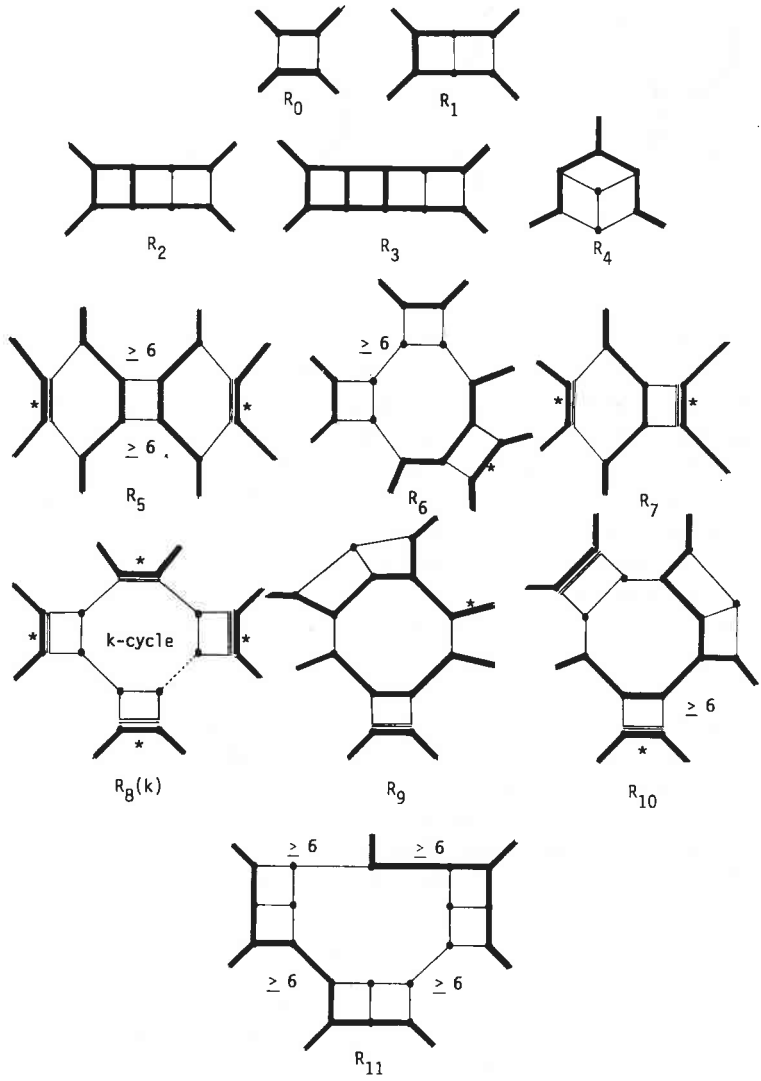

FIGURE 2

Deleting the non-bold edges of R_i and suppressing vertices of degree 2 produces a new graph G', a process called *reduction* by R_i, and denoted $G(R_i)G'$. The subgraph $R_8(k)$ is a k-cycle with 4-faces on every second edge, save one. The multiple edges shown in R_5, R_7, $R_8(k)$, R_9, and R_{10} indicate that the reduction extends to include any and all adjacent 4-faces. Thus R_7 may contain just one 4-face or it might contain several, as shown

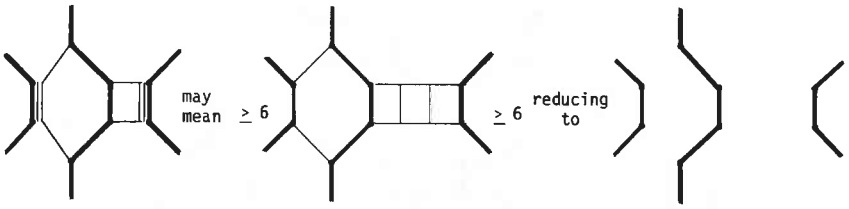

FIGURE 3

in Fig. 3. Where an asterisk appears on an edge, use of that edge in a Hamilonian cycle of G' assures that there is an extension of that Hamiltonian cycle to G. We will only prove that fact, and use it, for R_5, R_6, and $R_8(k)$, but it is easy to check in general. A Hamiltonian cycle in G' which uses no edge marked with an asterisk may or may not extend to a Hamiltonian cycle for G, except in the case of R_{11}. If $G(R_{11})G'$ and G' is Hamiltonian, then G is Hamiltonian, but we will not use that fact, either.

We use one further variation on the reductions R_i, $6 \leqslant i \leqslant 10$, which each involve some lone 4-faces adjacent to larger faces. In each case a larger subgraph can be made by replacing any of these 4-faces (but not the 4-faces in pairs in R_9 and R_{10}) by a triple of 4-faces, all adjacent to the large face. Figure 4 shows examples. Such an expansion of an R_i reduction will be called and R_i *triple reduction*, and denoted by T_i or, generically, by T. If the triple of 4-faces is adjacent to a 6-face, the reduction expands to include that 6-face and further 4-faces, as also shown in Fig. 4. The reductions R_7, T_7, R_{10}, and T_{10} shown are only examples, since the T's can generally have one or more triples, and there may or may not be 6-faces beside triples and other 4-faces beside those 6-faces, and so on. Notice that triple reductions have larger principal faces than the related non-triple reductions. For example R_7 involves a 6-face but the large face in T_7 is an 8-face.

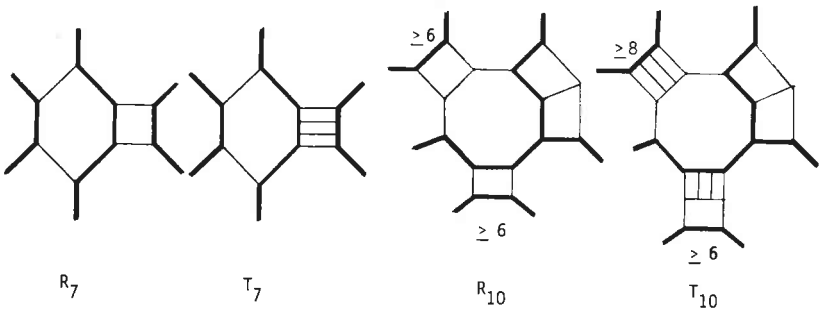

FIGURE 4

We need one more H-property, H^*, which we define after the next lemma. As we will explain in Section 4, a computer search has verified that all C3CBPs up to and including 40 vertices are H^{+-}. The search also verified that graphs on 42 and 44 vertices with no R_2 or R_4 subgraphs are H^{+-}. Any graph G on 42 or 44 vertices with an R_4 subgraph can be reduced by that R_4 to G', of size less than 40. Thus, as we will prove in Lemma 3, G itself is H^{+-}. Therefore the only graphs on up to 44 vertices which may not be H^{+-} are those containing an R_2 but no R_4 subgraph. In fact reduction by R_2 preserves most of the H^{+-} property, as we now show. A *central edge* of an R_2 is an edge such as d or e in Fig. 5.

LEMMA 1. *If G is a C3CBP containing R_2, $G(R_2)G'$, and G' is H^{+-}, then G is H^{+-} except that it may not be possible to find a Hamiltonian cycle in G on a specified central edge of R_2, avoiding a specified edge containing no vertex of R_2.*

Proof. The reduction replaces the subgraph R_2 by a 4-face, and Fig. 5 shows how Hamiltonian cycles of the 4-face extend to R_2. We find a Hamiltonian cycle through edge a in G, by using one in G' through edge A, and similarly for b, B and c, C. Unfortunately only cycles 1, 3, and 4 of Fig. 5 extend to a central edge, such as d, and no specified edge in G' can narrow the possibilities down to 1, 3, and 4. In order to avoid edges, to miss a miss A; b, B; c, C; d, A. Also if both the forced edge and the avoided edge contain at least one vertex of R_2, we can specify exactly which of the paths from 1 to 6 the Hamiltonian cycle uses in G', and so satisfy the condition in G. The specifications to attain given paths are: 1, miss B use F (we denote this by $-B\ +F$); 2, $-A\ +E$; 3, $-B\ +C$; 4, $+B\ -C$; 5, $-A\ +D$; 6, $+A\ -D$. ∎

This information leads us to the following definition. A graph G is H^* if (i) it is H^{+-} or (ii) it contains a unique subgraph R_2 and any pair of edges, one to use, one to avoid, can be specified for a Hamiltonian cycle in G, unless the edge to be used is a central edge of R_2 and the edge to be missed contains no vertex of that R_2 subgraph. In particular, if G is H^* then a Hamiltonian cycle can be found using any two specified edges. By Lemma 1, and the observations preceding it, we have the following result.

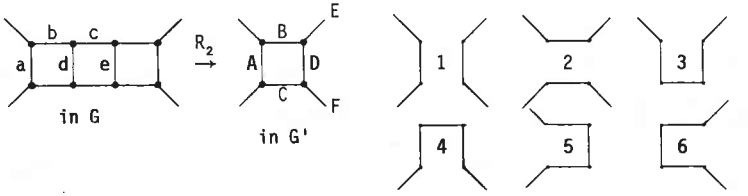

FIGURE 5

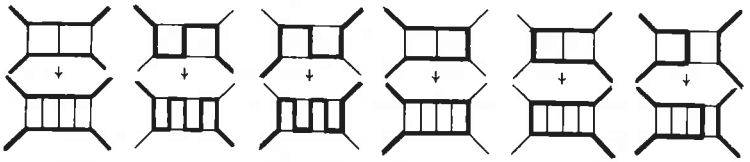

FIGURE 6

LEMMA 2. *Suppose all C3CBP containing no subgraphs R_2 or R_4 on up to n vertices are H^{+-}. Then $N^* = \min(n, 4 + N^{+-})$.*

We now prove a series of lemmas which show how the various reductions preserve the character and the Hamiltonicity of a C3CBP.

LEMMA 3. *If G is a C3CBP and $G(R_i)G'$, $i = 3$ or 4 then G' is a C3CBP. If $G(R_3)G'$ and G' is H^+, then G is H^+. If $G(R_4)G'$ and G' is H^+ (H^{+-}), then G is H^+ (H^{+-}).*

Proof. The first claim is obvious. Say that $G(R_3)G'$ and G' is H^+. The six possible ways a Hamiltonian cycle can visit R_3 in G', and the extensions to G, are shown in Fig. 6. (Actually there are four variations of the last way, since any two adjacent vertical edges can be used.) In each of the six ways, an edge in G can be included in a Hamiltonian cycle if the corresponding edge is included in the cycle in G'. For example, any cycle for G' which includes the central edge can be extended to a cycle for G which includes any desired central edge. It is equally simple to exclude edges, or specify inclusion of one, exclusion of another.

Now say $G(R_4)G'$ and G' is H^{+-} (the argument for H^+ is similar, and therefore omitted). The six possible ways a Hamiltonian cycle can visit R_4 in G are shown in Fig. 7. Notice that for each combination of entering and leaving edges, there are three edges used in both routes through the cube and six edges used in just one route or the other. Thus it is easy to use or avoid any edge or pair of edges. For example, if we want a Hamiltonian cycle for G which avoid the top right edge of the cube, and uses an edge elsewhere in G, we can force a cycle in G' which uses that other edge and avoids the bottom left edge leading to the cube. The only combination of

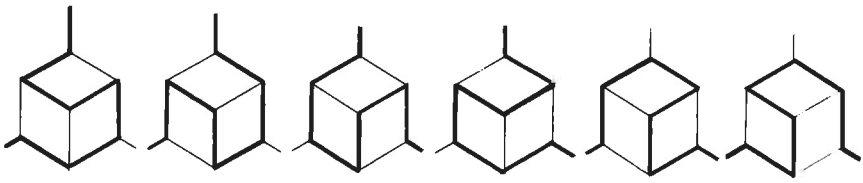

FIGURE 7

edges leading to the cube which requires the top right edge is the bottom two, which we have avoided. All other cases are equally simple. ▮

In proving the next two lemmas, we let $n(i, j)$ denote the number of adjacencies of i-faces and j-faces in G, and let f_k denote the numer of k-faces.

LEMMA 4. *Every C3CBP4 except C_1 contains at least one of the reductions R_5, R_6, or $R_8(k)$, $k \geqslant 8$.*

Proof. Suppose G contains no R_6 or $R_8(k)$. We will prove it contains an R_5. Our assumptions imply that $n(4, 4) = 0$, that each 8-face is adjacent to at most two 4-faces, and that each k-face ($k \geqslant 10$) is adjacent to at most $(k/2 - 1)$ 4-faces. Thus the obvious count on adjacencies of 4-faces,

$$4f_4 = 2n(4, 4) + \sum_{k \geqslant 6} n(4, k), \tag{1}$$

yields

$$4f_4 \leqslant n(4, 6) + 2f_8 + \sum_{k \geqslant 10} f_k(k/2 - 1). \tag{2}$$

But for any plane graph

$$\sum_{k \geqslant 4} f_k(6 - k) = 12,$$

so

$$4f_4 = 24 + 4f_8 + \sum_{k \geqslant 10} f_k 2(k - 6). \tag{3}$$

Combining (2) and (3),

$$n(4, 6) \geqslant 24 + 2f_8 + \sum_{k \geqslant 10} f_k(\tfrac{3}{2}k - 11). \tag{4}$$

Suppose the average number of 6-faces adjacent to each 4-face is x, so that $n(4, 6) = xf_4$. Then (4) becomes

$$xf_4 \geqslant 24 + 2f_8 + \sum_{k \geqslant 10} f_k(\tfrac{3}{2}k - 11). \tag{5}$$

Multiplying (3) by $x/4$ yields

$$xf_4 = 6x + \sum_{k \geqslant 8} xf_k(k - 6)/2. \tag{6}$$

Combining (5) and (6),

$$6(x-4) \geqslant (2-x) f_8 + \sum_{k \geqslant 10} f_k(k(3-x)/2 - (3x-11)). \tag{7}$$

For $x \leqslant 2$, the left side is negative, the right side non-negative, so $x > 2$. Thus some 4-face is adjacent to three or four 6-faces, and G contains an R_5. ∎

LEMMA 5. *If G is C3CBP4* then G contains a subgraph R_7, $R_8(k)$, R_9, R_{10}, R_{11}, or a related triple-reduction.*

Proof. We assume not, and derive a contradiction. The absence of R_7 implies $n(4, 6) = 0$, the absence of $R_8(k)$ $(k \geqslant 6)$, R_9, R_{10}, or related triple reductions or triple reduction related to R_7, implies $n(4, 8) \leqslant 3f_8$. Say that there are t triples of 4-faces, and $(f_4 - 3t)$ 4-faces not in triples, so that $n(4, 4) \leqslant 2t + (f_4 - 3t)/2$. Then

$$4f_4 = 2n(4, 4) + \sum_{k \geqslant 6} n(4, k) \tag{8}$$

yields

$$4f_4 \leqslant 2\left(2t + \frac{f_4 - 3t}{2}\right) + 3f_8 + \sum_{k \geqslant 10} n(4, k) \tag{9}$$

or

$$4f_4 \leqslant 4t + f_4 - 3t + 3f_8 + \sum_{k \geqslant 10} n(4, k), \tag{10}$$

so

$$3f_4 \leqslant t + 3f_8 + \sum_{k \geqslant 10} n(4, k). \tag{11}$$

But Euler's polyhedron formula implies

$$3f_4 = 18 + 3f_8 + \sum_{k \geqslant 10} \tfrac{3}{2}(k-6) f_k. \tag{12}$$

Using (11) and (12) we obtain

$$2t \geqslant 36 + \sum_{k \geqslant 10} 3(k-6) f_k - 2n(4, k). \tag{13}$$

But no triple of 4-faces can have its long side beside an 8-face, since we have no triple R_7 reduction (see T_7 in Fig. 4). So every triple has its two long sides beside faces of size 10 or more. A 10-face can be adjacent to only

one triple (we have no triple T_7 or $T_8(6)$), a 12-face can be adjacent to only two triples (no $T_8(6)$), a 14-face to at most 3 (each uses four spaces) and, in general, space dictates that for $k \geqslant 16$, a k-face can be adjacent to at most $[k/4]$ triples. Counting faces adjacent to the two long sides of the t triples we find

$$2t \leqslant f_{10} + 2f_{12} + \sum_{k \geqslant 14} [k/4] f_k. \tag{14}$$

Subtracting (14) from (13) we obtain

$$\begin{aligned} 0 \geqslant 36 &+ (11f_{10} - 2n(4, 10) + (16f_{12} - 2n(4, 12)) \\ &+ \sum_{k \geqslant 14} ((3(k-6) - [k/4]) f_k - 2n(4, k)). \end{aligned} \tag{15}$$

But that implies that some 10-face is adjacent to at least six 4-faces (and we have R_{11} or a T_9 or T_{10} triple reduction) or some 12-face is adjacent to at least nine 4-faces (and we have a $T_8(6)$ triple reduction) or a k-face ($k \geqslant 14$) is adjacent to more than $(\frac{11}{8}k - 9)$ 4-faces. For $k = 14$ that means there are eleven 4-faces, which is impossible without four 4-faces in a row. For larger k, $(\frac{11}{8}k - 9) > \frac{3}{4}k$, which is similarly impossible. Thus inequality (15) cannot be satisfied, and the lemma is proved. ∎

LEMMA 6. *If G is C3CBP4* and $G(R_i)G'$, $i = 5, 7, 8, 9, 10,$ or 11, or a related triple reduction, then G' is C3CBP. If G' is H* then G is H^+.*

Proof. The bold edges produced are all distinct from each other, because all faces of G are of size $\geqslant 4$, and some are restricted to size $\geqslant 6$ (as indicated). Furthermore, none of these reductions produce multiple edges. The bold edges on 4-faces (e.g., those marked with an asterisk in R_5, R_7, and R_{10}) are not double in G' because the reduction extends to include as many adjacent 4-faces as possible (see Fig. 3). The bold edges crossing the center in R_5, R_7, and R_9 do not duplicate an edge already present in G because that would, in each case, imply the presence in G of a cycle of length six with two or more vertices both inside and outside. This would imply G has a 3-cut. Finally, a bold central edge and a bold edge from a 4-face cannot join the same pair of vertices in G'. If they did, then G would contain a major 4-cut in every case. For the triple reductions a triple is extended to include an adjacent 6-face and then 4-faces to avoid the production of double edges. The argument about production of a cycle of length six does not work with triples but it is not needed because a triple cannot have a 4-face adjacent to either of its ends in a C3CBP4*. Thus no multiple edges are produced by any of these reductions or triple reductions.

The reduced graphs are clearly cubic, planar, and bipartite, so we need only check that they are 3-connected. Suppose, for example, that $G(R_5)G'$ and G' is not 3-connected. Then any 2-cut in G' must separate a component containing one of the bold edges of R_5 (or two adjacent bold edges) from a component containing the others. In every case such a 2-cut of G' can be combined with two non-bold edges of R_5 to yield a major 4-cut in G, which is impossible. The arguments for all the other reductions are exactly the same.

Now we suppose that G' is H^*, and show that G is H^{+}. Because we have only H^* rather then H^{+-}, we may occasionally have an edge in G' which cannot be forced into a Hamiltonian cycle, but it turns out that H^* is sufficient to force a Hamiltonian cycle through any edge of G. Our names R_i are shorthand for many reductions, since there may be 4-faces added at certain places. We display typical examples of R_i, $i = 5, 7, 8, 9, 10$, and 11, with edges labelled, in Fig. 8. The G edges are labelled with letters, the G' edges with numbers. The following list specifies the G' edges which must be forced or avoided in a Hamiltonian cycle for G' in order to assure a Hamiltonian cycle in G which includes the designated edge. Of course, H^{+-} allows us to force two edges, instead of forcing one and avoiding one:

R_5: *ad* (+1 +4), *b* (+1 −2), *cf* (+1 +2), *e* (−1 +4 or +1 −4), *h* (+1).

R_7: *a* (+1 +2), *cef* (+1 +3), *b* (+1 −2), *d* (+1 −3 or −1 +3).

R_8: *ab* (−1 +2), *cd* (+1 −2), *egh* (+1 +2), *f* (+1).

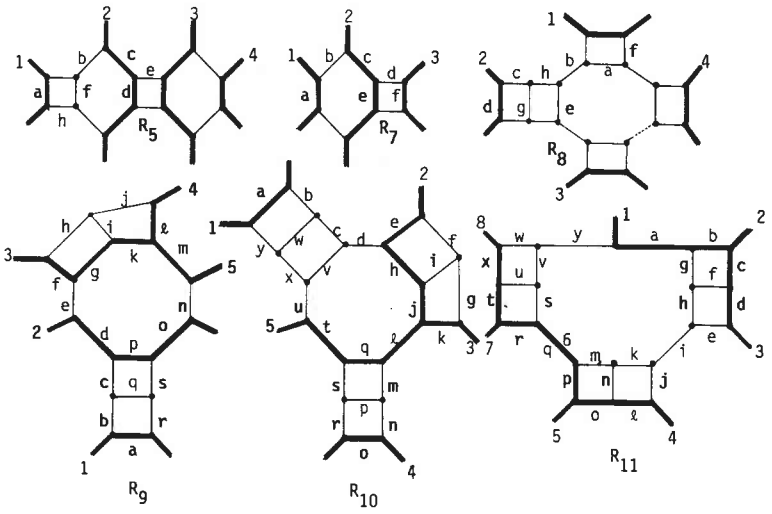

FIGURE 8

R_9: *apq* (+1 +5), *bcdmors* (+5), *ehkn* (−2 +5), *f* (+4 +5), *gjm* (+2 +5), *i* (−2 +4 or +2 −4), 1 (+3 +5).

R_{10}: *abcmnsrxy* (+4), *di* (+3 +4), *ehg* (+1 +5), *fjku* (+4 +5), *lopqtvw* (+1 +4).

R_{11}: *acdghilmprsxy* (+2 +3), *bkoqtv* (−2 +4), *wj* (+5 −8), *ef* (+6 −2), *n* (−5 +1), *u* (−7 +1).

In most cases we have two forced edges or a forced edge and an avoided edge on the same cycle, so H^* is sufficient. In some exceptional cases we have a choice, as in R_5, where we get *e* using −1 +4 or +1 −4. But having that choice means that either the edge to be forced can be chosen away from the center of a triple or the + and − edges both have a vertex in the subgraph R_2. The other exceptions are all in R_{11}, where no one of the forced edges can be the center edge of an R_2. Thus in every case H^* will suffice.

If we have a triple reduction based on one of these R_i, H^* still yields H^+. Figure 9 shows the three ways in which a Hamiltonian cycle can visit a 4-face around one of the R_i's in a graph G, and the corresponding coverage in G with a triple replacing that 4-face. Because the vertical edges of the 4-face must be covered by some Hamiltonian cycle, either Z_1 or Z_2 must exist in G. But since both the top and the bottom edge of the 4-face are covered by some cycle, either both Z_1 and Z_2 or one of them plus Z_3 must exist in G. In either case, all edges of the triple are in some Hamiltonian cycle. ∎

LEMMA 7. *If G is* C3C*BP*4 *and* $G(R_i)G'$, $i=5$, 6, *or* 8, *then* G' *is* C3*CBP. Furthermore, if* G' *is* H^* (H^+), *then G is* H^+ (*H*).

Proof. We have already shown that G' is C3CBP, and H^* yields H^+ in the cases $i=5$ and 8 in Lemma 6. We did not include R_6 in that lemma because there we were dealing with the class C3CBP4*, and if R_6 is extended to include a string of two 4-faces at its bottom right (where the *i* is in

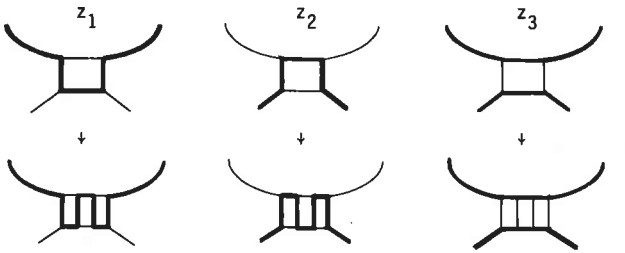

FIGURE 9

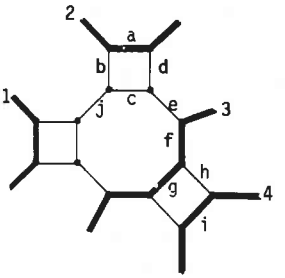

FIGURE 10

Fig. 10), then the resulting R_6 may not have the property that H^* implies H^+. That is not a problem here, since G is C3CBP4 and therefore has no adjacent 4-faces.

Note that $G(R_6)G'$ implies G' is C3CBP. G' is clearly cubic, planar, and bipartite. The arguments given in proving Lemma 6 suffice to show that G' has no multiple edges and is 3-connected. Now say G' is H^*. Figure 10 shows a labelling of R_6, and the following list tells how to find a Hamiltonian cycle for G through a given edge, given a Hamiltonian cycle for G' using and avoiding certain edges:

R_6: *aceh* $(+2\ +4)$, *bd* $(-2\ +4)$, *f* $(-3\ +4)$, *g* $(+3\ +4)$, *i* $(+1\ +2)$, *j* $(+2\ -3)$.

All of these combinations of $+$ and $-$ are assured by H^*. It remains to show that G' is H^+ implies G is H, for R_i, $i = 5$, 6, or 8. But it is easy to check that any Hamiltonian cycle for G' which uses one of the edges indicated with an asterisk in Fig. 2 extends to a Hamiltonian cycle for G. ∎

3. CUTS AND REDUCTIONS

We now begin our argument to show that every C3CBP with fewer than $(N^* + 22)$ vertices is H. Two types of argument are needed. A graph with a 3-cut or an essential 4-cut can be broken at that cut, and the Hamiltonian cycles of the pieces combined. A C3CBP4 must be reduced so that a Hamiltonian cycle of the smaller graph extends to a Hamiltonian cycle for the original graph.

We begin with graphs which have 3-cuts or essential 4-cuts. Suppose that in a cubic bipartite graph G the vertices of the two parts are white and black, and a cut separates G into G_1 and G_2. A simple edge-count implies that the numbers of white and black vertices in G_1 incident with the cut are

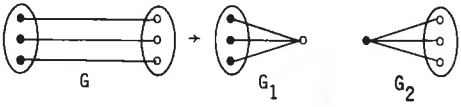

FIGURE 11

the same modulo 3. Thus in a 3-cut they must all be the same color and in a 4-cut two must be white and two black (see Figs. 11, 12, and 15).

THEOREM 1. *If G is the smallest non-Hamiltonian C3CBP then*

(a) *G has a 3-cut, and* $|G| \geq 2N^* + 2$, *or*

(b) *G has no 3-cut but has an essential 4-cut, and* $|G| \geq 2N^* - 4$, *or*

(c) *G is C3CBP4.*

Proof. (a) Separate G along the 3-cut, adding two new vertices to form G_1 and G_2, as in Fig. 11. Say $|G_1| \leq |G_2|$ and note that both G_1 and G_2 are H. If $|G_1| \leq N^*$, then a Hamiltonian cycle can be found for G_1 which uses the proper two edges to link up with a Hamiltonian cycle in G_2. Since G is not H, $|G_1| > N^*$, and we have $|G| = |G_1| + |G_2| - 2 \geq (N^* + 2) + (N^* + 2) - 2 = 2N^* + 2$.

(b) 4-cuts of a C3CBP can be of two types, depending on how the vertices of the cut are arranged in the plane. These two types are shown in Figs. 12 and 15.

Case (i). If G has an essential 4-cut as shown in Fig. 12 then we form graphs G_1 and G_2 as shown. We may suppose $|G_1| \leq |G_2| < |G|$, and note that G_1 and G_2 are H. Say G_2 has a Hamiltonian cycle using edge x and $|G_1| \leq N^*$. Then G_1 is H^*, and it will generally be possible to find a Hamiltonian cycle for G_1 which combines with that of G_2 to form a Hamiltonian cycle for G. To accommodate a cycle using edges 1 and 3, we select a cycle for G_1 using edge a and excluding edge e; for 1 and 4, we select a and exclude d; for 2 and 3, select b and exclude e. The only problems arise if an edge we want to force in G_1 is a central edge of a subgraph R_2 and so, by the definition of H^*, possibly unforceable. Figure 13 indicates the two cases which can arise.

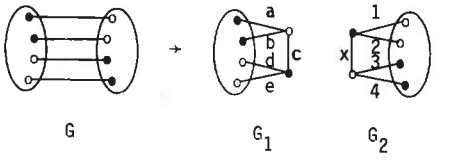

FIGURE 12

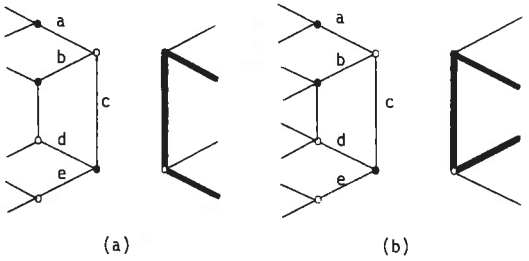

FIGURE 13

In Fig. 13a we want to force a Hamiltonian cycle in G_1 on edge b, avoiding edge d, or on edge e, avoiding edge a. But edges b and e cannot both be central edges of a unique subgraph R_2, so there is no problem.

Figure 13b shows the other possible case, requiring a Hamiltonian cycle in G_1 on edges b and d, avoiding edges a and e. But if b or d is a central edge of the R_2 subgraph, the other must be part of that subgraph, so there is no problem finding an appropriate cycle. Thus in every case with $|G_1| \leqslant N^*$, we find G is H. So we must have $|G| = |G_1| + |G_2| - 4 \geqslant (N^* + 2) + (N^* + 2) - 4 = 2N^*$.

Next we suppose that G_2 has no Hamiltonian cycle using edge x. Then replace G_1 and G_2 by G_1' and G_2' as shown in Fig. 14. Now if G_1' is H^*, every Hamiltonian cycle in G_2' can be extended to a Hamiltonian cycle in G by choosing a Hamiltonian cycle in G_1' using edge a and avoiding edge b. Since the face F above a is not a 4-face, there is no trouble with H^* here. Since G is not H, we must have

$$|G| = |G_1'| + |G_2'| - 8 \geqslant (N^* + 2) + (N^* + 2) - 8 = 2N^* - 4.$$

Case (ii). The essential 4-cut of G might not be as in Fig. 12, but rather as shown in Fig. 15. In that case, form graphs G_3 and G_4 as shown. We may suppose that $|G_3| \leqslant |G_4| < G$, so that both G_3 and G_4 are H. A Hamiltonian cycle in G_4 can traverse the added 4-face in two essentially different ways, one using two of the connecting edges, the other using all

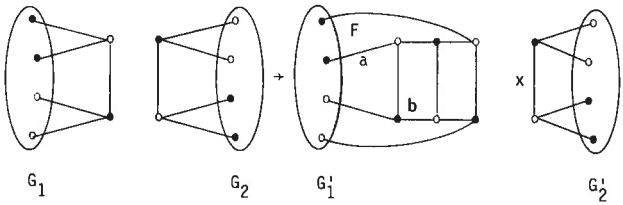

FIGURE 14

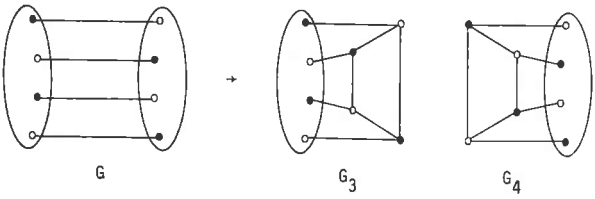

FIGURE 15

four. In either case, if G_3 is H^* a Hamiltonian cycle can be found for G. Again there is no trouble with H^*, because the edges to be forced and excluded are close together. Since G is not H we have $|G| = |G_3| + |G_4| - 8 \geq (N^* + 2) + (N^* + 2) - 8 = 2N^* - 4$. ∎

If G is C3CBP4, then we cannot use cuts but must rely on reductions. In fact we want to reduce in two stages, and we begin with the following theorem.

THEOREM 2. *Any C3CBP on at most $N^* + 8$ vertices is H^+. That is, $N^+ \geq N^* + 8$.*

Proof. If the given C3CBP contains any subgraphs R_3 or R_4, reduce by them, repeating as long as possible. By Lemma 3 the resulting graph is C3CBP and if it is H^+, then the original is also. So we may as well suppose that the graph G with which we begin contains no subgraphs R_3 or R_4. If G has a 3-cut, separate it into G_1 and G_2 as in Fig. 11. Since G contains no R_3 or R_4 neither G_1 nor G_2 can be one of the graphs of Fig. 1, so G_1 and G_2 each contain 16 or more vertices. Thus each contains at most $(N^* + 8) + 2 - 16 = N^* - 6$ vertices. By Lemma 2, G_1 and G_2 are both H^{+-}, so G is H^+.

If G is C3CBP4, then by Lemma 4 it contains a subgraph R_i, $i = 5$, 6, or 8. Reducing by that R_i we obtain G' with $|G'| \leq (N^* + 8) - 10 < N^*$. Thus G' is H^* and, by Lemma 7, G is H^+.

Finally, suppose G has no 3-cut but has an essential 4-cut. Say that 4-cut is as in Fig. 12. If the smaller side contains at least 12 vertices, then the larger side contains at most $(N^* + 8) + 4 - 12 = N^*$. So both sides are H^*, and G is H^+. The smaller side can contain fewer than 12 vertices only if it is the graph C_1 of Fig. 1. We are forced to examine that case only if all essential 4-cuts of G separate a pair of adjacent 4-faces from the rest of G.

Leaving that for a moment, consider the other possible type of 4-cut, shown in Fig. 15. If G has a 4-cut like that and the smaller side has at least 16 vertices, then the larger side contains at most $(N^* + 8) + 8 - 16 = N^*$ vertices. So both sides are H^*, and G is H^+. The smaller side will contain fewer than 16 vertices only if it is one of the graphs C_1, C_2, and C_3 of

Fig. 1. But C_1 cannot arise from an essential cut of this type, and C_3 would imply that G contains an R_4 subgraph, so the cut must begin with a triple of 4-faces to build C_2.

Thus with either kind of 4-cut we narrow the problem down from graphs with an essential 4-cut to those in which the only essential 4-cuts are non-major. But in that case Lemma 5 implies that we can reduce by R_7, $R_8(k)$, R_9, R_{10}, R_{11}, or a related triple reduction. Since R_7, the smallest of those reductions, removes 8 vertices, the reduced graph G' has at most $(N^*+8)-8=N^*$ vertices, and is H^*. By Lemma 6, G is H^+. ▮

THEOREM 3. *If G is the smallest non-Hamiltonian C3CBP and G is C3CBP4, $|G| \geqslant N^*+22$.*

Proof. Say $|G| \leqslant N^*+20$, and G is C3CBP4. We will prove G is H. Lemma 4 implies G contains an R_5, R_6, or $R_8(k)$, $k \geqslant 8$, each of which reduces by at least 12 vertices. If $G(R_i)G'$, $i=5$, 6, or 8, then $|G'| \leqslant (N^*+20)-12=N^*+8$, and G' is C3CBP by Lemma 7. Hence, by Theorem 2, G' is H^+ and, by Lemma 7, G is H. ▮

From Theorems 1 and 3 we have the following corollary.

COROLLARY 1. $N \geqslant \min\{N^*+20,\ 2N^*-6\}$.

4. COMPUTER GENERATION

In order to find a lower bound on N^* we have generated all C3CBPs with up to 40 vertices and those on 42 and 44 vertices without subgraphs R_2 or R_4. The method of generation was based on the following theorem.

THEOREM 4. *Let G be a C3CBP of order greater than 8. Then, for some C3CBP G' we have either $G(R_0)G'$ or $G(R_4)G'$.*

Proof. If G is cyclically 4-edge connected, then Lemma 1 of [9] shows that at least one of the two possible applications of reduction R_0 to any 4-face produces a C3CBP.

If G has a 3-cut, form G_1 and G_2 as in Fig. 11. If we choose the 3-cut so that $|G_1|$ is minimized, we ensure that G_1 is cyclically 4-edge connected. If G_1 is the graph C_1 of Fig. 1 then $G(R_4)G'$ and G' is a C3CBP. If not, we can apply at least one of the two possible applications of reduction R_0 to any 4-face in G which is also in G_1. Such a 4-face must exist, since G_1 has at least six 4-faces. ▮

Theorem 4 tells us that we can generate all C3CBPs by starting with C_1 of Fig. 1 and applying the reverses of reductions R_0 and R_4. In Table 1,

TABLE I

Counts of Nonisomorphic C3CBPs

n	$c_1(n)$	$c_2(n)$	$c_3(n)$
8	1	0	0
12	1	0	0
14	1	0	0
16	2	0	0
18	2	1	0
20	8	1	0
22	8	1	0
24	32	4	1
26	57	5	0
28	185	14	0
30	466	28	1
32	1543	86	3
34	4583	211	1
36	15374	648	5
38	50116	1878	4
40	171168	5941	11
42	?	18326	20
44	?	58746	46

$c_1(n)$ is the number of non-isomorphic C3CBPs with n vertices, $c_2(n)$ is the number of those without subgraphs R_2 or R_4, and $c_3(n)$ is the number of those without subgraphs R_1 or R_4. We believe that these classes of graphs have not been enumerated before, although Tutte [18] has enumerated labelled C3CBPs.

For each of the graphs generated in producing Table 1, the following properties were verified:

(i) If any two edges are chosen, there is a Hamiltonian cycle through one, avoiding the other (property H^{+-}).

(ii) If any three independent edges on the same face are chosen, there is a Hamiltonian cycle through all of them. This is not true for four edges (the smallest counterexample, Fig. 16a, is on 32 vertices) or if the edges are not required to be on the same face (e.g., any 3-cut). There is also a cyclically 4-edge connected counterexample on 16 vertices, Fig. 16b.

(iii) If any two edges are chosen which are an even distance apart on the same face, there is a Hamiltonian cycle which avoids both. This is not true for an odd distance apart. For a counterexample on 12 vertices see Fig. 16c.

(iv) If a maximum independent set of edges on any face is chosen, a Hamiltonian cycle can be found using all of them. The same set of edges

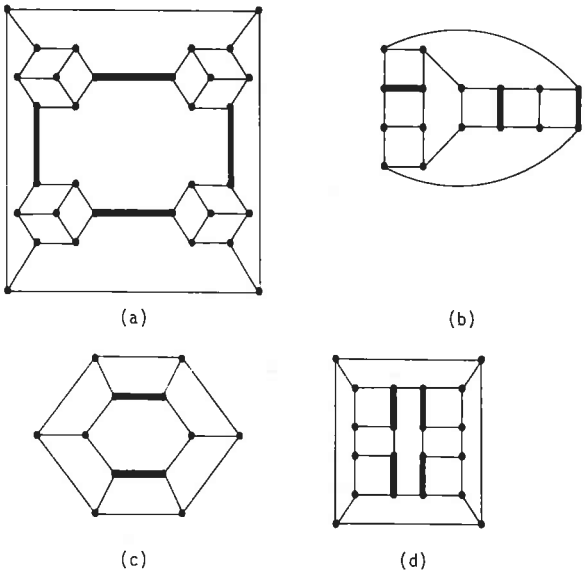

FIGURE 16

cannot necessarily be all avoided. For a counterexample on 20 vertices see Fig. 16d.

Production of all the cycles needed to verify (i)–(iv) proved to be a difficult computational problem, which was solved by finding a new algorithm [10].

THEOREM 5. *If G is a cubic 3-connected bipartite planar graph on n vertices then*

(a) $n \leqslant 64$ *implies G is Hamiltonian (i.e., $N \geqslant 64$);*

(b) $n \leqslant 52$ *implies every edge of G lies on some Hamiltonian cycle (i.e., $N^+ \geqslant 52$);*

(c) $n \leqslant 44$ *implies that for any two edges e and f of G, there is a Hamiltonian cycle through e avoiding f, except possibly if e is a central edge of a unique subgraph R_2 and f has no vertex in that subgraph R_2 (i.e., $N^* \geqslant 44$);*

(d) $n \leqslant 40$ *implies that for any two edges e and f of G, there is a Hamiltonian cycle through e avoiding f (i.e., $N^{+-} \geqslant 40$).*

Proof. The computational results in Table 1 show that $N^{+-} \geqslant 40$ and, by virtue of Lemma 2, $N^* \geqslant 44$. The other two bounds then follow from Theorem 2 and Corollary 1 of Theorem 3. ∎

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